On Vassiliev Invariants and Weight Systems of Classical and Welded Knots

Damian Lin

An essay submitted in fulfilment of
the requirements for the degree of
Master of Philosophy (Science)

Pure Mathematics
University of Sydney

February 26, 2026

Contents

Statement of originality

This statement is to certify that to the best of my knowledge, the content of this thesis is my own work, and that this thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Damian Lin

Acknowledgement of financial support

This research was supported by an Australian Government *Research Training Program (RTP) Fees Offset Scholarship*, a Sydney University School of Mathematics and Statistics *Postgraduate Research Scholarship in Mathematics and Statistics* and a Sydney University Grants–in–Aid *William and Catherine McIlrath Scholarship*.

Statement on use of generative artificial intelligence

Generative artifical intelligence was not used in the writing or creation of the intellectual content of this thesis.

Acknowledgements

My sincere thanks go to my advisor, Zsuzsanna Dancso for two years of sage advice, patient guidance and wholehearted encouragement. Further thanks for fostering a wholesome group environment, and to the members of that group for their support and company.

Some conversations with members of the mathematical community I am particularly grateful for, particularly those with Dror Bar-Natan, Edmund Heng, Tamara Hogan, Thorsten Hertl, Chandan Singh and Anderson Vera who gave insightful answers to my questions. Thanks also to many office mates for various mathematical discussions: Nelson Odins-Jones, Lewis Combes, Max Mikkelsen, Nick Bridger, Tao Qin and Jensen O'Sullivan. Special thanks in particular to Max Mikkelsen for proofreading a significant portion of a draft of this document and providing a great deal of typographical and editorial suggestions from a perspective outside this field.

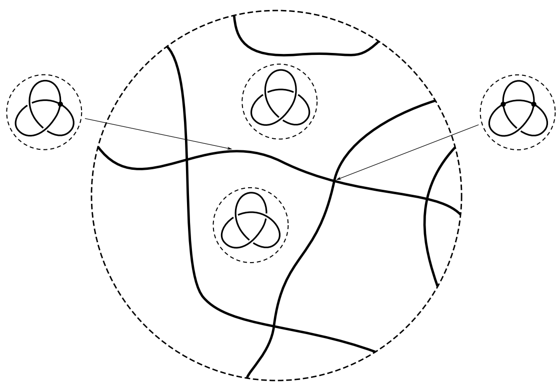
Thanks to the School of Mathematics and Statistics ICT support team for maintaining the machines which ran some of the computations in this thesis, and to the developers of Inkscape, VimTeX and Sagemath for creating excellent free software I used to write it.

Finally, thanks to my friends for helping me maintain my sanity these last two years.

Introduction

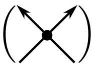
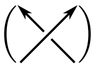
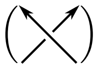
Vassiliev invariants are a class of knot invariant that have special properties, much like how polynomials are a special kind of function. They fit within a general framework of invariants of topological objects due to Thom, Arnold and Vassiliev. The general theory defines invariants of a class of topological objects by taking into account not only the objects themselves but also their singular versions and how they all fit within a larger topological space.

The example we discuss in this thesis is that of knots. The space of immersions $S^1 \to \mathbb{R}^3$ contains knots (which are the embeddings), but also proper immersions that have one or more intersection points in $\mathbb{R}^3$ and therefore fail to be knots. The singular knots form a space of codimension one within the space of immersions; taking its complement divides the space of immersions into connected components which are exactly the knots. Similarly, inside the space of immersions with one or more intersection points lies the codimension one space of immersions with two or more intersection points, and it again divides up the space of one-singular-point immersions into connected components. This continues, dividing the infinite-dimensional space of immersions into the stratification of the space of knots. A rough schematic illustration of the stratification might look like the image below:



Of course, this two-dimensional picture doesn't properly represent the infinite-dimensional stratification.

The Vassiliev invariants are functions on the chambers of the strata (the knots) which take into account the walls of the strata by changing by a predictable amount across a wall (or higher-dimensional stratum), therefore obeying a relation between

f  and f  $-$ f  .

Chapter 1 reviews the theory of Vassiliev invariants, with attention given to the particularly fruitful polynomial analogy first made by Bar–Natan in [Bar95]. The set of all Vassiliev invariants $\mathcal{V}$ has the structure of a filtered bialgebra, and its product and coproduct mirror those of the filtered bialgebra of polynomial functions on the real line.

The most important result of the chapter is the fundamental theorem of Vassiliev invariants — a corollary of the work of Kontsevich [Kon93] — which states that the bialgebra of Vassiliev invariants is effectively described by its associated graded bialgebra, the algebra $\mathcal{A}$ of chord diagrams. Once again, this fits neatly into an analogy with polynomials in which elements of $\mathcal{A}$ are likened with constants of integration. Indeed, any polynomial function of degree m on the real line can be constructed as a definite integral of the zero function m times by making some specific choice of the constant of integration in each of the m definite integrals. This choice of constants at every stage completely describes the polynomial. An arbitrary Vassiliev invariant is described by the algebra $\mathcal{A}$ of chord diagrams in an analogous way.

In Chapter 2, the structure of the algebra $\mathcal{A}$ is examined. The relations in this algebra have a Lie algebraic flavour, and as such a classical construction of Bar-Natan takes a metric Lie algebra and produces a functional on $\mathcal{A}$, known as a weight system. By the fundamental theorem of the previous chapter this yields a class of Vassiliev invariants. We showcase this construction by computing, following Yang [Yan24], the values of the functional associated with the exceptional Lie algebra $\mathfrak{g}_2$ on an infinite family of chord diagrams.

Somewhat surprisingly, not all Vassiliev invariants arise from this construction, and a more general construction is required which takes not just Lie (super)algebras but Lie algebra objects in arbitrary symmetric monoidal categories. We review a classical theorem of Hinich–Vaintrob from [HV00] that every Vassiliev invariant is recovered by this more general construction which takes as input some Lie algebra object in some symmetric monoidal category. However, this result does not shed light onto which type of symmetric monoidal categories' Lie algebra objects product more Vassiliev invariants than simply Lie algebras.

The focus of Chapter 3 is welded long knots, a type of two-dimensional knotted object in four-dimensional space. Welded long knots also have a theory of Vassiliev invariants and a corresponding version of $\mathcal{A}$, known as the bialgebra of arrow diagrams $\mathcal{A}_w$. Just as the structure of $\mathcal{A}$ is Lie algebraic in nature, in a similar way $\mathcal{A}_w$ has structure reminiscent of the cocommutative Drinfeld double of a Lie algebra. We prove a Hinich–Vaintrob style theorem in the theory of welded long knots. This gives an alternative proof of the fact just like for classical knots, that every Vassiliev invariant of welded long knots comes from a Lie algebra object in a symmetric monoidal category. This fact is also a corollary of the work Bar-Natan–Dancso [BD16], in fact it follows from their work that taking Lie algebra objects in the symmetric monoidal category **sVect** of supervector spaces suffices to construct all Vassiliev invariants of welded long knots.

Conventions

All knots are assumed to be oriented and framed unless specified otherwise. Where a statement holds for a knot which is drawn unoriented, it holds for both orientations of that knot.

1

Vassiliev invariants and chord diagrams

POLYNOMIALS are a special type of function: they are related in a natural way to the derivative, they are defined by a finite amount of combinatorial data, and they can be used to approximate any continuous function.

In a similar way, the Vassiliev knot invariants are a special type of knot invariant. This analogy, first made by Dror Bar-Natan in [Bar95] is in no way superficial. As we will come to see, Vassiliev invariants enjoy analogues of the first two properties above. They are also conjectured to enjoy the third, at the very least being able to approximate many well-known invariants, for example the Conway and Jones polynomials.

In this chapter we present the introductory theory of Vassiliev invariants with the aim of making the analogy above as explicit as possible. This also leads a natural interpretation of the defining relations of the algebra $\mathcal{A}$ of chord diagrams, which is the fundamental object of study in the field.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m-**singular**.

Remark 1.1.2 Immersions with other types of singularities are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular, immersions with

(a) triple points, and

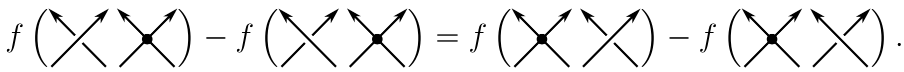
(b) points with vanishing derivative

are excluded from the definition.

A singular knot with one double point is very close to two other knots: the knot in which the singular point is replaced by a positive crossing, and the knot in which it is replaced by a negative crossing.

Just as we have notions of ambient isotopy for knots and knot invariants, we can have m-singular isotopy and m-singular knot invariants.

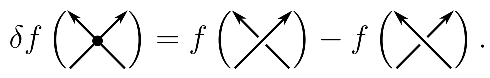
Definition 1.1.3 An invariant f of m-singular knots is **differentiable** if



(DIFF)

If an m-singular knot invariant is differentiable, we can extend it to an invariant of $(m+1)$-singular knots by a procedure analagous to taking its derivative.

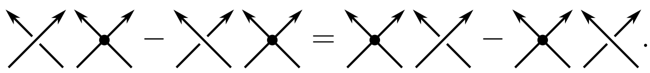
Definition 1.1.4 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



A knot invariant (which is an invariant of 0-singular knots) is vaccuously differentiable and its derivative is an invariant of 1-singular knots. Furthermore, if an invariant of m-singular knots is differentiable then so is its derivative, so it can be extended to any number of double points. In particular, knot invariants have derivatives of all orders.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

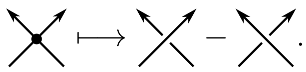
Definition 1.1.5 Define $\mathcal{K}_m^\bullet$ as the span of all m-singular knots, taken over $\mathbb{Q}$, modulo the following boundary relation (also known as a codifferentiability relation):



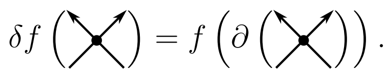
(DIFF*)

From now on, we will refer to elements of $\mathcal{K}_m^\bullet$ as m-**singular knots**, that is the `DIFF`* relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m^\bullet \to \mathcal{K}_{m-1}^\bullet$ defined by



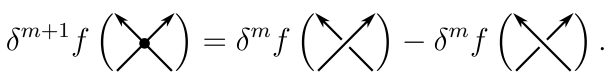
Remark 1.1.7 The derivative δ and the boundary ∂ are adjoint,



From Definition 1.1.4, any knot invariant f can be extended to an invariant $\delta^m f$ of m-singular knots

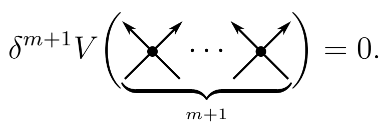
$$\delta^0 f = f$$

and



The Vassiliev skein relation extends a knot via its derivative, that is, defines a function which on an $(m+1)$-singular knot agrees with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to $(m+1)$-singular knots via its $(m+1)$st derivative, it vanishes



(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the greatest number of double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 The $(m+1)$st derivative of a degree m polynomial also vanishes.

1.2 The stratification of the space of knots and integration

To help see the bird's eye view we phrase the analogy between Vassiliev invariants and polynomials in terms of an integration theory, following [Hut98].

