On Vassiliev Invariants

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Contents

Acknowledgements

Thanks to ...

Introduction

The introduction goes here.
All knots are assumed to be oriented and framed unless specified otherwise.

1

Vassiliev invariants and chord diagrams

Something to maybe include somewhere:

The point of Vassiliev theory is to study the space of knots in the context of the singularities that lie between knots.

In view of this point, we spend Section 1.1 looking at a the stratification of the space of knots. This leads to a beautiful (and fruitful) classical analogy which we will explored in Section 1.2 and throughout this chapter.

In Section 1.3, with the context in mind, we introduce the main players in this theory.

Polynomial functions are a special type of function. They are related in a natural way to the derivative, they are defined by a finite amount of combinatorial data, and they can be used to approximate any continuous function.

In the same way, the Vassiliev knot invairants are a special type of knot invariant. And this analogy, first made by Dror Bar-Natan in [Bar95] is by no means superficial. As we will come to see, Vassiliev invariants enjoy analogues of the first two properties above, and conjeturally also the third.

In this chapter we give a version of the introductory theory of Vassiliev invariants in which the analogy above is made as explicitly as possible. This also leads a natural interpretation of the defining relations of the algebra $\mathcal{A}$ which is the fundamental object of study in the field.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m-**singular**.

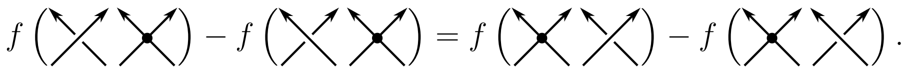
Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular immersions with

(a) triple points
(b) points with vanishing derivative
are excluded from the definition.

A singular knot with one double point is very close to two other knots. In one, the double point is replaced by a positive crosing, and in the other a negative crossing.

Just as we have notions of ambient isotopy for knots, and knot invairants, we can have m-singular isotopy and m-singular knot invariants.

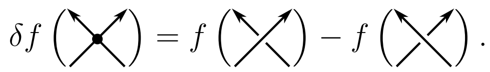
Definition 1.1.3 An invariant f of m-singular knots is **differentiable** if



(DIFF)

If an m-singular knot invariant is differentiable, we can extend it to an invariant of $(m+1)$-singular knots by a procedure analagous to taking its derivative.

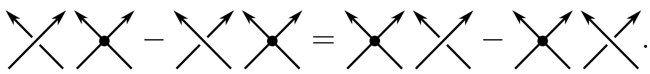
Definition 1.1.4 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



A regular knot invariant (which is an invariant of 0-singular knots) satisfies this condition vaccuously so is differentiable and its derivative is an invariant of 1-singular knots. Furthermore, if an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points. In-particular, regular knot invariants have derivatives of any order.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

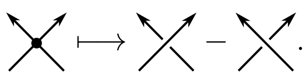
Definition 1.1.5 Define $\mathcal{K}_m^\bullet$ as the span of all m-singular knots, taken over $\mathbb{Q}$, modulo the following boundary relation (also known as a codifferentiability relation):



($\mathtt{DIFF}^*$)

From now on, we will refer to elements $\mathcal{K}_m^\bullet$ as m-**singular knots**, i.e. the $\mathtt{DIFF}^*$ relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m^\bullet \to \mathcal{K}_{m-1}^\bullet$ defined by

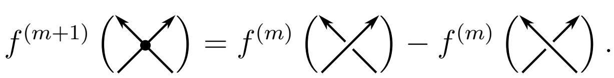


Remark 1.1.7 The derivative operation and the `DIFF` relation are dual to the boundary operation and the $\mathtt{DIFF}^*$ relation. For example, a differentiable invariant of knots is the same as an invariant of knots in $\mathcal{K}_m^\bullet$.

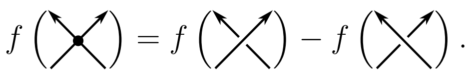
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

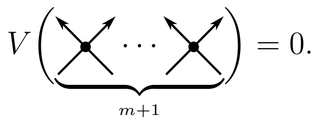


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation,



(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the most double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 In other words, Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

1.2 The stratification of the space of knots and integration

To help see the bird's eye view we phrase the analogy between Vassiliev invariants and polynomials in terms of an integration theory, following [Hut98].

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of objects in some category, for example they could be vector spaces, algebras, free abelian groups, modules, etc. Here we call them spaces, because we wish to use the word object for elements of the spaces. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The space $\mathcal{O}_0$ is the primary space we want to study. The spaces $\mathcal{O}_m$ can often be thought of as containing m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing another space X and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, X)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose (adjoint) of ∂_m. The maps δ behave like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular elements.

Questions 1.2.2 We wish to understand how to invert this process, namely:

(a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?

(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

(c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?
(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The answers to the above questions are given precisely by the following spaces1

Definitions 1.2.3 (a) The **primary obstructions to integration** are the space

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the space

$$C\mathcal{O}_m = \mathcal{O}_m / \partial \mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the space

$$S\mathcal{O}_m = \ker (\partial_{m+1} \partial_m) / P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.
(d) The **weights of integration** are the space

$$W\mathcal{O}_m = C\mathcal{O}_m / \pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the projection.
(e) The **finite type invariants** of order m are the space

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $\mathcal{O}_\star = \mathcal{K}_\star^\bullet$.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the m-th and $(m+1)$-st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

Figure: singular isotopy.

To rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}_\star = \mathcal{K}_\star^\bullet$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

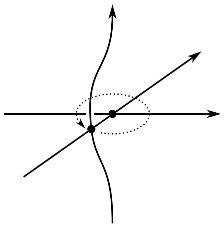
Due to the boundary relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

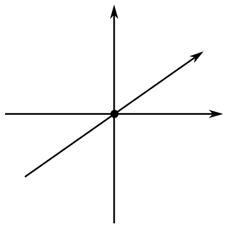
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a “path-integral” along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that for example, triple points are excluded from all levels of the stratification, leaving “holes” in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

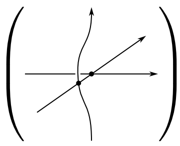
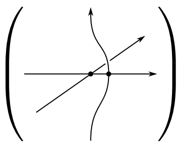
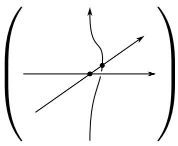
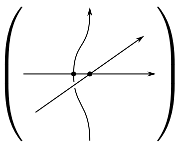
Definition 1.2.5 (a) The following closed singular isotopy around a triple point:



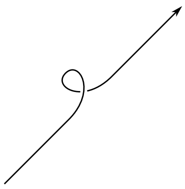
around



gives rise to a **topological four-term relation**

f  $-f$  $-f$  $+f$  $= 0.$ (T4T*)

(b) The singular isotopy



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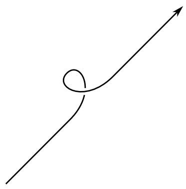
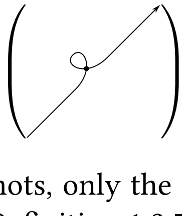


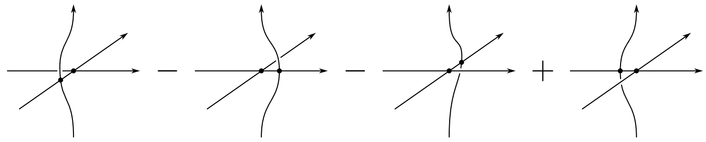
around a point with a vanishing derivative gives rise to a **topological one-term relation**

f  $= 0.$ (T1T*)

Remark 1.2.6 Note that for framed knots, only the singular isotopy in Definition 1.2.5 (a) is closed, whereas the singular isotopy Definition 1.2.5 (b) is not. So only T4T* holds. When considering unframed knots, the theory is the same as what we present below, but T1T* plays the same role as T4T*.

In particular, for every statement in the rest of this chapter about weight systems, chord diagrams, etc. there is a corresponding theorem for the unframed weight systems, etc. in which for every hypothesis involving a T4T* (or variant thereof), a similar one involving T1T* (or variant thereof) must also hold.

So, combinations of m-singular knots of the form



are in $P\mathcal{K}_m^\bullet = \ker \partial_m$. Any nonvanishing functionals on them can't integrate. But do combinations of these forms span the primary obstructions? Are they the only possible reasons that that f doesn't integrate?

Theorem 1.2.7 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $(m-1)$-singular knots if and only if it satisfies* `T4T`*.*

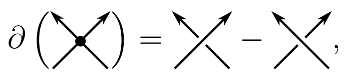
For example, the corresponding unframed version of Stanford's theorem reads: *An invariant f of m-singular unframed knots integrates to an invariant of $(m-1)$-singular unframed knots if and only if it satisfies* `T4T`* *and* `T1T`*.*

Proof (sketch) Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m-1$ double points, and additional worse singularities. The only events to consider are codimension $m+2$ (why $m+2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with `DIFF` relations, which we've said are implicit).

Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

Continuing with Questions 1.2.2 and Definitions 1.2.3, the constants of integration comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost, and when integrating, a choice has to be made.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}_m^\bullet = \mathcal{K}_m^\bullet / \partial\mathcal{K}_{m+1}^\bullet$. What do the classes of this quotient look like? Well since

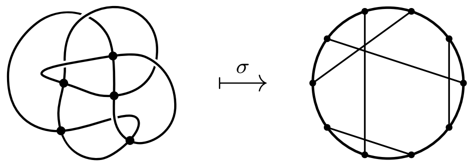


the difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. Thus, the objects of this quotient are m-singular knots modulo crossing changes. So, the only data needed to describe them is the order in which the m double points are traversed around the m-singular knot.

Definitions 1.2.8 (a) A **chord diagram** of order m is an element of $\mathcal{K}_m^\bullet / \partial\mathcal{K}_{m+1}^\bullet$. An equivalent combinatorial description of such an element is an oriented circle with a distinguished set of m unordered pairs of points, considered up to orientation-preserving diffeomorphism of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}_m^\bullet$. The oriented circle is called the **skeleton** of the chord diagram.

(b) The **chord diagram of an m-singular knot** k, denoted $\sigma(k)$ is the chord diagram formed by the following process. Place $2m$ points on the skeleton, two for each singular point of k. Traversing both k and the circle, label the points on the skeleton in the order in which the singular points of k are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(k)$.

Example 1.2.9



◊

Proposition 1.2.10 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}_m^*$ Then Q_0 is also an integral of P.

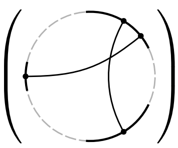
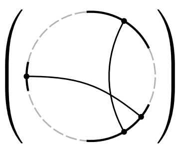
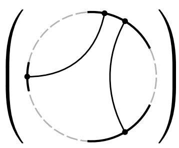
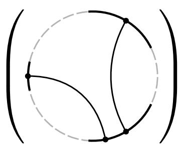
Proof The derivative of an n-singular invariant that factors through σ is zero: crossing changes do not change the chord diagram, so when a crossing change occurs, the function does not change. Hence Q and Q_0 have the same derivative: P. □

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

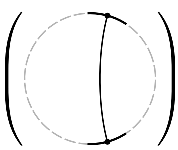
Definition 1.2.11 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

The T4T* (and T1T* for unframed) relations are what knot invariants must satisfy to be integrable. If we take their images under σ, we get the following relations in the space of chord diagrams.

Definitions 1.2.12 (a) A **four-term relation** is a relation of the kind

q  $- q$  $- q$  $+ q$  $= 0.$ (4T*)

(b) A **one-term relation** (relevant in the unframed case) is a relation of the kind

q  $= 0.$ (1T*)

where the chord shown is **isolated**, meaning no other chord intersects it (when suitably isotoped).

Just like T4T*, 4T* is not an individual relation, but a class of relations. For example, there's one 4T* relation for all ways of placing other chords on the dotted parts of the diagrams above, identically in all four diagrams.

Proposition 1.2.13 *A weight system is characterised as a constant of integration that satisfies* 4T*.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T*. Since crossing changes are free, this is equivalent to satisfying the projection of T4T* into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of an m-singular P is only defined up to constants of integration, so is of the form

$$Q + q \circ \sigma.$$

Since integration is linear, for this to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that the secondary obstructions $S\mathcal{K}_m^\bullet$ vanish. But this is in terms of the general integration theory – what does this mean concretely in this special case? It remains open but it's conjectured that the secondary obstructions always vanish.

Conjecture 1.2.14 *An invariant of m-singular knots satisfying* T4T* *integrates m times into a genuine knot invariant.*

Remarks 1.2.15
(a) At first glance, this conjecture looks like it follows from Theorem 1.2.7. The point is that it may not be possible to choose the integral to again satisfy T4T*, which is what $S\mathcal{K}^\bullet$ measures.
(b) Computing $S\mathcal{K}_m^\bullet$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.
(c) This conjecture is proven in certain cases. It holds in the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98].

The finite type invariants in $\mathcal{K}_\star^\bullet$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m+1$.

If we restrict Conjecture 1.2.14 to Vassiliev invariants, then we get the following.

Theorem 1.2.16 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* T4T* *and the additional condition that $\delta v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant of order m).*

There are various proofs of the fundamental theorem. They are listed in [BS97], and each proof is accompanied by a series of moral objections. In the words of Bar-Natan: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made".

Remark 1.2.17 We have the implication Conjecture 1.2.14 $\implies$ Theorem 1.2.16, and this is actually realised in the theory of braids. It is mysterious that the fate of the slightly stronger conjecture which comes from taking the natural topological approach to the fundamental theorem still remains unknown, and that there are grievances to be had with all known proofs of the theorem.

In Section 1.5 we will prove an equivalent formulation of the fundamental theorem.

1.3 Knots and Vassiliev invariants

Speaking broadly, the aim of Vassiliev theory is to study the space of knots using information from the stratification of knots introduced in the first two sections. This is done via the space of chord diagrams, which can be considered its linearisation or projectivisation. But these spaces are not just vector spaces. There is some further structure which we wish to incorporate.

Definitions 1.3.1 (a) The **space of knots**, denoted $\mathcal{K}$, is the vector space spanned, over $\mathbb{Q}$, by non-singular knots. Equivalently, $\mathcal{K} = \mathcal{K}_0^\bullet$.

(b) The space of knots is equipped with the **singular knot filtration**

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

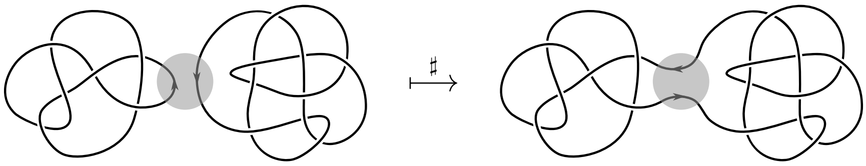
where the ith filtered component $\mathcal{K}_i$ is the span of resolutions of singular knots with i double points, equivalently the image of ∂^i, that is, $\mathcal{K}_i = \partial^i(\mathcal{K}_i^\bullet)$.

Proposition 1.3.2 *The singular knot filtration is indeed a descending filtration of vector spaces.*

Proof This being a filtration of vector spaces, the only thing to check is that if $i < j$, then $\mathcal{K}_i \supset \mathcal{K}_j$. If $k \in \mathcal{K}_j$, then $k = \partial^j(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}_j$. But then $k = \partial^j(k^\bullet) = \partial^i\partial^{j-i}(k^\bullet)$, so $k \in \partial^i(\mathcal{K}_i^\bullet)$. □

The algebraic structure on knots comes from the following operation.

Definition 1.3.3 The **connected sum** of two knots k_1 and k_2 is the knot obtained by removing a small arc from each of k_1 and k_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

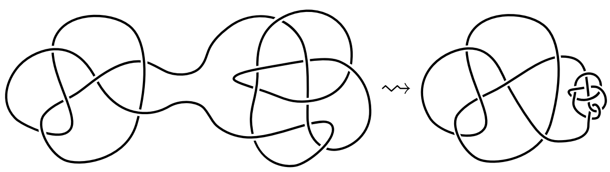


This definition is extended bilinearly to $\mathcal{K}$, i.e., to linear combinations of knots.

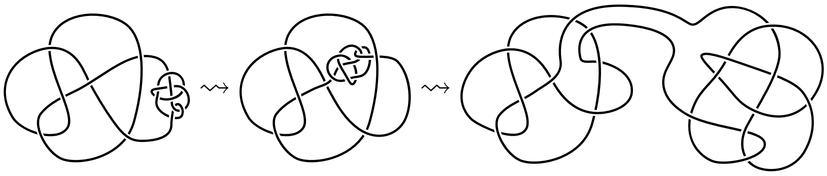
The connected sum of knots is not a-priori well-defined, as we have not specified where along either k_1 or k_2 the small arc is to be removed or along which path they should be connected. However, by a classical knot-theoretic argument, the result is independent of either choice.

Proposition 1.3.4 *The connected sum $\sharp : \mathcal{K}\otimes\mathcal{K} \to \mathcal{K}$ operations is well-defined: up to ambient isotopy, it does not matter where and along what path the connection was performed.*

Proof We exhibit an ambient isotopy starting at $k_1 \sharp k_2$ where the small arc is removed from k_1 as in the example above. The part of the connected sum coming from k_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of k_1.



Then, k_2 is then isotoped along k_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along k_1, and the same argument with the roles of k_1 and k_2 reversed completes the proof.

A similar agument also shows that the path along which the two knots are connected doesn't matter, as k_2 can always be shrunk to within a small tubular neighbourhood of k_1. $\square$

Proposition 1.3.5 *The connected sum respects the descending filtration:*

$$K_i \sharp K_j \subset K_{i+j}.$$

That is, the connected sum makes $(\mathcal{K}, \sharp)$ into a descending filtered algebra.

Proof Indeed, the connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. The question is whether the connected sum respects the filtration.

If $k \otimes \ell \in \mathcal{K}_i \otimes \mathcal{K}_j$, then there are $k^\bullet$ and $\ell^\bullet$ in $\mathcal{K}_i^\bullet$ and $\mathcal{K}_j^\bullet$ that resolve to k and ℓ, respectively. Similarly, the 'connected sum' $k^\bullet \sharp \ell^\bullet$ resolves by ∂^{i+j} to $k \sharp \ell$, which is therefore in $\partial^{i+j}(\mathcal{K}_{i+j})$.

