On Vassiliev Invariants

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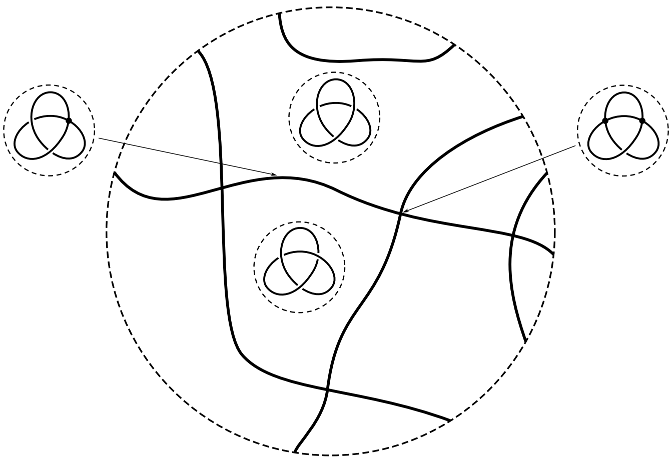
Contents

Acknowledgements

Thanks to ...

Introduction

THE space of knots is the disconnected space of embeddings of $\mathbb{S}^1$ into $\mathbb{R}^3$, in which the connected components, the "rooms", are the knot types. Vassiliev studied the space of knots by looking at its "walls" of a specific type: those immersions of $\mathbb{S}^1$ into $\mathbb{R}^3$ which fail to be embeddings by having a single point of trasverse intersection. The addition of these walls makes the space of objects we study a connected space. Any two knots are connected by a path that passes through finitely many of the walls. The space of walls is also disconnected, but it can be connected by allowing paths to pass through finitely many "cornices" where two walls meet. These are immersions that fail to be embeddings by having two points of transverse intersection. The cornices are again disconnected but can be connected by "corners" and so on, with any immersions with m double points being connected by a finite number of immersions with $m + 1$. This produces the "stratification" of the space: a very schematic illustration is given below. Note that this picture doesn't properly capture the infinite-dimensional nature of the stratification. Nor some other missing details which will appear in Chapter 1.



In [Vas90], Vassiliev makes a sequence of approximations of the cohomology ring of the space of knots, yielding a certain subring of the zeroth cohomology ring. Elements of this are locally constant functions on the space, so knot invariants. This subring of "Vassiliev" knot invariants can be computed on any knot by some procedure involving the homology group of the strata at a finite number of increasing depths.

Birman and Lin in [BL93] give an axiomatic definition of the Vassiliev invariants as those that respect the Vassiliev skein relation. It follows, as we will see in Chapter 1 that Vassiliev invariants can be described completely combinatorially by functions on chord diagrams obeying certain combinatorial rules.

By the work of Bar-Natan in [Bar95] the algebra of chord diagrams turns out to be equivalent to a different diagrammatic algebra, that of Jacobi diagrams, again up to a different set of combinatorial rules. This will be discussed in Chapter 2. This change of perspective introduces Lie theory in the following sense. A key relation in the algebra of Jacobi diagrams is a formal version of the relation in the universal enveloping algebra of a Lie algebra that the bracket is equal to the commutator. This is discussed in Chapter 3. A rigorous version of this statement is the work of Hinnich and Vaintrob [HV00] which constructs the algebra of Jacobi diagrams as the universal enveloping algebra object of some Lie algebra object in some tensor category.

A paragraph introducing welded knots: If knots have to do with the configuration space of some number of points, then welded knots have to do with the configuration space of some number of 'flying rings'. Arrowed Jacobi diagrams will need to be introduced. Kashiwara-Vergne may need to be mentioned.

In Chapter 5, we generalise the result that Jacobi diagrams are a universal enveloping algebra to directed(/welded/arrowed) Jacobi diagrams.

Some paragraphs talking about the rest of the Chapters.

1

Vassiliev invariants and chord diagrams

Vassiliev invariants are sophisticated to define in terms of the space of knots from the introduction, but the axiomatic definition of Birman-Lin [BL93] is much simpler. The definition also illustrates an analogy first made by Bar-Natan [Bar95] in which Vassiliev invariants are "polynomial invariants". This is not meant in the sense that Vassiliev invariants take values in a polynomial ring (like say, the Jones polynomial), but rather that Vassiliev invariants have special properties not shared by all invariants, just as polynomial functions have special properties not shared by all functions.

1.1 Singular Knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are double-points and transverse. When a singular knot has m such singularities, we call it m**-singular**.

Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular immersions with

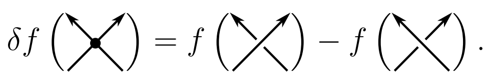
(a) triple points

(b) points with vanishing derivative

are excluded from the definition.

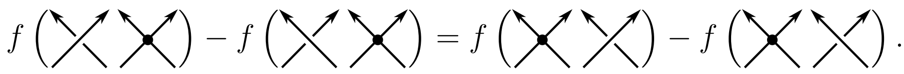
A singular knot with one double point is very close to two other knots, one where it's replaced by a positive crossing and one by a negative. If the conditions are right, we can extend a knot invariant to an invariant of singular knots by "taking its derivative".

Definition 1.1.3 The **derivative** δ of a differentiable m-singular knot invariant f is



What are the conditions? For this to be a well-defined operation, it mustn't matter which double point we choose.

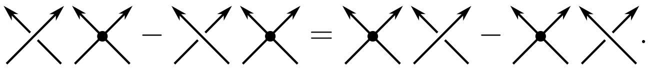
Definition 1.1.4 An invariant f of m-singular knots is **differentiable** if


(DIFF)

If an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points.