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of objects and morphisms in some category. For example, they could be vector spaces, algebras, bialgebras, free abelian groups or modules. Here we call them spaces, because we wish to use the word object for elements of the spaces. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The primary interest is often the space $\mathcal{O}_0$, which contains the objects we want to understand. The spaces $\mathcal{O}_m$ can often be thought of as containing m-singular versions of the same kind of objects as $\mathcal{O}_0$. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing another space X and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, X)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is adjoint to ∂_m. The maps δ behave like derivatives: if $f \in \mathcal{O}_m^*$, then $\delta f \in \mathcal{O}_{m+1}^*$ is equal to some combination of values of f on "close" m-singular elements.

Questions 1.2.2 We wish to understand how to invert this process, namely:

(a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?

(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there multiple solutions when integrating (akin to the constant of integration " $+C$" on the real line)?

(c) When does such a functional integrate multiple times, in particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are multiple solutions when integrating, does this choice affect whether the new functional is integrable again?

(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The answers to the above questions are given precisely by the following spaces.

Definitions 1.2.3 (a) The **primary obstructions to integration** are the space

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the space

$$C\mathcal{O}_m = \mathcal{O}_m / \partial \mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the space

$$S\mathcal{O}_m = \ker (\partial_{m+1}\partial_m) / P\mathcal{O}_m,$$

and the **order k obstructions to integration** are defined analogously.

(d) The **weights of integration** are the space

$$W\mathcal{O}_m = C\mathcal{O}_m/\pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the natural projection.

(e) The **finite type invariants** of order m are the space

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

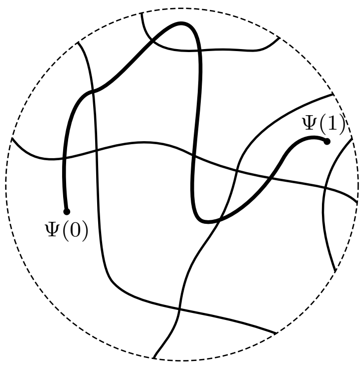
It may not be entirely obvious how these definitons provide answers to the questions above, but in the rest of this section we will illustrate this for the case $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$.

Recall that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy** Ψ of m-singular knots is a path $\Psi(t), t \in [0,1]$ in the union of the mth and $(m+1)$st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : s \in I\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

A singular isotopy is **closed** if $\Psi(0) = \Psi(1)$.

For example in terms of our pictorial representation of the strata from the introduction, a singular isotopy might look like the following.



Let us rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$. Differentiation takes an invariant Q of m-singular knots and defines an invariant Q' of $(m+1)$-singular knots. The values of Q' on the $(m+1)$-singular knot $k^\bullet$ is the difference of values of Q on the combination of m-singular knots $\partial k^\bullet$. Therefore if $\Psi(t)$ is a singular isotopy of m-singular knots from $k^\bullet_0$ to $k^\bullet_1$, with singularity set $\{\Psi_s : s \in I\}$, we have

$$Q(k^\bullet_0) - Q(k^\bullet_1) = \sum_{s \in I} \varepsilon_s Q'(\Psi_s).$$

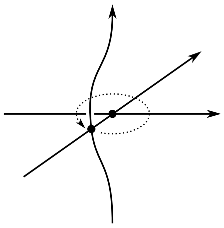
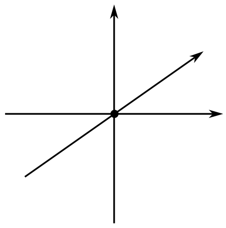
Note that due to the `DIFF*` relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knots such that

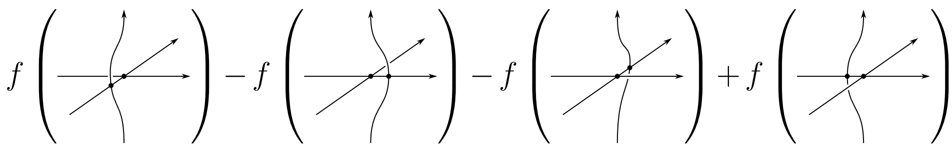
$$Q(k^\bullet_0) - Q(k^\bullet_1) = \sum_{s \in I} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a "path-integral" along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that for example, triple points are excluded from all levels of the stratification, leaving "holes" in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate.

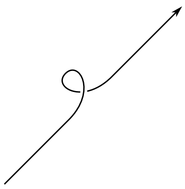
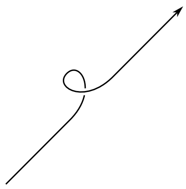
Definition 1.2.5 (a) The closed singular isotopy around a triple point given by

 around 

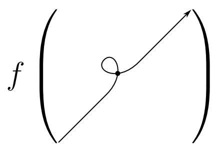
gives rise to a **topological four-term relation**

 $= 0.$ (T4T*)

(b) The singular isotopy

 $\rightsquigarrow$ 

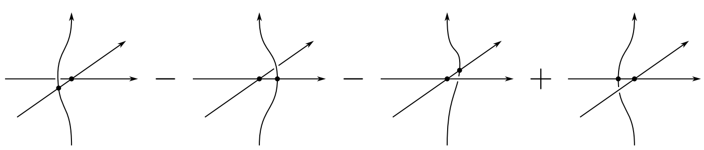
around a point with a vanishing derivative gives rise to a **topological one-term relation**

 $= 0.$ (T1T*)

Remark 1.2.6 Since we are considering framed knots, only the singular isotopy in Definition 1.2.5 (a) is closed, whereas the singular isotopy in Definition 1.2.5 (b) is not; so only T4T* holds. When considering unframed knots, the theory is the same as what we present below, but T1T* plays the same role as T4T*.

In particular, for every statement in the rest of this chapter about weight systems, chord diagrams and indeed every space in Definition 1.2.3, a corresponding statement holds for an unframed version of that space. In the unframed version of the statement, hypotheses involve only T4T* (or a variant thereof), rather than both T4T* and T1T* like in the framed version. We give a few examples of the unframed statements as we go. For all of the statements, and the relationship between the framed and unframed spaces, we refer the reader to [CDM12].

Indeed, the definition of the primary obstructions to integration as $P\mathcal{O}_m = \ker \partial_m$ is justified, as a cycle in $\ker \partial_m$ is a combination of $(m+1)$-singular knots with signs through which a singular isotopy could pass. From Definition 1.2.5, we have a cycle of the kind



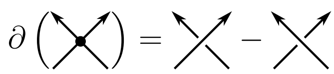
in $P\mathcal{K}^\bullet_m = \ker \partial_m$. However, we it is not clear whether combinations of this form span $P\mathcal{O}$, that is whether obstructions of this kind are the only type of obstruction.

Theorem 1.2.7 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $(m-1)$-singular knots if and only if it satisfies* T4T*.

Remark 1.2.8 The corresponding unframed version of Stanford's theorem reads: *An invariant f of m-singular unframed knots integrates to an invariant of $(m-1)$-singular unframed knots if and only if it satisfies* T4T* *and* T1T*.

To answer the question about whether the primary obstructions are the only obstructions, it's helpful to define the constants of integration, which comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost, and when integrating, a choice has to be made.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}^\bullet_m = \mathcal{K}^\bullet_m/\partial\mathcal{K}^\bullet_{m+1}$. Let us examine the classes of this quotient space. From the defintion of the derivative,

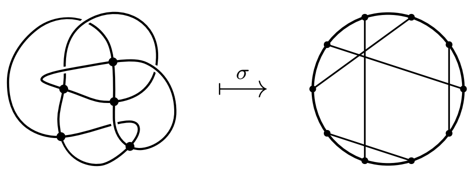


so the difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. The classes of this quotient are therefore m-singular knots modulo crossing changes. The only data needed to describe such an equivalence class is the order in which the m double points are traversed around the m-singular knot.

Definitions 1.2.9 (a) A **chord diagram** of order m is an element of $\mathcal{K}^\bullet_m/\partial\mathcal{K}^\bullet_{m+1}$. An equivalent combinatorial description of such an element is an oriented circle with a distinguished set of m unordered pairs of points, considered up to orientation-preserving diffeomorphisms of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}^\bullet_m$. The oriented circle is called the **skeleton** of the chord diagram.

(b) The **chord diagram of an m-singular knot** k, denoted $\sigma(k)$ is the chord diagram formed by the following process. Place $2m$ points on the skeleton, two for each singular point of k. Traversing both k and the skeleton, label the points on the skeleton in the order in which the singular points of k are traversed. Each label is given twice, and points with the same label are paired, forming $\sigma(k)$.

Example 1.2.10



◊

Proposition 1.2.11 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function q on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}^*_m$. Then Q_0 is also an integral of P.

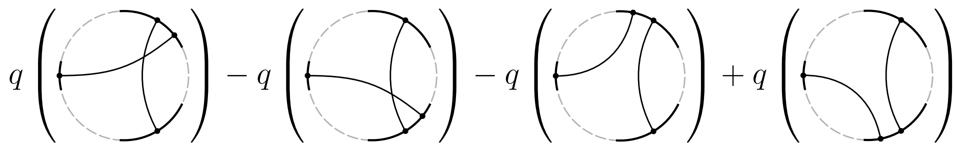
Proof The derivative of an n-singular invariant that factors through σ is zero: crossing changes do not change the chord diagram, so when a crossing change occurs, the function does not change. Hence Q and Q_0 have the same derivative P. $\square$

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.12 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

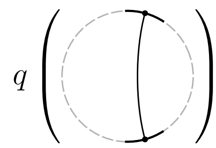
The T4T* relations are what knot invariants must satisfy to be integrable (also T1T* for unframed invariants). If we take their images under σ, we get the following relations in the space of chord diagrams.

Definitions 1.2.13 (a) A **four-term relation** is a relation of the kind



$= 0.$ (4T*)

(b) A **one-term relation** (relevant only in the unframed case) is a relation of the kind



$= 0.$ (1T*)

where the chord shown is **isolated**, meaning no other chord intersects it (when suitably isotoped).

Just like T4T*, 4T* is not an individual relation, but a class of relations. For example, there's one 4T* relation for all ways of placing other chords on the dotted parts of the diagrams above, identically in all four diagrams.

Proposition 1.2.14 *Weight systems are exactly constants of integration that satisfy* 4T*.

Proof Recall that a weight system is an integrable $w \in \mathcal{D}_m^*$. A functional $w : \mathcal{D}_m \to \mathbb{Q}$ defines an m-singular invariant $w \circ \sigma$ by precomposing with σ. Such an m-singular invariant depends only on the chord diagram and so it is unchanged by crossing changes. To be integrable, $w \circ \sigma$ must satisfy T4T*, but this is equivalent to w satisfying the projection of T4T* into chord diagrams, which is 4T*. $\square$

We return now to the secondary (and higher) obstructions. A general integral of an m-singular P is only defined up to constants of integration, so is of the form

$$Q + q \circ \sigma.$$

For this to be integrable again, both Q and $q \circ \sigma$ need to be integrable since integration is linear. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that the secondary obstructions $S\mathcal{K}_m^\bullet$ vanish. In our current case of $\mathcal{O} = \mathcal{K}$, the secondary obstructions are conjectures to vanish.