Here, 'connected sum' is enclosed in inverted commas due to the following technicality. Connected sums of singular knots with singular knots were not part of Definition 1.3.3. Even if we ensure that the small arcs removed from a singular knot do not contain a singular point, still, this is ill-defined. The ambiguity is that the resulting singular knot may depend on from which side of the singular point the arc was removed i.e. the repositioning argument in the proof of Proposition 1.3.4 fails due to the presence of singular points. After taking the resolution under ∂^{i+j} however, the repositioning argument now works again, so any of the choices of connected sum in $k^\bullet \sharp \ell^\bullet$ produce a singular knot which resolves to $k \sharp \ell$. $\square$

Definition 1.3.6 The coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is defined on knots k as

$$\Delta(k) = k \otimes k$$

and extended bilinearly to $\mathcal{K}$.

Proposition 1.3.7 *The triple $(\mathcal{K}, \sharp, \Delta)$ forms a bialgebra. In other words, the connected sum and coproduct are compatible.*

Proof A bialgeba is a vector space which is both a coalgebra and an algebra algebra with compatible product and coproduct. That $(\mathcal{K}, \Delta)$ forms a coalgebra is trivial (the counit is the augmentation map denoted ε). And we have already seen that $(\mathcal{K}, \sharp)$ is an algebra. So it remains only to check the compatibility conditions. We check that the product and coproduct are compatible:

$$\begin{aligned} \Delta(k \sharp \ell) &= k \sharp \ell \otimes k \sharp \ell \\ &= (k \otimes k) \sharp^{\otimes 2} (\ell \otimes \ell) \\ &= \Delta(k) \sharp^{\otimes 2} \Delta(\ell) \end{aligned}$$

where $\sharp^{\otimes 2}$ denotes the component-wise tensor product on $\mathcal{K} \otimes \mathcal{K}$.

Checking the unit and counit is trivial. $\square$

So far, $\mathcal{K}$ is a bialgebra whose product respects the filtration. The same is true of the coproduct.

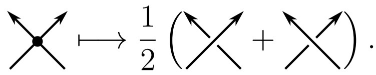
Proposition 1.3.8 *The coproduct Δ also respects the filtration,*

$$\Delta(\mathcal{K}_j) \subset \sum_{i=0}^{j} \mathcal{K}_i \otimes \mathcal{K}_{j-i}.$$

That is, with the singular knot filtration, $\mathcal{K}$ is a filtered bialgebra.

We give a proof, due to Willerton which follows directly from Lemma of 1.3.10 of [Wil96]. The lemma is a formula for the coproduct of an element of $k \in \mathcal{K}_m$ that comes from some $k^\bullet \in \mathcal{K}^\bullet_m$. The formula is terms of the 2^m ways of resolving some of singular points in one cofactor and the rest in the other, but first we need some notation.

If I is a subset of the singular points of a singular knot, let ∂^I be the operator that resolves the singular points I. Let μ^I be the operator that averages singular points in I, where averaging a singular point is sending



Remark 1.3.9 There is one technicality. The operations δ and ∂ were defined without the need to specify a specific double point (this was the point of the `DIFF` and `DIFF*` relations). But the definitions of μ^I and ∂^I are specific about double points. So for this Lemma 1.3.10, its proof and the proof of Proposition 1.3.8, we write $\mathcal{K}^\bullet_m$ even though we mean the lift of the $\mathcal{K}^\bullet_m$'s, without the `DIFF` and `DIFF*` relations. We choose not to alter the notation as projecting the formulas back down to the quotient properly proves that Δ respects the filtration, as intended.

Lemma 1.3.10 (Willerton) *Suppose $k^\bullet \in \mathcal{K}^\bullet_m$, and let S denote the set of singular points of $k^\bullet$.*

$$\Delta(\partial^S(k^\bullet)) = \sum_{I \subset S} \mu^{\bar{I}} \partial^I(k^\bullet) \otimes \mu^I \partial^{\bar{I}}(k^\bullet)$$

where $\bar{I} = S \setminus I$.

Proof We proceed by induction on m. In the base case of $m = 0$, $S = \emptyset$, and $k^\bullet = k$ is a genuine knot, so

$$\begin{aligned}
\Delta(\partial^0(k)) &= \Delta(k) \\
&= k \otimes k \\
&= \sum_{I \subset \emptyset} \mu^{\bar{I}} \partial^I(k) \otimes \mu^I \partial^{\bar{I}}(k).
\end{aligned}$$

The inductive step is as follows. Let $k^\bullet \in \mathcal{K}^\bullet_{m+1}$. Let J denote all singular points of $k^\bullet$, and $x \in J$ denote a specific singular point. Furthermore, let $k^{\bullet+}$ (resp. $k^{\bullet-}$) denote the m-singular knots obtained from $k^\bullet$ when x is replaced by a positive (resp. negative) crossing, so that $\partial^{\{x\}}(k^\bullet) = k^{\bullet+} - k^{\bullet-}$. We examine

$$\sum_{I \subset J} \mu^{\bar{I}} \partial^I(k^\bullet) \otimes \mu^I \partial^{\bar{I}}(k^\bullet).$$

Decomposing the sum based on whether $x \in I$ yields

$$\sum_{x\in I\subset J} \mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^\bullet)\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^\bullet)+\sum_{x\notin I\subset J}\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^\bullet)\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^\bullet),$$

then resolving either ∂ or μ on x,

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J}\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet+}-k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet+}+k^{\bullet-})\\ +&\frac{1}{2}\sum_{x\notin I\subset J}\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^{\bullet+}+k^{\bullet-})\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^{\bullet+}-k^{\bullet-}).\end{aligned}$$

Expanding, yields the cumbersome formula

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J}\Big(\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet+})+\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet-})\\ &\qquad\qquad-\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet+})-\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet-})\Big)\\ +&\frac{1}{2}\sum_{x\notin I\subset J}\Big(\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^{\bullet+})\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^{\bullet+})-\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^{\bullet+})\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^{\bullet-})\\ &\qquad\qquad+\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^{\bullet-})\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^{\bullet+})-\mu^{\bar{I}\smallsetminus\{x\}}\partial^{I}(k^{\bullet-})\otimes\mu^{I}\partial^{\bar{I}\smallsetminus\{x\}}(k^{\bullet-})\Big).\end{aligned}$$

Since $\{I \mid I \subset J, x \notin I\}$ is equal to $\{I \smallsetminus \{x\} \mid I \subset J, x \in I\}$, in each of the above sums, the corresponding terms have the same indices. Hence, the first and last terms in each sum combine, and the second and third terms cancel out to give

$$\sum_{x\in I\subset J}\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet+})-\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}(k^{\bullet-}).$$

Since neither $\mu^{\bar{I}}\partial^{I\smallsetminus\{x\}}$ or $\mu^{I\smallsetminus\{x\}}\partial^{\bar{I}}$ are with respect to x, this can be written

$$\sum_{I\subset J\smallsetminus\{x\}}\mu^{\bar{I}}\partial^{I}(k^{\bullet+})\otimes\mu^{I}\partial^{\bar{I}}(k^{\bullet+})-\mu^{\bar{I}}\partial^{I}(k^{\bullet-})\otimes\mu^{I}\partial^{\bar{I}}(k^{\bullet-})$$

which by the inductive hypothesis is

$$\begin{aligned}\Delta(\partial^{J\smallsetminus\{x\}}(k^{\bullet+}))-\Delta(\partial^{J\smallsetminus\{x\}}(k^{\bullet-}))&=\Delta(\partial^{J\smallsetminus\{x\}}(k^{\bullet+})-\partial^{J\smallsetminus\{x\}}(k^{\bullet-}))\\ &=\Delta(\partial^{J}(k^{\bullet})).\end{aligned}$$

□

Proof of Proposition 1.3.8 The operators ∂^I and $\mu^{\bar{I}}$ commute since they are evaluating different singular points. Let I be an arbitrary subset of S, and let $|I| = i$ and $|S| = j$, then the left cofactor is in $\mathcal{K}_i$ and the right in $\mathcal{K}_{j-i}$. □

Not all knot invariants respect the singular knot filtration, as we will see. The point of the Vassiliev invariants is that they're the ones that are natural to consider with respect to the singular knot filtration. Indeed, the Vassiliev invariants are obtained directly from $\mathcal{K}$ via the following construction.

A decreasing filtration on a bialgebra induces an ascending filtration on its dual bialgebra. The dual bialgebra's mth filtered component is the space of functionals on the original bialgebra that vanish on the $(m+1)$st filtered component. In the case of $\mathcal{K}$, this is the set of knot invriants that vanish on $\mathcal{K}_{m+1}$: exactly the Vassiliev invariants of order m. The product/coproduct in the dual bialgebra are those operations adjoint to the coproduct/product in the original bialgebra. The construction is standard and the details can be found in [CDM12, Appendix A.2.4].

Definition 1.3.11 The **filtered bialgebra of Vassiliev invariants**, denoted $\mathcal{V}$, is the vector space of Vassiliev invariants with an ascending filtration by degree

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots, \qquad \mathcal{V} = \bigcup_{m=0}^{\infty} \mathcal{V}_m.$$

The product is given by pointwise multiplication

$$V_1 \cdot V_2(k) = V_1(k)V_2(k),$$

and the coproduct, η, is given by

$$\eta(V)(k_1 \otimes k_2) = V(k_1 \sharp k_2).$$

Proposition 1.3.12 *The filtered bialgebraic dual of the descending filtered bialgebra of singular knots $\mathcal{K}$ is the ascending filtered bialgebra of Vassiliev invariants $\mathcal{V}$.*

We don't prove this textbook fact, but let us sketch the main points. Indeed, by definition the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$ is $\mathcal{V}_m$.

Pulling $V_1 \otimes V_2$ back along Δ gives,

$$(V_1 \otimes V_2) \circ \Delta : k \mapsto V_1(k)V_2(k),$$

which recovers the formula for the product $V_1 \cdot V_2$ in $\mathcal{V}$. Similarly, pulling V back along $\sharp$ yields

$$V \circ \sharp(k_1 \otimes k_2) \mapsto V(k_1 \sharp k_2)$$

which recovers the formula for the coproduct, η.

Here we rely on the fact that $\mathcal{K}$ is of finite type (finite-dimensional in each filtered component).

Remark 1.3.13 Alternatively, this can be proved directly in $\mathcal{V}$. Proving that $\sharp$ respects the filtration on $\mathcal{K}$ was easy, and is just as easy in the dual case. However, the proof that Δ respects the filtration on $\mathcal{K}$ was cumbersome, and so is its dual. But it is worth looking into how it can be understood by a continuation of the polynomial analogy due to Willerton [Wil96].

The generalised Leibniz theorem of multivariable calculus says that if f and $g : \mathbb{R}^m \to \mathbb{R}$ are differentiable, then (in similar derivative notation to as above)

$$\frac{\partial^{|I|}(fg)}{\partial_{x_I}} = \sum_{J \subset \{1,\cdots,i\}} \frac{\partial^{|J|} f}{\partial_{x_J}} \cdot \frac{\partial^{|\bar{J}|} g}{\partial_{x_{\bar{J}}}}.$$

This says that the derivative of a product of f and g with respect to some variables is the sum of every way of taking some of those derivatives with respect to f and some with respect to g.

A kind of dual theorem follows from this. Let $c \in \mathbb{R}$, and suppose that c comes from some pair of functions f and $g : \mathbb{R}^m \to \mathbb{R}$ by taking their derivatives with respect to some of the variables and evaluating all remaining variables in the result. Then, we get a cofactorisation for c in $\mathbb{R} \otimes \mathbb{R}$: i.e. if

$$c = \left.\frac{\partial^{|I|}(fg)}{\partial_{x_I}}\right|_{\{x_I = a_I\}}$$

then

$$\mu\left(\sum_{J\subset\{1,\cdots,i\}} \left.\frac{\partial^{|J|}f}{\partial_{x_J}}\right|_{\{x_J=a_{\bar{J}}\}} \otimes \left.\frac{\partial^{|\bar{J}|}g}{\partial_{x_{\bar{J}}}}\right|_{\{x_J=a_J\}}\right) = c,$$

where $\mu : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is multiplication.

As it turns out, this generalised "co-Leibniz theorem" is pretty useless in the multivariable calculus case. The filtration on $\mathbb{R}$ coming from being the derivative of some function evaluated at some point is trivial, and so every $c \in \mathbb{R}$ comes from some such f and g, and it's easy to construct such f and g. But the co-Leibniz theorem in the case of knots is exactly Willerton's Lemma 1.3.10, where the averaging map plays the role of evaluation. Recall that this was used to show that the coproduct respects the filtration.

Furthermore, the knot version of the generealised Leibniz theorem [Wil96] is that if $k^\bullet \in \mathcal{K}^\bullet_m$ and S the set of singular points of $k^\bullet$, then

$$(V_1 \cdot V_2)(\partial^S(k^\bullet)) = \sum_{I\subset S} V_1(\mu^{\bar{I}}\partial^I(k^\bullet)) \otimes V_2(\mu^I\partial^{\bar{I}}(k^\bullet)).$$

It follows directly from this that the product of two Vassiliev invariants respects the filtration: if V_1 is of type m and V_2 of type n, then $V_1 \cdot V_2$ is of type $m+n$: for if $k^\bullet \in \mathcal{K}^\bullet_{m+n+1}$, then either $|I| > m$ or $|\bar{I}| > n$, so in each summand, one of the cofactors is a Vassiliev invariant being evaluated above its order, so zero. Hence $(V_1 \cdot V_2)(\partial^S(k^\bullet)) = 0$.

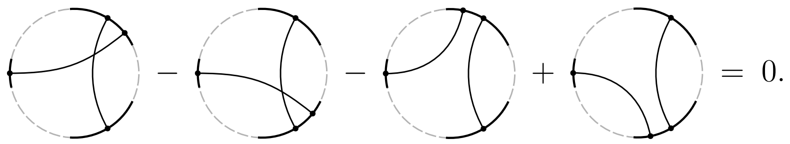
How is this argument in analogy with some property of polynomials? Translating it from the knot-theoreic setting back into the original setting of multivariable calculus, it becomes a proof that polynomials are filtered by degree. So, in summary the cumbersome proof that Δ respects the filtration on $\mathcal{K}$ is a dual-version in a knot-theoretic setting of the fact that polynomials are filtered by degree.

Example 1.3.14 In view of Remark above about non-Vassiliev invariants, let's define a knot invariant. (Define f_{3_1}.)

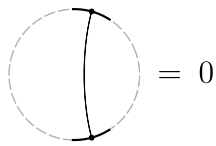
1.4 Chord diagrams and weight systems

This bialgebra strucutre on knots is closely related to a similar bialgebra structure on chord diagrams. Knots are complicated and chord diagrams are much simpler, and the general idea is to study the former via the latter.

In Section 1.2 we saw how functions on chord diagrams specify Vassiliev invariants, so long as the functions satisfy 4T* (and unframed Vassiliev invariants if they further satisfy 1T*). We can instead encode this directly into the algebra of chord diagrams by the following relations.



(4T)



(1T)

Definition 1.4.1 We define $\mathcal{A}_m$, the **space of chord diagrams** of degree m as

$$\mathcal{A}_m = \mathcal{D}_m/\mathtt{4T}$$

and $\mathcal{A}$, the **space of chord diagrams** as

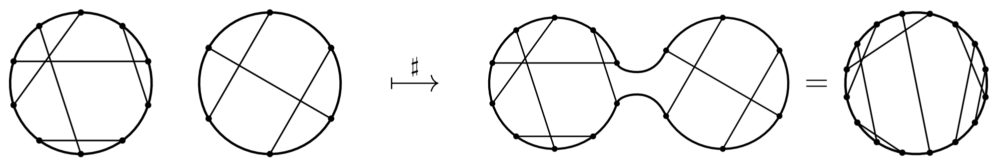
$$\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m.$$

(The **space of unframed chord diagrams**, $\mathcal{A}'$ is defined similarly from $\mathcal{A}'_m = \mathcal{D}_m/\mathtt{4T}, \mathtt{1T}$.)

Warning 1.4.2 Both elements of $\mathcal{A}$ and $\mathcal{D}$ are known as chord diagrams. From now on when we say "a chord diagram", we mean an element of $\mathcal{A}$ unless otherwise specified.

The algebra $\mathcal{A}$ has multiplication and coproduct operations that mirror those in $\mathcal{K}$.

Definition 1.4.3 The **connected sum of two chord diagrams** A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



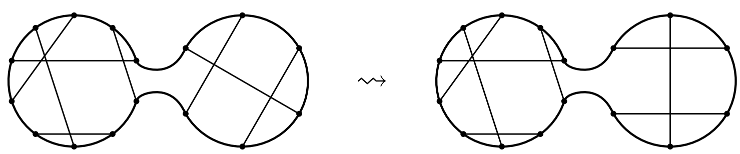
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed in the algebra $\mathcal{D}$ this is ill-defined. However the 4T relation in $\mathcal{A}$ takes care of this.

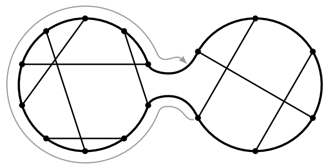
Proposition 1.4.4 *The connected sum operation $\sharp : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is well-defined.*

Proof We will prove that the connected sums of two chord diagrams, given any two choices of connection locations, are equal modulo 4T.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one 'click', like so:

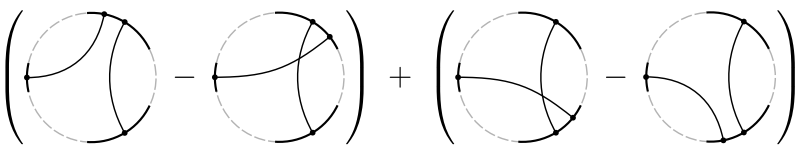


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow.



Which we show can be achieved by a series of 4T relations.

