On Vassiliev Invariants

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An essay submitted in fulfilment of
the requirements for the degree of
Master of Philosophy (Science)

Pure Mathematics
University of Sydney

October 3, 2025

Contents

Acknowledgements

Thanks to …

Introduction

THE introduction goes here.

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Vassiliev invariants and chord diagrams

Vassiliev invariants are sophisticated to define in terms of the space of knots from the introduction, but the axiomatic definition of Birman-Lin [BL93] is much simpler. The definition also illustrates an analogy made by Bar-Natan [Bar95] in which Vassiliev invariants are “polynomial invariants”. This is not meant in the sense that Vassiliev invariants take values in a polynomial ring (like say, the Jones polynomial), but rather that Vassiliev invariants have special properties not shared by all invariants, just as polynomial functions have special properties not shared by all functions.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m**-singular**.

Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word “singular” in “singular knot” refers specifically to double point singularities. In particular immersions with

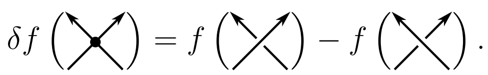
(a) triple points

(b) points with vanishing derivative

are excluded from the definition.

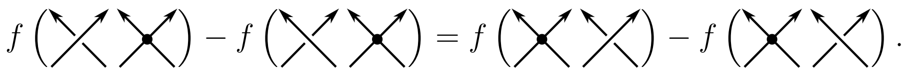
A singular knot with one double point is very close to two other knots. In one, the double point is replaced by a positive crosing, and in the other a negative crossing. If the conditions are right, we can extend a knot invariant to an invariant of singular knots by a procedure analagous to taking its derivative.

Definition 1.1.3 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



What are the conditions? For this to be a well-defined operation, it mustn’t matter which double point we choose.

Definition 1.1.4 An invariant f of m-singular knots is **differentiable** if

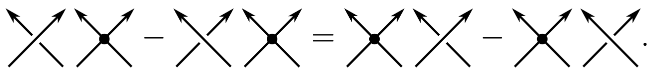


(DIFF)

If an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

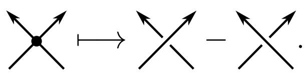
Definition 1.1.5 Define $\mathcal{K}_m = \operatorname{span}(\{m\text{-singular knots}\})$ (taken over $\mathbb{Q}$) modulo the following boundary relation (also known as a codifferentiability relation):



(DIFF*)

From now on, we will refer to elements $\mathcal{K}_m$ as m-**singular knots**, i.e. the DIFF* relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m \to \mathcal{K}_{m-1}$ defined by

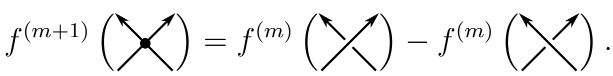


Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as an invariant of knots in $\mathcal{K}_m$.

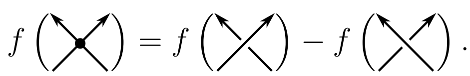
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

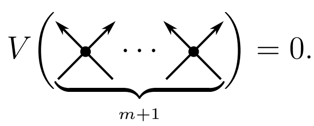


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation,



(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the most double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 In other words, Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

1.2 Integration and the fundamental theorem

To help see the bird's eye view, and following [Hut98], we phrase this in terms of an integration theory.

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group $\mathcal{O}_0$ is typically free abelian, and in our case this is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behave like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

We wish to understand how to invert this process, namely:

Questions 1.2.2 (a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?

(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

(c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?

(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The following modules provide the tautological answers to the above questions in the general theory.

Definitions 1.2.3 (a) The **primary obstructions to integration** are the module

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the module

$$C\mathcal{O}_m = \mathcal{O}_m / \partial \mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the module

$$S\mathcal{O}_m = \ker (\partial_{m+1}\partial_m) / P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.

(d) The **weights of integration** are the module

$$W\mathcal{O}_m = C\mathcal{O}_m/\pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the projection.