Conjecture 1.2.15 *An invariant of m-singular knots satisfying* T4T* *integrates m times into a genuine knot invariant.*

Remarks 1.2.16 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.7. The point is that it may not be possible to choose the integral to again satisfy T4T*, which is what $S\mathcal{K}^\bullet$ measures.

(b) Computing $S\mathcal{K}^\bullet_m$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (a similar thing was seen with the primary obstructions). Computing $\ker \partial^2$ is the hard part — some elements of this space are found in [Hut98], but whether they suffice to generate the kernel remains open.

(c) This conjecture is known in certain cases. It holds in the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98a].

The finite type invariants in $\mathcal{K}^\bullet_\star$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m+1$.

Restricting Conjecture 1.2.15 to Vassiliev invariants, yields the following.

Theorem 1.2.17 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* T4T* *and the additional condition that $\delta v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant of order m).*

Remark 1.2.18 There are various proofs of the fundamental theorem, listed in [BS97]. Unfortunately none of them are fully understood. In the words of Bar-Natan: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made".

In particular, it is mysterious that the natural topological approach of computing $S\mathcal{K}^\bullet_m$ works for braids but remains intractable for knots.

In Section 1.5 we will prove an equivalent formulation of the fundamental theorem.

1.3 Knots and Vassiliev invariants

Speaking broadly, the aim of Vassiliev theory is to study the space of knots using information from the stratification of knots introduced in the first two sections. This is done via the space of chord diagrams, which can be considered its linearisation or projectivisation. But these spaces are not just vector spaces — there is further multiplicative structure which we wish to incorporate.

Definitions 1.3.1 (a) The **space of knots** denoted $\mathcal{K}$ is the vector space spanned over $\mathbb{Q}$ by non-singular knots. Equivalently, $\mathcal{K} = \mathcal{K}^\bullet_0$.

(b) The space of knots is equipped with the **singular knot filtration**

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

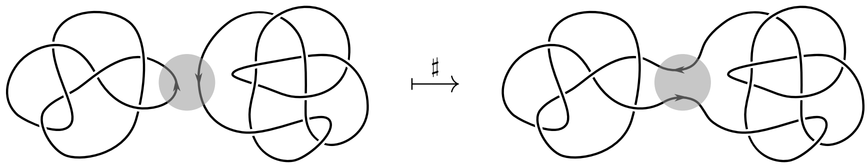
where the ith filtered component $\mathcal{K}_i$ is the span of resolutions of singular knots with i double points. Equivalently $\mathcal{K}_i = \partial^i(\mathcal{K}^\bullet_i)$.

Proposition 1.3.2 *The singular knot filtration is indeed a descending filtration of vector spaces.*

Proof This being a filtration of vector spaces, the only thing to check is that if $i < j$, then $\mathcal{K}_i \supset \mathcal{K}_j$. If $k \in \mathcal{K}_j$, then $k = \partial^j(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}_j$. But then $k = \partial^j(k^\bullet) = \partial^i\partial^{j-i}(k^\bullet)$, so $k \in \partial^i(\mathcal{K}^\bullet_i)$. $\square$

The algebraic structure on knots comes from the following operation.

Definition 1.3.3 The **connected sum** of two knots k_1 and k_2 is the knot $k_1 \sharp k_2$ obtained by removing a small arc from each of k_1 and k_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

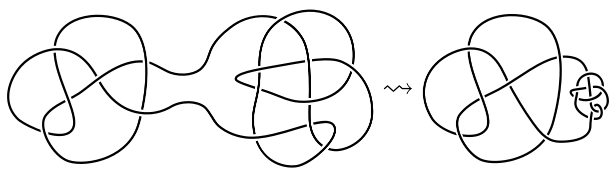


This definition is extended bilinearly to $\mathcal{K}$.

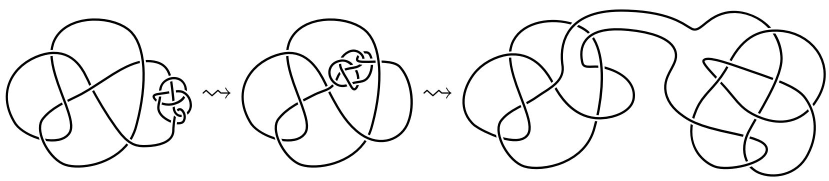
The connected sum of knots is not a priori well-defined, as we have not specified where along either k_1 or k_2 the small arc is to be removed or along which path they should be connected. However, by a classical knot-theoretic argument, the result is independent of either choice.

Proposition 1.3.4 *The connected sum operation* $\sharp : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ *is well-defined — up to ambient isotopy, it does not matter where and along what path the connection was performed.*

Proof We exhibit an ambient isotopy starting at $k_1 \sharp k_2$ where the small arc is removed from k_1 as in the example above. The part of the connected sum coming from k_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of k_1.



Then, k_2 is isotoped along k_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along k_1, and the same argument with the roles of k_1 and k_2 reversed completes the proof.

A similar agument also shows that the path along which the two knots are connected doesn't matter, as k_2 can always be shrunk to within a small tubular neighbourhood of k_1. □

Proposition 1.3.5 *The connected sum respects the descending filtration, so that*

$$K_i \sharp K_j \subset K_{i+j} \quad \textit{for all } i, j \geq 0.$$

That is, the connected sum makes $(\mathcal{K}, \sharp)$ *into a descending filtered algebra.*

Proof Indeed, the connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. The question is whether the connected sum respects the filtration.

If $k \otimes \ell \in \mathcal{K}_i \otimes \mathcal{K}_j$, then there are $k^\bullet$ and $\ell^\bullet$ in $\mathcal{K}_i^\bullet$ and $\mathcal{K}_j^\bullet$ that resolve to k and ℓ, respectively. Similarly, the 'connected sum' $k^\bullet \sharp \ell^\bullet$ resolves by ∂^{i+j} to $k \sharp \ell$, which is therefore in $\partial^{i+j}(\mathcal{K}_{i+j})$.

Here, 'connected sum' is enclosed in inverted commas due to the following technicality. Connected sums of singular knots with singular knots were not part of Definition 1.3.3. Even if we ensure that the small arcs removed from a singular knot do not contain a singular point, this is still ill-defined. The ambiguity is that the resulting singular knot may depend on from which side of the singular point the arc was removed, i.e. the repositioning argument in the proof of Proposition 1.3.4 fails due to the presence of singular points. After taking the resolution under ∂^{i+j} the repositioning argument now works again, so any of the choices of connected sum in $k^\bullet \sharp \ell^\bullet$ produce a singular knot which resolves to $k \sharp \ell$. $\square$

Definition 1.3.6 The coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is defined on knots k as

$$\Delta(k) = k \otimes k$$

and extended bilinearly to $\mathcal{K}$.

Proposition 1.3.7 *The triple $(\mathcal{K}, \sharp, \Delta)$ forms a bialgebra. In other words, the connected sum and coproduct are compatible.*

Proof A bialgeba is a vector space which is both an algebra and a coalgebra with compatible product and coproduct. That $(\mathcal{K}, \Delta)$ forms a coalgebra is trivial (the counit is the augmentation map denoted ε), and we have already seen that $(\mathcal{K}, \sharp)$ is an algebra. So it remains only to check the compatibility conditions. Indeed, we have

$$\begin{aligned}\Delta(k \sharp \ell) &= k \sharp \ell \otimes k \sharp \ell \\ &= (k \otimes k) \sharp^{\otimes 2} (\ell \otimes \ell) \\ &= \Delta(k) \sharp^{\otimes 2} \Delta(\ell)\end{aligned}$$

where $\sharp^{\otimes 2}$ denotes the component-wise tensor product on $\mathcal{K} \otimes \mathcal{K}$.

Checking the unit and counit is trivial. $\square$

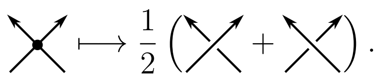
Proposition 1.3.8 *The coproduct Δ also respects the filtration,*

$$\Delta(\mathcal{K}_j) \subset \sum_{i=0}^{j} \mathcal{K}_i \otimes \mathcal{K}_{j-i}.$$

That is, $\mathcal{K}$ is a filtered bialgebra with the singular knot filtration.

We give a proof due to Willerton which follows directly from Lemma of 1.3.10 of [Wil96]. The lemma is a formula for the coproduct of an element $k \in \mathcal{K}_m$ that comes from some $k^\bullet \in \mathcal{K}_m^\bullet$. The formula is in terms of the 2^m ways of resolving some of singular points in one cofactor and the rest in the other, but first we need some notation.

If I is a subset of the singular points of a singular knot, let ∂^I be the operator that resolves the singular points I. Let μ^I be the operator that averages singular points in I, where averaging a singular point defined by the map



Remark 1.3.9 There is one technicality. The operations δ and ∂ were defined without the need to specify a specific double point (this was the point of the `DIFF` and `DIFF*` relations), but the definitions of μ^I and ∂^I are specific about double points. So for Lemma 1.3.10, its proof and the proof of Proposition 1.3.8, we write $\mathcal{K}^\bullet_m$ even though we mean the lift of the $\mathcal{K}^\bullet_m$'s, without the `DIFF` and `DIFF*` relations. We choose not to alter the notation as projecting the formulas back down to the quotient properly proves that Δ respects the filtration as intended.

Lemma 1.3.10 (Willerton) *Suppose $k^\bullet \in \mathcal{K}^\bullet_m$, and let S denote the set of singular points of $k^\bullet$. Then we have*

$$\Delta(\partial^S(k^\bullet)) = \sum_{I\subset S} \mu^{\bar{I}}\partial^I(k^\bullet) \otimes \mu^I\partial^{\bar{I}}(k^\bullet)$$

where $\bar{I} = S \setminus I$.