We can rewrite 4T as



$= 0.$

A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T. So, the two connected sums $a_1 \sharp a_2$ differ by a sum of 4T relations, completing the proof. $\square$

Proposition 1.4.5 *The connected sum operation makes $\mathcal{A}$ into a graded algebra.*

Proof No chords are lost during the connected sum: the 4T relation is homogenous with respect to degree, so the connected sum of a chord diagram of order i and a chord diagram of order j is a chord diagram of order $i + j$. $\square$

Just as there is a connected sum operation in $\mathcal{A}$ reminiscent to that in $\mathcal{K}$, there is a coproduct too.

Definition 1.4.6 The **coproduct of a chord diagram** A is the sum of ways of partitioning its chords between two subdiagrams. Specifically, if S is the set of chords of A, and $J \subset S$, let $\widehat{J} = S \setminus J$. Denote by A_J the chord diagram A but with only the chords in $J \subset S$, and the rest deleted. Then

$$\Delta(A) = \sum_{J \subset S} A_J \otimes A_{\overline{J}}.$$

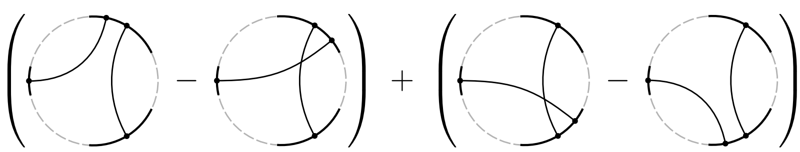
Proposition 1.4.7 *The coproduct Δ is well-defined in $\mathcal{A}$, and makes $\mathcal{A}$ into a graded bialgebra.*

Proof We need to check: that the coproduct factors through the quoitents, that the coproduct respects the grading, and that the coproduct is an algebra morphism (one way of writing the compatibility condition).

That in the unframed case, Δ factors through 1T is easy: an isolated chord in A remains isolated and appears in one cofactor of every term of $\Delta(A)$.

Also, Δ factors through 4T. Suppose that $K = A_1 - A_2 + A_3 - A_4$ is some combination of chord diagrams to be killed by 4T. This means that K looks like

$K =$



where there may be other chords O that the above diagrams have in common, as well as those shown. Note that there is one moving chord in the above diagram and one stationary chord. Let us label these m and s. Take the same partition $J \sqcup \overline{J}$ of S for all of the four chord diagrams at once, and write as the resulting coproduct $\Delta(A_i) = C_i \otimes D_i$. Suppose without loss of generality that m was partitioned into the C_i's. Then $D_1 = D_2 = D_3 = D_4$, so this term of the coproduct factors as $(C_1 - C_2 + C_3 - C_4) \otimes D_1$ and either:

- s was also partitioned into the C_i's, and the relation remains a 4T, or
- s was partitioned into the D_i's, and so $C_1 = C_2$ and $C_3 = C_4$,

and in either case, that term of the coproduct is killed.

The coproduct clearly satisfies

$$\Delta(\mathcal{A}_m) \subset \bigoplus_{i+j=m} \mathcal{A}_i \otimes \mathcal{A}_j = (\mathcal{A} \otimes \mathcal{A})_m,$$

so it is graded.

The compatibility condition holds. If A has chord set S and B has chord set T, then

$$\begin{aligned}\Delta(A) \sharp^{\otimes 2} \Delta(B) &= \left(\sum_{J' \subset S} A_{J'} \otimes A_{\overline{J'}}\right) \sharp^{\otimes 2} \left(\sum_{J'' \subset T} B_{J''} \otimes B_{\overline{J''}}\right) \\ &= \sum_{J \subset S \sqcup T} (A \sharp B)_J \otimes (A \sharp B)_{\overline{J}} \\ &= \Delta(A \sharp B).\end{aligned}$$

□

We have shown that $\mathcal{A}$ is a graded bialgebra of finite type. In fact $\mathcal{A}$ is an even more specific structure.

Definition 1.4.8 A **connected, commutative, cocommutative** graded bialgebra of finite type is a graded bialgebra, A, of finite type for which

- The unit map $\mathcal{Q} \to A_0$ is an isomorphism (**connectedness**)
- The product is commutative, $m \circ \tau = m$
- The coproduct is cocommutative, $\tau \circ \Delta = \Delta$

where $\tau : A \otimes A \to A \otimes A$ sends $x \otimes y \mapsto y \otimes x$.

Proposition 1.4.9 *The bialgebra $\mathcal{A}$ is a connected, commutative, cocommutative graded bialgebra of finite type.*

Proof The connetedness isomorphism is given by associating a scalar k to k times the empty chord diagram.

We have already shown the connected sum to be commutative, and the coproduct can clearly be seen to be cocommutative from the symmetry of Definition 1.4.6. □

Connected, commutative, cocommutative graded bialgebras of finite type are very rigid structures. In-particular, a classical strucutral theorem applies, and such a bialgebra can be understood in terms of its primitive elements.

Definition 1.4.10 An element x is **primitive** in a coalgebra (so in-particular in a bialgebra) with coproduct Δ if it satisfies

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

The set of primitive elements of a bialgebra A is denoted $\mathcal{P}(A)$, and if A is graded, then let $\mathcal{P}_n(A)$ denote the set of primitive elements of degree n.

Theorem 1.4.11 (Milnor-Moore) *Let H be a connected, commutative, cocommutative bialgebra of finite type, over a field of characteristic zero. Then, as an algebra, H is isomorphic to the symmetric algebra of $\mathcal{P}(H)$. In other words, H is a polynomial algebra in $\mathcal{P}(H)$.*

We refer to [Les24] or [CDM12] for a proof. This fact leads to important consequences about the structure of $\mathcal{A}$ and its relation to Lie algebras, which will the subject of Chapter 2.

Definition 1.4.12 The weight systems (Definition 1.2.11), denoted $\mathcal{W}$, form a graded bialgebra, the **graded bialgebra of weight systems** with grading given by degree m, product given by pointwise multiplication

$$W_1 \cdot W_2(a) = W_1(a)W_2(a),$$

and coproduct, η given by

$$\eta(W)(a_1 \otimes a_2) = W(a_1 \sharp a_2).$$

In fact, every graded object is a filtered object with the naturally induced filtration. Considering a graded object as such allows us to take its dual filtered bialgebra (which we refer to as its dual graded bialgebra in this case).

Proposition 1.4.13 *The graded bialgebra $\mathcal{W}$ is the dual graded bialgebra of the graded bialgebra $\mathcal{A}$.*

Proof That $\mathcal{W} = \mathcal{A}^*$ as sets follows from Definition 1.2.11. A weight system is a functionals on chord diagrams that is integrable. So a functional satisfying 4T* and 1T*. That is exactly an element of $\mathcal{A}^*$. The rest of the proof uses the same arguments as Proposition 1.3.12. □

1.5 The fundamental theorem of Vassiliev invariants

In Section 1.2, we gave the fundamental theorem of Vassiliev invariants. The point of this theorem is that is establishes a particular relationship between the algebras of the previous two chapters, $\mathcal{K}$ and $\mathcal{A}$ (or equivalently, between $\mathcal{W}$ and $\mathcal{V}$). But admittedly, in the form of Theorem 1.2.16, it's not a-priori obvious why this is the case. Here we give a restatement of that theorem that makes the relationship explicit.

Definition 1.5.1 The **associated graded** bialgebra of a filtered bialgebra A is the bialgebra formed by the direct sum of the succesive quotients of the filtered components of A. For a bialgebra with a descending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m+1},$$

and for a bialgebra with an ascending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m/A_{m-1},$$

where $A_{-1} = \{0\}$.

The operations on $\operatorname{gr} A$ are those induced from the operations on A. For example, on a bialgebra with a descending filtration, if multiplication $\mu : A_m \otimes A_n \to A_{m+n}$ sends $a \otimes b \mapsto ab$, then the induced operation is

$$\begin{array}{rccc} \operatorname{gr}\mu : & A_m/A_{m+1} \otimes A_n/A_{n+1} & \longrightarrow & A_{m+n}/A_{m+n+1} \\ & a + A_{m+1} \otimes b + A_{n+1} & \longmapsto & ab + A_{m+n+1}. \end{array}$$

For the comultiplication Δ that sends $a \mapsto \bigoplus_{i+j=m} a'_i \otimes a''_i$ the induced operation is

$$\begin{array}{rccc} \operatorname{gr}\Delta : & A_m/A_{m+1} & \longrightarrow & \bigoplus_{i+j=m} A_i/A_{i+1} \otimes A_j/A_{j+1} \\ & a + A_{m+1} & \longmapsto & \bigoplus_{i+j=m} a'_i + A_{i+1} \otimes a''_i + A_{j+1}. \end{array}$$

Theorem 1.5.2 (Fundamental theorem) *The algebra of weight systems is isomorphic to the associated graded algebra of the algebra of Vassiliev invariants, $\mathcal{W} \cong \operatorname{gr}\mathcal{V}$, or on the level of graded components,*

$$\bigoplus_{m=0}^{\infty} \mathcal{W}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{V}_m/\mathcal{V}_{m-1}.$$

Equivalently, this can be stated in the dual setting as follows. The algebra of chord diagrams is isomorphic to the associated graded algebra of the algebra of knots, $\mathcal{A} \cong \operatorname{gr}\mathcal{K}$, or on the level of graded components,

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{K}_m/\mathcal{K}_{m+1}.$$

The equivalence of this version of the theorem with the version (Theorem 1.2.16) given in Section 1.2 will be proven at the end of this section.

We can break the fundamental theorem up into two parts.

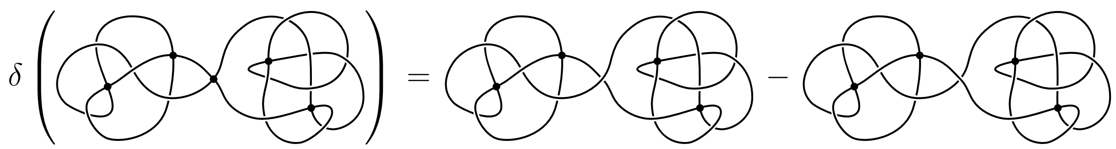
Vassiliev	Every Vassiliev invariant modulo Vassiliev invariants of higher order gives a Weight system, so a map $$\mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{W}_m.$$	Every chord diagram gives an element of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$, so a map $$\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}.$$
Kontsevich	Every weight system gives a Vassiliev invariant modulo Vassiliev invariants of higher order, so a map $$\mathcal{W}_m \to \mathcal{V}_m/\mathcal{V}_{m-1}$$ which is inverse to the above.	Every equivalence class of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$ gives a chord diagram, so a map $$\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m.$$ which is inverse to the above.

The point of the part of this theorem due to Vassiliev [Vas90; Vas92] is that the relations in $\mathcal{A}$ are compatible with the relations of $\mathcal{K}_n/\mathcal{K}_{n+1}$. Indeed, $\mathcal{A}$ was constructed in this way, and much of that work was already done in Section 1.2, and the following proof involves nothing new.

Proof of Theorem 1.5.2 (Vassiliev) The aim is to define a map $\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$. Let us start with a map $\mathcal{D}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$. Given $d \in \mathcal{D}_m$, take a singular knot $k^\bullet \in \mathcal{K}_m^\bullet$ whose chord diagram is d. Then, d is sent to $[\partial^m(k^\bullet)] \in \mathcal{K}_m/\mathcal{K}_{m+1}$.

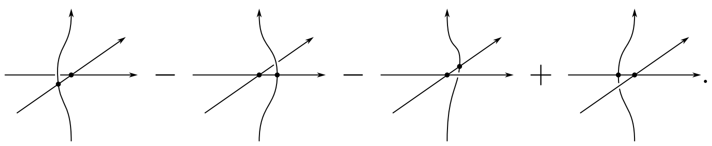
This is well-defined: any other $\ell^\bullet$ with chord diagram d also resolves to $[\partial^m(k^\bullet)]$. Indeed, any two m-singular knots with the same chord diagram differ by some crossing changes. But if $k^\bullet$ and $\ell^\bullet$ differ by a crossing change, then $\partial^m(k^\bullet)$ and $\partial^m(\ell^\bullet)$ differ by an element of $\mathcal{K}_{m+1}$, so $[\partial^m(k^\bullet)] = [\partial^m(\ell^\bullet)]$ in the quotient. This argument that any $k^\bullet \in \mathcal{K}_m^\bullet$ can be chosen to represent $[\partial^m(k^\bullet)]$ as long as it has chord diagram d will be used again below, and we refer to it as the crossing-change argument.

Recalling that $\mathcal{A}_m = \mathcal{D}_m/4\mathrm{T}$ (and $\mathcal{A}'_m = \mathcal{D}_m/1\mathrm{T}, 4\mathrm{T}$), we need to show that the map factors through the quotient. Indeed in the unframed case, 1T is in the kernel: a type 1T chord diagram is sent to a singular knot with a singular point that is passed through twice in a row when the knot is traversed. The resolution of that singular point, by the crossing-change argument, can be chosen to be a difference of two $(m-1)$-singular unframed knots that are singular isotopic:



$$= 0.$$

Similarly, 4T is also in the kernel. A combination of chord diagrams appearing in a 4T relation is sent to a combination of singular knots which by the crossing-change argument can all be chosen to be identical except near a small region, where



Resolving around these four terms about the singular point that they don't have in common yields eight terms which cancel out in pairs.

The isomorphism respects the algebra structure as $a_1 \sharp a_2$ is sent to $[k_{a_1 \sharp a_2}] \in \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ that comes from the resolution of some singular knot we shall denote $k^\bullet_{a_1 \sharp a_2}$ with chord diagram $a_1 \sharp a_2$. But likewise a_1 and a_2 map to $[k_{a_1}]$ and $[k_{a_2}]$ which are resolutions of singular knots $k^\bullet_{a_1}$ and $k^\bullet_{a_2}$. The induced operation from the connect sum $\mathcal{K}_m/\mathcal{K}_{m+1} \otimes \mathcal{K}_n/\mathcal{K}_{n+1} \to \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ takes these to $[k_{a_1} \sharp k_{a_2}]$, which comes from the resolution of $k^\bullet_{a_1} \sharp k^\bullet_{a_2}$. These singular knots may not a-priori be the same, but they have the same chord diagram, so by the crossing-change argument they can be chosen to be the same. Hence resolving then projecting into $\mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ gives the same equivalence class.

Using similar arguments, from the formula of Willerton's Lemma, the isomorphism can be shown to respect the bialgebra structure.

The dual version of the statement follows from the regular version. We also give a more illustrative direct proof of the dual theorem later. $\square$

For now, let's turn to the other part of the fundamental theorem, due to Maxim Kontsevich which is much more involved. In [Kon93], Konsevich constructed an integral invariant which proves the fundamental theorem, as well as containing the information of all Vassiliev invariants at the same time. We will not give a detailed exposition of Kontsevich's invariant here, but they abound in the literature, for example [BS97; CD05; CDM12] in order of increasing level of detail. Rather we will boil the Kontsevich integral down to a single universal property.

Definition 1.5.3 The **completion** of a descending-filtered algebra A is the descending-filtered algebra

$$\widehat{A} = \varprojlim_{m \to \infty} A/A_{m+1}.$$

This is the inverse system

$$0 = A/A_0 \twoheadleftarrow A/A_1 \twoheadleftarrow A/A_2 \twoheadleftarrow \cdots$$

whose mth filtered component is the set of sequences that vanish in A/A_{m+1} (and by the properties of inverse systems, all terms thereafter).

If A is graded, then it is also filtered with the natural descending filtration

$$\bigoplus_{i=0}^{\infty} A_i \supseteq \bigoplus_{i=1}^{\infty} A_i \supseteq \cdots .$$

The degree completion of a graded algebra A if a descending-filtered algebra that coincides with the completion $\widehat{A}$ taken with respect to the natural descending filtration on A.

Definition 1.5.4 A **universal Vassiliev invariant** is a knot invariant $Z : \mathcal{K} \to \widehat{\mathcal{A}}$ with the following property. If $k \in \mathcal{K}_m$ is a linear combination of knots with $k = \partial^m(k^\bullet)$ and $k^\bullet$ has chord diagram $a \in \mathcal{A}_m$, then

$$Z(k) = a + \text{higher degree terms}.$$

Phrased in terms of σ and ∂ this reads

$$Z(\partial^m(k^\bullet)) = \sigma(k^\bullet) + \text{higher degree terms}.$$

Remark 1.5.5 Another equivalent way of defining a universal Vassiliev invariant is as follows. If f is a descending-filtration-respecting map $f : A \to B$, then define the associated graded map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ that sends $a_m + A_{m+1} \mapsto f(a_m) + B_{m+1}$. In other words, a graded map coming from the filtered map f that forgets information about higher degrees. A universal Vassiliev invariant is a map $Z : \mathcal{K} \to X$ whose associated graded $\operatorname{gr} Z : \operatorname{gr} \mathcal{K} \to \operatorname{gr} X$ is an isomorphism.

A universal Vassiliev invariant is such a map with $X = \widehat{\mathcal{A}}$. In particular, note that $\operatorname{gr} \widehat{\mathcal{A}} = \mathcal{A}$ (because the direct sum implicit in gr means $a \in \operatorname{gr} \widehat{\mathcal{A}}$ cannot have infinitely many non-zero terms). So since Z is $\mathcal{K} \to \widehat{\mathcal{A}}$ and satisfies this property, then $\operatorname{gr} \mathcal{K} \cong \mathcal{A}$.

Theorem 1.5.6 (Kontsevich Integral) *There exists a universal Vassiliev invariant, denoted $Z(k)$, called the Kontsevich integral.*

Proof of Theorem 1.5.2 (Kontsevich) Take the map $k \in \mathcal{K}_m \to \mathcal{A}_m$ coming from killing the higher degree terms in the Kontsevich integral, and taking the lowest order non-zero chord diagram. This factors through the quotient to a map $\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m$ since by Theorem 1.5.6, any additional $k' \in \mathcal{K}_{m+1}$ contributes only higher degree terms, which get killed. It is easy to see that the two maps are inverses. □

Again, it's worth looking at the proof in the dual setting too.

Lemma 1.5.7 *Post-composing the Kontsevich integral with a weight system of order* m,

$$k \longmapsto W \circ Z(k)$$

gives a Vassiliev invariant of order m.

