On Vassiliev Invariants

Damian Lin

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Contents

Acknowledgements

Thanks to ...

Introduction

The introduction goes here.

1

Vassiliev invariants and chord diagrams

Something to maybe include somewhere:

The point of Vassiliev theory is to study the space of knots in the context of the singularities that lie between knots.

In view of this point, we spend Section 1.1 looking at a the stratification of the space of knots. This leads to a beautiful (and fruitful) classical analogy which we will explored in Section 1.2 and throughout this chapter.

In Section 1.3, with the context in mind, we introduce the main players in this theory.

Vassiliev invariants are sophisticated to define in terms of the space of knots from the introduction, but the axiomatic definition of Birman-Lin [BL93] is much simpler. The definition also illustrates an analogy made by Bar-Natan [Bar95] in which Vassiliev invariants are "polynomial invariants". This is not meant in the sense that Vassiliev invariants take values in a polynomial ring (like say, the Jones polynomial), but rather that Vassiliev invariants have special properties not shared by all invariants, just as polynomial functions have special properties not shared by all functions.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m**-singular**.

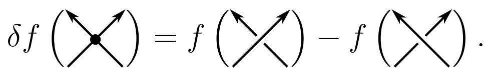
Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular immersions with

(a) triple points
(b) points with vanishing derivative

are excluded from the definition.

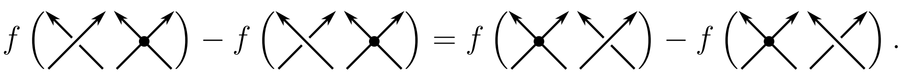
A singular knot with one double point is very close to two other knots. In one, the double point is replaced by a positive crosing, and in the other a negative crossing. If the conditions are right, we can extend a knot invariant to an invariant of singular knots by a procedure analagous to taking its derivative.

Definition 1.1.3 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



What are the conditions? For this to be a well-defined operation, it mustn't matter which double point we choose.

Definition 1.1.4 An invariant f of m-singular knots is **differentiable** if

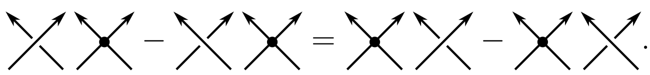


(DIFF)

If an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

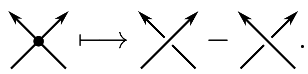
Definition 1.1.5 Define $\mathcal{K}_m^\bullet$ as the span of all m-singular knots, taken over $\mathbb{Q}$, modulo the following boundary relation (also known as a codifferentiability relation):



(DIFF*)

From now on, we will refer to elements $\mathcal{K}_m^\bullet$ as m**-singular knots**, i.e. the DIFF* relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m^\bullet \to \mathcal{K}_{m-1}^\bullet$ defined by

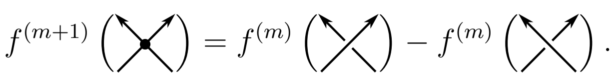


Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as an invariant of knots in $\mathcal{K}_m^\bullet$.

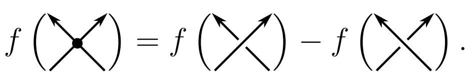
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

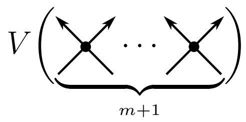


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation,



$= 0.$

(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the most double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 In other words, Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

1.2 The stratification of the space of knots and integration

To help see the bird's eye view, and following [Hut98], we phrase this in terms of an integration theory.

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group $\mathcal{O}_0$ is typically free abelian, and in our case this is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behave like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

We wish to understand how to invert this process, namely:

Questions 1.2.2 (a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?

(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

(c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?

(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The following modules provide the tautological answers to the above questions in the general theory.

Definitions 1.2.3 (a) The **primary obstructions to integration** are the module

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the module

$$C\mathcal{O}_m = \mathcal{O}_m/\partial\mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the module

$$S\mathcal{O}_m = \ker(\partial_{m+1}\partial_m)/P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.

(d) The **weights of integration** are the module

$$W\mathcal{O}_m = C\mathcal{O}_m/\pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the projection.

(e) The **finite type invariants** of order m are the module

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$ which provides good intuition for the general case.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the m-th and $(m+1)$-st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

To rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

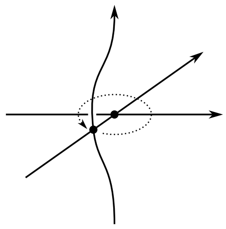
Due to the boundary relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

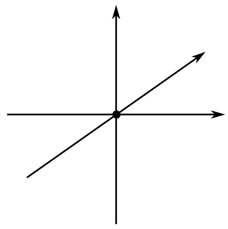
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a "path-integral" along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving "holes" in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

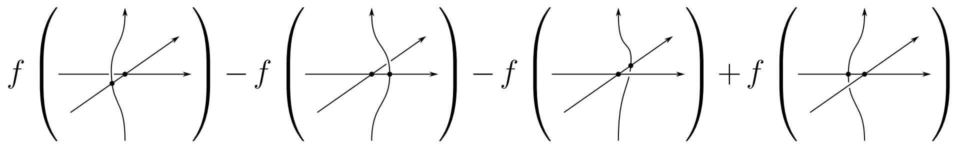
Definitions 1.2.5 (a) The following closed singular isotopy around a triple point:



around

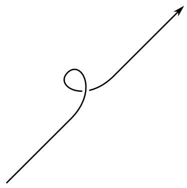


gives rise to the **topological four-term relation**



$= 0.$ (T4T*)

(b) The closed singular isotopy



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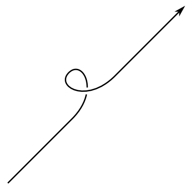
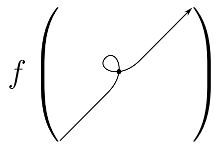
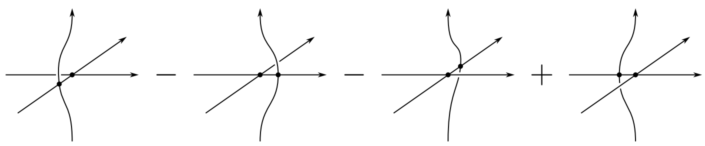


around a point with a vanishing derivative gives rise to the **topological one-term relation**

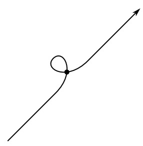


$= 0.$ (T1T*)

The necessity of the above relations for an invariant of m-singular knots to integrate is equivalent to the assertion that



and



are in $P\mathcal{K}_m^\bullet = \ker \partial_m$. Let us denote arbitrary such singular knots $\mathbf{T4T}$ and $\mathbf{T1T}$.

This raises the question of whether we have found all primary obstructions. Do $\mathbf{T4T}$ and $\mathbf{T1T}$ span $\ker \partial$?

Theorem 1.2.6 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $m-1$-singular knots if and only if it satisfies* T4T* *and* T1T*.

Proof (sketch) Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m-1$ double points, and additional worse singularities. The only events to consider are codimension $m+2$ (why $m+2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with `DIFF` relations, which we've said are implicit).

Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

In the order of Questions 1.2.2 and Definitions 1.2.3, we ought to talk next about the secondary and further obstructions to integration. However, let us postpone this until after we have discussed constants of integration, which will be a more natural place.

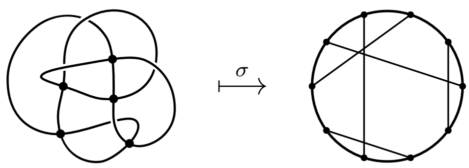
The constants of integration comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost. When integrating, values on these can be chosen freely.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}^\bullet_m = \mathcal{K}^\bullet_m / \partial\mathcal{K}^\bullet_{m+1}$. The difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. Thus, we see that constants of integration are singular knots which are blind to crossing changes, so the only data needed to describe them is the order in which the double points are traversed around the knot.