Proof We proceed by induction on m. In the base case of $m = 0$, $S = \emptyset$, and $k^\bullet = k$ is a genuine knot, so

$$\begin{aligned}\Delta(\partial^0(k)) &= \Delta(k)\\ &= k\otimes k\\ &= \sum_{I\subset\emptyset} \mu^{\bar{I}}\partial^I(k)\otimes \mu^I\partial^{\bar{I}}(k).\end{aligned}$$

The inductive step is as follows. Let $k^\bullet \in \mathcal{K}^\bullet_{m+1}$, let J denote all singular points of $k^\bullet$, and $x \in J$ be a specific singular point. Furthermore, let $k^{\bullet+}$ (resp. $k^{\bullet-}$) denote the m-singular knots obtained from $k^\bullet$ when x is replaced by a positive (resp. negative) crossing, so that $\partial^{\{x\}}(k^\bullet) = k^{\bullet+} - k^{\bullet-}$. We proceed by examining

$$\sum_{I\subset J} \mu^{\bar{I}}\partial^I(k^\bullet) \otimes \mu^I\partial^{\bar{I}}(k^\bullet).$$

Decomposing the sum based on whether $x \in I$ yields

$$\sum_{x\in I\subset J} \mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^\bullet)\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^\bullet) + \sum_{x\notin I\subset J}\mu^{\bar{I}\setminus\{x\}}\partial^I(k^\bullet)\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^\bullet),$$

after which resolving either ∂ or μ on x,

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J}\mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^{\bullet+}-k^{\bullet-})\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^{\bullet+}+k^{\bullet-})\\ &+\frac{1}{2}\sum_{x\notin I\subset J}\mu^{\bar{I}\setminus\{x\}}\partial^I(k^{\bullet+}+k^{\bullet-})\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^{\bullet+}-k^{\bullet-}).\end{aligned}$$

Expanding, yields the cumbersome formula

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J}\Big(\mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^{\bullet+})\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^{\bullet+}) + \mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^{\bullet+})\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^{\bullet-})\\ &\qquad - \mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^{\bullet-})\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^{\bullet+}) - \mu^{\bar{I}}\partial^{I\setminus\{x\}}(k^{\bullet-})\otimes\mu^{I\setminus\{x\}}\partial^{\bar{I}}(k^{\bullet-})\Big)\\ &+\frac{1}{2}\sum_{x\notin I\subset J}\Big(\mu^{\bar{I}\setminus\{x\}}\partial^I(k^{\bullet+})\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^{\bullet+}) - \mu^{\bar{I}\setminus\{x\}}\partial^I(k^{\bullet+})\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^{\bullet-})\\ &\qquad + \mu^{\bar{I}\setminus\{x\}}\partial^I(k^{\bullet-})\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^{\bullet+}) - \mu^{\bar{I}\setminus\{x\}}\partial^I(k^{\bullet-})\otimes\mu^I\partial^{\bar{I}\setminus\{x\}}(k^{\bullet-})\Big).\end{aligned}$$

Since $\{I \mid I \subset J, x \notin I\}$ is equal to $\{I \setminus \{x\} \mid I \subset J, x \in I\}$, in each of the above sums, the corresponding terms have the same indices. Hence, the first and last terms in each sum combine, and the second and third terms cancel out to give

$$\sum_{x \in I \subset J} \mu^{\bar{I}} \partial^{I \setminus \{x\}}(k^{\bullet +}) \otimes \mu^{I \setminus \{x\}} \partial^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \partial^{I \setminus \{x\}}(k^{\bullet -}) \otimes \mu^{I \setminus \{x\}} \partial^{\bar{I}}(k^{\bullet -}).$$

Since neither $\mu^{\bar{I}} \partial^{I \setminus \{x\}}$ or $\mu^{I \setminus \{x\}} \partial^{\bar{I}}$ are with respect to x, this can be written

$$\sum_{I \subset J \setminus \{x\}} \mu^{\bar{I}} \partial^{I}(k^{\bullet +}) \otimes \mu^{I} \partial^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \partial^{I}(k^{\bullet -}) \otimes \mu^{I} \partial^{\bar{I}}(k^{\bullet -})$$

which by the inductive hypothesis is

$$\begin{aligned} \Delta(\partial^{J \setminus \{x\}}(k^{\bullet +})) - \Delta(\partial^{J \setminus \{x\}}(k^{\bullet -})) &= \Delta(\partial^{J \setminus \{x\}}(k^{\bullet +}) - \partial^{J \setminus \{x\}}(k^{\bullet -})) \\ &= \Delta(\partial^{J}(k^{\bullet})). \end{aligned}$$ □

Proof of Proposition 1.3.8 The operators ∂^I and $\mu^{\bar{I}}$ commute since they are evaluating different singular points. Let I be an arbitrary subset of S, and let $|I| = i$ and $|S| = j$, then the left cofactor is in $\mathcal{K}_i$ and the right in $\mathcal{K}_{j-i}$. □

Not all knot invariants respect the singular knot filtration, as we will see. The point of the Vassiliev invariants is that they are the ones that are natural to consider with respect to the singular knot filtration. Indeed, the Vassiliev invariants are obtained directly from $\mathcal{K}$ via the following construction.

A decreasing filtration on a bialgebra induces an ascending filtration on its dual bialgebra. The dual bialgebra's mth filtered component is the space of functionals on the original bialgebra that vanish on the $(m+1)$st filtered component. In the case of $\mathcal{K}$, this is the set of knot invriants that vanish on $\mathcal{K}_{m+1}$: exactly the Vassiliev invariants of order m. The product/coproduct in the dual bialgebra are those operations adjoint to the coproduct/product in the original bialgebra. The construction is standard and the details can be found in [CDM12, Appendix A.2.4].

Definition 1.3.11 The **filtered bialgebra of Vassiliev invariants**, denoted $\mathcal{V}$, is the vector space of Vassiliev invariants with an ascending filtration by degree

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \quad \text{and} \quad \mathcal{V} = \bigcup_{m=0}^{\infty} \mathcal{V}_m.$$

The product is given by pointwise multiplication

$$V_1 \cdot V_2(k) = V_1(k) V_2(k)$$

and the coproduct η is given by

$$\eta(V)(k_1 \otimes k_2) = V(k_1 \sharp k_2).$$

Proposition 1.3.12 *The filtered bialgebraic dual of the descending filtered bialgebra of singular knots $\mathcal{K}$ is the ascending filtered bialgebra of Vassiliev invariants $\mathcal{V}$.*

We will not prove this standard fact, but let us sketch the main points. Firstly, by definition the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$ is $\mathcal{V}_m$.

Pulling $V_1 \otimes V_2$ back along Δ gives

$$(V_1 \otimes V_2) \circ \Delta : k \mapsto V_1(k)V_2(k),$$

which recovers the formula for the product $V_1 \cdot V_2$ in $\mathcal{V}$. Similarly, pulling V back along $\sharp$ yields

$$V \circ \sharp(k_1 \otimes k_2) \mapsto V(k_1 \sharp k_2)$$

which recovers the formula for the coproduct, η.

Here we rely on the fact that $\mathcal{K}$ is of finite type (finite-dimensional in each filtered component).

Remark 1.3.13 Alternatively, this can be proved directly in $\mathcal{V}$. Showing that $\sharp$ respects the filtration on $\mathcal{K}$ was easy, and is just as easy in the dual case. However, the proof that Δ respects the filtration on $\mathcal{K}$ was cumbersome, and so is that of its dual. But it is worth looking into how it can be understood by a continuation of the polynomial analogy due to Willerton [Wil96].

The generalised Leibniz theorem of multivariable calculus says that if f and $g : \mathbb{R}^m \to \mathbb{R}$ are differentiable, then (in similar derivative notation to as above)

$$\frac{\partial^{|I|}(fg)}{\partial_{x_I}} = \sum_{J \subset \{1,\cdots,i\}} \frac{\partial^{|J|} f}{\partial_{x_J}} \cdot \frac{\partial^{|\bar{J}|} g}{\partial_{x_{\bar{J}}}}.$$

This says that the derivative of a product of f and g with respect to some variables is the sum of every way of taking some of those derivatives with respect to f and some with respect to g.

A kind of dual theorem follows from this. Let $c \in \mathbb{R}$, and suppose that c comes from some pair of functions f and $g : \mathbb{R}^m \to \mathbb{R}$ by taking their derivatives with respect to some of the variables and evaluating all remaining variables in the result. Then, we get a cofactorisation for c in $\mathbb{R} \otimes \mathbb{R}$: i.e. if

$$c = \left.\frac{\partial^{|I|}(fg)}{\partial_{x_I}}\right|_{\{x_I = a_I\}}$$

then

$$\mu\left(\sum_{J \subset \{1,\cdots,i\}} \left.\frac{\partial^{|J|} f}{\partial_{x_J}}\right|_{\{x_J = a_{\bar{J}}\}} \otimes \left.\frac{\partial^{|\bar{J}|} g}{\partial_{x_{\bar{J}}}}\right|_{\{x_J = a_J\}} \right) = c$$

where $\mu : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is multiplication.

As it turns out, this generalised "co-Leibniz theorem" is pretty useless in the multivariable calculus case. The filtration on $\mathbb{R}$ coming from being the derivative of some function evaluated at some point is trivial, and so every $c \in \mathbb{R}$ comes from some such f and g, and it's easy to construct such f and g. But the co-Leibniz theorem in the case of knots is exactly Willerton's Lemma 1.3.10, where the averaging map plays the role of evaluation. Recall that this was used to show that the coproduct respects the filtration.

Furthermore, the knot version of the generalised Leibniz theorem [Wil96] is that if $k^\bullet \in \mathcal{K}^\bullet_m$ and S the set of singular points of $k^\bullet$, then

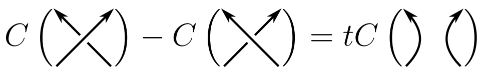
$$(V_1 \cdot V_2)(\partial^S(k^\bullet)) = \sum_{I \subset S} V_1(\mu^{\bar{I}} \partial^I(k^\bullet)) \otimes V_2(\mu^I \partial^{\bar{I}}(k^\bullet)).$$

It follows directly from this that the product of two Vassiliev invariants respects the filtration. Indeed if V_1 is of type m and V_2 of type n, then $V_1 \cdot V_2$ is of type $m+n$: if $k^\bullet \in \mathcal{K}^\bullet_{m+n+1}$, then either $|I| > m$ or $|\bar{I}| > n$. Therefore in each summand, one of the cofactors is a Vassiliev invariant being evaluated above its order, so zero. Hence $(V_1 \cdot V_2)(\partial^S(k^\bullet)) = 0$.

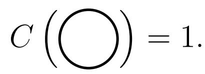
How is this argument in analogy with some property of polynomials? Translating it from the knot-theoreic setting back into the original setting of multivariable calculus, it becomes a proof that polynomials are filtered by degree. In summary the cumbersome proof that Δ respects the filtration on $\mathcal{K}$ is a dual-version (in a knot-theoretic setting) of the fact that polynomials are filtered by degree.

We have not yet given any examples of non-Vassiliev invariants.

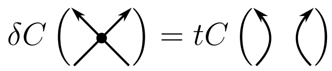
Example 1.3.14 The **Conway polynomial** $C(k)$ is an invariant $C : \mathcal{K} \to Z[t]$ defined by the skein relation



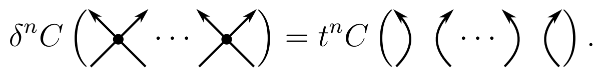
and the relation



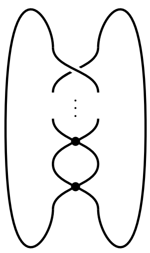
The Conway polynomial itself is not a Vassiliev invariant: its extension to singular knots is



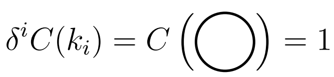
and for multiple double points



Take for example, the sequence $\{k_i\}_{i\in\mathbb{N}}$ of singular knots, where

$k_i =$  has i double points and a single crossing.

Then evaluating the Conway polynomials of this sequence of singular knows, we have

 for all $i \in \mathbb{N}$

and therefore the Conway polynomial is not a Vassiliev invariant.

However fix $\ell \in \mathbb{N}$. An argument of [Bar95] shows that the coefficient c_ℓ of t^ℓ in C is a Vassiliev invariant. Indeed, by the iterated skein relation, $t^{\ell+1}$ is a factor of $\delta^{\ell+1}C$, so $c_\ell = 0$, so the coefficient of t^ℓ vanishes. As such, while $C(k)$ is not a Vassiliev invariant, it is approximated by Vassiliev invariants, specifically it is a power series Vassiliev invariant which we now define. $\Diamond$

Definition 1.3.15 A **power series Vassiliev invariant** is an element of the space

$$\prod_{n=0}^{\infty} \mathcal{V}_n.$$

The power series Vassiliev invariants are those with a particular topological interpretation in terms of the stratification of the space of knots. The details of this perspective are rather technical, so we give only a brief idea of this interpretation: roughly speaking, they arise as linking numbers with cycles in the homology of the complement of the stratification of the space of knots truncated to a certain level. We refer the reader to [CDM12, Ch. 15] for a summary.