(e) The **finite type invariants** of order m are the module

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $\mathcal{O}^\star = \mathcal{K}^\star$ which provides good intuition for the general case.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the m-th and $(m+1)$-st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

To rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}^\star = \mathcal{K}^\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

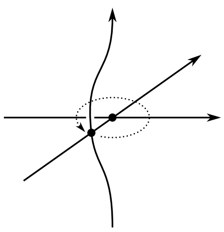
Due to the boundary relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

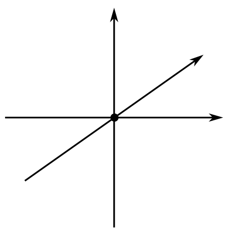
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a “path-integral” along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving “holes” in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

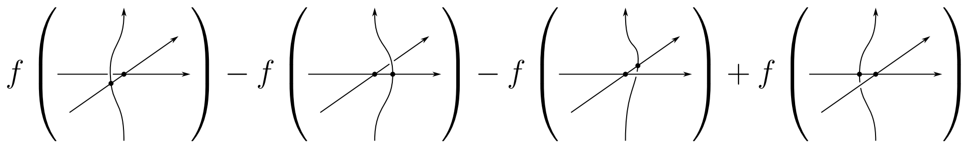
Definitions 1.2.5 (a) The following closed singular isotopy around a triple point:



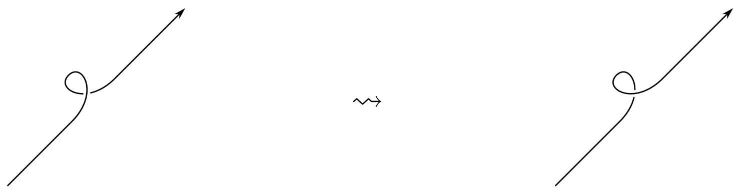
around



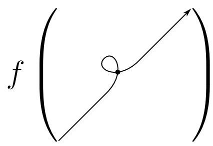
gives rise to the **topological four-term relation**


$= 0.$ (T4T)

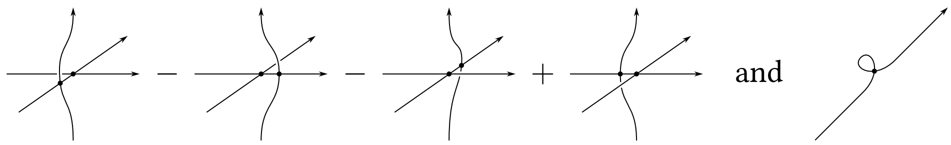
(b) The closed singular isotopy



around a point with a vanishing derivative gives rise to the **topological one-term relation**


$= 0.$ (T1T)

The necessity of the above relations for an invariant of m-singular knots to integrate is equivalent to the assertion that



are in $P\mathcal{K}_m = \ker \partial_m$. Let us denote arbitrary such singular knots $\mathbf{T4T}$ and $\mathbf{T1T}$.

This raises the question of whether we have found all primary obstructions. Do $\mathbf{T4T}$ and $\mathbf{T1T}$ span $\ker \partial$?

Theorem 1.2.6 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $m-1$-singular knots if and only if it satisfies* `T4T` *and* `T1T`.

Proof (sketch) Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m-1$ double points, and additional worse singularities. The only events to consider are codimension $m+2$ (why $m+2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with `DIFF` relations, which we've said are implicit).

Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

In the order of Questions 1.2.2 and Definitions 1.2.3, we ought to talk next about the secondary and further obstructions to integration. However, let us postpone this until after we have discussed constants of integration, which will be a more natural place.

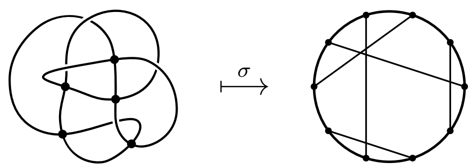
The constants of integration comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost. When integrating, values on these can be chosen freely.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}_m = \mathcal{K}_m/\partial\mathcal{K}_{m+1}$. The difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. Thus, we see that constants of integration are singular knots which are blind to crossing changes, so the only data needed to describe them is the order in which the double points are traversed around the knot.