Definitions 1.2.7 (a) A **chord diagram** of order m is an element of $\mathcal{K}^\bullet_m / \partial\mathcal{K}^\bullet_{m+1}$. This is equivalent to an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}^\bullet_m$.

(b) The **chord diagram of an m-singular knot** k, denoted $\sigma(k)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of k. Traversing both k and the circle, label the points on the circle in the order in which the singular points of k are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(k)$.

Example 1.2.8



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Proposition 1.2.9 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}^*_m$ Then Q_0 is also an integral of P.

Proof The derivative of a function of chord diagrams is zero, as two knots which differ by a crossing change have the same chord diagram. Hence Q and Q_0 have the same derivative: P. $\square$

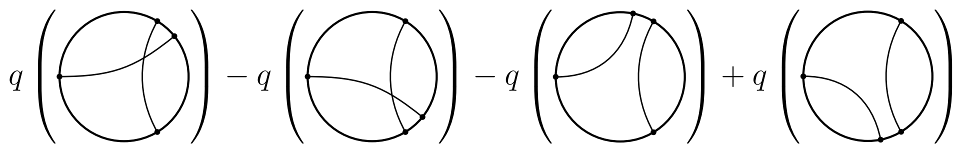
Remark 1.2.10 We defined a constant of integration as a set of objects (e.g. $c \in \mathcal{D}_m$) but perhaps it would have been more accurate to define it as a functional (e.g. $q \in \mathcal{D}^*_m$). After all the constant of integration on the real line is "$+\,C$" rather than the set $\{*\}$ with one object. There is no real risk of confusion, so let us be slightly loose and use the terminology for either.

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.11 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

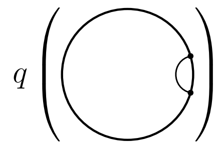
If we take the relations for a knot invariant to be integrable and project them into the space of chord diagrams, we get the following relations.

Definitions 1.2.12 (a) The **four-term relation** is the relation



$= 0.$ (4T*)

(b) The **one-term relation** is the relation



$= 0.$ (1T*)

Just like T4T* and T1T*, 4T* and 1T* are not individual relations, but kinds of relations. There's one 4T* relation for all ways of placing other chords in-between the three far-apart chord ends, in all of the four diagrams. There's a 1T* relation for all ways of placing chord diagrams between either of the two far-apart chord ends. In other words, any chord with an 'isolated chord' that doesn't cross any other chords in the diagram is killed by a 1T* relation.

Proposition 1.2.13 *A weight system is characterised as a constant of integration that satisfies* 4T* *and* 1T*.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T* and 1T*. Since crossing changes are free, this is equivalent to satisfying the projection of T4T* and T1T* into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of on m-singular P is of the form

$$Q + q \circ \sigma.$$

Since integration is linear, to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that $S\mathcal{K}_m^\bullet$ vanishes. But this is a tautological statement of the general theory — it doesn't mean much if we don't know what $S\mathcal{K}_m^\bullet$ is.

Conjecture 1.2.14 *An invariant of m-singular knots satisfying* T4T* *and* T1T* *integrates m times into a genuine knot invariant.*

Remarks 1.2.15 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.6. The point is that it may not be possible to choose the integral to again satisfy T4T* and T1T*, which is what $S\mathcal{K}^\bullet$ measures.

(b) Computing $S\mathcal{K}_m^\bullet$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.

(c) This conjecture is proven in certain cases. It holds the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98].

The finite type invariants in $\mathcal{K}^\bullet_\star$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m+1$.

If we restrict Conjecture 1.2.14 to Vassiliev invariants, then we get the following.

Theorem 1.2.16 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* T4T*, T1T* *and the additional condition that $\delta^k v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant).*

There are various proofs of the fundamental theorem. They are listed in [BS97], and each proof is accompanied by a series of moral objections. To quote their introduction: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made". To summarise their philosophy:

Remark 1.2.17 We have the implication Conjecture 1.2.14 $\Longrightarrow$ Theorem 1.2.16, and this is actually realised in the theory of braids. It is mysterious that the fate of the slightly stronger conjecture which comes from taking the natural topological approach to the fundamental theorem still remains unknown, and that there are grievances to be had with all known proofs.

In Section 1.5 we will look at equivalent formulation of the fundamental theorem.

1.3 Knots and Vassiliev invariants

Speaking broadly, the aim of Vassiliev theory is to study the space of knots via the space of chord diagrams, using information from the stratification of knots introduced in the first two sections. But this space is not just a vector space; it has some further structure which we wish to incorporate. In this section, we synthesise all of this information into one algebraic structure.

Definitions 1.3.1 (a) The **space of knots**, denoted $\mathcal{K}$ is the vector space spanned (over $\mathbb{Q}$) by non-singular knots. Equivalently, $\mathcal{K} = \mathcal{K}^\bullet_0$.

(b) The space of knots is equipped with the **singular knot filtration**

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

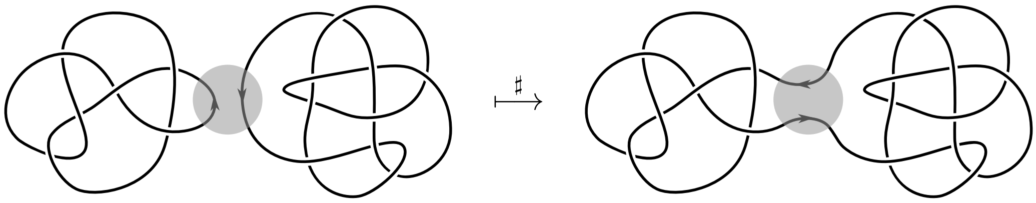
where the ith filtered component $\mathcal{K}_i$ is the span of resolutions of singular knots with i double points, equivalently the image of δ^i; $\mathcal{K}_i = \delta^i(\mathcal{K}^\bullet_i)$.

Proposition 1.3.2 *The singular knot filtration is indeed a descending filtration of vector spaces.*

Proof A filtration of vector spaces is uninteresting. The only thing to check is that if $i < j$, then $\mathcal{K}_i \supset \mathcal{K}_j$. If $k \in \mathcal{K}_j$, then $k = \delta^j(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}_j$. But then $k = \delta^j(k^\bullet) = \delta^i\delta^{j-i}(k^\bullet)$, so $k \in \delta^i(\mathcal{K}^\bullet_i)$. $\square$

The algebraic structure on knots comes from the following operation.

Definition 1.3.3 The **connected sum** of two knots k_1 and k_2 is the knot obtained by removing a small arc from each of k_1 and k_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

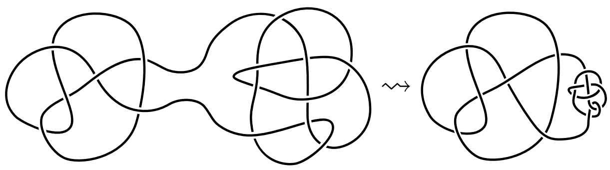


This definition is extended bilinearly to $\mathcal{K}$, i.e. linear combinations.

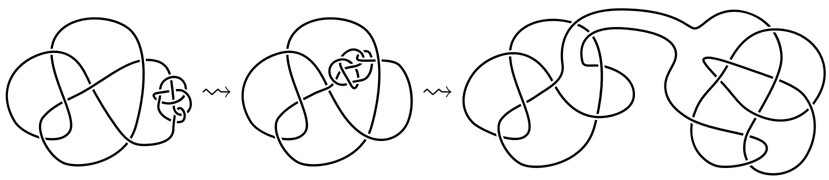
This is not a-priori well-defined, as we have not specified where along either k_1 or k_2 the small arc is to be removed. However, by a classical knot-theoretic argument, the result is independent of the choice.