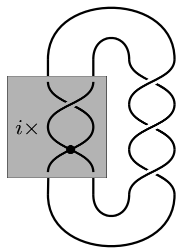
The Conway and Jones polynomials and all quantum invariants [BL93] are power series Vassiliev invariants. Let us now examine an invariant which is not obviously a power series Vassiliev invariant.

Example 1.3.16 Let 3_1 denote the right-handed trefoil knot. The trefoil indicator invariant f_{3_1} is defined

$$f_{3_1}(k) = \begin{cases} 1 & \text{if } k = 3_1 \\ 0 & \text{otherwise} \end{cases}.$$

Consider the sequence of singular knots

$\{k_i\}_{i\in\mathbb{N}}$ where $k_i =$



For $i \in \mathbb{N}$, $\partial^i(k_i)$ is a sum of the 2^i terms coming from resolving each double point. Exactly one of these terms is 3_1, with a coefficient of 1. So $\delta^i(k_i) = 1 \neq 0$ for all $i \in \mathbb{N}$. Hence f_{3_1} is not Vassiliev.

It is unknown whether the set of Vassiliev invariants detects the right-handed trefoil knot. But if so, then this invariant is a power series Vassiliev invariant. $\Diamond$

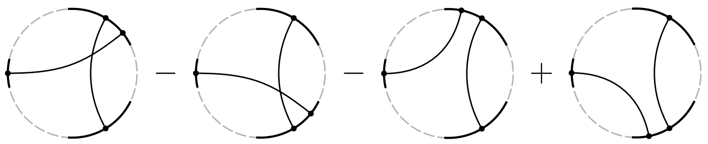
More examples of non-Vassiliev invariants are the unknotting number, crossing number, signature, genus and bridge number [Dea94; Tra94]. The proofs of these being non-Vassiliev touch on yet another dimension of the polynomial analogy, reviewed in [Wil98b]. It is unknown whether these invariants are power series Vassiliev invariants.

In particular some of these invariants are defined as invariants tautologically, as the minimum of some property of a knot diagram across all diagrams of a knot. Knowing whether those invariants are power series Vassiliev invariants or not would be insightful as to whether they have some topological interpretation in terms of the stratification of the space of knots.

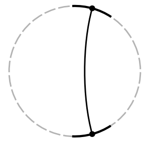
1.4 Chord diagrams and weight systems

This bialgebra structure on knots is closely related to a similar bialgebra structure on chord diagrams. Knots are complicated and chord diagrams are much simpler, and the general idea is to study the former via the latter.

In Section 1.2 we saw how functions on chord diagrams specify Vassiliev invariants, so long as the functions satisfy 4T* (and unframed Vassiliev invariants if they further satisfy 1T*). We can instead encode this directly into the algebra of chord diagrams by the following relations.



$$= 0. \qquad \text{(4T)}$$



$$= 0 \qquad \text{(1T)}$$

Definition 1.4.1 We define $\mathcal{A}_m$, the **space of chord diagrams** of degree m as

$$\mathcal{A}_m = \mathcal{D}_m / 4\mathtt{T}$$

and $\mathcal{A}$, the **space of chord diagrams** as

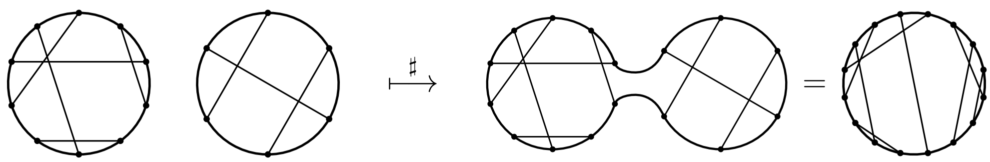
$$\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m.$$

The **space of unframed chord diagrams**, $\mathcal{A}'$ is defined similarly from $\mathcal{A}'_m = \mathcal{D}_m / 4\mathtt{T}, 1\mathtt{T}$.

Warning 1.4.2 Both elements of $\mathcal{A}$ and $\mathcal{D}$ are known as chord diagrams. From now on when we say “a chord diagram”, we mean an element of $\mathcal{A}$ unless otherwise specified.

The algebra $\mathcal{A}$ has multiplication and coproduct operations that mirror those in $\mathcal{K}$.

Definition 1.4.3 The **connected sum of two chord diagrams** A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



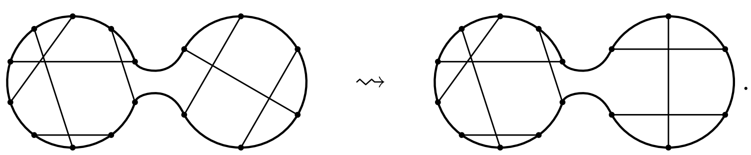
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed this is ill-defined in $\mathcal{D}$. However the 4T relation in $\mathcal{A}$ rectifies this issue.

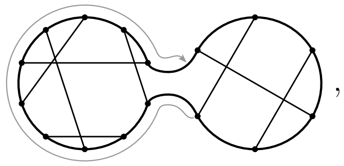
Proposition 1.4.4 *The connected sum operation* $\sharp : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ *is well-defined.*

Proof We will prove that the connected sums of two chord diagrams, given any two choices of connection locations, are equal modulo 4T.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one ‘click’, like so:

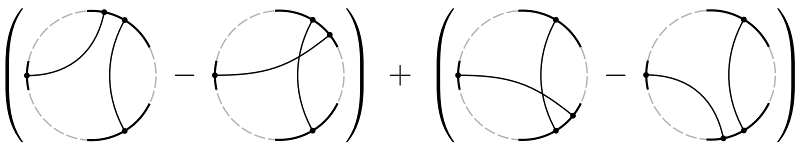


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow



which we show can be achieved by a series of 4T relations.

Note that we can rewrite 4T as



$= 0.$

A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T. As such, the two connected sums $a_1 \sharp a_2$ differ by a sum of 4T relations, completing the proof. $\square$

Proposition 1.4.5 *The connected sum operation makes $\mathcal{A}$ into a graded algebra.*

Proof No chords are lost during the connected sum: the 4T relation is homogenous with respect to degree, so the connected sum of a chord diagram of order i and a chord diagram of order j is a chord diagram of order $i + j$. $\square$

Just as there is a connected sum operation in $\mathcal{A}$ reminiscent to that in $\mathcal{K}$, there is a coproduct too.

Definition 1.4.6 The **coproduct of a chord diagram** A is the sum of ways of partitioning its chords between two subdiagrams. Specifically, if S is the set of chords of A, and $J \subset S$, let $\widehat{J} = S \smallsetminus J$. Denote by A_J the chord diagram A but with only the chords in $J \subset S$, and the rest deleted. Then

$$\Delta(A) = \sum_{J \subset S} A_J \otimes A_{\overline{J}}.$$

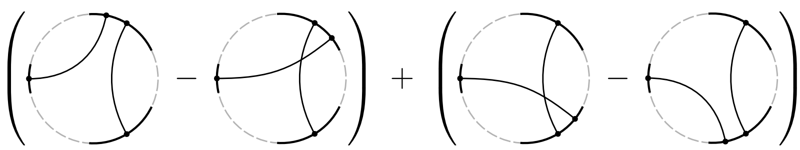
Proposition 1.4.7 *The coproduct Δ is well-defined in $\mathcal{A}$, and makes $\mathcal{A}$ into a graded bialgebra.*

Proof We need to check: that the coproduct factors through the quotients, that the coproduct respects the grading, and that the coproduct is an algebra morphism (one way of writing the compatibility condition).

That in the unframed case, Δ factors through 1T is easy: an isolated chord in A remains isolated and appears in one cofactor of every term of $\Delta(A)$.

Also, Δ factors through 4T. Suppose that $K = A_1 - A_2 + A_3 - A_4$ is some combination of chord diagrams to be killed by 4T. This means that K looks like

$K =$



where there may be other chords that the above diagrams have in common, as well as those shown. Note that there is one moving chord in the above diagram and one stationary chord. Let us label these m and s respectively. Take the same partition $J \sqcup \overline{J}$ of S for all of the four chord diagrams at once, and write as the resulting coproduct $\Delta(A_i) = C_i \otimes D_i$. Suppose without loss of generality that m was partitioned into the C_i's. Then $D_1 = D_2 = D_3 = D_4$, so this term of the coproduct factors as $(C_1 - C_2 + C_3 - C_4) \otimes D_1$ and either:

- s was also partitioned into the C_i's and the relation remains a 4T, or
- s was partitioned into the D_i's, and so $C_1 = C_2$ and $C_3 = C_4$.

In either case, that term of the coproduct is killed.

The coproduct is clearly graded, as

$$\Delta(\mathcal{A}_m) \subset \bigoplus_{i+j=m} \mathcal{A}_i \otimes \mathcal{A}_j = (\mathcal{A} \otimes \mathcal{A})_m.$$

Finally, we check the compatibility condition. If A has chord set S and B has chord set T, then

$$\begin{aligned}
\Delta(A) \sharp^{\otimes 2} \Delta(B) &= \left(\sum_{J' \subset S} A_{J'} \otimes A_{\overline{J'}} \right) \sharp^{\otimes 2} \left(\sum_{J'' \subset T} B_{J''} \otimes B_{\overline{J''}} \right) \\
&= \sum_{J \subset S \sqcup T} (A \sharp B)_J \otimes (A \sharp B)_{\overline{J}} \\
&= \Delta(A \sharp B).
\end{aligned}$$

□

We have shown that $\mathcal{A}$ is a graded bialgebra of finite type. In fact $\mathcal{A}$ is an even more specific structure.

Definition 1.4.8 A **connected, commutative, cocommutative** graded bialgebra of finite type is a graded bialgebra A of finite type for which

- The unit map $\mathcal{Q} \to A_0$ is an isomorphism (**connectedness**)
- The product is commutative, that is $m \circ \tau = m$
- The coproduct is cocommutative, that is $\tau \circ \Delta = \Delta$

where $\tau : A \otimes A \to A \otimes A$ sends $x \otimes y \mapsto y \otimes x$.

Proposition 1.4.9 *The bialgebra $\mathcal{A}$ is a connected, commutative, cocommutative graded bialgebra of finite type.*

Proof The connetedness isomorphism is given by associating a scalar k to k times the empty chord diagram.

We have already shown the connected sum to be commutative, and the coproduct can clearly be seen to be cocommutative from the symmetry of Definition 1.4.6. □

Connected, commutative, cocommutative graded bialgebras of finite type are very rigid structures. In particular, a classical structural theorem applies, and such a bialgebra can be understood in terms of a particular class of elements.

Definition 1.4.10 An element x is **primitive** in a coalgebra (so in-particular in a bialgebra) with coproduct Δ if it satisfies

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

The set of primitive elements of a bialgebra A is denoted $\mathcal{P}(A)$, and if A is graded, then we write $\mathcal{P}_n(A)$ to denote the set of primitive elements of degree n.