Warning 1.5.8 More precisely, this is the following composition

$$\mathcal{K} \xrightarrow{Z} \widehat{\mathcal{A}} \xrightarrow{\pi_m} \mathcal{A}_m \xrightarrow{W} \mathbb{Q}$$

but we will drop the map π_m from the notation. This is equivalent to asserting that a weight system of order m is zero on chord diagrams of order other than m.

Proof The map $W \circ Z$ is clearly an invariant, since Z is an invariant. By the universal property of Z, if $k \in \mathcal{K}_{m+1}$, so then $k = \partial(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}^\bullet_{m+1}$,

$$Z(k) = \sigma(k^\bullet) + \begin{matrix}\text{terms of} \\ \text{order } (m+2)\end{matrix}$$

$Z(k)$ is zero in degrees up to and including m, so composing so composing with a weight system of degree m gives zero. □

Proof of Theorem 1.5.2 (dual) The map defined by Lemma 1.5.7, $\mathcal{W}_m \to \mathcal{V}_m$ is injective, as only the zero weight system gives the zero invariant.

However, it is not surjective. We show that the map, written as

$$\mathcal{W}_m \xrightarrow{Z^*} \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}$$

forces a choice of Vassiliev invariant of degree $m-1$. Being as explicit as possible, let

$$\Omega(k) = \begin{cases} (W \circ Z)(k) & k \in \mathcal{K}_m \\ 0 & k \notin \mathcal{K}_m \end{cases}, \quad \text{and} \quad \Theta(k) = \begin{cases} 0 & k \in \mathcal{K}_m \\ (W \circ Z)(k) & k \notin \mathcal{K}_m \end{cases},$$

then $W \circ Z = \Omega + \Theta$, where Ω recovers the weight system W, when $k \in \mathcal{K}_m$, and Θ is some finite type invariant of order $m-1$, determined by W.

In essence, we have found that the cokernel of Z^* is $\mathcal{V}_{m-1}$, so we get the desired isomorphism $\mathcal{W}_m \cong \mathcal{V}_m/\mathcal{V}_{m-1}$. □

The natural question arises, what was this summand Θ? Fixing $n < m$ and a knot $k \in \mathcal{K}_n$ with chord diagram a_k, it has Kontsevich integral

$$Z(k) = a_k + \begin{matrix}\text{terms of} \\ \text{order } (n+1)\end{matrix} + \cdots + \begin{matrix}\text{terms of} \\ \text{order } m\end{matrix} + \cdots.$$

Applying the projection π_m, all that remains are some chord diagrams of order m, with coefficients depending on the intricacies of the Kontsevich integral for that particular knot. Composing with the weight system, this is a $\mathbb{Q}$-valued Vassiliev invariant of order $m-1$ determined by W.

The name 'universal Vassiliev invariant' we gave to the Kontsevich integrals and invariants of its kind is indeed justified. Every Vassiliev invariant can be obtained through the Kontsevich integral.

Theorem 1.5.9 *If Z is a universal Vassiliev invariant, then every Vassiliev invariant factors through Z.*

Proof Let $V \in \mathcal{V}_m$. Following the proof above, we can project V to $\mathcal{V}_m/\mathcal{V}_{m-1}$ to get a weight system W_m. Subtracting $W_m \circ Z$ leaves a Vassiliev invariant of lesser degree. In other words, via the isomorphism

$$\begin{aligned}\mathcal{V}_m &\cong \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0 \\ &\cong \mathcal{W}_m \oplus \mathcal{W}_{m-1} \oplus \cdots \oplus \mathcal{W}_1 \oplus \mathcal{W}_0,\end{aligned}$$

V can be written as a sequence of weight systems of degree from 1 to m such that V factors through the Kontsevich integral

$$V = \sum_{i=0}^{m} (W_m \circ Z) = \left(\bigoplus_{i=0}^{m} W_m \right) \circ Z.$$

□

Corollary 1.5.10 *A universal Vassiliev invariant (in-particular, the Kontsevich integral Z) is exactly as strong as the set of Vassiliev invariants.*

Definition 1.5.11 Taking the projection $\mathcal{V}_m \to \mathcal{V}_m/\mathcal{V}_{m+1} \cong \mathcal{W}_m$ yields a weight system. The **canonical Vassiliev invariants** are those Vassiliev invariants whose weight systems W recover them completely via $W \circ Z$.

In other words, not all bases of $\mathcal{V}_m$ are created equal. The canonical Vassiliev invariants are those that are homogenous with respect to the splitting of $\mathcal{V}_m$

$$\mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0$$

induced by the Kontsevich integral. Canonical Vassiliev invariants were first defined in [BG96] and used to prove the Melvin-Morton-Rozansky conjecture relating the coefficients of Alexander polynomial of a knot to those of the coloured Jones polynomials. This is a good example of how the theory of Vassiliev invariants is useful for probing the structure of knots.

Question 1.5.12 (Bar-Natan–Garoufalidis) There are a few known from-the-bottom-up constructions of universal Vassiliev invariants of knots. However they are all equivalent to, or conjecturally equivalent to the Kontsevich integral. Yet the Kontsevich integral is not the only invariant that can satisfy the defining degree property of a universal Vassiliev invariant.

Is there a reason why the Kontsevich integral, or equivalently, this splitting appears to be canonical?

Finally, we return to the equivalence of the two fundamental theorems.

Proof (Equivalence of Theorems 1.2.16 and 1.5.2) For the forward direction, suppose Theorem 1.2.16 holds. This states that every invariant $v^\bullet$ of m-singular knots satisfying T4T* and further that $\delta v^\bullet = 0$, then $v^\bullet$ integrates to an invariant v of 0-singular knots.

First we prove that the Kontsevich part of the fundamental theorem holds. Let W be a weight system of order m. Then W defines an invariant $v_W^\bullet$ of m-singular knots by $v_W^\bullet(k) = W(\sigma(k))$. The derivative of $v_W^\bullet$ is zero. Indeed,

$$\delta v_W^\bullet(k) = W(\sigma(k^+)) - W(\sigma(k^-))$$

for some knots k^+ and k^- that differ by crossing changes, but σ is invariant under crossing changes. Also, since W is a weight system it satisfies 4T*, and so W' satisfies T4T*. Therefore the hypotheses of Theorem 1.2.16 hold and integrates into a Vassiliev invariant.

The Vassiliev part of the theorem is independent of the original version and was proven separately in Section 1.2.

Now for the reverse direction, suppose Theorem 1.5.2 holds. Take a m-singular knot invariant $v^\bullet$ satisfying 4T* and that $\delta v^\bullet = 0$. This defines a weight system $W_{v^\bullet}$, which by the Kontsevich part of Theorem 1.5.2 gives an invariant class in $\mathcal{V}_m/\mathcal{V}_{m-1}$. Explicitly, this invariant is $W_{v^\bullet} \circ Z$. But by definition of δ,

$$\delta^m(W_{v^\bullet} \circ Z)(k^\bullet) = (W_{v^\bullet} \circ Z)(\partial^m k^\bullet)$$

which by the definition of a universal Vassiliev invariant is just $v^\bullet$.

Thus, $W_{v^\bullet} \circ Z$ is the mth derivative of $v^\bullet$, so $v^\bullet$ integrates into a Vassiliev invariant of order m, completing the proof.

Without loss of generality, the proof works also in the unframed case. $\square$

2

Lie theory and Jacobi diagrams

The fundamental theorem of Vassiliev invariants states that the bialgebra of Vassiliev invariants can be broken up into nice combinatiorial weight systems. So to understand $\mathcal{V}$ it suffices to understand $\mathcal{W}$, or equivalently its dual $\mathcal{A}$. There is a hint that the structure of $\mathcal{A}$ may relate to Lie algebras.

2.1 Jacobi diagrams

This side of the story reframes the bialgebra $\mathcal{A}$ as an isomorphic bialgebra known as the algebra of Jacobi diagrams to illuminate the Lie theory connections.

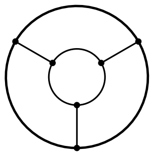
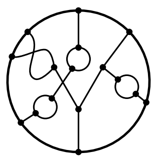
Definition 2.1.1 A **unitrivalent diagram** is a unitrivalent graph (with loops and multiple edges allowed) with the following additional data:

- each trivalent vertex has a fixed cyclic order of incident edge-connections,
- the set of univalent vertices has a fixed cyclic order.

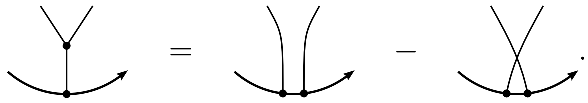
The vector space of unitrivalent diagrams is denoted $\mathcal{T}$.

When drawing unitrivalent diagrams, there are two notation conventions. Firstly, the fixed cyclic order of the univalent edges is specified by drawing them connected to a circle (the cyclic order is induced by traversing the circle anticlockwise). Secondly, all the trivalent vertices are taken with the anticlockwise cyclic ordering unless an arrow around that vertex indicates otherwise.

In particular, from the first point, all chord diagrams are unitrivalent diagrams with only univalent vertices (the chord ends). Further examples of unitrivalent diagrams would be

 ,  $\in \mathcal{T}$.

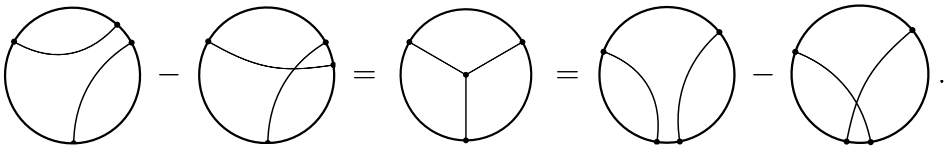
Definition 2.1.2 The **STU relation** is the relation



(STU)

As usual, this is not an individual relation but a class of relations, true in any diagrams that are identical except for the parts shown.

Note that the for the chord diagrams inside the algebra of Jacobi diagrams, the STU relations imply the 4T relations, as

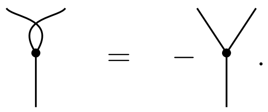


Definition 2.1.3 The algebra $\mathcal{J}$ of Jacobi diagrams is the vector space $\mathcal{T}/\mathtt{STU}$, with the product $\sharp$ defined the same way as it was for chord diagrams.

This is well-defined: the proof of Proposition 1.3.4 showed that the product $\sharp$ being well-defined on $\mathcal{A}$ was a consequence of the 4T relations, which are implied by the STU relations. From the STU relations, we may deduce the following other relations which hold in $\mathcal{J}$.

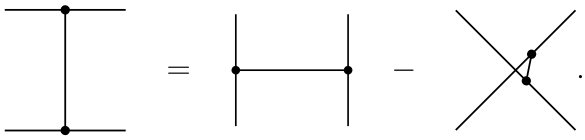
Proposition 2.1.4 *The following relations are consequences of the* STU *relation in* $\mathcal{J}$*:*

*(a) The **AS relation** (antisymmetry relation),*



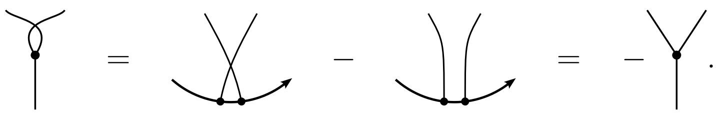
(AS)

*(b) The **IHX relation**,*



(IHX)

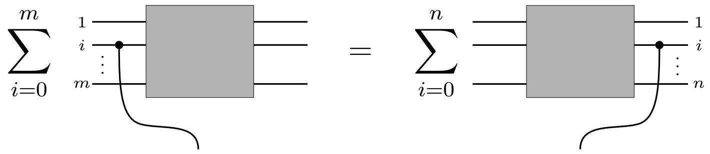
Proof (a) Take two diagrams which differ only by AS at one (trivalent) vertex. If the vertex at which the AS relation resides is adjacent to a univalent vertex (i.e. touches the outer circle), then this is immediate from applying STU to both diagrams at that vertex.



If the vertex is not immediately adjacent to a univalent vertex, then it has some d vertices 'in the way'. By applying STU to those vertices yields a sum of 2^d diagrams, all identical except for differing by AS, now on a vertex adjacent to a univalent vertex.

(b) A similar argument applies. If one of the two vertices of the IHX is adjacent to the circle, then the result is a direct consequence of an STU on each of the vertices, then some applications of AS. Otherwise, some STUs are required first. □

Proposition 2.1.5 (Generalised IHX) *The following holds in $\mathcal{J}$ for any subgraph consisting of trivalent vertices that can be inserted into the grey box.*

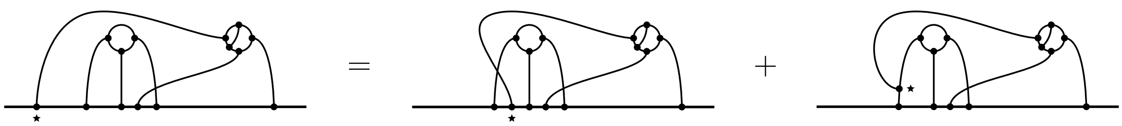


The result is standard, see Chapter 5.2 of [CDM12] for a proof. A corollory is the following result, which will be important later.

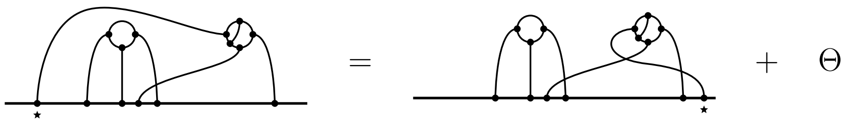
Proposition 2.1.6 *If the univalent vertices of a Jacobi diagram are ordered linearly rather than cyclically, all linear orders that respect a given cyclic order are equivalent.*

Proof Let us draw the linearly ordered Jacobi diagrams on a line rather than a circle and let the univalent edge ordering be given by the order on the line. It suffices to prove that we can move the univalent vertex in the first place to the last place via the relations in $\mathcal{A}$. Let us call this first univalent vertex of the original diagram the marked vertex, as its position will later change when relations are applied.

By STU, the original diagram is equal to the diagram with the marked vertex moved into the second place, plus a diagram in which the marked vertex is now a trivalent vertex and attached above what was the second (but is now the first) univalent vertex.



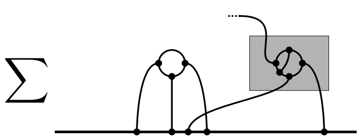
Repeatedly applying STU to move the the marked vertex until reaches the last place, we get



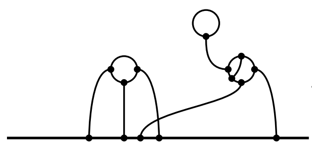
where Θ is a sum of terms with the marked vertex now a trivalent vertex attached above each of the other univalent vertices. It suffices to show that Θ vanishes.

We can split Θ up based on which connected component the marked vertex now connects to. For connected components other than the connected component of the marked vertex, apply a vertical version of the the generalised IHX, where the the whole connected component is inside the grey box except for where the univalent vertices connect to the bottom line. This is the case of the generalised IHX where $n = 0$ and no vertices leave. Hence, the sum vanishes for that connected component.

This leaves only the terms where the marked vertex connects back to its own connected component. By a generalised IHX of the form



, this is equal to the diagram



.

But any diagram with a “balloon” vanishes, as applying the AS relation to the vertex on the balloon, it is equal to its negative.

Hence $\Theta = 0$, completing the proof. ☐

We have already spoiled the surprise that in the end, $\mathcal{A}$ and $\mathcal{J}$ will be isomorphic as bialgebras. In fact, as algebras, this is clearly true so far, as $\mathcal{J}$ is just a change of basis from $\mathcal{A}$. Since $\mathcal{A}$ spans $\mathcal{J}$, we can attempt to lift the coproduct from $\mathcal{A}$ directly onto $\mathcal{J}$.

Proposition 2.1.7 *The coproduct Δ on $J \in \mathcal{J}$ defined by taking a Jacobi diagram, representing it as a chord diagram via* STU*, taking the coproduct in $\mathcal{A}$, then interpreting the result as a Jacobi diagram via the inclusion of $\mathcal{A}$ into $\mathcal{J}$, is also given by the following formula.*

$$\Delta(J) = \sum_{C \subset S} J_C \otimes J_{\overline{C}},$$

where S is the set of connected components of J, and $\overline{C} = S \smallsetminus C$.

Proof Note that this has the same symbolic form as the coproduct in $\mathcal{A}$ given in Definition 1.4.6, but with chords replaced by connected components of Jacobi diagrams. However, when working in $\mathcal{A} \subset \mathcal{J}$, there are only univalent vertices, so the connected components are exactly the chords. Since $\mathcal{A}$ forms a basis for $\mathcal{J}$, and the formula is linear, it extends to all of $\mathcal{J}$. $\square$

Corollary 2.1.8 *The primitive elements $\mathcal{P}(\mathcal{A})$ are the connected Jacobi diagrams.*

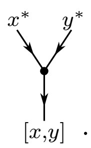
Corollary 2.1.9 *The bialgebras $\mathcal{A}$ and $\mathcal{J}$ are isomorphic.*

Warning 2.1.10 Justified by this isomorphism, we henceforth write $\mathcal{A}$, for both chord diagrams and Jacobi diagrams.

2.2 Lie algebra weight systems

Similar diagrammatic relations to STU, AS and IHX satisfied in $\mathcal{A}$ are appear also in the context of a graphical notation for multilinear maps, a fact which we may exploit to probe $\mathcal{A}$. Before seeing how, let us review this graphical notation following [Thu00; RW06]. This diagrammatic calculus is well-known but it goes by many names: string diagram calculus, Penrose calculus, tensor calculus, diagrammatic calculus for tensors, etc. We call them string diagrams.

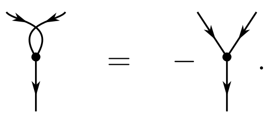
A tensor is a multilinear map $X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$, or equivalently (via the canonical isomorphism) an element of the vector space $X_1^* \otimes X_2^* \otimes \cdots X_n^* \otimes Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$. Such a tensor can be represented as a vertex with $m + n$ unbound directed edges: m incoming edges decorated by the corresponding vector spaces (in the example above, $X_1, \cdots$), and n outgoing edges decordated by $Y_1, \cdots$. For example, the bracket in a Lie algebra $\mathfrak{g}$ is an element $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, expressed as



.

It will be obvious from each string diagram what tensor it represents so we can drop the edge labels.