Definitions 1.2.7 (a) A **chord diagram** of order m is an element of $\mathcal{K}_m/\partial\mathcal{K}_{m+1}$. This is equivalent to an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}_m$.

(b) The **chord diagram of an** m**-singular knot** K, denoted $\sigma(K)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of K. Traversing both K and the circle, label the points on the circle in the order in which the singular points of K are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(K)$.

Example 1.2.8



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Proposition 1.2.9 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}_m^*$. Then Q_0 is also an integral of P.

Proof The derivative of a function of chord diagrams is zero, as two knots which differ by a crossing change have the same chord diagram. Hence Q and Q_0 have the same derivative: P. □

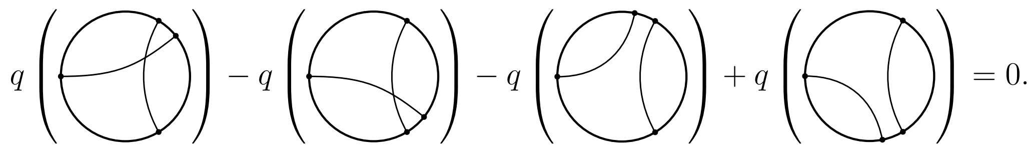
Remark 1.2.10 We defined a constant of integration as a set of objects (e.g. $c \in \mathcal{D}_m$) but perhaps it would have been more accurate to define it as a functional (e.g. $q \in \mathcal{D}_m^*$). After all the constant of integration on the real line is "$+\,C$" rather than the set $\{*\}$ with one object. There is no real risk of confusion, so let us be slightly loose and use the terminology for either.

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.11 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

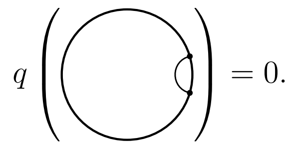
If we take the relations for a knot invariant to be integrable and project them into the space of chord diagrams, we get the following relations.

Definition 1.2.12 (a) The **four-term relation** is the relation



(4T)

(b) The **one-term relation** is the relation



(1T)

Remark 1.2.13 Just like T4T and T1T, 4T and 1T are not individual relations, but kinds of relations. To satisfy them, $q \in \mathcal{D}_m^*$ needs to satisfy them for all ways additional chords can be placed into the diagram to make a diagram of degree m, so long as they don't go between the close chord-ends.

Proposition 1.2.14 *A weight system is characterised as a constant of integration that satisfies* 4T *and* 1T.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T and 1T. Since crossing changes are free, this is equivalent to satisfying the projection of T4T and T1T into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of on m-singular P is of the form

$$Q + q \circ \sigma.$$

Since integration is linear, to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that $S\mathcal{K}_m$ vanishes. But this is a tautological statement of the general theory — it doesn't mean much if we don't know what $S\mathcal{K}_m$ is.

Conjecture 1.2.15 *An invariant of m-singular knots satisfying* T4T *and* T1T *integrates m times into a genuine knot invariant.*

Remarks 1.2.16 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.6. The point is that it may not be possible to choose the integral to again satisfy T4T and T1T, which is what $S\mathcal{K}$ measures.

(b) Computing $S\mathcal{K}_m$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.

(c) This conjecture is proven in certain cases. It holds the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98].

The finite type invariants in $\mathcal{K}^\star$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m + 1$.

If we restrict Conjecture 1.2.15 to Vassiliev invariants, then we get the following.