Proposition 1.3.4 *The connected sum $\sharp \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ forms a well-defined operation. It does not matter where along either knot the small arc was removed, the results are ambient-isotopic.*

Proof We exhibit an ambient isotopy starting at $k_1 \sharp k_2$ where the small arc is removed from k_1 as in the example above. The part of the connected sum coming from k_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of k_1.



Then, k_2 is then isotoped along k_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along k_1, and the same argument with the roles of k_1 and k_2 reversed completes the proof. □

Proposition 1.3.5 *The connected sum respects the descending filtration,*

$$K_i \sharp K_j \subset K_{i+j}.$$

That is, the connected sum makes $(\mathcal{K}, \sharp)$ into an ascending filtered algebra.

Proof Indeed, the connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. The question is whether the connected sum respects the filtration.

If $k \otimes \ell \in \mathcal{K}_i \otimes \mathcal{K}_j$, then there are $k_\bullet$ and $\ell_\bullet$ in $\mathcal{K}_i^\bullet$ and $\mathcal{K}_j^\bullet$ that resolve to k and ℓ, respectively. Similarly, the 'connected sum' $k_\bullet \sharp \ell_\bullet$ resolves by δ^{i+j} to $k \sharp \ell$, which is therefore in $\delta^{i+j}(\mathcal{K}_{i+j})$.

Here, 'connected sum' is enclosed in inverted commas due to the following technicality. Connected sums of singular knots with singular knots were not part of Definition 1.3.3. Even if we ensure that the small arcs removed from a singular knot do not contain a singular point, still, this is ill-defined. The ambiguity is that the resulting singular knot may depend on from which side of the singular point the arc was removed i.e. the repositioning argument in the proof of Proposition 1.3.4 fails due to the presence of singular points. After taking the resolution under δ^{i+j} however, the repositioning argument now works again, so any of the choices of connected sum in $k_\bullet \sharp \ell_\bullet$ produce a singular knot which resolves to $k \sharp \ell$. □

Say something intelligent to introduce this definition.

I guess it's something like: We assert that knots are (semi)group-like. This must be somehow related to (some?) chord diagrams being like exponentials (primitives?).

Definition 1.3.6 The coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is defined on knots k as

$$\Delta(k) = k \otimes k$$

and extended bilinearly to $\mathcal{K}$.

Proposition 1.3.7 *The connected sum and coproduct are compatible. That is,* $(\mathcal{K}, \sharp, \Delta)$ *forms a bialgebra.*

Proof A bialgeba is a coalgebra that is also an algebra with compatible product and coproduct. That $(\mathcal{K}, \Delta)$ forms a coalgebra is trivial (the counit is the augmentation map denoted ε). And we have already seen that $(\mathcal{K}, \sharp)$ is an algebra. So it remains only to check the compatibility conditions. We check that the product and coproduct are compatible:

$$\begin{aligned} \Delta(k \sharp \ell) &= k \sharp \ell \otimes k \sharp \ell \\ &= (k \otimes k) \sharp^{\otimes 2} (\ell \otimes \ell) \\ &= \Delta(k) \sharp^{\otimes 2} \Delta(\ell) \end{aligned}$$

where $\sharp^{\otimes 2}$ denotes the component-wise tensor product on $\mathcal{K} \otimes \mathcal{K}$.

Checking the unit and counit is trivial. □

So far, $\mathcal{K}$ is a bialgebra whose product respects the filtration. The same is true of the coproduct.

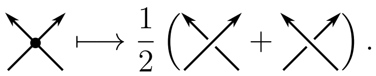
Proposition 1.3.8 *The coproduct* Δ *also respects the filtration,*

$$\Delta(\mathcal{K}_j) \subset \sum_{i=0}^{j} \mathcal{K}_i \otimes \mathcal{K}_{j-i}.$$

That is, with the singular knot filtration, $\mathcal{K}$ *is a filtered bialgebra.*

We give a proof, due to Willerton [Wil96] which follows directly from his Lemma 1.3.10. The lemma is a formula for the coproduct of an element of $k \in \mathcal{K}_n$ that comes from some $k^\bullet \in \mathcal{K}_m^\bullet$. The formula is terms of the 2^n ways of resolving some of singular points in one cofactor and the rest in the other, but first we need some notation.

If I is a subset of the singular points of a singular knot, let δ^I be the operator that resolves the singular points I. Let μ^I be the operator that averages singular points in I, where averaging a singular point is sending



Remark 1.3.9 There is one technicality. To make the above definitions of μ^I and δ^I, we must forget the DIFF* relation on $\mathcal{K}_m^\bullet$ (or work in the appropriate lift) as we wish to resolve with respect to specific double points. This provides no mathematical difficulty, so we chose not to do the reader the disservice of altering the notation. But for Lemma 1.3.10, its proof and the proof of Proposition 1.3.8, let $\mathcal{K}_m^\bullet$ not contain the quotient by the DIFF* relation.

Lemma 1.3.10 (Willerton) *Suppose $k^\bullet \in \mathcal{K}_m^\bullet$, and let S denote the set of singular points of $k^\bullet$.*

$$\Delta(\delta^S(k^\bullet)) = \sum_{I \subset S} \mu^{\bar{I}} \delta^I(k^\bullet) \otimes \mu^I \delta^{\bar{I}}(k^\bullet)$$

where $\bar{I} = S \setminus I$.

Proof We proceed by induction on m. In the base case of $m = 0$, $S = \emptyset$, and $k^\bullet = k$ is a genuine knot, so

$$\begin{aligned}
\Delta(\delta^0(k)) &= \Delta(k) \\
&= k \otimes k \\
&= \sum_{I \subset \emptyset} \mu^{\bar{I}} \delta^I(k) \otimes \mu^I \delta^{\bar{I}}(k).
\end{aligned}$$

The inductive step is as follows. Let $k^\bullet \in \mathcal{K}_{m+1}^\bullet$. Let J denote all singular points of $k^\bullet$, and $x \in J$ denote a specific singular point. Furthermore, let $k^{\bullet +}$ (resp. $k^{\bullet -}$) denote the m-singular knots obtained from $k^\bullet$ when x is replaced by a positive (resp. negative) crossing, so that $\delta^{\{x\}}(k^\bullet) = k^{\bullet +} - k^{\bullet -}$. We examine

$$\sum_{I \subset J} \mu^{\bar{I}} \delta^I(k^\bullet) \otimes \mu^I \delta^{\bar{I}}(k^\bullet).$$

Decomposing the sum based on whether $x \in I$ yields

$$\sum_{x \in I \subset J} \mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^\bullet) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^\bullet) + \sum_{x \notin I \subset J} \mu^{\bar{I} \setminus \{x\}} \delta^I(k^\bullet) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^\bullet),$$

then resolving either δ or μ on x,

$$\begin{aligned}
&\frac{1}{2} \sum_{x \in I \subset J} \mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^{\bullet +} - k^{\bullet -}) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^{\bullet +} + k^{\bullet -}) \\
+&\frac{1}{2} \sum_{x \notin I \subset J} \mu^{\bar{I} \setminus \{x\}} \delta^I(k^{\bullet +} + k^{\bullet -}) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^{\bullet +} - k^{\bullet -}).
\end{aligned}$$