Such a notation is useful because composition of tensors can be expressed graphically by connecting outgoing and incoming legs with the same decoration. Relations can therefore be expressed graphically, for example, the antisymmetry of the bracket $[y, x] = -[x, y]$ becomes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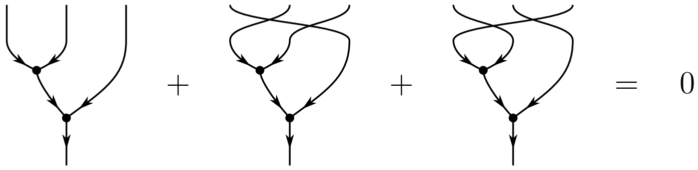
the Jacobi relation $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ becomes

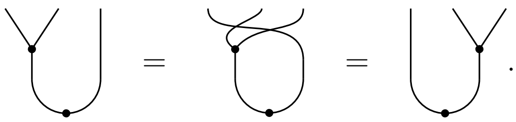


Looking at the relations these relations in the tensor algebra $\mathcal{T}(\mathfrak{g})$, the first solid evidence of Lie-theoretic structure in this story emerges. The antisymmetry of the bracket, drawn as a string diagram looks like a directed version of `AS`. Similarly the string diagram Jacobi relation can be arranged into a directed version of `IHX`.

Furthermore, suppose $\mathfrak{g}$ is a metric Lie algebra. Then it has an invariant, nondegenerate, bilinear form $\langle\cdot,\cdot\rangle \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. Being nondegenerate, it can be inverted to an element $c \in \mathfrak{g} \otimes \mathfrak{g}$. This element is known as the casimir. These tensors can be diagrammatically represented as additional bivalent vertices.

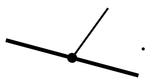
The bilinear form induces an isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$. Diagramatically, this can be used to change the arrow direction on any edge, allowing us to drop the edge arrows from the notation.

Moreover, the invariance of the metric can be written as $\langle[x, y], z\rangle = \langle[y, z], x\rangle$ which graphically can be represented as cyclic invariance of the contraction of the bracket and the metric



A similar argument works for the casimir.

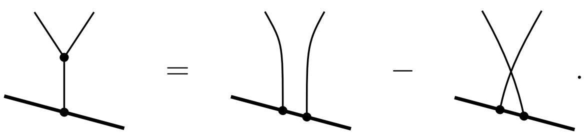
A representation of $\mathfrak{g}$ on a finite-dimensional vector space V can be written as a tensor $\rho \in \mathfrak{g}^* \otimes V^* \otimes V$. This takes a new kind of input an output, namely a $v \in V$ which we denote by a thick line at a shallow angle



That this action be a Lie action,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

is graphically



Again, arrows are unnecessary on the thick edges corresponding to inputs and outputs of V, as cup and cap vertices similar to the metric and casimir for $\mathfrak{g}$ are given by the maps

$$f \otimes v \longmapsto f(v) \quad \text{and} \quad 1 \longmapsto \sum_i e_i \otimes e_i^*.$$

The famous construction of Bar-Natan [Bar95] uses this diagrammatic calculus to take metric Lie algebras and produce weight systems.

Construction 2.2.1 The construction takes a metric Lie algebra $\mathfrak{g}$, and produces a map $W_\mathfrak{g} : \mathcal{A} \to \mathcal{U}(\mathfrak{g})$ which is a $\mathcal{U}(\mathfrak{g})$-valued weight system. Given further a representation ρ of $\mathfrak{g}$ it produces a map $W_\mathfrak{g} : \mathcal{A} \to k$ which is a k-valued weight system. That is, given $\mathfrak{g}$ a metric Lie algebra, $J \in \mathcal{A}$ it produces an element of $\mathcal{U}(\mathfrak{g})$, and if also given a representation ρ it produces a scalar. We write v and u for the number of trivalent and univalent vertices of J.

To each trivalent vertex of J, associate a copy of the tensor $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ (see Warning 2.2.2). For each edge between trivalent vertices, contract the corresponding tensors along the components corresponding to those half-edges. Where the signature of the components doesn't allow for contraction (both components are covariant, i.e. inputs or both are contravariant, i.e. outputs), contract one of them first with either the metric or the casimir (whichever is allowed by its variance). The resulting tensor has u components (they may be co- or contra-variant).

Contract this tensor with a copy of the casimir along all remaining covariant components. We define the result to be the tensor $T_\mathfrak{g}(J) \in \mathfrak{g}^{\otimes u}$. The linear order of its components must be a linear order that agrees with the cyclic order of the corresponding univalent vertices in J. Define $W_\mathfrak{g}(J)$ to be the projection of this tensor into $\mathcal{U}(\mathfrak{g})$,

$$W_\mathfrak{g}(J) = [T_\mathfrak{g}(J)] \in \mathcal{U}(\mathfrak{g}).$$

A representation $\rho : \mathfrak{g} \to \mathrm{Hom}(V)$ of a Lie algebra extends uniquely to a representation of its universal enveloping algebra $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{Hom}(V)$. Define $W_{\mathfrak{g},\rho}$ as the trace of $W_\mathfrak{g}(J)$ with respect to this representation,

$$W_{\mathfrak{g},\rho}(J) = \mathrm{tr}(\rho(W_\mathfrak{g}(J))) \in \mathbb{Q}.$$

Warning 2.2.2 In Construction 2.2.1 when constructing $T_\mathfrak{g}(m)$, the tensor factors in the tensor corresponding to the bracket need to have the unusual cyclic order $(y^*, x^*, [x,y]_\mathfrak{g})$. This is because its projection into $\mathcal{U}(\mathfrak{g})$ should obey `STU`, and this is the cyclic order of the trivalent vertex in `STU` (this will become evident in the proof that this construction works).

Example 2.2.3 Take the metric Lie algebra $(\mathfrak{sl}_2, \langle\cdot,\cdot\rangle)$ where $\mathfrak{sl}_2$ is defined by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h,$$

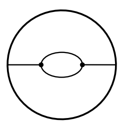
and the metric is

$$\langle h,h\rangle = 2, \quad \langle e,f\rangle = 1, \quad \langle f,e\rangle = 1.$$

Here we have chosen the normalisation of the metric that agrees with the trace in the adjoint representation. So,

$$[\cdot,\cdot] = 2e^* \otimes h^* \otimes e - 2f^* \otimes h^* \otimes f + f^* \otimes e^* \otimes h.$$

We do the computations for

$J =$  .

Taking another copy $[\cdot,\cdot]'$ of the bracket tensor, one way to compute $W_{\mathfrak{sl}_2}(J)$ is to let $[\cdot,\cdot]$ take the left trivalent vertex, and associate the upward facing half-edge to the first component. Let $[\cdot,\cdot]'$ take the right trivalent vertex and associate the downward facing half-edge to the first component. The cyclic orders determine the rest. Then the computation is to take the contraction of $[\cdot,\cdot]$ along components 1 and 3 with $[\cdot,\cdot]'$ along components 3 and 1. This gives

$$2h^* \otimes h^* + e^* \otimes f^* + f^* \otimes e^*,$$

and contracting along each component with a casimir to make them contravariant,

$$T_{\mathfrak{sl}_2}(J) = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e.$$

Projecting this into $\mathcal{U}(\mathfrak{g})$ and writing it in the PBW-basis,

$$W_{\mathfrak{sl}_2}(J) = \frac{1}{2} h \otimes h - h + 2e \otimes f.$$

Finally, if we use the adjoint representation ad,

$$\mathrm{ad}(h) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathrm{ad}(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{ad}(f) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we get

$$W_{\mathfrak{sl}_2,\mathrm{ad}}(J) = 3.$$

$\Diamond$

Theorem 2.2.4 *In Construction 2.2.1, $W_{\mathfrak{g}}$ is a well-defined $\mathcal{U}(\mathfrak{g})$-valued weight system, and $W_{\mathfrak{g},\rho}$ is a well-defined $\mathbb{Q}$-valued weight system.*

Proof ([Bar95]) To prove that $W_{\mathfrak{g},\rho}$ is well-defined, there are two points to prove. Jacobi diagrams are defined up to certain internal relations, namely `IHX` and `AS`, so the first point is that $T_{\mathfrak{g}}(J)$ is invariant under `IHX` and `AS`. For `AS`, it follows from the antisymmetry of $[\cdot,\cdot]$, and for `IHX` it follows from the Jacobi relation. Alternatively, it follows from the second point as `IHX` and `AS` are consequences of `STU`.

The second thing to prove is that $W_{\mathfrak{g}}(J)$ is invariant under `STU`. If two chord diagrams differ by `STU`, on some univalent vertices associated with adjacent tensor factors y and x, the construction gives

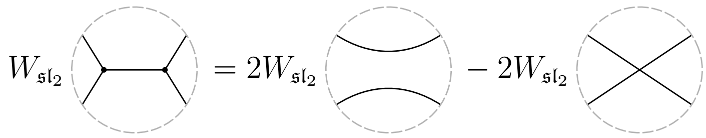
$$\cdots \otimes [x,y]_{\mathfrak{g}} \otimes \cdots \qquad \text{and} \qquad (\cdots \otimes x \otimes y \otimes \cdots) - (\cdots \otimes y \otimes x \otimes \cdots),$$

but this equality is exactly the defining relation of $\mathcal{U}(\mathfrak{g})$.

We come to the third point. There was one other arbitrary choice we made. The cyclic order of the univalent vertices of J induce a cyclic order on the components of the components of the tensor $T_{\mathfrak{g}} \in \mathfrak{g}^{\otimes u}$. However, in the construction, a linear order which respects that cyclic order was chosen. We must show that any choice of linear order respecting the cyclic order produces the same result. This is no problem for the well-definition of $W_{\mathfrak{g},\rho}(J) = \mathrm{tr}(\rho(W_{\mathfrak{g}}(J)))$, as the trace is invariant under cyclic permutation. However for $W_{\mathfrak{g}}(J) = [T_{\mathfrak{g}}(J)] \in \mathcal{U}(\mathfrak{g})$ it remains to prove cyclic permutation invariance. In fact a stronger statement is true. The diagram J itself is invariant under a cyclic permutation by Proposition 2.1.6, so of course this is true for $W_{\mathfrak{g}}(J)$. $\square$

Let's look at a specific weight system for the Lie algebra $\mathfrak{sl}_2$ [Bar95; CV97].

Example 2.2.5 (Weight system for $\mathfrak{sl}_2$) We use the metric Lie algebra from Example 2.2.3. In [CV97], the following skein relation is derived:

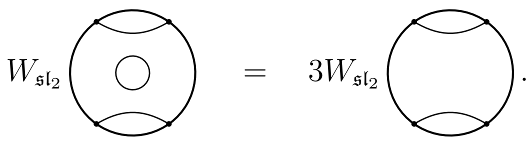


◊

Proof Compute both sides by contracting and permuting tensors for the bracket, and casimir/metric in $\mathfrak{sl}_2$. Both give

$$\begin{aligned} &- h\otimes e\otimes h\otimes f + h\otimes e\otimes f\otimes h - h\otimes f\otimes h\otimes e + h\otimes f\otimes e\otimes h \\ &+ e\otimes h\otimes h\otimes f - e\otimes h\otimes f\otimes h + 2e\otimes f\otimes e\otimes f - 2e\otimes f\otimes f\otimes e \\ &+ f\otimes h\otimes h\otimes e - f\otimes h\otimes e\otimes h - 2f\otimes e\otimes e\otimes f + 2f\otimes e\otimes f\otimes e. \end{aligned}$$ □

Remark 2.2.6 When computing via the $\mathfrak{sl}_2$ skein relation above, it's possible to create a "bubble" (part of a diagram without any trivalent vertices). Since we are computing via contractions in the tensor algebra, this is to be interpreted as the contraction of the metric with the casimir. In a finite-dimensional Lie algebra, this is just the dimension, so for $\mathfrak{sl}_2$, the factor 3. For example



Question 2.2.7 In [CDM12, Remark 16.9] it is noted that the $\mathfrak{sl}_2$ skein relation is an analogue of the vector triple product rule for the cross product in $\mathbb{R}^3$. The relation has been further studied in [MS17] to determine a basis for $\mathcal{W}_{\mathfrak{sl}_2}$. There is also a cross product in $\mathbb{R}^7$, related to the exceptional Lie algebra $\mathfrak{g}_2$. It also obeys a variant of the vector triple product rule. Is there a similar skein relation for $\mathcal{W}_{\mathfrak{g}_2}$? Are there other skein relations that the weight systems for the exceptional Lie algebras obey?

Construction 2.2.1 yields a way of extracting some information from $\mathcal{A}$ by plugging in a metric Lie algebra – doing so constructs some quotient of $\mathcal{A}$. This naturally begs the question whether all of the information in $\mathcal{A}$ can be extracted by metric Lie algebras in this way.

A Computer enumeration of [Bar95] proves this for order $m \leq 9$:

m	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{W}_m$	1	1	2	3	6	10	19	33	60	104
$\dim(\mathcal{W}_{\text{Lie}})_m$	1	1	2	3	6	10	19	33	60	104

Here $\mathcal{W}_{\text{Lie}}$ denotes the dimension of the subspace of $\mathcal{W}_m$ spanned by weight systems coming from Construction 2.2.1. In other words

$$\mathcal{W}_{\text{Lie}} = \operatorname{span}\{W_{\mathfrak{g},\rho} \mid \mathfrak{g} \text{ a Lie algebra}, \rho \text{ a representation of } \mathfrak{g}\}.$$

In [Bar95], this is computed up to degree 9 only using the span of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ Lie algebras, providing a lower bound on $\dim(\mathcal{W}_{\text{Lie}})_m$. The upper bound comes from $\dim \mathcal{A}_m$. So in fact, up to degree 9, $\mathcal{W}_m$ is spanned even by only the $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ weight systems.

Conjecture 2.2.8 (Bar-Natan) *All weight systems are obtained as Lie algebra weight systems. In other words the set* $\{W_{\mathfrak{g},\rho} \mid \mathfrak{g}$ *a Lie algebra,* ρ *a representation of* $\mathfrak{g}\}$ *spans* $\mathcal{W}$.

Indeed the Lie action relation $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ as it was drawn graphically looks exactly like the `STU` relation in $\mathcal{A}$. However, looks can be decieving and quite surprisingly this conjecture is false. The counterexample was found by Pierre Vogel [Vog97] in an attempt to answer the following related question:

Question 2.2.9 (Vogel) Is there some single universal Lie algebra object whose weight system spans W_{Lie}, the span of all Lie-algebraic weight systems?

2.3 Non-Lie algebraic weight systems

Conjecture 2.2.8 being false implies that `STU` is more general than $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$. The same is true for `IHX` and `AS` compared to their strictly Lie-theoretic counterparts. In fact, Construction 2.2.1 is just one example of a more general construction introduced by Vogel and Vaintrob to construct weight systems coming from metric Lie super-algebras.

The most general type of objects these constructions apply to are called by Vaintrob [Vai94] 'Lie S-algebras', but we will follow the more modern approach of [RW06; Rob01] and they will be known as Lie algebra objects in a symmetric monoidal category.

Definition 2.3.1 A **(weak) monoidal category** is a category $\mathcal{C}$ equipped with a functor

$$\begin{array}{rccc} \otimes: & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C} \\ & (A,B) & \longmapsto & A\otimes B, \end{array}$$

a **unit** object $k \in \mathcal{C}$, and natural isomorphisms

$$\otimes\circ(\otimes\times\mathrm{id}) \longrightarrow \otimes\circ(\mathrm{id}\times\otimes) \quad\text{and}\quad \otimes \longrightarrow \mathrm{id}$$

satisfying some relations known as the pentagon and triangle relations [Lei04, Sec. 1.2]. The natural isomorphisms give isomorphisms

$$(A\otimes B)\otimes C \cong A\otimes(B\otimes C), \qquad k\otimes A\cong A\cong A\otimes k$$

for every tuple of objects A, B and C in $\mathcal{C}$. If these isomorphisms are equalities, then $\mathcal{C}$ is a **strict** monoidal category.

Remark 2.3.2 Omitting the details, we assume that these natural isomorphisms are equalities, for example that $(A\otimes B)\otimes C = A\otimes(B\otimes C)$. This is acceptable by the coherence theorem for monoidal categories which says that every monoidal category is equivalent to a strict monoidal category, and it's why we omit the pentagon and triangle relations in the definition above. We refer to [Lei04, Sec. 1.2] for details.

Definitions 2.3.3 (a) The **flip functor** is the functor

$$\begin{array}{rccc} \sigma: & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C}\times\mathcal{C} \\ & (A,B) & \longmapsto & (B,A). \end{array}$$

(b) A **symmetric monoidal category** is a monoidal category $\mathcal{C}$ equipped with a **symmetry natural isomorphism** τ

$$\otimes \longrightarrow \otimes \circ \sigma$$

satisfying the hexagon relation. The **hexagon relation** is the relation that the isomorphisms

$$\tau_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

coming from the natural isomorphism τ obey

$$\tau_{A,B\otimes C} = (\mathrm{id}_B \otimes \tau_{A,C}) \circ (\tau_{A,B} \otimes \mathrm{id}_C)$$

for every pair of objects A, and B in $\mathcal{C}$.

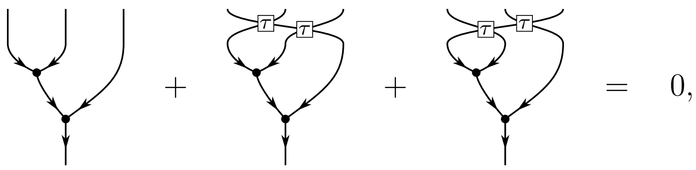
The hexagon relations have look like they only have three terms instead of the six one might expect from the name. This is because we have omited the reassociation natural isomorphisms that we can assume are identites by Remark 2.3.2.

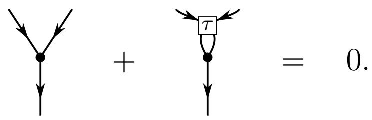
If a monoidal category $\mathcal{C}$ is additionally additive, we can define Lie algebra objects internal to $\mathcal{C}$.