Theorem 1.4.11 (Milnor-Moore) *Let H be a connected, commutative, cocommutative bialgebra of finite type, over a field of characteristic zero. Then, as an algebra, H is isomorphic to the symmetric algebra of $\mathcal{P}(H)$. In other words, H is a polynomial algebra in $\mathcal{P}(H)$.*

We refer the reader to [Les24] or [CDM12] for a proof. This fact leads to important consequences about the structure of $\mathcal{A}$ and its relation to Lie algebras, which will be the subject of Chapter 2.

In the previous chapter we saw a duality between a space $\mathcal{K}$ of objects (specifically knots) and a space V of (a certain class of) functionals on knots. There is a similar relation between the graded bialgebra of chord diagrams and weight systems.

Definition 1.4.12 The space of weight systems (Definition 1.2.12) denoted $\mathcal{W}$ forms a graded bialgebra, the **graded bialgebra of weight systems** with grading given by degree m, product given by pointwise multiplication

$$W_1 \cdot W_2(a) = W_1(a)W_2(a),$$

and coproduct, η given by

$$\eta(W)(a_1 \otimes a_2) = W(a_1 \sharp a_2).$$

Every graded object is a filtered object with the naturally induced filtration. Considering a graded object as such allows us to take its dual filtered bialgebra (which we refer to as its dual graded bialgebra in this case).

Proposition 1.4.13 *The graded bialgebra $\mathcal{W}$ is the dual graded bialgebra of the graded bialgebra $\mathcal{A}$.*

Proof That $\mathcal{W} = \mathcal{A}^*$ as sets follows from Definition 1.2.12. Indeed, a weight system is a functional on chord diagrams that is integrable, that is a functional satisfying 4T*. Recall that by Definition 1.4.1, this is exactly an element of $\mathcal{A}^*$. The rest of the proof uses the same arguments as Proposition 1.3.12. □

1.5 The fundamental theorem of Vassiliev invariants

In Section 1.2, we gave the fundamental theorem of Vassiliev invariants. The point of this theorem is that it establishes a particular relationship between the algebras of the previous two sections, $\mathcal{K}$ and $\mathcal{A}$ (or equivalently, between $\mathcal{W}$ and $\mathcal{V}$). Admittedly, in the form of Theorem 1.2.17, it's not a-priori obvious why this is the case. Here we give a restatement of the theorem which makes the relationship explicit.

Definition 1.5.1 The **associated graded** bialgebra of a filtered bialgebra A is the bialgebra $\operatorname{gr} A$ formed by the direct sum of the succesive quotients of the filtered components of A. For a bialgebra with a descending filtration we have

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m/A_{m+1}$$

and for a bialgebra with an ascending filtration we have

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m/A_{m-1}$$

where $A_{-1} = \{0\}$.

The operations on $\operatorname{gr} A$ are those induced from the operations on A. For example, on a bialgebra with a descending filtration, if multiplication $\mu : A_m \otimes A_n \to A_{m+n}$ sends $a \otimes b \mapsto ab$, then the induced operation is

$$\begin{array}{rccc} \operatorname{gr}\mu : & A_m/A_{m+1} \otimes A_n/A_{n+1} & \longrightarrow & A_{m+n}/A_{m+n+1} \\ & a + A_{m+1} \otimes b + A_{n+1} & \longmapsto & ab + A_{m+n+1}. \end{array}$$

For the comultiplication Δ that sends $a \mapsto \bigoplus_{i+j=m} a_i' \otimes a_i''$ the induced operation is

$$\begin{array}{rccc} \operatorname{gr}\Delta : & A_m/A_{m+1} & \longrightarrow & \bigoplus_{i+j=m} A_i/A_{i+1} \otimes A_j/A_{j+1} \\ & a + A_{m+1} & \longmapsto & \bigoplus_{i+j=m} a_i' + A_{i+1} \otimes a_i'' + A_{j+1}. \end{array}$$

Theorem 1.5.2 (Fundamental theorem) *The algebra of weight systems is isomorphic to the associated graded algebra of the algebra of Vassiliev invariants. That is we have $\mathcal{W} \cong \operatorname{gr} \mathcal{V}$, or on the level of graded components*

$$\bigoplus_{m=0}^{\infty} \mathcal{W}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{V}_m/\mathcal{V}_{m-1}.$$

Equivalently, this can be stated in the dual setting as follows. The algebra of chord diagrams is isomorphic to the associated graded algebra of the algebra of knots. That is we have $\mathcal{A} \cong \operatorname{gr} \mathcal{K}$, or on the level of graded components

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{K}_m/\mathcal{K}_{m+1}.$$

The equivalence of Theorem 1.5.2 with Theorem 1.2.17 will be proven at the end of this section.

We can break the fundamental theorem up into two parts based on the direction of the map in Theorem 1.5.2. The two maps being inverses gives the isomorphism. Historically one of the directions was shown first, by Vassiliev [Vas90; Vas92], and then later the direction due to Kontsevich [Kon93], proving the theorem.

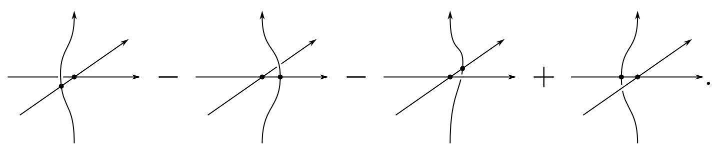
Vassiliev	Every Vassiliev invariant modulo Vassiliev invariants of higher order gives a Weight system. $$\mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{W}_m$$	Every chord diagram gives an element of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$. $$\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$$
Kontsevich	Every weight system gives a Vassiliev invariant modulo Vassiliev invariants of higher order. $$\mathcal{W}_m \to \mathcal{V}_m/\mathcal{V}_{m-1}$$ and this is inverse to the above.	Every equivalence class of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$ gives a chord diagram. $$\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m.$$ and this is inverse to the above.

We discuss and prove the direction due to Vassiliev, then return to the direction of Kontsevich. The point of the Vassiliev direction is that the relations in $\mathcal{A}$ are compatible with the relations of $\mathcal{K}_n/\mathcal{K}_{n+1}$. Indeed, $\mathcal{A}$ was constructed in this way in Section 1.2, and the proof doesn't involve anything technical.

Proof of Theorem 1.5.2 (Vassiliev) The aim is to define a map $\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$. Let us start with a map $\mathcal{D}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$. Given $d \in \mathcal{D}_m$, take a singular knot $k^\bullet \in \mathcal{K}_m^\bullet$ whose chord diagram is d. Then, d is sent to $[\partial^m(k^\bullet)] \in \mathcal{K}_m/\mathcal{K}_{m+1}$.

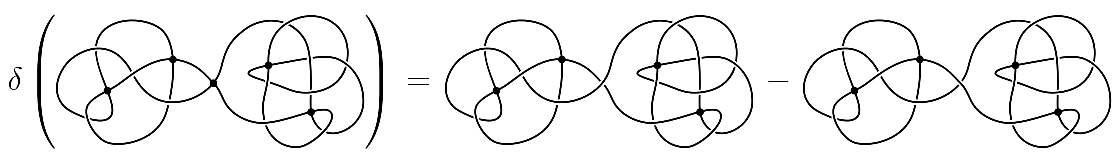
This is well-defined: any other $\ell^\bullet$ with chord diagram d also resolves to $[\partial^m(k^\bullet)]$. Indeed, any two m-singular knots with the same chord diagram differ by some crossing changes. But if $k^\bullet$ and $\ell^\bullet$ differ by a crossing change, then $\partial^m(k^\bullet)$ and $\partial^m(\ell^\bullet)$ differ by an element of $\mathcal{K}_{m+1}$, so $[\partial^m(k^\bullet)] = [\partial^m(\ell^\bullet)]$ in the quotient. This argument that any $k^\bullet \in \mathcal{K}_m^\bullet$ can be chosen to represent $[\partial^m(k^\bullet)]$ as long as it has chord diagram d will be used again below, and we refer to it as the crossing-change argument.

Recalling that $\mathcal{A}_m = \mathcal{D}_m/4\mathrm{T}$ (and $\mathcal{A}'_m = \mathcal{D}_m/1\mathrm{T}, 4\mathrm{T}$), we need to show that the map factors through the quotient. Indeed, 4T is in the kernel. A combination of chord diagrams appearing in a 4T relation is sent to a combination of singular knots. By the crossing-change argument these can all be chosen to be identical except near a small region, where the have the form



Resolving around these four terms about the singular point that they don't have in common yields eight terms which cancel out in pairs.

We also include the proof that 1T is in the kernel of the unframed version of this map. A type 1T chord diagram is sent to an unframed singular knot with a singular point that is passed through twice in a row when the unframed knot is traversed. The resolution of that singular point can be chosen to be a difference of two unframed $(m-1)$-singular knots that are singular isotopic by the crossing change argument. For example,



$$= 0.$$

The isomorphism respects the algebra structure as $a_1 \sharp a_2$ is sent to $[k_{a_1 \sharp a_2}] \in \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ which comes from the resolution of some singular knot we shall denote $k^\bullet_{a_1 \sharp a_2}$ with chord diagram $a_1 \sharp a_2$. Likewise a_1 and a_2 map to $[k_{a_1}]$ and $[k_{a_2}]$ which are resolutions of singular knots $k^\bullet_{a_1}$ and $k^\bullet_{a_2}$. The induced operation from the connected sum $\mathcal{K}_m/\mathcal{K}_{m+1} \otimes \mathcal{K}_n/\mathcal{K}_{n+1} \to \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ takes these to $[k_{a_1} \sharp k_{a_2}]$, which comes from the resolution of $k^\bullet_{a_1} \sharp k^\bullet_{a_2}$. These singular knots may not be the same a priori, but they have the same chord diagram, so by the crossing-change argument they can be chosen to be the same without loss of generality. Hence resolving then projecting into $\mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ gives the same equivalence class.

Using similar arguments as in the formula of Willerton's Lemma, the isomorphism can be shown to respect the bialgebra structure.

The dual version of the statement follows from the regular version. □

We also give a more illustrative direct proof of the dual theorem later.

Let's turn to the other part of the fundamental theorem, due to Maxim Kontsevich (which is much more involved). In [Kon93], Konsevich constructed an integral invariant which proves the fundamental theorem, as well as containing the information of all Vassiliev invariants at the same time. We will not give a detailed exposition of Kontsevich's invariant here, such treatments are abundant in the literature, for example [BS97; CD05; CDM12]. Rather, we will boil the Kontsevich integral down to a single universal property.

Definition 1.5.3 The **completion** of a descending-filtered algebra A is the descending-filtered algebra

$$\widehat{A} = \varprojlim_{m\to\infty} A/A_{m+1}.$$

This is the inverse system

$$0 = A/A_0 \twoheadleftarrow A/A_1 \twoheadleftarrow A/A_2 \twoheadleftarrow \cdots$$

whose mth filtered component is the set of sequences that vanish in A/A_{m+1} (and by the properties of inverse systems, all terms thereafter).

If A is graded, then it is also filtered with the natural descending filtration

$$\bigoplus_{i=0}^{\infty} A_i \supseteq \bigoplus_{i=1}^{\infty} A_i \supseteq \cdots .$$

The degree completion of a graded algebra A if a descending-filtered algebra that coincides with the completion $\widehat{A}$ taken with respect to the natural descending filtration on A.