Theorem 1.2.17 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* `T4T`, `T1T` *and the additional condition that $\delta^k v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant).*

There are various proofs of the fundamental theorem. They are listed in [BS97], and each proof is accompanied by a series of moral objections. To quote their introduction: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made". To summarise their philosophy:

Remark 1.2.18 We have the implication Conjecture 1.2.15 $\implies$ Theorem 1.2.17, and this is actually realised in the theory of braids. It is mysterious that the fate of the slightly stronger conjecture which comes from taking natural topological approach to the fundamental theorem still remains unknown, and that there are grievances to be had with all known proofs.

In the Section 1.4 we will look at equivalent formulation of the fundamental theorem, and then in Chapter 2 we present some progress towards Conjecture 1.2.15.

1.3 The algebra of chord diagrams

Scaffold for this chapter.

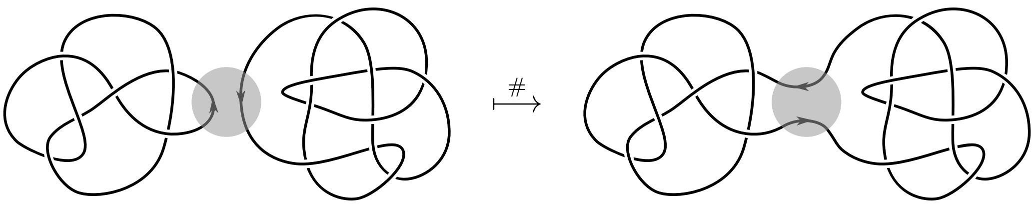
· Begin with the connected sum on plain knots (don't talk about modding by 1T 4T yet).

· This induces a connected sum on singular knots (define via commutative diagram).

· This nearly induces a conntected sum on chord diagrams... except need to mod out by 4T and 1T which we were gonna do anyway.

There is also an interesting algebraic structure on the space of knots, $\mathcal{K}_0$ which extends to chord diagrams.

Definition 1.3.1 The **connected sum** of two knots K_1 and K_2 is the knot obtained by removing a small arc from each of K_1 and K_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

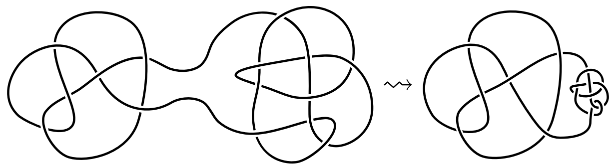


This definition is extended bilinearly to $\mathcal{K}_0$, i.e. linear combinations.

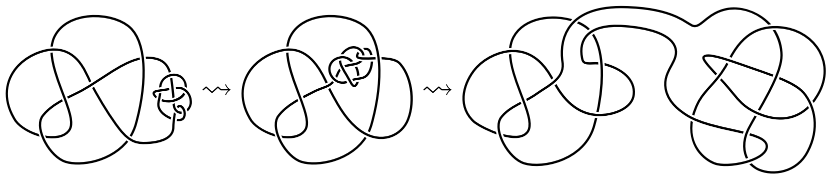
This in not a-priori well-defined: we have not specified where along either K_1 or K_2 the small arc is to be removed. However, by a classical knot-theoretic argument, the result is independent of the choice.

Proposition 1.3.2 *The connected sum $\# : \mathcal{K}_0 \otimes \mathcal{K}_0 \to \mathcal{K}_0$ forms a well-defined operation. It does not matter where along either knot the small arc was removed, the results are ambient-isotopic.*

Proof We exhibit an ambient isotopy starting at $K_1 \# K_2$ where the small arc is removed from K_1 as in the example above. The part of the connected sum coming from K_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of K_1.



Then, K_2 is then isotoped along K_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along K_1, and the same argument with the roles of K_1 and K_2 reversed completes the proof. □

Proposition 1.3.3 *There is a descending filtration of ideals on the space* $\mathcal{K}_0$

$$\mathcal{K}_0 \supset \delta^1(\mathcal{K}_1) \supset \delta^2(\mathcal{K}_2) \supset \cdots$$

with ith step of the filtration spanned by all resolutions of i-singular knots. Furthermore, the connected sum preserves the filtration,

$$\# : \delta^i(\mathcal{K}_i) \otimes \delta^j(\mathcal{K}_j) \to \delta^{i+j}(\mathcal{K}_{i+j}),$$

giving $\mathcal{K}_0$ the structure of a filtered algebra.