Definitions 2.3.4 (a) A **Lie algebra object** in an additive symmetric tensor category $\mathcal{C}$ is an object L equipped with a bracket morphism $\beta : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ such that

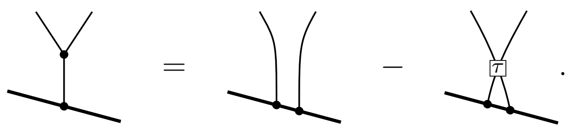
$$(\beta \circ (\beta \otimes \mathrm{id})) \circ (1 + \tau_{123} + (\tau_{123})^2) = 0 \qquad \text{and} \qquad \beta + \beta \circ \tau = 0.$$

Graphically, in terms of the string diagrams of the previous section, this is





(b) A **representation of a Lie algebra object** L in $\mathcal{C}$ into an object V in $\mathcal{C}$ is a morphism $\rho : L \otimes V \to V$, such that



(c) A **metric Lie algebra object** L in $\mathcal{C}$ is a Lie algebra object in $\mathcal{C}$, further equipped with the following modules over L...

We will give various concrete examples of Lie algebra objects in different symmetric tensor categories later. But for now, let's show that this data can still be used to construct weight systems, generalising Construction 2.2.1.

Theorem 2.3.5 ([Vai94]) *Let $\mathcal{C}$ be a rigid, additive, symmetric monoidal category, L a metric Lie algebra in $\mathcal{C}$, and η a dualisable representation. Then, there is a weight system*

$$W_{L,\eta} : \mathcal{A} \longrightarrow \mathcal{C}(k,k) \cong k.$$

Proof Similar to the proof of the construction... □

The difference between this construction and Bar-Natan's original one is the treatment of the symmetry natural isomorphism τ. This tells us which isomorphism to use when rearranging the tensor factors. The most obvious isomorphism would be the identity, as it was in the original construction, corresponding to when $\mathcal{C}$ is a strict symmetric (strict) monoidal category. However unlike for general monoidal categories, not every symmetric monoidal category is equivalent to a strict symmetric monoidal category. At a down-to-earth level, Lie algebra objects with non-trivial symmetry isomorphisms are necessary to pick up all the structure in $\mathcal{A}$.

Example 2.3.6 If we take $\mathcal{C} = \mathbf{sVect}$, the symmetric monoidal category of super vector spaces, the Lie algebra objects are the following.

A **Lie superalgebra** $\mathfrak{g}$ is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading, equipped with a bracket $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ satisfying some axioms to follow. The grading induces the splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the direct summand $\mathfrak{g}_0$ is known as the **even** part and the summand $\mathfrak{g}_1$ is known as the **odd** part.

The axioms a Lie superalgebra must satisfy are the following analogues of the usual lie algebra axioms. Let x, y, z be homogeneous elements, so in $\mathfrak{g}_i$, $\mathfrak{g}_j$, and $\mathfrak{g}_k$ respectively, then **super symmetry** axiom is

$$[x, y] = -(-1)^{ij}[y, x],$$

and the **super Jacobi identity** is the axiom

$$(-1)^{ik}[x, [y, z]] + (-1)^{ji}[y, [z, x]] + (-1)^{kj}[z, [x, y]] = 0. \qquad \diamondsuit$$

Shortly after Conjecture 2.2.8 was made, the following results were achieved based on constructions with the exceptional Lie superalgebra $\mathfrak{D}(2, 1; \alpha)$.

Theorem 2.3.7 *There are primitive Jacobi diagrams of order at least 17 [Vog97] and at least 15 [Lie99] which vanish under all Lie algebra weight systems.*

Furthermore, as was known about shortly after but not published until significantly later,

Theorem 2.3.8 *Vogel's diagram of order 17 vanishes also on all Lie superalgebra weight systems [Vog11].*

Corollary 2.3.9 *The set of Lie (super)algebra weight systems does not span $\mathcal{W}$.*

In general, it is still unknown to what exact level of generality one needs to go (what type of symmetric monoidal categories need to be considered) in order to generate all weight systems. We hereby provide a succinct review the current state of the literature on the subject.

Roberts and Willerton in [Rob01] and [RW06] examine weight systems constructructed from Lie algebra objects in the derived category of complex manifolds. Such weight systems are candidates for being able to detect knot orientation, which Lie algebra weight systems cannot [Bar95]. However, computing these weight systems is difficult, and to our knowledge, no computations exist in the literature.

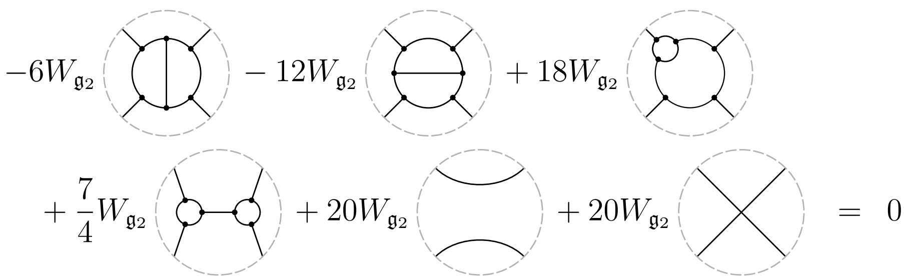
More recently Aizawa-Kimura [AK25], have conducted some preliminary investigations into the class of colour Lie algebras (also known as ϵ-Lie algebras). This class generalises the $\mathbb{Z}/2\mathbb{Z}$ grading on Lie superalgebras to a more general type of group and sign rule. The example they present lies within the span of the $\mathfrak{sl}_2$ and $\mathfrak{gl}_{1|1}$ weight systems.

2.4 Some weight systems at the exceptional Lie algebras

The original motivation for the work of Vogel [Vog97; Vog11] (and it can be seen more explicity in [Vog99]) was to construct a universal object generalising all simple Lie algebras. Such an object would parametrise the relations in tensor powers of the adjoint representation of all simple Lie algebras, and its weight systems would span $\mathcal{W}$. For example, the relation for $W_{\mathfrak{sl}_2}$ given in Example 2.2.5 is entirely internal to the Lie algebra; it doesn't involve projecting to the universal enveloping algebra or choosing a representation (since the adjoint representation is natural).

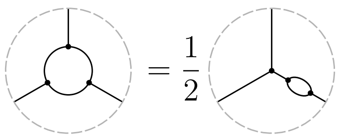
There seems to be a recent renewed interest in Vogel's methods [KLS25; BM25; KMS25]. In-particular, present implicitly in [Vog11] is a way to determine local relations internal to any simple Lie algebra in terms of two independent so-called Vogel parameters, and some complicated 'marked' elements of $\mathcal{A}$ defined recursively. These were written out explicitly in the preprint [KLS25, Appendix]. The use of different normalisations of the Vogel parameters between this and [Vog11] makes computing explicit relations tedious. Furthermore, in Vogel's original work, the relations are general enough to hold for any Lie superalgebra. However when restricting to Lie algebras, the diagrammatic formulae can be simplified.

Theorem 6.3(2) of [Vog11] gives a relation in $W_{\mathfrak{g}_2}$.

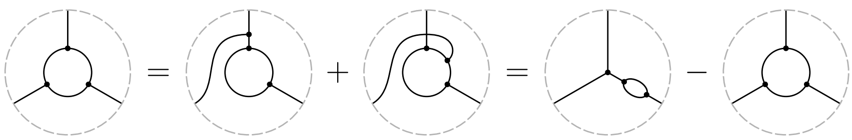


However, by some well-known identies internal to Jabobi diagrams, these diagrams can be simplified.

Lemma 2.4.1 *In $\mathcal{A}$ the 'trivalent-bubble' vertex is proprtional to the regular trivalent vertex with a bivalent bubble inserted on any one of the half-edges.*

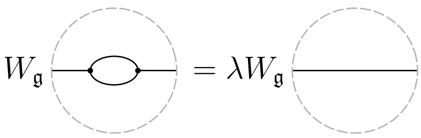


Proof Apply `IHX` to any pair of vertices, then three applications of `AS`.



□

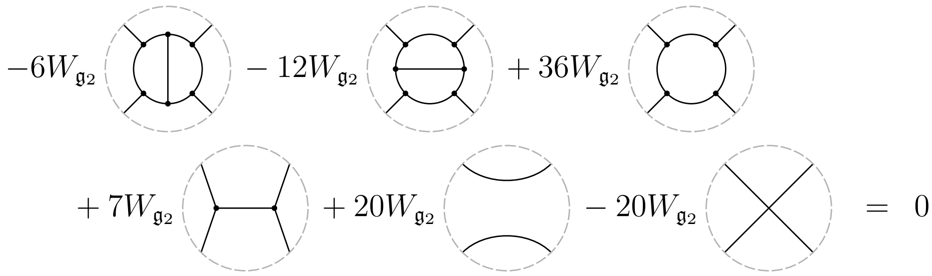
Lemma 2.4.2 *In Lie algebraic weight systems, the bubble is proportional to the line.*



This is proven in [CDM12, Lem. 6.15]. It would be interesting to have a purely categotical proof that doesn't refer to any specific representation, and uses the fact that Lie algebras are Lie algebra objects in the category **Vect**.

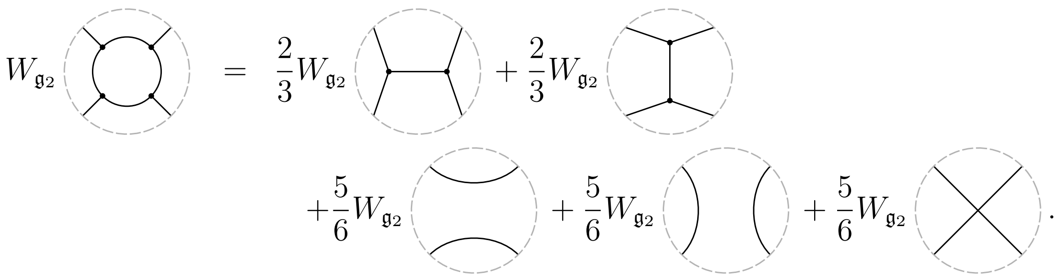
Simplifying Vogel's relation via these lemmas , gives the following relation. (We use the metric in which $\lambda = 1/4$, i.e. our metric is 4 times $\langle \cdot, \cdot \rangle_K$, the Killing form.)

Proposition 2.4.3



This relation is true for any diagram into which each term is inserted. In-particular, it remains true after rotating all terms a quarter rotation clockwise. However this symmetry is not obvious in the formula. This suggests that perhaps the relation is a consequence of a similar, simpler relation.

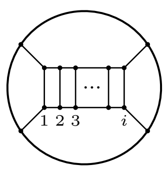
Theorem 2.4.4 *For the Lie algebra $\mathfrak{g}_2$, the weight systems obey*



Proof Compute the tensors corresponding to both sides. □

This is an analogue of a relation for $\mathfrak{sl}_3$ found by Yoshizumi and Kuga in [YK99]. This was used recently by Yang in [Yan24] to compute the value of the $\mathfrak{sl}_3$ weight systems on the following infinite sequence of chord diagrams.

Definition 2.4.5 The ith **bookshelf diagram**, B_i, is the Jacobi diagram

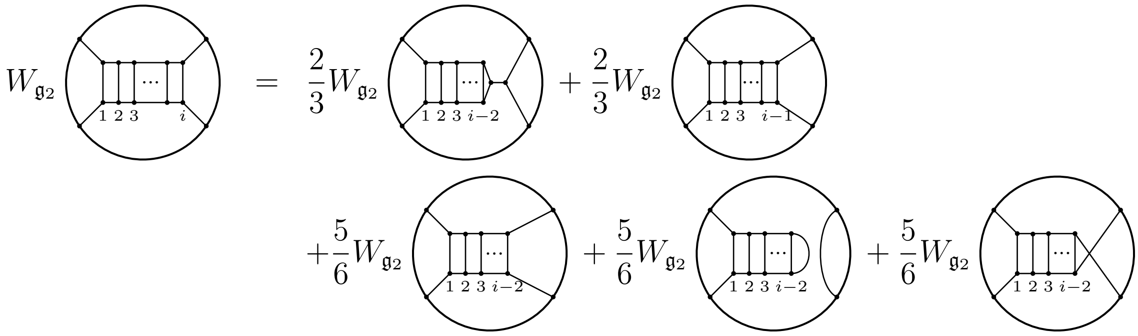


In [Yan24], a third-order recursion relation for the values of the $\mathfrak{sl}_3$ weight system on the bookshelf diagrams is found. We use the same idea with the new relation of Theorem 2.4.4 to compute their values under the $\mathfrak{g}_2$ weight system.

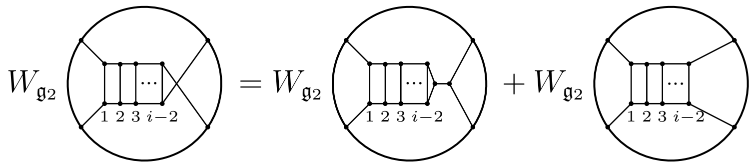
Theorem 2.4.6 *The values of the bookshelf diagram B_i are at most quadratic in the (quadratic) casimir element c of $\mathfrak{g}_2$. In particular,*

$$W_{\mathfrak{g}_2}(B_i) = c^2\left(\frac{1}{14}4^n + \frac{27}{112}(5/3)^n + \frac{11}{16}(-1)^n\right) + c\left(2^n + \frac{8}{3}(5/3)^n - \frac{11}{8}(-1)^n\right).$$

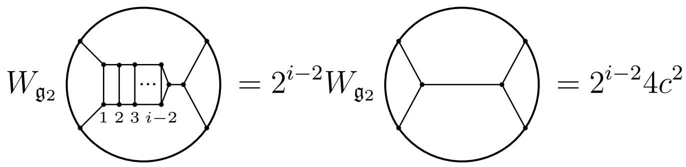
Proof Apply Theorem 2.4.4, to the rightmost box on a bookshelf diagram.



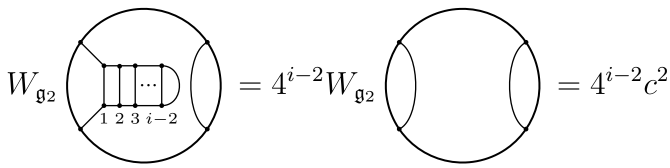
Applying STU to the final term gives



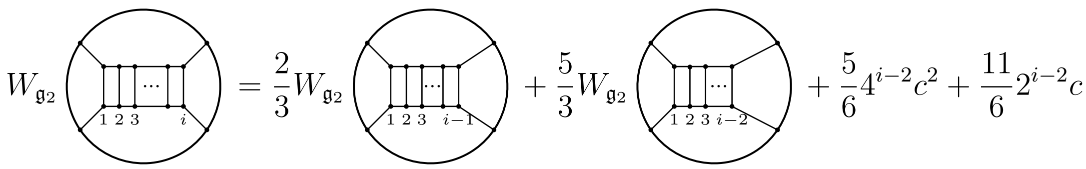
By Lemma 2.4.1, we have



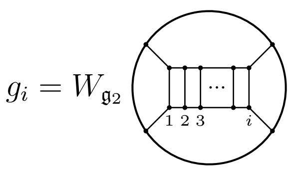
This calculation of the $\mathfrak{g}_2$ weight system on this diagram first appears in the literature in the PhD thesis of A. Kaishev which we have been unable to source, however it also appears in [CDM12, p. 181]. Our computer program also verifies it. By Lemma 2.4.2,



All in all,



so setting



we have the second-order non-homogeneous recursion relation for g_i

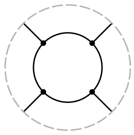
$$g_{i+2} = \frac{2}{3} g_{i+1} + \frac{5}{3} g_i + \frac{5}{6} 4^i c^2 + \frac{11}{6} 2^i c.$$

Its fourth-order homogeneous form is

$$g_{i+4} - \frac{20}{3}g_{i+3} + \frac{31}{3}g_{i+2} + \frac{14}{3}g_{i+1} - \frac{40}{3}g_i = 0.$$

To solve this we need the inital conditions g_1, g_2 and g_3. The first two come from the same computations of Kaishev [CDM12, p. 181], whereas the third we have to compute ourselves.

Computing in sagemath the tensor corresponding to the diagram



then projecting it to $\mathcal{U}(\mathfrak{g}_2)$ gives the value, but expressed in the PBW basis rather than as a polynomial in the casimir. There are two generating casimirs of $\mathfrak{g}_2$. The other is of degree 6, but g_3 is a degree 4 element of $Z(\mathfrak{g}_2)$, so it will be a polynomial in c of degree two or less. Evaluating $\rho(g_3)$ with ρ being the trivial representation, the seven-dimensional standard representation, and the fourteen-dimensional adjoint representation gives the following values.

$$\begin{aligned} \rho_{\text{tr.}}(c) &= 0 & \qquad \rho_{\text{tr.}}(g_3) &= 0 \\ \rho_{\text{st.}}(c) &= 2 & \rho_{\text{st.}}(g_3) &= \frac{380}{9} \\ \rho_{\text{ad.}}(c) &= 4 & \rho_{\text{ad.}}(g_3) &= \frac{1120}{9} \end{aligned}$$

So by the Lagrange interpolation formula,

$$g_3 = 5c^2 + \frac{100}{9}c.$$

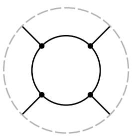
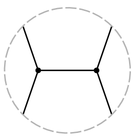
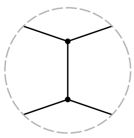
Solving the recurrence relation with these inital conditions yields the formula

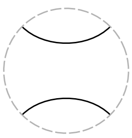
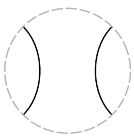
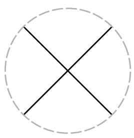
$$g_i = c^2\left(\frac{1}{14}4^i + \frac{27}{112}(5/3)^i + \frac{11}{16}(-1)^i\right) + c\left(2^i + \frac{8}{3}(5/3)^i - \frac{11}{8}(-1)^i\right).$$

□

We use the same method to compute the a relation in the weight system $W_{\mathfrak{g}}$ for the exceptional Lie algebra $\mathfrak{g} = \mathfrak{f}_4$.