Expanding, yields the cumbersome,

$$\begin{aligned}
&\frac{1}{2} \sum_{x \in I \subset J} \Big(\mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^{\bullet +}) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^{\bullet +}) + \mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^{\bullet +}) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^{\bullet -}) \\
&\qquad - \mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^{\bullet -}) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \delta^{I \setminus \{x\}}(k^{\bullet -}) \otimes \mu^{I \setminus \{x\}} \delta^{\bar{I}}(k^{\bullet -}) \Big) \\
+&\frac{1}{2} \sum_{x \notin I \subset J} \Big(\mu^{\bar{I} \setminus \{x\}} \delta^I(k^{\bullet +}) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^{\bullet +}) - \mu^{\bar{I} \setminus \{x\}} \delta^I(k^{\bullet +}) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^{\bullet -}) \\
&\qquad + \mu^{\bar{I} \setminus \{x\}} \delta^I(k^{\bullet -}) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^{\bullet +}) - \mu^{\bar{I} \setminus \{x\}} \delta^I(k^{\bullet -}) \otimes \mu^I \delta^{\bar{I} \setminus \{x\}}(k^{\bullet -}) \Big)
\end{aligned}$$

but since $\{I \mid I \subset J, x \notin I\}$ is equal to $\{I \smallsetminus \{x\} \mid I \subset J, x \in I\}$, in each of the above sums, the corresponding terms have the same indices. Hence, the first and last terms in each sum combine, and the second and third terms cancel out to give

$$\sum_{x \in I \subset J} \mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}(k^{\bullet +}) \otimes \mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}(k^{\bullet -}) \otimes \mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}(k^{\bullet -}).$$

Since neither $\mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}$ or $\mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}$ are with respect to x, this can be written

$$\sum_{I \subset J \smallsetminus \{x\}} \mu^{\bar{I}} \delta^{I}(k^{\bullet +}) \otimes \mu^{I} \delta^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \delta^{I}(k^{\bullet -}) \otimes \mu^{I} \delta^{\bar{I}}(k^{\bullet -})$$

which by the inductive hypothesis is

$$\begin{aligned} \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet +})) - \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet -})) &= \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet +}) - \delta^{J \smallsetminus \{x\}}(k^{\bullet -})) \\ &= \Delta(\delta^{J}(k^{\bullet})). \end{aligned}$$ □

Proof of Proposition 1.3.8 The operators δ^I and $\mu^{\bar{I}}$ commute since they are evaluating different singular points. Let I be an arbitrary subset of S, and let $|I| = i$ and $|S| = j$, then the left cofactor is in $\mathcal{K}_i$ and the right in $\mathcal{K}_{j-i}$. □

Not all knot invariants respect the singular knot filtration, as we will see. The point of the Vassiliev invariants is that they're the ones that are natural with respect to the singular knot filtration. Indeed, the Vassiliev invariants are obtained naturally from $\mathcal{K}$ via the following construction.

The dual filtered bialgebra construction makes a filtered bialgebra whose mth filtered component is the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$. The result is the Vassiliev invariants, along with a product transpose to the coproduct in $\mathcal{K}$ and coproduct transpose to the product in $\mathcal{K}$.

Definition 1.3.11 The **filtered bialgebra of Vassiliev invariants**, denoted $\mathcal{V}$, is the vector space of Vassiliev invariants with an ascending filtration by degree

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots, \qquad \mathcal{V} = \bigcup_{m=0}^{\infty} \mathcal{V}_m.$$

The product is given by pointwise multiplication

$$V_1 \cdot V_2(k) = V_1(k) V_2(k),$$

and the coproduct, η, is given by

$$\eta(V)(k_1 \otimes k_2) = V(k_1 \sharp k_2).$$

Proposition 1.3.12 *The filtered bialgebraic dual of the descending filtered bialgebra of singular knots $\mathcal{K}$ is the ascending filtered bialgebra of Vassiliev invariants $\mathcal{V}$.*

We don't prove this textbook fact, but let us sketch the main points. Indeed, the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$ is $\mathcal{V}_n$.

Pulling $V_1 \otimes V_2$ back along Δ,

$$(V_1 \otimes V_2) \circ \Delta : k \mapsto V_1(k)V_2(k),$$

recovers the formula for the product $V_1 \cdot V_2$ in $\mathcal{V}$. Similarly, pulling V back along $\sharp$ is

$$V \circ \sharp k_1 \otimes k_2 \mapsto V(k_1 \sharp k_2)$$

recovers the formula for the coproduct, η. Here we rely on the fact that $(\mathcal{K}\otimes\mathcal{K})^*$ is canoncially isomorphic to $\mathcal{K}^* \otimes \mathcal{K}^*$, which follows from $\mathcal{K}$ being finite type (finite-dimensional in each filtered component).

We didn't prove that the product and coproduct in $\mathcal{V}$ respect the ascending filtration. However the filtered algebra dual of a decreasing filtered bialgebra is always an increasing filtered bialgebra (for the proof, as well as the full details of the dual filtered bialgebra construction, the reader is invited to consult, say [CDM12, Appendix A.2.4]).

Remark 1.3.13 Alternatively, this can be proved directly in $\mathcal{V}$. Proving that $\sharp$ respects the filtration on $\mathcal{K}$ was easy, and is just as easy in the dual case. However, recall the proof that Δ respects the filtration on $\mathcal{K}$ was cumbersome, and so is its dual. But it is worth looking into how it can be understood by a continuation of the polynomial analogy due to Willerton [Wil96].

The generalised Leibniz theorem of multivariable calculus says that if f and $g : \mathbb{R}^m \to \mathbb{R}$ are differentiable, then (in similar derivative notation to as above)

$$\frac{\partial^{|I|}(fg)}{\partial_{x_I}} = \sum_{J\subset\{1,\cdots,i\}} \frac{\partial^{|J|}f}{\partial_{x_J}} \cdot \frac{\partial^{|\bar{J}|}g}{\partial_{x_{\bar{J}}}}.$$

This says that the derivative of a product f and g with respect to some variables is the sum of every way of taking some of those derivatives with respect to f and some with respect to g.

A kind of dual theorem follows from this. For $c \in \mathbb{R}$, if given that c comes from some f and $g : \mathbb{R}^m \to \mathbb{R}$ by taking their derivatives with respect to all the variables and evaluating all remaining variables in the result, then we get a cofactorisation for c in $\mathbb{R} \otimes \mathbb{R}$: i.e.

$$c = \left.\frac{\partial^{|I|}(fg)}{\partial_{x_I}}\right|_{\{x_I=a_I\}} \text{ implies } \left(\sum_{J\subset\{1,\cdots,i\}} \left.\frac{\partial^{|J|}f}{\partial_{x_J}}\right|_{\{x_J=a_{\bar{J}}\}} \otimes \left.\frac{\partial^{|\bar{J}|}g}{\partial_{x_{\bar{J}}}}\right|_{\{x_J=a_J\}} \right) \xrightarrow{\mu} c,$$

where $\mu : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is multiplication.

As it turns out, this theorem is pretty useless in the multivariable calculus case. The filtration on $\mathbb{R}$ coming from being the derivative of some function evaluated at some point is trivial, and so every $c \in \mathbb{R}$ comes from some such f and g, and it's easy to construct such f and g.

But this dual theorem in the case of knots is exactly Willerton's Lemma 1.3.10, where the averaging map plays the role of evaluation.

Furthermore, the knot version of the generealised Leibniz theorem [Wil96] is that if $k^\bullet \in \mathcal{K}^\bullet_m$ and S the set of singular points of $k^\bullet$, then

$$(V_1 \cdot V_2)(\delta^S(k^\bullet)) = \sum_{I\subset S} V_1(\mu^{\bar{I}}\delta^I(k^\bullet)) \otimes V_2(\mu^I\delta^{\bar{I}}(k^\bullet)).$$

It follows directly from this that if V_1 is of type m and V_2 of type n, then $V_1 \cdot V_2$ is of type $m+n$: for if $k^\bullet \in \mathcal{K}^\bullet_{m+1}$, then either $|I| > m$ or $|\hat{I}| > n$, so in each summand, one of the cofactors is a Vassiliev invariant being evaluated above its order, so zero. Hence $(V_1 \cdot V_2)(\delta^S(k^\bullet)) = 0$.