Definition 1.5.4 A **universal Vassiliev invariant** is a knot invariant $Z : \mathcal{K} \to \widehat{\mathcal{A}}$ with the following property. If $k \in \mathcal{K}_m$ is a linear combination of knots with $k = \partial^m(k^\bullet)$ and $k^\bullet$ has chord diagram $a \in \mathcal{A}_m$, then

$$Z(k) = a + \text{higher degree terms}.$$

Phrased in terms of σ and ∂ this reads

$$Z(\partial^m(k^\bullet)) = \sigma(k^\bullet) + \text{higher degree terms}.$$

Remark 1.5.5 An equivalent way of defining a universal Vassiliev invariant is as follows. If f is a descending-filtration-respecting map $f : A \to B$, then define the associated graded map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ that sends $a_m + A_{m+1} \mapsto f(a_m) + B_{m+1}$. In other words, $\operatorname{gr} f$ is a graded map coming from the filtered map f that forgets information about higher degrees. A universal Vassiliev invariant is a map $Z : \mathcal{K} \to X$ whose associated graded $\operatorname{gr} Z : \operatorname{gr} \mathcal{K} \to \operatorname{gr} X$ is an isomorphism.

A universal Vassiliev invariant is such a map with $X = \widehat{\mathcal{A}}$. In particular, note that $\operatorname{gr} \widehat{\mathcal{A}} = \mathcal{A}$ (because the direct sum implicit in gr means $a \in \operatorname{gr} \widehat{\mathcal{A}}$ cannot have infinitely many non-zero terms). So since Z is $\mathcal{K} \to \widehat{\mathcal{A}}$ and satisfies this property, then $\operatorname{gr} \mathcal{K} \cong \mathcal{A}$.

Theorem 1.5.6 (Kontsevich integral) *There exists a universal Vassiliev invariant, $Z(k)$; it is called the Kontsevich integral.*

Proof of Theorem 1.5.2 (Kontsevich) Take the map $k \in \mathcal{K}_m \to \mathcal{A}_m$ coming from killing the higher degree terms in the Kontsevich integral, and taking the lowest order non-zero chord diagram. This factors through the quotient to a map $\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m$ since by Theorem 1.5.6 any additional $k' \in \mathcal{K}_{m+1}$ contributes only higher degree terms (which get killed). It is easy to see that the two maps are inverses. □

Again, it is worth looking at the proof in the dual setting as well.

Lemma 1.5.7 *Post-composing the Kontsevich integral with a weight system of order m via*

$$k \longmapsto W \circ Z(k)$$

gives a Vassiliev invariant of order m.

Warning 1.5.8 More precisely, this is the following composition

$$\mathcal{K} \xrightarrow{Z} \widehat{\mathcal{A}} \xrightarrow{\pi_m} \mathcal{A}_m \xrightarrow{W} \mathbb{Q}$$

but we will drop the map π_m from the notation. In other words, implicit in our notation is the notion that a weight system of order m does not see chord diagrams of order other than m.

Proof The map $W \circ Z$ is clearly an invariant, since Z is an invariant. By the universal property of Z, if $k \in \mathcal{K}_{m+1}$, so then $k = \partial(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}^\bullet_{m+1}$ and

$$Z(k) = \sigma(k^\bullet) + \begin{matrix}\text{terms of} \\ \text{order} \geq (m+2)\end{matrix}.$$

Now, $Z(k)$ is zero in degrees up to and including m, so composing with a weight system of degree m gives zero. □

Proof of Theorem 1.5.2 (dual) The map $\mathcal{W}_m \to \mathcal{V}_m$ defined by Lemma 1.5.7 is injective, as only the zero weight system gives the zero invariant.

However, it is not surjective: the map $\mathcal{W}_m \to \mathcal{V}_m$, written as

$$\begin{aligned} \mathcal{W}_m &\xrightarrow{Z^*} \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1} \\ W &\longrightarrow W \circ Z \end{aligned}$$

forces a choice of Vassiliev invariant in $\mathcal{V}_{m-1}$. Being as explicit as possible, let

$$\Omega(k) = \begin{cases} (W \circ Z)(k) & k \in \mathcal{K}_m \\ 0 & k \in \mathcal{K}_{<m} \end{cases} \quad \text{and} \quad \Theta(k) = \begin{cases} 0 & k \in \mathcal{K}_m \\ (W \circ Z)(k) & k \in \mathcal{K}_{<m} \end{cases}.$$

The case $k \in \mathcal{K}_{>m}$ needn't be considered because $W \circ Z$ is Vassiliev. As such, we have $W \circ Z = \Omega + \Theta$. Observe that Ω recovers the weight system W when $k \in \mathcal{K}_m$. Finally, Θ is some Vassiliev invariant of order $m-1$ because it vanishes when $k \in \mathcal{K}_m$.

In essence, we have found that the cokernel of Z^* is $\mathcal{V}_{m-1}$, so we get the desired isomorphism $\mathcal{W}_m \cong \mathcal{V}_m/\mathcal{V}_{m-1}$. □

A natural question arises — what is the summand Θ? Fixing $n < m$ and a knot $k \in \mathcal{K}_n$ with chord diagram a_k, k has Kontsevich integral

$$Z(k) = a_k + \begin{matrix}\text{terms of} \\ \text{order } (n+1)\end{matrix} + \cdots + \begin{matrix}\text{terms of} \\ \text{order } m\end{matrix} + \cdots .$$

Applying the projection π_m, all that remains are some chord diagrams of order m, with coefficients depending on the intricacies of the Kontsevich integral for that particular knot. Composing with the weight system, this is a $\mathbb{Q}$-valued Vassiliev invariant of order $m-1$ determined by choice of W and the structure of Z.

Calling the Kontsevich integral and invariants of its kind is indeed justified: every Vassiliev invariant can be obtained through the Kontsevich integral.

Theorem 1.5.9 *If Z is a universal Vassiliev invariant, then every Vassiliev invariant factors through Z.*

Proof Let $V \in \mathcal{V}_m$. Following the proof above, we can project V to $\mathcal{V}_m/\mathcal{V}_{m-1}$ to get a weight system W_m. Subtracting $W_m \circ Z$ leaves a Vassiliev invariant of lesser degree. In other words, we have the isomorphism

$$\begin{aligned} \mathcal{V}_m &\cong \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0 \\ &\cong \mathcal{W}_m \oplus \mathcal{W}_{m-1} \oplus \cdots \oplus \mathcal{W}_1 \oplus \mathcal{W}_0 \end{aligned}$$

and via this isomorphism, V can be written as a sequence of weight systems of degree from 0 to m. Since each W_m factors through Z, so does V:

$$V = \sum_{i=0}^{m} (W_m \circ Z) = \left(\bigoplus_{i=0}^{m} W_m \right) \circ Z.$$

□

Corollary 1.5.10 *A universal Vassiliev invariant (in particular, the Kontsevich integral Z) is exactly as strong as the set of Vassiliev invariants.*

As we have seen, taking the projection $\mathcal{V}_m \to \mathcal{V}_m/\mathcal{V}_{m-1} \cong \mathcal{W}_m$ yields a weight system, but loses the additional information of a Vassiliev invariant of order $m-1$.

Definition 1.5.11 The **canonical Vassiliev invariants** are those Vassiliev invariants whose weight systems W recover them completely via $W \circ Z$.

In other words, not all bases of $\mathcal{V}_m$ are created equal. The canonical Vassiliev invariants are those that are homogenous with respect to the splitting of $\mathcal{V}_m$

$$\mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0$$

of $\mathcal{V}_m$ induced by the Kontsevich integral. Canonical Vassiliev invariants were first defined in [BG96] and used to prove the Melvin-Morton-Rozansky conjecture relating the coefficients of the Alexander polynomial to those of the coloured Jones polynomials. This is one example of how the theory of Vassiliev invariants is useful for probing the structure of knots.

There are a few known from-the-bottom-up constructions of universal Vassiliev invariants of knots. However they are all equivalent to (or conjecturally equivalent to) the Kontsevich integral. Yet the Kontsevich integral is not the only invariant that can satisfy the defining degree property of a universal Vassiliev invariant.

Question 1.5.12 (Bar-Natan–Garoufalidis) Is there a reason why the Kontsevich integral, or equivalently, this splitting appears to be canonical?

Finally, we return to the equivalence of the two fundamental theorems.

Proof (Equivalence of Theorems 1.2.17 and 1.5.2) For the forward direction, suppose Theorem 1.2.17 holds: if $v^\bullet$ is an invariant of m-singular knots satisfying T4T* and further that $\delta v^\bullet = 0$, then $v^\bullet$ integrates to an invariant v of 0-singular knots.

First we prove that the Kontsevich part of the fundamental theorem holds. Let W be a weight system of order m. Then W defines an invariant $v^\bullet_W$ of m-singular knots by

$$v^\bullet_W(k) = W(\sigma(k)).$$

We wish to apply Theorem 1.2.17 to $v^\bullet_W$, so let us check the hypotheses. Firstly, we check that the derivative of $v^\bullet_W$ is zero. Indeed

$$\delta v^\bullet_W(k) = W(\sigma(k^+)) - W(\sigma(k^-))$$

for some knots k^+ and k^- that differ by crossing changes. But σ is invariant under crossing changes, hence $\delta v^\bullet_W = 0$. For the second hypothesis we check that $v^\bullet_W$ satisfies T4T*. Now, W is a weight system so it satisfies 4T*, and therefore $v^\bullet_W$ satisfies T4T*. As such, both hypotheses of Theorem 1.2.17 hold and $v^\bullet_W$ integrates into a Vassiliev invariant. The Vassiliev part of the theorem is independent of the original version and was proven separately in Section 1.2.

Now for the reverse direction, suppose Theorem 1.5.2 holds: $\mathcal{W}_m \cong \mathcal{V}/\mathcal{V}_{m-1}$. Take an m-singular knot invariant $v^\bullet$ satisfying 4T* and $\delta v^\bullet = 0$. This defines a weight system $W_{v^\bullet}$, which by the Kontsevich part of Theorem 1.5.2 gives an invariant class in $\mathcal{V}_m/\mathcal{V}_{m-1}$. Explicitly, this invariant is $W_{v^\bullet} \circ Z$. But by definition of δ,

$$\delta^m(W_{v^\bullet} \circ Z)(k^\bullet) = (W_{v^\bullet} \circ Z)(\partial^m k^\bullet)$$

which by the definition of a universal Vassiliev invariant is just $v^\bullet$. Thus, $W_{v^\bullet} \circ Z$ is the mth derivative of $v^\bullet$, so $v^\bullet$ integrates into a Vassiliev invariant of order m, completing the proof.

Without loss of generality, the proof works also in the unframed case. □

2

Lie theory and Jacobi diagrams

THE fundamental theorem of Vassiliev invariants states that the bialgebra of Vassiliev invariants can be broken up into "nice" combinatorial weight systems. So to understand $\mathcal{V}$ it suffices to understand $\mathcal{W}$ (or equivalently its dual $\mathcal{A}$). As we will see in this chapter, the structure of $\mathcal{A}$ is related to Lie theory.