Proof Indeed the filtration is descending, since if $k \in \delta^j(\mathcal{K}_j)$, then it is the resolution of j double points of some singular knot with at least j double points. For any $i < j$, by resolving only i of the double points, it can be seen that $k \in \delta^i(\mathcal{K}_i)$.

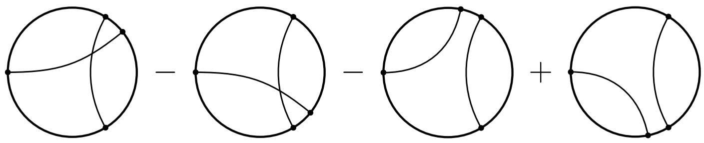
The connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. It remains to prove that $\delta^i(\mathcal{K})$ are ideals, and that the connected sum respects the filtration.

For the former, suppose $k \in \delta^i(\mathcal{K}_i)$. Then there is i-singular $k_\bullet \in \mathcal{K}_i$ such that $\delta^i(k_\bullet) = k$. Let $a \in \mathcal{K}_{\prime}$. To show that $k \# a \in \delta^i(\mathcal{K}_i)$, it suffices to exhibit an element of $\mathcal{K}_i$ that resolves under δ^i to $k \# a$. The "connected sum" $k_\bullet \# a$ is such a singular knot (see Remark 1.3.4 below).

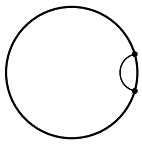
For the latter, if $k \otimes \ell \in \delta^i(\mathcal{K}_i) \otimes \delta^j(\mathcal{K}_j)$, then there are $k_\bullet$ and $\ell_\bullet$ in $\mathcal{K}_i$ and $\mathcal{K}_j$ that resolve to k and ℓ, respectively. Similarly, the "connected sum" $k_\bullet \# \ell_\bullet$ resolves by δ^{i+j} to $k \# \ell$, which is therefore in $\delta^{i+j}(\mathcal{K}_{i+j})$. □

Remark 1.3.4 Parts of the proof involve the connected sums of singular knots with regular knots, or singular knots with singular knots. This was not part of Definition 1.3.1, but the definition can easily be extended by ensuring that the small arcs removed from a singular knot do not contain a singular point. Still, this is ill-defined, as the resulting singular knot may depend on from which side of the singular point the arc was removed. However, for the sake of the proof, any choice suffices.

The algebra strucutre on knots is heavily related to an algebra of chord diagrams. Recall the `4T` and `1T` relations from the previous section. Just like with the `DIFF` and `DIFF`* relations, instead of studying functions on chord diagrams satisfying `4T` and `1T`, we prefer to study functions on spaces quotiented by relations on the objects.



$$= 0. \qquad (\mathtt{4T}^*)$$



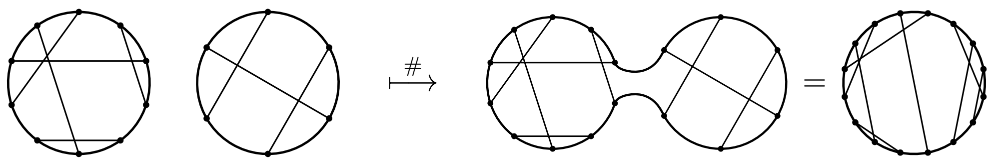
$$= 0 \qquad (\mathtt{1T}^*)$$

Definition 1.3.5 We define $\mathcal{A}$, the **space of chord diagrams** as

$$\mathcal{A} = \mathcal{D}/\mathtt{4T}^*, \mathtt{1T}^*.$$

Warning 1.3.6 Both $\mathcal{A}$ and $\mathcal{D}$ are known as the space of chord diagrams. From now on when we refer to a chord diagram, we mean an element of $\mathcal{A}$ unless otherwise specified.