Theorem 2.4.7 *For the Lie algebra $\mathfrak{f}_4$, the weight systems obey*

$W_{\mathfrak{f}_4}$  $= \frac{1}{2}W_{\mathfrak{f}_4}$  $+ \frac{1}{2}W_{\mathfrak{f}_4}$ 

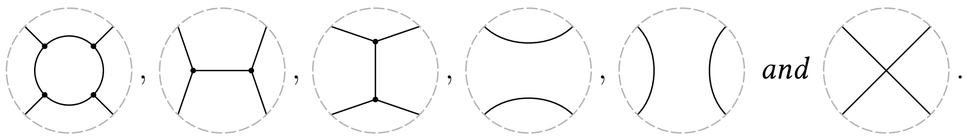
$+\frac{5}{36}W_{\mathfrak{f}_4}$  $+\frac{5}{36}W_{\mathfrak{f}_4}$  $+\frac{5}{36}W_{\mathfrak{f}_4}$  .

Proof Direct computation with sagemath. □

The dimension of $\mathfrak{g}_2$ is 14, and tensor contraction with sagemath is computable on a laptop in a few seconds. One the other hand, $\dim \mathfrak{f}_4 = 52$, and calculation takes several hours on a server with two Intel Xeon E5-2667v3 CPUs. The main bottleneck is tensor contraction, and indeed for $\mathfrak{e}_6$ ($\dim \mathfrak{e}_6 = 78$), the computation cannot complete even in several days. There is some hope that this can be done by using more efficient tensor contraction algorithms than are offered by sagemath, but this has not yet been explored.

Finally, we note that for $\mathfrak{sl}_4$ the arguments given above do not work.

Theorem 2.4.8 *In the weight system corresponding to the Lie algebra $\mathfrak{sl}_4$, there is no linear relation between the diagrams*



Proof By computer verification with sagemath, the span of the corresponding tensors in $(\mathfrak{sl}_4)^{\otimes 4}$ is six-dimensional. □

2.5 A universal metric Lie algebra object

We return to the universal Lie algebra of Vaintrob which was discussed in the previous section. It is yet uncertain to what degree such an algebra exists. The obstruction are the zero divisors in Vogel's algebra Λ, [KMS25; BM25; KLS25] give more details.

There is, however, a theoretical construction of Hinnich-Vaintrob [HV00]. They construct a Lie algebra object in a symmetric monoidal category with a weight system which is isomorphic to $\mathcal{A}$. By constructing an algebra isomorphic to $\mathcal{A}$, this guarantees that every Vassiliev invariant can be constructed from appling Construction 2.2.1 to some Lie algebra object in some symmetric monoidal category. However, it doesn't give any hints as to which symmetric monoidal category one needs to work with to construct a particular Vassiliev invariant. Thus it is useful theoretically, but not computationally. We review this construction of Hinich and Vaintrob. The purpose is that in Chapter 3, we will generalise this construction from Vassiliev invariants of classical knots to those of welded long knots.

The operations on a set can always be composed by taking as input the output of another operation. These compositions alyways obey certain rules. If the set is a vector space, the operations too form a vector space. An operad captures this concept of operations on some object of a category abstractly.

Warning 2.5.1 For all of the below, we assume categories are k-linear, k a field of characteristic zero.

Definition 2.5.2 An **operad** $\mathcal{O}$ in monoidal category $\mathcal{C}$ consists of:

- a collection of objects in $\mathcal{C}$, $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$, with $\mathcal{O}(n)$ known as the n-**ary operations**
- an element $\mathrm{id} \in \mathcal{O}(1)$.
- an action of the symmetric group S_n on each $\mathcal{O}(n)$

- a series of composition morphisms: for each n and $k_1, k_2, \cdots, k_n \in \mathbb{N}$

$$\begin{array}{rccc} \circ_n : & \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \otimes \cdots \otimes \mathcal{O}(k_n) & \longrightarrow & \mathcal{O}(k_1 + k_2 + \cdots + k_n) \\ & \theta \otimes \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n & \longmapsto & \theta \circ (\theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n) \end{array}$$

satisfying the following axioms. (Note: the axioms are listed for completeness, but won't be used heavily below.)

(a) That for any $\theta \in \mathcal{O}(n)$, $\theta \circ_n (\mathrm{id}, \dots, \mathrm{id}) = \theta = \mathrm{id} \circ_1 \theta$. (The **identity** axiom.)

(b) **Associativity of composition**, that for $\theta \in \mathcal{O}(n)$, and $\theta_1 \in \mathcal{O}(k_1)$, $\theta_2 \in \mathcal{O}(k_2)$, up to $\theta_n \in \mathcal{O}(k_n)$, that the two ways of forming the composition of the $\theta_{i,j}$ into θ_i into θ are the same:

$$\theta \circ_n \Big(\theta_1 \circ_{k_1} (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ_{k_n} (\theta_{n,1}, \dots, \theta_{n,k_n})\Big)$$

is equal to

$$\big(\theta \circ_n (\theta_1, \dots, \theta_n)\big) \circ_{k_1 + \cdots + k_n} (\theta_{1,1}, \dots, \theta_{1,k_n}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n}).$$

(c) An **equivariance** axiom, that for $\sigma \in S_n$, and all $\theta \in \mathcal{O}(n)$, and all $\theta_1, \dots, \theta_n$,

$$(\theta\sigma) \circ_n (\theta_1, \dots, \theta_n) = \big(\theta \circ_n (\theta_{\sigma^{-1}(1)}, \dots, \theta_{\sigma^{-1}(n)})\big)\sigma_{\text{block}}$$

where $\sigma_{\text{block}} \in S_{k_1 + \cdots + k_n}$ is the permutation that block-permutes blocks of size k_1 through to k_n.

(d) A further **equivariance** axiom, that for permutations $\sigma_i \in S_{k_i}$, and all $\theta \in \mathcal{O}(n)$, and all $\theta_1, \dots, \theta_n$,

$$\theta \circ_n (\theta_1\sigma_1, \dots, \theta_n\sigma_n) = \big(\theta \circ_n (\theta_1, \dots, \theta_n)\big)(\sigma_1 \sharp \cdots \sharp \sigma_n)$$

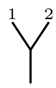
where $(\sigma_1 \sharp \cdots \sharp \sigma_n) \in S_{k_1 + \cdots + k_n}$ is the block-wise permutation of block i by σ_i.

(e) That the action of the symmetric group agrees with its action in the symmetric monoidal category $\mathcal{C}$.

Definition 2.5.3 An **algebra** A **over the operad** $\mathcal{O}$ **(in** $\mathcal{C}$**)** is an object of $\mathcal{C}$ satisfying certain natural compatibility conditions.

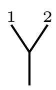
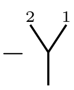
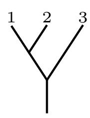
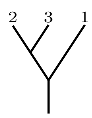
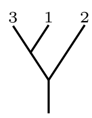
In essence, an algebra over an operad is an object in $\mathcal{C}$ whose operations obey the rules defined by $\mathcal{O}$.

Example 2.5.4 Let Γ denote the free operad functor (for details we refer the reader to [Mar08]). Let

$\alpha =$ 

denote a generator operation with two inputs.

The **Lie operad**, $\mathcal{O}_{\text{Lie}}$ then has the following presentation

$\mathcal{O}_{\text{Lie}} \;=\; \Gamma(\alpha) \;/$  $= -$  ,  $+$  $+$  $= 0.$

◊

Recall that the goal is an algebra isomorphic to $\mathcal{A}$. The first step is to create an object that encodes the internal structure of $\mathcal{A}$, i.e. the AS and IHX relations, but not yet with a choice of representation or the STU relations. An important feature of $\mathcal{A}$ that enabled the construction of Lie algebra weight systems was that contraction of tensors corresponding to the Lie bracket or metric/casimir elements was akin to joining parts of diagrams in $\mathcal{A}$. So a contraction-like operation would be a good start.

Composition in operads is something like contraction in a symmetric monoidal category, but with restictions on what can be contracted: only outputs with inputs. Furthermore, in an operad, no object has two outputs, for example there is no natural notion of tensor product in an operad.

A step towards describing the internal structure of $\mathcal{A}$ without specifying a Lie algebra is to allow multiple outputs, thereby considering not just operations but, for example in $\mathcal{C} = \mathbf{Vect}$, general multilinear functions. Operads were abstractions of the operations on some object (say, A), i.e. morphisms $A^{\otimes n} \to A$. Props will be abstractions of all maps $A^{\otimes n} \to A^{\otimes m}$.

Definition 2.5.5 A **prop** is a symmetric monoidal category where every object is of the form $A^{\otimes n}$ for some object A. Equivalently, it's a symmetric monoidal category generated by a single object.

Definition 2.5.6 An **algebra A (in $\mathcal{C}$) over the prop $\mathbf{P}$** is tensor functor $\alpha : \mathbf{P} \to \mathcal{C}$.

Remark 2.5.7 Under suitable definitions of morphisms of props and operads, there are the categories $\mathbf{Prop}$ and $\mathbf{Oper}$, and a pair of adjoint functors.

$$\begin{tikzcd} \mathbf{Oper} \arrow[d, shift right, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Prop} \arrow[u, shift right, "F"] \end{tikzcd}$$

The forgetful functor F takes a prop $\mathbf{P}$ to the operad whose operations are the $m, 1$-ary elements of the prop (i.e. have one output):

$$F(\mathbf{P})(m) = \mathbf{P}(m, 1).$$

The free functor U takes an operad to the prop whose m, n-ary morphisms are tensor products of operations from the operad:

$$U(\mathcal{O})(m, n) = \bigoplus_{f:[m]\to[n]} \bigotimes_{i=1}^{n} \mathcal{O}(|f^{-1}(i)|).$$

Proposition 2.5.8 *The notions of algebra over a prop and algebra over an operad are equivalent. $A \in C$ is an algebra over the operad $\mathcal{O}$ if and only if it is an algebra over the prop $\mathbf{P}(\mathcal{O})$.*

Now the main difference between contraction and partial composition in a prop is the marking of inputs and outputs. With metric Lie algebras, co- and contravariant indices could be contracted with the metric or casimir to invert their variance, but in a general prop there's no way to turn an input into an output.

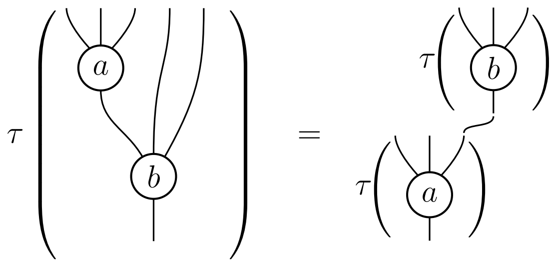
Recall that the action of the symmetric group comes from the symmetry of the symmetric monoidal category $\mathcal{C}$. But this only permutes inputs amongst themselves, rather than permuting inputs with outputs. In a certain kind of operad, this can be achieved.

Definition 2.5.9 A **cyclic operad** in $\mathcal{C}$ is a pair $(\mathcal{O}, \tau_+)$ where τ_+ is an action of each symmetric group S_{n+1} on $\mathcal{O}(n)$, extending the action coming from the symmetry τ of $\mathcal{C}$, and satisfying the axiom

$$(a \circ_1 b)\tau_+ = (b\tau_+) \circ_n (a\tau_+)$$

for any $a \in \mathcal{O}(m)$, $b \in \mathcal{O}(n)$. Here to specify the action of S_{n+1}, the action of $S_n \subset S_{n+1}$ is predetermined by the action of the symmetry τ.

This has the following graphical interpretation.



We denote an arbitrary cyclic operad $\mathcal{O}_{\tau_+}$ to reflect that whether an operad is cyclic depends both on $\mathcal{O}$ and the choice of action extending the action τ of $\mathcal{C}$ on $\mathcal{O}$.

The definition of a metric Lie algebra involved axioms on both the bracket, but also about the symmetry on the bracket and the form.

Definition 2.5.10 For a cyclic operad $(\mathcal{O}, \tau_+)$ in category $\mathcal{C}$, a **metric algebra** over $(\mathcal{O}, \tau_+)$ is a pair (A, b). That is, a choice of object $A \in \mathcal{C}$ with together with a symmetric morphism $m : A \otimes A \to k$, such that:

- there exists a morphism $c : k \to A \otimes A$ such that the composition

  $$A \xrightarrow{c \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes m} A$$

  is the identity (m is **invertible**).

- the composition

  $$\mathcal{O}(n) \otimes A^{\otimes n+1} \longrightarrow A \otimes A \xrightarrow{m} k$$

  is S_{n+1}-invariant (m is $\mathcal{O}$-**invariant**).

(REALLY? Surely it just has to agree with the S_n-action. What about swapping!)

The conditions above can alternatively and more naturally be phrased in terms of props.

Definition 2.5.11 We define the **prop for metric algebras over the cyclic operad** $(\mathcal{O}, \tau_+)$, denoted $\mathbf{P}_m(\mathcal{O}, \tau_+)$ as the prop generated by the prop $\mathbf{P}(\mathcal{O})$ along with two elements $m \in \mathbf{P}_m(\mathcal{O}, \tau_+)(2, 0)$ and $c \in \mathbf{P}_m(\mathcal{O}, \tau_+)(0, 2)$, satisfying the following conditions:

- The morphisms b and c are are symmetric and mutually inverse (in the sense of Definition 2.5.10).

- For each $f \in \mathcal{O}(n)$, the composition

  $$A^{\otimes n} \xrightarrow{c \otimes \mathrm{id}} A^{\otimes n+2} \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} A^{\otimes 3} \xrightarrow{\mathrm{id} \otimes b} A$$

  is equal to $f\tau_+$.

In other words, in $\mathbf{P}_m(\mathcal{O}, \tau_+)$, the action of τ_+ corresponds to the cyclic permutation of inputs and outputs by means of the metric and the casimir.

Proposition 2.5.12 *Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{P}_m(Lie, \tau_+)$ are exactly the metric Lie algebra objects in the category $\mathcal{C}$.*

Proof Kind of by definition

This generalises the definition of metric Lie algebra to an arbitrary symmetric monoidal category. Indeed any prop for metric algebras over a cyclic operad in any symmetric monoidal category gives rise to a weight system. But furthermore, this permits the following slight *further* generalisation of the type of objects that give rise to weight systems, which is necessary for the universal construction of Hinnich and Vaintrob.

Definition 2.5.13 The **prop for casimir algebras over the cyclic operad** $(\mathcal{O}, \tau_+)$, denoted $\mathbf{P}_c(\mathcal{O}, \tau_+)$ is the prop generated by the prop $\mathbf{P}(\mathcal{O})$ along with the element $c \in \mathbf{P}_c(\mathcal{O}, \tau_+)$ such that:

- The morphism c is symmetric.
- For each $f \in \mathcal{O}(n)$, the diagram

$$\begin{tikzcd}
A^{\otimes n-1} \arrow[r, "c\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes c"'] & A^{\otimes n+1} \arrow[d, "\mathrm{id}\otimes f"] \\
A^{\otimes n+1} \arrow[r, "f\tau\otimes\mathrm{id}"'] & A^{\otimes 2}
\end{tikzcd}$$

  commutes.

Proposition 2.5.14 *Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{P}_c(Lie, \tau_+)$ are exactly the casimir Lie algebra objects in the category $\mathcal{C}$.*

Proof Kind of by definition

This generalises the notion of a Lie algebra (object) with a casimir element (but not necessarily with a metric). We call such algebra (object) a casimr Lie algebra (object).

The reason we did not see an analagous definition earlier in this chapter is the following. In the case of finite-dimensional Lie algebras, every casimir Lie algebra is a metric Lie algebra. This is because in finite-dimensional vector spaces, a casimir, being an invariant symmetric map $c : k \to A \otimes A$ is always invertible, and its inverse is a metric. So, any finite-dimensional casimir Lie algebra (superalgebra, etc.) is a metric one.

The reason we need to consider casimir Lie algebra objects as opposed to just metric Lie algebra objects to make the universal construction is the following.

Consider the tensor category of a specific finite-dimensional (casimir) hence metric Lie algebra object in a specific symmetric tensor category. In $\mathcal{C}(k,k)$ we have the object formed by the contraction of the tensor with the metric. Since this is finite-dimensional, this is just the trace, therefore the dimension of the algebra. So, in the tensor category we have the relation $m \circ c = \dim \mathfrak{g}$. If we are trying to find an algebra which contains all the relations true in the tensor category of any metric Lie algebra object, then we arrive at a problem. All the relations of this form are contradictory.

Two obvious choices present themselves. Removing all relations, therefore allowing symbolic objects of the form $circle$, we construct an algebra bigger that $\mathcal{A}$. The other option is to not allow any circles by instead generalising the notion of casimir Lie algebra objects, (or something – this could use some thought)

Definition 2.5.15 The object $\mathbb{L}_m \in \mathbf{P}(Lie)$ Hard question: what do I put as the symmetry action here? Usually the symmetry is determined by the category. But here we defined the category as generated by the prop $\mathbf{P}(\mathcal{O})$ with an additional element. Does this all go round in circles?

Theorem 2.5.16 (Hinnich-Vaintrob) *Under [certain mild conditions] on $\mathcal{O}$, $\mathcal{C}$ and A, there exists an algebra $U(\mathcal{O}, A)$, the quotient of the external tensor algebra, obeying some important relation analogous to the defining property of the external enveloping algebra of a Lie algebra, but general to any operad.*

Theorem 2.5.17 (Hinnich-Vaintrob) $U(\mathcal{O} = Lie, A = 1 = \mathbb{L}_c \in \mathbf{P}_c(Lie))$ *is isomorphic as a Hopf algebra to $\mathcal{A}$.*

Proof We appeal to Hinnich and Vaintob a lot. □

In the next chapter we look at a generalisation of this statement in the context of welded knots.

3

Welded knots, isometry Lie algebras and arrow diagrams

Welded knots, or w-knots are a generalisation of knots first introduced by [who?].

On one hand, topologically, welded knots are more complicated objects, relating to ribbon embeddings of tori in four dimensions. On the other hand, their finite type invariant theory is much more manageable than the classical case, so studying the simpler, welded version of $\mathcal{A}$ may shed light on how to study the algebra $\mathcal{A}$.