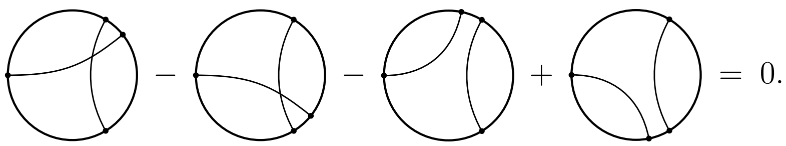
The analogy is that the multivariable calculus analogue of this argument is a proof that polynomials are filtered by degree.

Example 1.3.14 In view of Remark above about non-Vassiliev invariants, let's define a knot invariant. (Define f_{3_1}.)

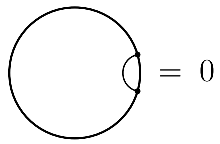
1.4 Chord diagrams and weight systems

This bialgebra strucutre on knots is heavily related to a Hopf algebra structure on chord diagrams. The general idea is to study the former via the latter.

In Section 1.2 we saw how functions on chord diagrams specify Vassiliev invariants, so long as the functions satisfy 4T* and 1T*. We can instead encode this directly into the algebra of chord diagrams by the following relations.



(4T)



(1T)

Definition 1.4.1 We define $\mathcal{A}_m$, the **space of chord diagrams** of degree m as

$$\mathcal{A}_m = \mathcal{D}_m / \mathtt{4T}, \mathtt{1T},$$

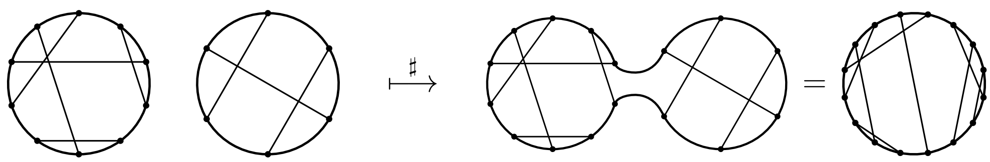
and $\mathcal{A}$, the **space of chord diagrams** as

$$\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m.$$

Warning 1.4.2 Both elements of $\mathcal{A}$ and $\mathcal{D}$ are known as chord diagrams. From now on when we say "a chord diagram", we mean an element of $\mathcal{A}$ unless otherwise specified.

The algebra $\mathcal{A}$ has multiplication and coproduct operations that mirror those in $\mathcal{K}$.

Definition 1.4.3 The **connected sum of two chord diagrams** A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



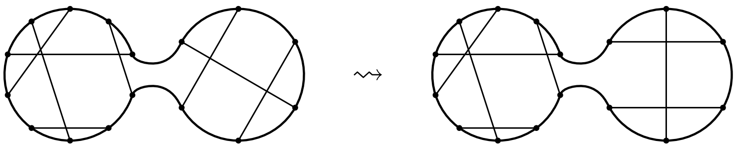
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed in the algebra $\mathcal{D}$ this is ill-defined. However the 4T relation in $\mathcal{A}$ takes care of this.

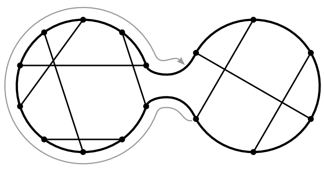
Proposition 1.4.4 *The connected sum operation $\sharp : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is well-defined.*

Proof We will prove that the connected sums, given any two choices of connection locations, are equal modulo 4T.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one 'click', like so:

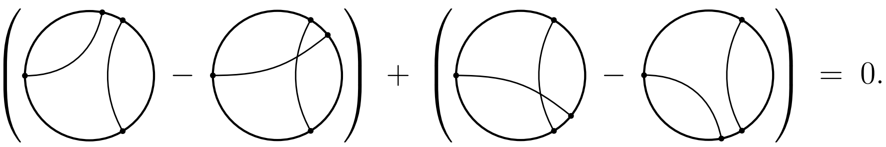


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow.



Which we show can be achieved by a series of 4T relations.

We can rewrite 4T as



A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T. So, the two connected sums $a_1 \sharp a_2$ differ by a sum of 4T relations, completing the proof. □

Proposition 1.4.5 *The connected sum operation makes $\mathcal{A}$ into a graded algebra.*

Proof No chords are lost during the connected sum: the 4T relation is homogenous with respect to degree, and the connected sum of a chord diagram containing a 1T chord relation still contains a 1T chord. So, the connected sum of a chord diagram of order i and a chord diagram of order j is a chord diagram of order $i + j$. □

Just as there is a connected sum operation in $\mathcal{A}$ reminiscent to that in $\mathcal{K}$, there is a coproduct too.

Definition 1.4.6 The **coproduct of a chord diagram** A is the sum of ways of partitioning its chords between two subdiagrams.

If S is the set of chords of A, and $J \subset S$, let $\widehat{J} = S \setminus J$. Denote by A_J the chord diagram A but with only the chords in $J \subset S$, and the rest deleted. Then

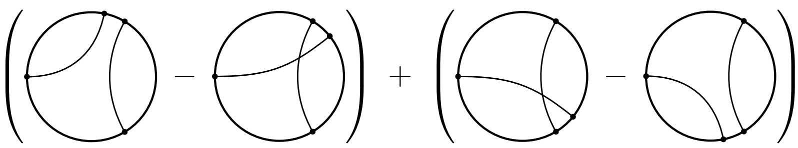
$$\Delta(A) = \sum_{J \subset S} A_J \otimes A_{\overline{J}}.$$

Proposition 1.4.7 *The coproduct Δ is well-defined in $\mathcal{A}$, and makes $\mathcal{A}$ into a graded bialgebra.*

Proof We need to check: that the coproduct factors through the quoitents, that the coproduct respects the grading, and that the compatibility condition holds.

That Δ factors through 1T is easy: an isolated chord in A remains isolated and appears in one cofactor of every term of $\Delta(A)$.

Also, Δ factors through 4T. Suppose that $K = A_1 - A_2 + A_3 - A_4$ is some combination of chord diagrams to be killed by 4T. This means that K looks like

$K =$ 

where there may be other chords O that the above diagrams have in common, as well as those shown. Note that there is one moving chord in the above diagram and one stationary chord. Let us label these m and s. Take the same partition $J \sqcup \overline{J}$ of S for all of the four chord diagrams at once, and write as the resulting coproduct $\Delta(A_i) = C_i \otimes D_i$. Suppose without loss of generality that m was partitioned into the C_i's. Then $D_1 = D_2 = D_3 = D_4$, so this term of the coproduct factors as $(C_1 - C_2 + C_3 - C_4) \otimes D_1$ and either:

- s was also partitioned into the C_i's, and the relation remains a 4T, or
- s was partitioned into the D_i's, and so $C_1 = C_2$ and $C_3 = C_4$,

and in either case, that term of the coproduct is killed.

The coproduct clearly satisfies

$$\Delta(\mathcal{A}_m) \subset \bigoplus_{i+j=m} \mathcal{A}_i \otimes \mathcal{A}_j = (\mathcal{A} \otimes \mathcal{A})_m,$$

so it is graded.

The compatibility condition holds. If A has chord set S and B has chord set T, then

$$\begin{aligned}\Delta(A) \sharp^{\otimes 2} \Delta(B) &= \left(\sum_{J' \subset S} A_{J'} \otimes A_{\overline{J'}}\right) \sharp^{\otimes 2} \left(\sum_{J'' \subset T} B_{J''} \otimes B_{\overline{J''}}\right)\\ &= \sum_{J \subset S \sqcup T} (A \sharp B)_J \otimes (A \sharp B)_{\overline{J}}\\ &= \Delta(A \sharp B). \end{aligned}$$

□

We have shown that $\mathcal{A}$ is a graded bialgebra of finite type. It also satisfies some technical, mild condition connected connectedness. A connected, graded bialgebra of finite type is sometimes called a Hopf algebra. Note that the modern definition of a Hopf algebra is a more general structure, of which $\mathcal{A}$ is also an example.