Definition 1.3.7 The **connected sum** of two chord diagrams A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



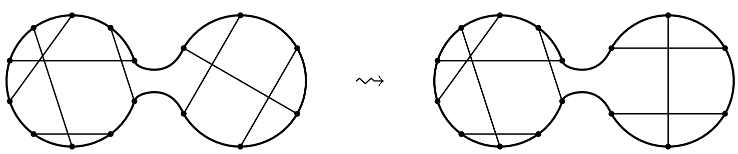
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed in the algebra $\mathcal{D}$ this is ill-defined. However the $\mathtt{4T}^*$ relation in $\mathcal{A}$ takes care of this.

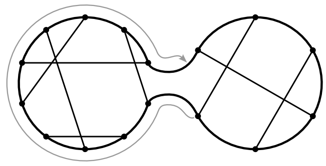
Proposition 1.3.8 *The connected sum operation* $\# : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ *is well-defined.*

Proof We will prove that the connected sums, given any two choices of connection locations, are equal modulo $\mathtt{4T}^*$.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one 'click', like so:

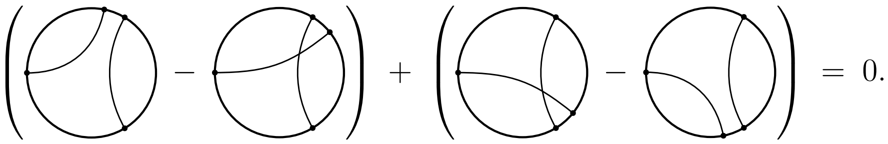


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow.



Which we show can be achieved by a series of 4T* relations.

We can rewrite 4T* as



A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T*. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T*. So, the two connected sums $a_1 \# a_2$ differ by a sum of 4T* relations, completing the proof. $\square$

1.4 Chord diagrams and the universal Vassiliev invariant

Either way.

Fundamental theorem is the same as having a universal invariant, which is the same as the space being the associated graded. In full detail.

For knots this is the Knotsevich integral. It comes about roughly as follows.

The point of the fundamental theorem of Vassiliev invariants is that it allows us to establish a relationship between Vassiliev invariants and weight systems. In this section we will reformulate the fundamental theorem in a way that makes this apparent. At the same time we will describe the algebra structure on the spaces at play.

Definitions 1.4.1 (a) A **filtration** on a vector space X is a descending sequence of nested subspaces

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

indexed by natural numbers

Notation 1.4.2 (a) Let $\mathcal{V}_m$ be the vector space of Vassiliev invariants of degree m. The space $\mathcal{V}$ will be the space of all Vassiliev invariants, and it is filtered by degree.

(b) Let W_m be the vector space of weight systems of degree m. Define $W = \bigoplus_{i=0}^{\infty} W_m$.

Corollary 1.4.3 *The space of weight systems is the associated graded space of the space of Vassiliev invariants:*

$$W = \bigoplus_{i=0}^{\infty} \mathcal{V}_m / \mathcal{V}_{m-1},$$

or

$$W = \operatorname{gr} \mathcal{V}$$

2

Combinatoral integration of weight systems

See the .tex comments about questions I still have about this section. (some now answered - will go in the previous section.)

2.1 Willerton's "half"-integration

2.2 Setting some more of the constants

2.3 Computational analysis

3

Lie theory and Jacobi diagrams

3.1 First subsection

3.2 Second section

3.3 Third subsection

4

Jacobi diagrams as a universal enveloping algebra

4.1 First subsection

4.2 Second section

4.3 Third subsection

5

Welded knots and arrow diagrams

5.1 First subsection

5.2 Second section

5.3 Third subsection

6

Arrowed Jacobi diagrams as a universal enveloping algebra

6.1 First subsection

6.2 Second section

6.3 Third subsection

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