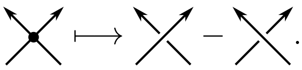
Rather than thinking about functions on knots satisfying certain relations, the modern version of this subject takes the philosophy of imposing relations on the objects directly.

Definition 1.1.5 Define $\mathcal{K}_m = \operatorname{span}(\{m\text{-singular knots}\})$ modulo the following boundary relation (also known as a codifferentiability relation):


(DIFF*)

From now on, we will refer to elements $\mathcal{K}_m$ as m**-singular knots**, i.e. the DIFF* relation will be implicit in everything.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m \to \mathcal{K}_{m-1}$ defined by

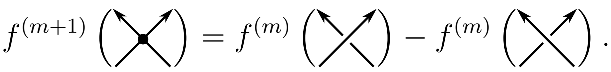


Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as a invariant of knots in $\mathcal{K}_m$.

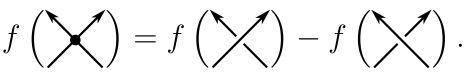
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

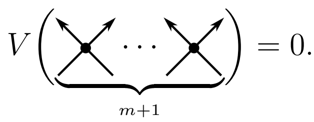


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation, there is an integer m such that



(b) The **order** of a Vassiliev invariant V it the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the highest number of double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

There are many other similar remarks to be made about the analogy between Vassiliev invariants and polynomials. To help see the bird's eye view, and following [Hut98], we phrase this in terms of an integration theory.

Definition 1.1.10 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group is $\mathcal{O}_0$ is typically free abelian, and in our case is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behaves like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

We wish to understand how to invert this process, namely:

Questions 1.1.11 (a) When does a functional in $\mathcal{O}_m^*$ "integrate" into a functional in $\mathcal{O}_{m-1}^*$?

(b) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(c) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

(d) Which functions on the non-singular objects are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The following abelian groups provide the tautological answers to the above questions in the general theory.

Definitions 1.1.12 (a) The **primary obstructions to integration** are the group

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **secondary obstructions to integration** are the group

$$S\mathcal{O}_m = \ker\,(\partial_{m+1}\partial_m)/P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.

(c) The **constants of integration** are the group

$$C\mathcal{O}_m = \mathcal{O}_m/\partial\mathcal{O}_{m+1}.$$

(d) The **finite type invariants** of order m are the group

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

The aim of the rest of this chapter will be to develop a geometric answer to the above questions in the case $\mathcal{O}^\star = \mathcal{K}^\star$ through a geometric understanding of the above definitions in that case.

1.2 Chord diagrams

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.1 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the mth and $(m+1)$st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

To rephrase the definitions in section 1.1, in the integration theory $\mathcal{O}^\star = \mathcal{K}^\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

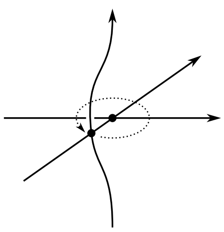
Due to the boundary relation this is always well-defined.

Integration, is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

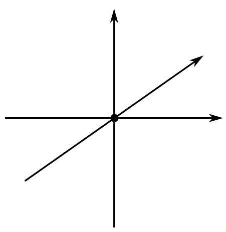
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a "path-integral" along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving "holes" in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

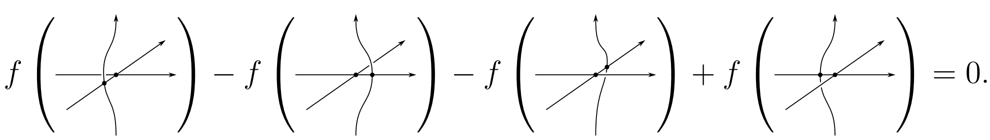
Definitions 1.2.2 (a) The following singular isotopy around a triple point:



around



gives rise to the **topological four-term relation**



(T4T)

(b) The singular isotopy around a singular point with vanishing derivative

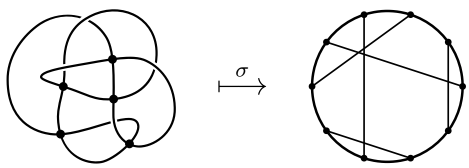
The constants of integration are the information that is lost when differentiating, or the information to be chosen when integrating. In the case of knots, $C\mathcal{K}_m = \mathcal{K}_m/\partial\mathcal{K}_{m+1}$, so m-singular knots modulo (the image of) $(m+1)$-singular knots. Since the objects already have m singular points, the quotient asserts that additional crossing changes are free. For example, the following 2-singular knots differ by the image of a 3-singular knot, so are equivalent in $C\mathcal{K}_2$:

Thus, elements of $C\mathcal{K}_m$ are described only by the combinatorial data of the cyclic order in which the singular points are traversed on the knot.

Definitions 1.2.3 (a) A **chord diagram** of order m is an element of $\mathcal{K}_m/\partial\mathcal{K}_{m+1}$, or equivalently an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. Chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$.

(b) The **chord diagram of an m-singular knot** K, denoted $\sigma(K)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of K. Traversing both K and the circle, label the points on the circle in the order in which the singular points of K are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(K)$.

Example 1.2.4



Given a functional

1.3 Third subsection

2

Lie theory and Jacobi diagrams

2.1 First subsection

2.2 Second section

2.3 Third subsection

3

Jacobi diagrams as a universal enveloping algebra

3.1 First subsection

3.2 Second section

3.3 Third subsection

4

Welded knots and arrow diagrams

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Arrowed Jacobi diagrams as a universal enveloping algebra

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Expansions and associators

6.1 First subsection

6.2 Second section

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7

Emergent knotting

7.1 First subsection

7.2 Second section

7.3 Third subsection

8

Emergent welded associators

8.1 First subsection

8.2 Second section

8.3 Third subsection

References

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