This fact leads to important consequences about the structure of $\mathcal{A}$ and its relation to Lie algebras that will the subject of Chapter 2.

Proposition 1.4.8 *THIS SHOULD BECOME A DEFINITION AND A PROPOSITION*
The graded bialgebra dual of $\mathcal{A}$ is the the graded bialgebra $\mathcal{W}$ of weight systems.

Proof We have constructued it as so. A map $f : \mathcal{A} \to \mathbb{F}$ satisfies relations that make it a weight system.

1.5 The fundamental theorem of Vassiliev invariants

In Section 1.2, we gave the fundamental theorem of Vassiliev invariants. The point of this theorem is that is establishes a particular relationship between the algebras of the previous two chapters, $\mathcal{K}$ and $\mathcal{A}$ (or equivalently, between $\mathcal{W}$ and $\mathcal{V}$). But admittedly, in the form of Theorem 1.2.16, it's not a-priori obvious why this is the case. Here we give a restatement of that theorem that makes the relationship explicit.

Definition 1.5.1 The **associated graded** algebra of a filtered algebra A is the algebra formed by the direct sum of the succesive quotient by the filtered components of A. For an algebra with a descending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m+1},$$

and for an algebra with an ascending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m-1},$$

where $A_{-1} = \{0\}$.

Theorem 1.5.2 (Fundamental theorem) *The algebra of weight systems is isomorphic to the associated graded algebra of the algebra of Vassiliev invariants, $\mathcal{W} \cong \operatorname{gr} \mathcal{V}$, or on the level of graded components,*

$$\bigoplus_{m=0}^{\infty} \mathcal{W}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{V}_m / \mathcal{V}_{m-1}.$$

Equivalently, it can be stated in the dual setting as follows. The algebra of chord diagrams is isomorphic to the associated graded algebra of the algebra of knots, $\mathcal{A} \cong \operatorname{gr} \mathcal{K}$, or on the level of graded components,

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{K}_m / \mathcal{K}_{m+1}.$$

We can break the fundamental theorem up into two parts.

Vassiliev	Every Vassiliev invariant modulo Vassiliev invariants of higher order gives a Weight system, so a map $$\mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{W}_m.$$	Every chord diagram gives an element of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$, so a map $$\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}.$$
Kontsevich	Every weight system gives a Vassiliev invariant modulo Vassiliev invariants of higher order, so a map $$\mathcal{W}_m \to \mathcal{V}_m/\mathcal{V}_{m-1}$$ which is inverse to the above.	Every equivalence class of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$ gives a chord diagram, so a map $$\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m.$$ which is inverse to the above.

It is actually only the second part of this theorem, due to Kontesevich that is equivalent to Theorem 1.2.16. The first part, due to Vassiliev, is additional (so in reality, this 'reformulation' is stronger), but considered "easy" relative to the second part. If the goal is to describe $\mathcal{K}$ by $\mathcal{A}$, it states that the relations in $\mathcal{A}$, 4T and 1T are truly compatible with $\mathcal{K}_n/\mathcal{K}_{n+1}$. Indeed this was the point of Section 1.2, so most of the work is already done.

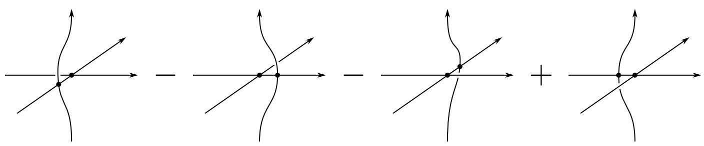
Proof of Theorem 1.5.2 (Vassiliev) For the map $a \in \mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$ we take the following. Given $a \in \mathcal{A}_m$, take a singular knot $k^\bullet \in \mathcal{K}_m^\bullet$ whose chord diagram is a, then resolving it to an element of $k \in \mathcal{K}_m$, and projecting that into the quotient $\mathcal{K}_m/\mathcal{K}_{m+1}$.

This is well-defined since any other $k^{\bullet\prime}$ also has chord diagram a. Any two m-singular knots with the same chord diagram differ by some crossing changes, in particular this applies to $k^\bullet$ and $k^{\bullet\prime}$. But if $k^\bullet$ and $k^{\bullet\prime}$ differ by a crossing change, then $\delta^m(k^\bullet)$ and $\delta^m(k^{\bullet\prime})$ differ by an element of $\mathcal{K}_{m+1}$, so $[\delta^m(k^\bullet)] = [\delta^m(k^{\bullet\prime})]$. This argument that any $k^\bullet \in \mathcal{K}_m^\bullet$ can be chosen to represent $[\delta^m(k^\bullet)]$as long as it has chord diagram a will be used again below, and we refer to it as the crossing-change argument.

Recalling that $\mathcal{A}_m = \mathcal{D}_m/\mathtt{1T}, \mathtt{4T}$, we need to show that the map factors through the quotient. Indeed 1T is the kernel, as a 1T type chord diagram is sent to a singular knot with a singular point that is passed through twice in a row when the knot is traversed. When this double point is resolved, it yields a difference of two isotopic knots.

figure of general case of one-term resolving to two isotopic (singular) knots.

Similarly, 4T is also in the kernel. A combination of chord diagrams appearing in a 4T relation is sent to a combination of singular knots which by the crossing-change argument can all be chosen to be identical except near a small region, where



but after resolving around the singular point that they don't have in common this becomes

figure of the 8 terms that cancel out.

The isomorphism respects the algebra structure as $a_1 \sharp a_2$ is sent to $[k_{a_1 \sharp a_2}] \in \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ that comes from the resolution of a singular knot $k^\bullet_{a_1 \sharp a_2}$ with chord diagram $a_1 \sharp a_2$. But likewise a_1 and a_2 map to $[k_{a_1}]$ and $[k_{a_2}]$ which are resolutions of singular knots $k^\bullet_{a_1}$ and $k^\bullet_{a_2}$. The induced operation from the connect sum $\mathcal{K}_m/\mathcal{K}_{m+1} \otimes \mathcal{K}_n/\mathcal{K}_{n+1} \to \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ takes these to $[k_{a_1} \sharp k_{a_2}]$, which comes from the resolution of $k^\bullet_{a_1} \sharp k^\bullet_{a_2}$. These singular knots may not a-priori be the same, but they have the same chord diagram, so by the crossing-change argument they can be chosen to be the same. Hence their resolutions are the same.

Using similar arguments, from the formula of Willerton's Lemma, the isomorphism can be shown to respect the bialgebra structure.

The dual version of the statement follows from the regular version, but direct proofs are illustrative and we return to them later. □

For now, let's turn to the Vassiliev part of the fundamental theorem, with the following definition.

Definition 1.5.3 The **completion** of a filtered algebra A is the filtered algebra

$$\widehat{A} = \varprojlim_{m \to \infty} A_m/A_{m+1},$$

and in the cases we care about, this will be equal to the degree-completion of $\operatorname{gr} A$, though I am not sure if that is true in general.

It is only filtered and not graded since infinitely many terms are non-zero and this is a technical restriction on direct sums.

The part of Theorem 1.5.2 due to Maxim Kontsevich [Kon93] is much more involved. Konsevich constructed an integral invariant which proves the fundamental theorem, as well as containing the information of all Vassiliev invariants at the same time. We will not give a detailed exposition of Kontsevich's invariant here, but they abound in the literature, for example [BS97; CD05; CDM12] in order of increasing level of detail. Rather we will boil the Kontsevich integral down to the following key property.