3.1 Welded knots

A good and thorough exposition of the theory of welded knots is [BD16; BD17]. Their presentation is in terms of virtual knots and circuit algebras.

With classical knots, the definition is first and foremost topological; they are ambient isotopy classes of embeddings of oriented circles into $\mathbb{R}^3$. It is due to the Reidemeister theorem that we may instead work with equivalence classes of planar knot diagrams under the Reidemeister moves. With welded knots, there is a similar topological interpretation from [Sat00], thought its details remain unclear. Welded knot diagrams relate to ambient isotopy classes of ribbon-embedded tori in $\mathbb{R}^3$. However, there are two problems with this classical approach of defining the knot as the ambient isotopy class of the relevant topological objects.

Firstly, for welded knots, there is a global orientation reversal contianed in the map from the diagrammatic object to the topological object.

To avoid this trouble, we choose to work instead with welded long knots. In the classical case, a long knot is the same as a knot, but instead of being $S^1 \hookrightarrow \mathbb{R}^3$, the embedding is $k : \mathbb{R} \hookrightarrow \mathbb{R}^3$, with the condition that $k(t) = (t, 0, 0)$ for all large $|t|$.

For the finite type theory, the algebra $\mathcal{A}$ of Jacobi diagrams (and chord diagrams), is replaced with $\mathcal{A}(|)$. The relations are the exact same (either `IHX` or `4T`), but instead of the univalent vertices having a cyclic order (and being drawn on a circle, they have a linear order and are drawn on a line.

Secondly, the relation between welded long knots and long ribbon-knotted tori in $\mathbb{R}^4$ is only conjectural. There is a tube map from welded long knots to long ribbon-knotted tori in $\mathbb{R}^4$ that is known to be surjective but its kernel remains unknown. A good exposition is [Aud16].

The result is a definition of welded knots via diagrams, though it is widely believed that either that welded knots correspond either to long ribbon-knotted tori, or to some small quotient thereof by something akin to framing or rotation number.

Definition 3.1.1 A **welded knot** is an equivalence class of four-valent planar diagram, where each four-valent planar vertex is decorated by a vertex of one of the following types, or rotated versions thereof.

The third crossing type is called a **virtual crossing**. Along each edge of the diagram, the orientations given by the adjacent vertices agree, and such that when opposite planar vertices are considered connected, there is a single connected component.

The equivalence relations are planar isotopy and the following moves.

- Reidemeister moves
- Virtual reidemeister moves
- Overcrossings commute

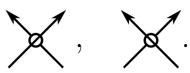
Remark 3.1.2 Note that we do not allow moves of the form undercrossings commute. Allowing such moves would trivialise welded knots.

Definition 3.1.3 A **welded long knot** is ...

3.2 Arrow diagrams

The finite type theory of welded long knots is similar to that of classical long knots.

Definition 3.2.1 A **singular welded long knot** is a welded long knot whose crossings can also be of the following types

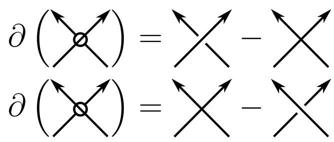


They are known as **positive semi-virtual crossings** and **negative semi-virtual crossings**, and they correspond to topological singularities of ribbon-knotted tori.

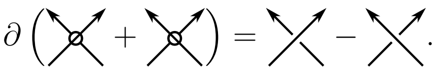
The set of m-singular welded long knots forms an algebra with the product given by connected sum given by concatenation and coproduct $\Delta(k) = k \otimes k$. This is called the **algebra of m-singular welded long knots**, $\mathcal{K}^\bullet_{w,m}(|)$.

The algebra of 0-singular welded long knots, or just the algebra of welded long knots $\mathcal{K}^\bullet_{w,0}(|) = \mathcal{K}_w(|)$ is filtered by the images of succesive powers of ∂, just like in the classical case, and the filtration is compatible with the bialgebra structure.

Just as in the classical case, the notions of invariants of m-singular welded long knots, their differentiability, and the derivative and boundary operators transfer over the the welded case. Here, the derivative and boundary depend on the type of singularity.



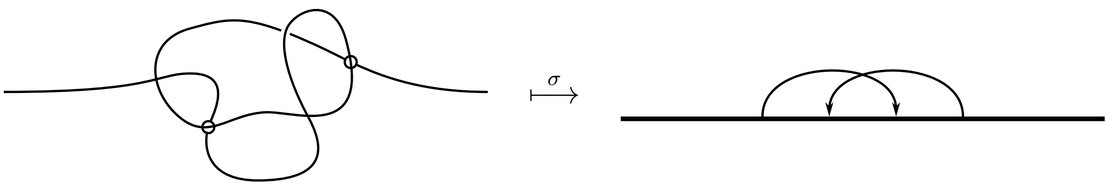
And correspondingly for m-singular invariants and the operator δ. As a consequence of these definitons, we can express the positive-to-negative crossing change via a sum of semivirtuals:



Definitions 3.2.2 (a) A **long arrow diagram** of order m is an oriented line (the **skeleton**) with a distinguished set of m (ordered) pairs of points, considered up to orientation-preserving diffeomorphism of the line. The set of all long arrow diagrams forms an algebra $\mathcal{D}_w(|)$. The product is induced by the natural connected sum operation on long knots, and the coproduct is given by the same formula as in the classical case. This is called the **bialgebra of long arrow diagrams**, and it is also graded by degree.

(b) The **long arrow diagram of an m-singular welded long knot**, denoted $\sigma(k)$ is the arrow diagram formed as follows. Traverse the welded long knot, and whenever a singularity is encountered, place a point on the skeleton. If the hollow part of the singularity is traversed, place a tail, and if the filled part is traversed place a head. Points on the skeleton corresponding to the same singularity are paired, and the order of the pair is tail first, then head.

Example 3.2.3



◊

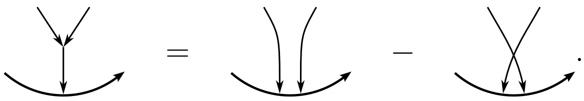
Just as in the classical case, there are natural relations to consider on arrow diagrams. The operations and maps defined above factor through the quotient, and we obtain the algebra $\mathcal{A}_w(|)$. We also refer to arrow diagrams as elements of this algebra. However, we are more interested in the algebra $\mathcal{J}_w(|)$.

Definition 3.2.4 A **long arrow Jacobi diagram** is an oriented unitrivalent graph where every trivalent vertex has at least one vertex oriented inward, and at least one oriented outwards, with the following additional data:

- at each trivalent vertex, a cyclic order of the incident edges,
- a fixed linear order on the univalent vertices.

modulo:

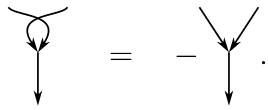
(a) Six **directed STU** relations. For example,



(STU)

Relations hold for both choices of vertex type (two-heads-one-tail, one-head-two-tails) and all three choices of orientation of the trivalent vertex. Note that even though the skeleton is the line rather than the circle, we still draw it curved to distinguish it.

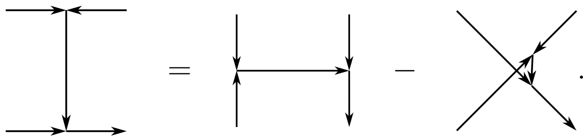
(b) Any **directed AS** relation. For example,



(AS)

Again, for any choice of vertex type and rotation.

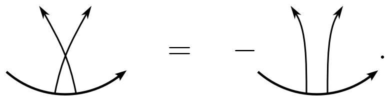
(c) Any **directed IHX** relations



(IHX)

These hold for any compatible choice of both vertices in the diagrams.

(d) The **TC relation** (tails commute relation)



(TC)

The main difference between $\mathcal{J}_w(|)$ and $\mathcal{J}(|)$ is the tails commute relation. This relation is a consequence of the overcrossings commute relation (in a sense it's the shadow of that relation on the associated graded side). This new relation is very important and dramatically reduces the complexity of the algebra.

Proposition 3.2.5 (Two-in-one-out rule) *In $\mathcal{J}_w(|)$, any diagram containing a trivalent vertex with one vertex oriented inwards and two vertices oriented outward vanishes.*

Proof Take such a diagram with a one-in-two-out vertex. Apply STU's to all other vertices of the arrow diagram. The result is a sum of arrow diagrams, each containing a trivalent one-in-two-out vertex. We show that each such arrow diagram vasnishes. Since there is only one trivalent vertex in each diagram, the one-in-two-out vertex's incoming edge has its tail on the skeleton. Therefore applying the relevant STU relation equates the diagram with a difference of two diagrams that differ only by the order of placement of two tails on the skeleton. By TC, the difference is zero. □

Resultingly, only two versions of STU, one of AS and one of IHX remain nontrivial relations in $\mathcal{J}_w(|)$.

Warning 3.2.6 From this point it is clear when talking about welded objects we mean long welded objects. So, we write $\mathcal{A}_w(|)$ as $\mathcal{A}_w$, and don't necessarily specify that they are long.

The existence of a finite type invariant Z for welded knots establishes a welded version of the fundamental theorem.

Theorem 3.2.7 *There exists a universal Vassiliev invariant $Z_w : \mathcal{K}_w \to \mathcal{A}_w$.*

Unlike the incredible involved construction for Z, the construction of Z_w is relatively simple. The TC relation does all of the heavy lifting. A proof can be found in [BD16].

Corollary 3.2.8 *We have*

$$\mathcal{A}_w \cong \operatorname{gr} \mathcal{A}_w.$$

Equivalently,

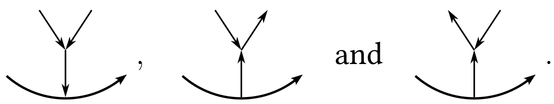
$$\mathcal{V}_w \cong \operatorname{gr} \mathcal{W}_w.$$

3.3 Weight systems and Lie algebras

There is a variant of Construction 2.2.1 for welded knots taking a Lie algebra and constructing a weight system. The difference is that vertices in string diagrams now have incoming edges ("heads") and outgoing edges ("tails"), and diagramatically, only edges of the opposite orientation can be contracted. This was not true in the classical case, and the need to fix this is what made us restrict to metric Lie algebras. Hence, in the welded case we can consider any finite-dimensional Lie algebra.

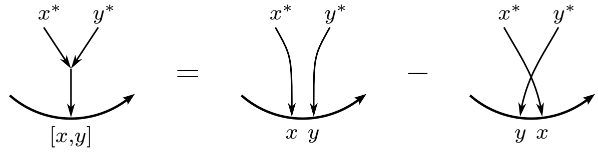
Note an imporant difference to the classical case. There are now two types of univalent vertices that can connect to the external line: heads and tails. Heads correspond to elements of $\mathfrak{g}$ and tails to elements of $\mathfrak{g}^*$. Therefore, the role of $\mathcal{U}(\mathfrak{g})$ in the classical case must be played here by some other associative algebra whose basis elements are words in $\mathfrak{g} \sqcup \mathfrak{g}^*$.

Analysing the relations involving the external line should tell us what kind of object replaces $\mathcal{U}(\mathfrak{g})$. From the two-in-one-out rule, there are three non-trivial cases to examine:



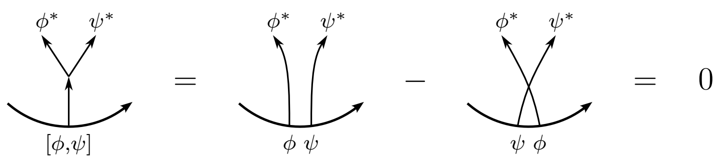
In all three cases, they obey commutator-like relations (the versions of STU), so the correct associative algebra target for the weight system is still the universal enveloping algebra of some Lie algebra. However since the connections to the external line can now be either heads or tails, representing elements of $\mathfrak{g}$ and $\mathfrak{g}^*$, the Lie algebra as a vector space is $\mathfrak{g}^* \oplus \mathfrak{g}$.

Interpreting tails and heads in these STU relations as elements of $\mathfrak{g}^* \oplus 0$ or $0 \oplus \mathfrak{g}$ rather than $\mathfrak{g}^*$ or $\mathfrak{g}$ determines the bracket on $\mathfrak{g}^* \oplus \mathfrak{g}$. For example, the STU with two heads on the external line implies that the bracket on $0 \oplus \mathfrak{g}$ should be inherited from the bracket on $\mathfrak{g}$.



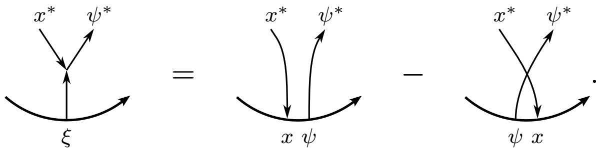
So, $[0 \oplus x, 0 \oplus y] = 0 \oplus [x, y]_{\mathfrak{g}}$.

To determine the bracket on $\mathfrak{g}^* \oplus 0$, we need to examine one of the trivial cases not shown above. Indeed, the two-in-one-out rule neccecitates that the bracket on $\mathfrak{g}^* \oplus 0$ be the zero-bracket. (Recall this rule was a consequence of the TC relation, and that TC stands for tails commute.)



This gives $[\phi \oplus 0, \psi \oplus 0] = 0 \oplus 0$.

The bracket structure is known on the $0 \oplus \mathfrak{g}$ and $\mathfrak{g}^* \oplus 0$ direct summands, so the total Lie algebra structure will be a semidirect product $\mathfrak{g}^* \rtimes \mathfrak{g}$, and what remains is to specify an action of $\mathfrak{g}$ on $\mathfrak{g}^*$. With a bit of work, the remaining 'mixed' STU relations determine this. One such relation is



Let us determine the functional ξ in terms of x and ψ. There is only one type of nonzero trivalent vertex in $\mathcal{A}_w$, so this is the same tensor as $0 \oplus \mathfrak{g}$ case, just viewed from a different component. That tensor is $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, so ξ satisfies the equation $[\cdot,\cdot] = x^* \otimes \xi \otimes \phi^*$, whereby

$$[x, \xi^*] = \phi^* \qquad \text{so} \qquad \phi([x, \xi^*]) = 1.$$

So, $\xi \in \mathfrak{g}^*$ is the functional

$$\xi : t \longmapsto \phi([x, t]).$$

This is also known as the **coadjoint action**, $\xi = \mathrm{ad}_x^*(\phi)$, the functional adjoint to $\mathrm{ad}_x(\phi)$. Writing this in terms of the bracket on $\mathfrak{g}^* \otimes \mathfrak{g}$,

$$[0 \oplus x, \psi \oplus 0] = \mathrm{ad}_x^*(\psi) \oplus 0.$$

The AS relation gives $[\phi \oplus 0, 0 \oplus y] = -\mathrm{ad}_y^*(\phi) \oplus 0$. Hence, the total Lie algebra structure on $\mathfrak{g}^* \oplus \mathfrak{g}$ is given by $I\mathfrak{g}$ in the following definition.

Definition 3.3.1 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{g}^*$ denote the vector space $\mathfrak{g}^*$ with the structure of an abelian Lie algebra. Then let $I\mathfrak{g}$ denote the Lie algebra $\mathfrak{g}^* \rtimes \mathfrak{g}$ with the bracket

$$[\phi \oplus x, \psi \oplus y] = \mathrm{ad}_x^*(\psi) - \mathrm{ad}_y^*(\phi) \oplus [x, y].$$

Remark 3.3.2 This construction coincides with the Drinfeld double [Dri88] of the Lie bialgebra $\mathfrak{g}$, in the case where $\mathfrak{g}$ is co-commutative.

Proposition 3.3.3 *There is a well-defined algebra homomorphism $\mathcal{A}_w \to \mathcal{U}(I\mathfrak{g})$ given by a variant of Construction 2.2.1.*

Proof The relations in $\mathcal{A}_w$ are generated by STU and TC. We have shown that these relations are also true in $\mathcal{U}(I\mathfrak{g})$. □

3.4 The universal welded weight system

The aim of this section is to prove a Hinnich-Vaintrob style statement for $\mathcal{A}_w$.

Definitions 3.4.1 An n-**coloured prop** is a symmetric monoidal category generated by n objects.

The definition of an algebra over a prop (Definition 2.5.6) extends naturally to coloured props as well.

Definition 3.4.2 Let $X = (X, \beta)$ be a Lie algebra object in the k-linear symmetric monoidal category $\mathcal{C}$. We define a 2-coloured prop $\mathbf{\Theta}(X, \beta)$, or just $\mathbf{\Theta}(X)$. Its objects are generated by X and another object Y. Let $I_{\mathbf{\Theta}} = X \otimes Y$.

The morphism β is one generating morphism, hence (X, β) is a Lie algebra object in $\mathbf{\Theta}(X, \beta)$. There is another generating morphism $\gamma : X \otimes Y \otimes X \otimes Y = I_{\mathbf{\Theta}} \otimes I_{\mathbf{\Theta}} \to k$.

The two generating morphisms are subject to some relations. Let $\zeta : Y \otimes Y \to Y$ denote the zero morphism $Y^{\otimes 2} \to Y$. Note that (Y, ζ) and therefore $(I_{\mathbf{\Theta}}, \beta \otimes \zeta)$ a Lie algebra object in $\mathbf{\Theta}(X, \beta)$, and so the following makes sense. The relations are that for each $f \in \mathcal{O}_{\text{Lie}}(n)$, the following diagram commutes.

$$\begin{tikzcd}
(X \otimes Y)^{\otimes n-1} \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\gamma"'] & (X \otimes Y)^{\otimes n+1} \arrow[d, "\mathrm{id}\otimes f"] \\
(X \otimes Y)^{\otimes n+1} \arrow[r, "f\tau\otimes\mathrm{id}"'] & (X \otimes Y)^{\otimes 2}
\end{tikzcd}$$

In the above diagram, τ is the cyclic permutation in S_{n+1} as it acts on the algebra $(X \otimes Y, \beta \otimes \zeta)$ over the operad $\mathcal{O}_{\text{Lie}}$: blockwise on pairs.

Theorem 3.4.3 *The bialgebra $\mathcal{A}_w$ of arrow diagrams is naturally isomorphic as a bialgebra to the external enveloping algebra $U(\mathcal{O}_{Lie}, I_{\mathbf{\Theta}})$.*

Proof Follow Hinich and Vaintrob. □

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