Definition 1.5.4 A **universal Vassiliev invariant** is a knot invariant $Z : \mathcal{K} \to \widehat{\mathcal{A}}$ with the following property. If $k \in \mathcal{K}_m$ is a linear combination of knots with $k = \delta^m(k^\bullet)$ and $k^\bullet$ has chord diagram $a \in \mathcal{A}_m$, then

$$Z(k) = a + \text{higher degree terms}.$$

Remark 1.5.5 Another equivalent way of defining a universal Vassiliev invariant is as follows. If f is a descending-filtration-respecting map $f : A \to B$, then define the associated graded map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ that sends $a_m + A_{m+1} \mapsto f(a_m) + B_{m+1}$. In other words, a graded map coming from the filtered map f that forgets information about higher degrees. A universal Vassiliev invariant is a map $Z : \mathcal{K} \to X$ whose associated graded $\operatorname{gr} Z : \operatorname{gr} \mathcal{K} \to \operatorname{gr} X$ is an isomorphism.

The Kontsevich integral is such a map with $X = \widehat{\mathcal{A}}$. In particular, note that $\operatorname{gr} \widehat{\mathcal{A}} = \mathcal{A}$ (because the direct sum implicit in gr means $a \in \operatorname{gr} \widehat{\mathcal{A}}$ cannot have infinitely many non-zero terms). So since Z is $\mathcal{K} \to \widehat{\mathcal{A}}$ and satisfies this property, then $\operatorname{gr} \mathcal{K} \cong \mathcal{A}$.

Theorem 1.5.6 (Kontsevich Integral) *There exists a universal Vassiliev invariant, denoted $Z(k)$, called the Kontsevich integral.*

Proof of Theorem 1.5.2 (Kontsevich) Take the map $k \in \mathcal{K}_m \to \mathcal{A}_m$ coming from killing the higher degree terms in the Kontsevich integral, and taking the lowest order non-zero chord diagram. This factors through the quotient to a map $\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m$ since by Theorem 1.5.6, any additional $k' \in \mathcal{K}_{m+1}$ contributes only higher degree terms, which get killed. It is easy to see that the two maps are inverses. □

Again, it's worth looking at the proof in the dual setting too.

Lemma 1.5.7 *Post-composing the Kontsevich integral with a weight system of order m,*

$$k \longmapsto W(Z(k))$$

(or more precisely, the following composition)

$$\mathcal{K} \xrightarrow{Z} \widehat{\mathcal{A}} \xrightarrow{\pi_m} \mathcal{A}_m \xrightarrow{W} \mathbb{Q}$$

gives a Vassiliev invariant of order m.

Proof The map $W \circ Z$ is clearly an invariant, since Z is an invariant. It's Vassiliev since on linear combinations $k \in \mathcal{K}_{m+1}$, $Z(k)$ has nothing in degree m, so composing with a weight system of degree m gives zero. □

Proof of Theorem 1.5.2 (dual) The map defined by Lemma 1.5.7, $\mathcal{W}_m \to \mathcal{V}_m$ is injective, as only the zero weight system gives the zero invariant.

However, it is not surjective. We show that the map, written as

$$\mathcal{W}_m \xrightarrow{Z^*} \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}$$

forces a choice of Vassiliev invariant of degree $m-1$. Being as explicit as possible, let

$$\Omega(k) = \begin{cases} k \in \mathcal{K}_m & (W \circ Z)(k) \\ k \notin \mathcal{K}_m & 0 \end{cases}, \quad \text{and} \quad \Theta(k) = \begin{cases} k \in \mathcal{K}_m & 0 \\ k \notin \mathcal{K}_m & (W \circ Z)(k) \end{cases},$$

then $W \circ Z = \Omega + \Theta$, where Ω recovers the weight system W, when $k \in \mathcal{K}_m$, and Θ is some finite type invariant of order $m-1$, determined by W.

In essence, we have found that the cokernel of Z^* is $\mathcal{V}_{m-1}$, so we get the desired isomorphism $\mathcal{W}_m \cong \mathcal{V}_m/\mathcal{V}_{m-1}$, at least on the level of vector spaces. □

The natural question arises, what was this summand Θ? Fixing $n < m$ and a knot $k \in \mathcal{K}_n$ with chord diagram a_k, it has Kontsevich integral

$$Z(k) = a_k + \begin{array}{c}\text{terms of}\\ \text{order } (n+1)\end{array} + \cdots + \begin{array}{c}\text{terms of}\\ \text{order } m\end{array} + \cdots.$$

Applying the projection π_m, all that remains are some chord diagrams of order m, with some coefficients depending on the intricacies of the Kontsevich integral for that particular knot. Composing with the weight system, this is a $\mathbb{Q}$-valued Vassiliev invariant of order $m-1$ determined by W.

The name 'universal Vassiliev invariant' we gave to the Kontsevich integrals and invariants of its kind is indeed justified. Every Vassiliev invariant can be obtained through the Kontsevich integral.

Theorem 1.5.8 *If Z is a universal Vassiliev invariant, then every Vassiliev invariant factors through Z.*

Proof Let $V \in \mathcal{V}_m$. Following the proof above, we can project V to $\mathcal{V}_m/\mathcal{V}_{m-1}$ to get a weight system W_m. Subtracting $W_m \circ Z$ leaves a Vassiliev invariant of one less degree. In other words, via the isomorphism

$$\begin{aligned}\mathcal{V}_m &\cong \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0 \\ &\cong \mathcal{W}_m \oplus \mathcal{W}_{m-1} \oplus \cdots \oplus \mathcal{W}_1 \oplus \mathcal{W}_0,\end{aligned}$$

V can be written as a sequence of weight systems of degree from 1 to m such that V factors through the Kontsevich integral

$$V = \sum_{i=0}^{m} (W_m \circ Z) = \left(\bigoplus_{i=0}^{m} W_m \right) \circ Z.$$

□

Corollary 1.5.9 *A universal Vassiliev invariant (in-particular, the Kontsevich integral Z) is exactly as strong as the set of Vassiliev invariants.*

Definition 1.5.10 Taking the projection $\mathcal{V}_m \to \mathcal{V}_m/\mathcal{V}_{m+1} \cong \mathcal{W}_m$ yields a weight system. The **canonical Vassiliev invariants** are those Vassiliev invariants whose weight systems W recover them completely via $W \circ Z$.

In other words, not all bases of $\mathcal{V}_m$ are created equal. The canonical Vassiliev invariants are those that are homogenous with respect to the splitting of $\mathcal{V}_m$

$$\mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0$$

induced by the Kontsevich integral. Canonical Vassiliev invariants were first defined in [BG96] and used to prove the Melvin-Morton-Rozansky conjecture relating the coefficients of Alexander polynomial of a knot to those of the coloured Jones polynomials. This is a good example of how the theory of Vassiliev invariants is useful for probing the structure of knots.

Question 1.5.11 (Bar-Natan–Garoufalidis) There are a few known from-the-bottom-up constructions of universal Vassiliev invariants of knots. However they are all equivalent to, or conjecturally equivalent to the Kontsevich integral. Yet the Kontsevich integral is not the only invariant that can satisfy the defining degree property of a universal Vassiliev invariant.

Is there a reason why the Kontsevich integral, or equivalently, this splitting appears to be canonical?

Why the reformulation was a reformulation. Or: go back and lessen the first formulation to just a related theorem.

2

Lie theory and Jacobi diagrams

The fundamental theorem of Vassiliev invariants states that the bialgebra of Vassiliev invariants can be broken up into nice combinatiorial weight systems. So to understand $\mathcal{V}$ it suffices to understand $\mathcal{W}$, or equivalently its dual $\mathcal{A}$. There is a hint that the structure of $\mathcal{A}$ may relate to Lie algebras.

The hint is that Hopf algebras have a lie algebra as their primitive elements $\mathcal{P}(H)$. However I clearly am confused here, for the following reason. By the Milnor-Moore theorem, applying this to $\mathcal{A}$, it should be the symmetric algebra (and therefore the universal enveloping algebra, as it's abelian?) of $\mathcal{P}(\mathcal{A})$.

However this clashes with my understanding of the raison d'etre of Hinnich-Vaintrob [HV00]. According to them, the point of their convoluted construction is to realise $\mathcal{A}$ as the universal enveloping algebra of some Lie algerba. And a key point is that that's not technically true, hence the need to move to more general tensor categroies.

Further evidence in [Kri11]: "The STU relation is a formal analogue of the relation in a Universal Enveloping Algebra which equates a commutator with the corresponding bracket" (the word formal implying that this isn't literally true).

Page 141 on [CDM12] furthermore says that [HV00] shows $\mathcal{A}$ (algebra of Jacobi diagrams on the circle) is isomorphic to the center of the universal enveloping algebra of a Casimir Lie algebra in a certain tensor category. Indeed that's my interpretation of HV, but why do if we already know that $\mathcal{A}$ is already the center (which in this case equals the whole thing) of $\mathcal{P}(\mathcal{A})$?!

2.1 Jacobi diagrams, AS, STU and IHX

This side of the story reframes the bialgebra $\mathcal{A}$ as an isomorphic bialgebra known as the algebra of Jacobi diagrams to illuminate the Lie theory connections.

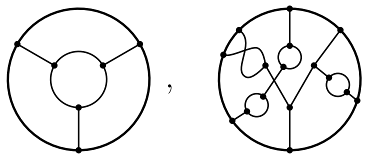
Definition 2.1.1 A **unitrivalent diagram** is a unitrivalent graph (with loops and multiple edges allowed) with the following additional data:

- each trivalent vertex has a fixed cyclic order of incident edge-connections,

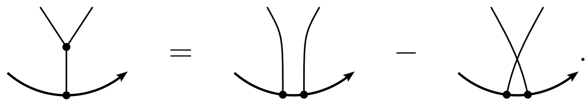
- the set of univalent vertices has a fixed cyclic order.

The vector space of unitrivalent diagrams is denoted $\mathcal{T}$.

When drawing unitrivalent diagrams, we specify the fixed cyclic order of the univalent edges of a unitrivalent diagram by drawing them connected to a circle. In particular all chord diagrams are Jacobi diagrams with only univalent vertices (the chord ends). Further examples of non-chord diagram Jacobi diagrams would be

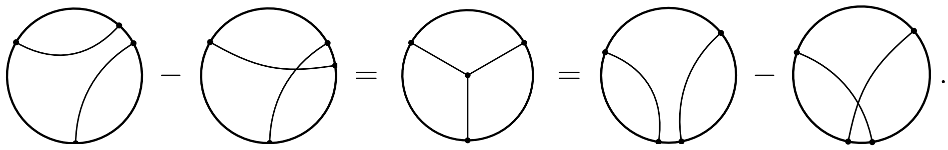
 $\in \mathcal{T}$.

Definition 2.1.2 The **STU relation** is the relation

 (STU)

As usual, this is not an individual relation but a type of relations, true in any diagrams that are identical except for the subdiagrams being as shown.

Note that the for the chord diagrams inside the algebra of Jacobi diagrams, the STU relations imply the 4T relations, as

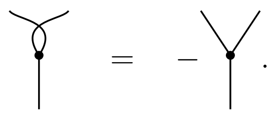


Definition 2.1.3 The algebra $\mathcal{J}$ of Jacobi diagrams is the vector space $\mathcal{T}/$STU, with the product $\sharp$ defined the same way as it was for chord diagrams.

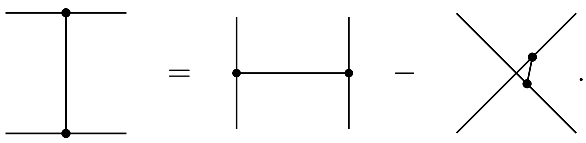
This is well-defined. The proof of Proposition 1.3.4 showed that the product $\sharp$ being well-defined on $\mathcal{A}$ was a consequence of the 4T relations, which are implied by the STU relations. From the STU relations, we may deduce the following other relations which hold in $\mathcal{J}$.

Proposition 2.1.4 *The following relations are consequences of the* STU *relation in* $\mathcal{J}$*:*

*(a) The **AS relation** (antisymmetry relation),*

 (AS)

*(b) The **IHX relation**,*

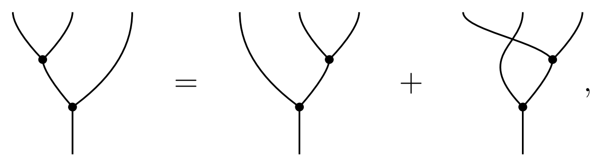
 (IHX)

Proof (a) Take two diagrams which differ only by AS at one (trivalent) vertex. If the vertex at which the AS relation resides is adjacent to a univalent vertex (i.e. touches the outer circle), then this is immediate from applying STU to both diagrams at that vertex.

If the vertex is not immediately adjacent to a univalent vertex, then it is some d vertices 'in the way'. By applying STU to those vertices yields a sum of 2^d diagrams, all identical except for differing by AS, now on a vertex adjacent to a univalent vertex.

(b) A similar argument applies. If one of the two vertices of the IHX is adjacent to the circle, then the result is a direct consequence of an STU on each of the vertices. Otherwise, some STUs may be required first. □

Looking at the relations STU, AS and IHX, the first solid evidence of Lie-theoretic structure in this story emerges. Explicit connections will be the subject of the next subsection, but for now notice that interpreting the trivalent vertex in STU (which has two 'input' edges above and one 'output' edge below) as a bracket, this bears resembelance to the relation that defines the universal enveloping algebra of a Lie algebra $[x, y] = xy - yx$. Similarly, AS looks like the antisymmetry property of the bracket $[x, y] = -[y, x]$. For IHX perhaps the relation isn't quite so obvious until the diagrams are rearranged into the form

 (IHX)

(here also one application of AS was used) whereby the IHX looks like the Jacobi identity, in the form $[[x, y], z] = [x, [y, z]] + [y, [x, z]]$.

We have already spoiled the surprise that in the end, $\mathcal{A}$ and $\mathcal{J}$ will be isomorphic as bialgebras. In fact, as algebras, this is clearly true so far, as $\mathcal{J}$ is just a change of basis from $\mathcal{A}$. Since $\mathcal{A}$ spans $\mathcal{J}$, we can attempt to lift the coproduct from $\mathcal{A}$ directly onto $\mathcal{J}$.

Proposition 2.1.5 *The coproduct Δ on $J \in \mathcal{J}$ defined by taking a Jacobi diagram, representing it as a chord diagram via* STU*, taking the coproduct in $\mathcal{A}$, then interpreting the result as a Jacobi diagram via the inclusion of $\mathcal{A}$ into $\mathcal{J}$, is also given by the following formula.*

$$\Delta(J) = \sum_{C \subset S} J_C \otimes J_{\overline{C}},$$

where S is the set of connected components of J, and $\overline{C} = S \setminus C$.

Proof Note that this has the same symbolic form as the coproduct in $\mathcal{A}$ given in Definition 1.4.6, but with chords replaced by connected components of Jacobi diagrams. However, when working in $\mathcal{A} \subset \mathcal{J}$, there are only univalent vertices, so the connected components are exactly the chords. Since $\mathcal{A}$ forms a basis for $\mathcal{J}$, and the formula is linear, it extends to all of $\mathcal{J}$. □

Corollary 2.1.6 *The bialgebras $\mathcal{A}$ and $\mathcal{J}$ are isomorphic.*

Warning 2.1.7 From here on, justified by this isomorphism which is just a change of basis, we write $\mathcal{J} = \mathcal{A}$, and $\mathcal{A}$ is the preferred choice of notation for both chord diagrams and Jacobi diagrams.

2.2 Lie algebra weight systems

2.3 Non-Lie algebraic weight systems

2.4 Jacobi diagrams as a universal enveloping algebra

3

Welded knots, isometry Lie algebras and arrow diagrams

test text

3.1 Welded knots

3.2 Arrow diagrams

3.3 A universal welded weight system

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