2.1 Jacobi diagrams

To see the Lie theory connections, we see that $\mathcal{A}$ is isomorphic as a bialgebra of Jacobi diagrams, whose elements are represented by graphs with one- and three-valent vertices.

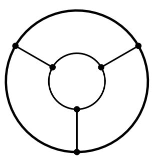
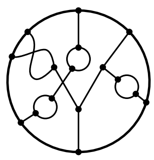
Definition 2.1.1 A **unitrivalent diagram** is a unitrivalent graph (with loops and multiple edges allowed) with the following additional data:

- each trivalent vertex has a fixed cyclic order of incident edge-connections,
- the set of univalent vertices has a fixed cyclic order.

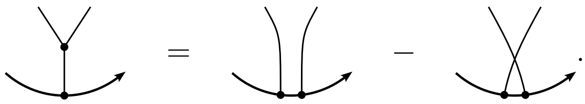
The vector space of unitrivalent diagrams is denoted $\mathcal{T}$.

Warning 2.1.2 When drawing unitrivalent diagrams, there are two notational conventions we use. Firstly, the fixed cyclic order of the univalent edges is specified by drawing them connected to a circle (where the cyclic order corresponds to traversing the circle anticlockwise). Like for chord diagrams, this circle is called the **skeleton**. Secondly, the trivalent vertices are assumed to have the cyclic order corresponding to traversing anticlockwise around the vertex. It is always possible to draw them this way because planarity is not a concern of abstract graphs.

In particular, all chord diagrams are unitrivalent diagrams with only univalent vertices (the chord ends). Further examples of unitrivalent diagrams would be

 ,  $\in \mathcal{T}.$

Definition 2.1.3 An **STU relation** is a relation of the form

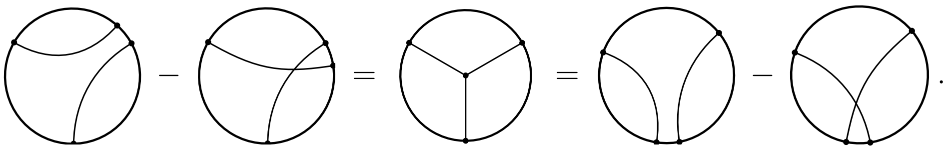


(STU)

Here the bottom line, drawn thicker and curved must be a part of the skeleton.

As usual, this is not an individual relation but a class of relations, true in any diagrams that are identical except for the parts shown.

Note that the for the chord diagrams inside the algebra of Jacobi diagrams, the STU relations imply the 4T relations, as

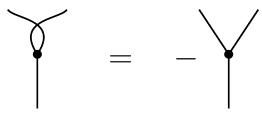


Definition 2.1.4 The algebra $\mathcal{J}$ of Jacobi diagrams is the vector space $\mathcal{T}/\mathtt{STU}$, with the product $\sharp$ defined the same way as it was for chord diagrams.

This is well-defined: the proof of Proposition 1.3.4 showed that the product $\sharp$ being well-defined on $\mathcal{A}$ was a consequence of the 4T relations, which are implied by the STU relations. As such the STU relations suffice to determined the structure of $\mathcal{J}$, but the following auxilliary relations which can be deduced from STU make the relations to Lie algebras more apparent.

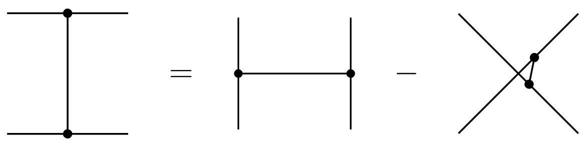
Proposition 2.1.5 *The following relations are consequences of the* STU *relation in* $\mathcal{J}$*:*

*(a) The **AS relation** (antisymmetry relation)*



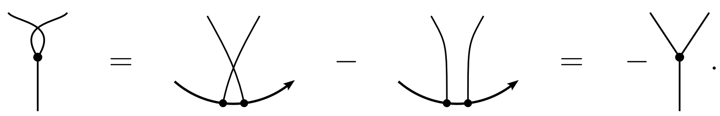
(AS)

*(b) The **IHX relation***



(IHX)

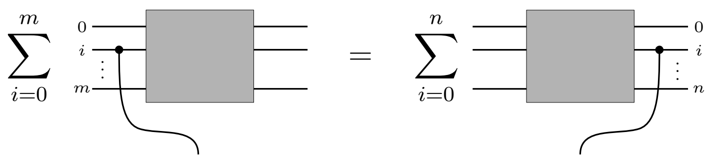
Proof (a) Take two diagrams which differ only by AS at one (trivalent) vertex. If the vertex at which the AS relation resides is adjacent to a univalent vertex (i.e. is adjacent to the skeleon), then this is immediate from applying STU to both diagrams at that vertex:



If the vertex is not immediately adjacent to a univalent vertex, then it has some d trivalent vertices 'in the way'. By applying STU to those vertices yields a sum of 2^d diagrams, all identical except for differing by AS, now on a vertex adjacent to a univalent vertex.

(b) A similar argument applies. We give a sketch. If one of the two vertices of the `IHX` is adjacent to the skeleton, then the result is a direct consequence of an `STU` on each of the vertices in each of the diagrams in the `IHX`, followed by a few applications of `AS` (computing the twelve diagrams from applying two `STU`s to the three diagrams in an `IHX` verifies this; the computation is given in [CDM12, Lem. 5.2.6]). Otherwise, there is a path of trivalent vertices in the way, and some `STU`s yield a sum of 2^d diagrams in which at least one of the two vertices involved in the `IHX` is adjacent to the skeleton, and so the argument above applies. □

Proposition 2.1.6 (Generalised `IHX`) *The following holds in $\mathcal{J}$ for any subgraph consisting of trivalent vertices that can be inserted into the grey box.*

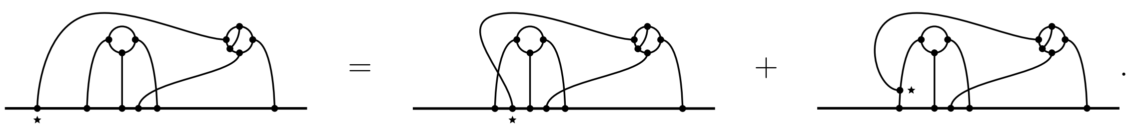


The result is standard — see for example Chapter 5.2 of [CDM12] for a proof. A corollary is the following result, which will be important later.

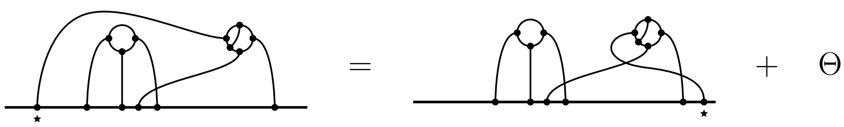
Proposition 2.1.7 *If the univalent vertices of a Jacobi diagram are ordered linearly rather than cyclically, all linear orders that respect a given cyclic order are equivalent.*

Proof Let us draw the linearly ordered Jacobi diagrams on a line rather than a circle and let the univalent edge ordering be given by the order on the line. It suffices to prove that we can move the univalent vertex in the first place to the last place via the relations in $\mathcal{A}$. Let us call this first univalent vertex of the original diagram the marked vertex, as its position will later change when relations are applied. We will mark it with a star.

By `STU`, the original diagram is equal to the diagram with the marked vertex moved into the second place, plus a diagram in which the marked vertex is now a trivalent vertex and attached above what was the second (but is now the first) univalent vertex:

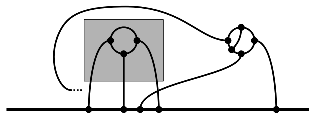


Repeatedly applying `STU` to move the the marked vertex until reaches the last place, we get

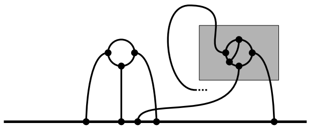
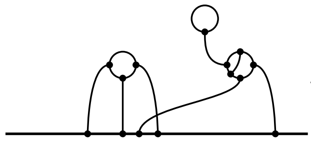


where Θ is a sum of terms with the marked vertex now a trivalent vertex attached above each of the other univalent vertices. It suffices to show that Θ vanishes.

We can split Θ up based on which connected component the marked vertex now connects to. For connected components other than the connected component of the marked vertex, apply a vertical version of the generalised `IHX` relation (Proposition 2.1.6), to the terms in which the marked vertex connects to that component. This is an `IHX` in which the grey box in Proposition 2.1.6 is around the whole connected component, except for where the univalent vertices connect to the skeleton. As such, this is the case of the generalised IHX in which $n = 0$:

$\sum$  $= \quad 0.$

This leaves only the terms where the marked vertex connects back to its own connected component. The sum of such terms also vanishes: by a generalised IHX relation, we have

$\sum$  $=$  .

But any diagram with a "balloon" vanishes, as applying the AS relation to the vertex on the balloon, it is equal to its negative.

Hence $\Theta = 0$, completing the proof. □

We have already spoiled the surprise that $\mathcal{A}$ and $\mathcal{J}$ are isomorphic as bialgebras. This is clearly true as $\mathcal{J}$ is just a change of basis from $\mathcal{A}$. Since $\mathcal{A}$ spans $\mathcal{J}$, we can attempt to lift the coproduct from $\mathcal{A}$ directly onto $\mathcal{J}$.

Proposition 2.1.8 *The coproduct Δ on $J \in \mathcal{J}$ defined by taking a Jacobi diagram, representing it as a chord diagram via* STU, *taking the coproduct in $\mathcal{A}$, then interpreting the result as a Jacobi diagram via the inclusion of $\mathcal{A}$ into $\mathcal{J}$ is given by the formula*

$$\Delta(J) = \sum_{C \subset S} J_C \otimes J_{\overline{C}}$$

where S is the set of connected components of J, and $\overline{C} = S \setminus C$.

Proof Note that this has the same symbolic form as the coproduct in $\mathcal{A}$ given in Definition 1.4.6, but with chords replaced by connected components of Jacobi diagrams. However, when working in $\mathcal{A} \subset \mathcal{J}$ there are only univalent vertices, so the connected components are exactly the chords. Since $\mathcal{A}$ forms a basis for $\mathcal{J}$ and the formula is linear, it extends to all of $\mathcal{J}$. □

Corollary 2.1.9 *The primitive elements $\mathcal{P}(\mathcal{A})$ are the connected Jacobi diagrams.*

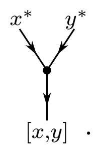
Corollary 2.1.10 *The bialgebras $\mathcal{A}$ and $\mathcal{J}$ are isomorphic.*

Warning 2.1.11 Justified by this isomorphism, we henceforth write $\mathcal{A}$ for both chord diagrams and Jacobi diagrams.

2.2 Lie algebra weight systems

Similar diagrammatic relations to STU, AS and IHX satisfied in $\mathcal{A}$ appear also in the context of a graphical notation for multilinear maps, a fact which can be exploited to probe $\mathcal{A}$. Before seeing how, let us review this graphical notation following [Thu00; RW06]. The diagrammatic calculus is well-known but it goes by many names: string diagram calculus, Penrose calculus, tensor calculus, diagrammatic calculus for tensors to name a few. We call them string diagrams.

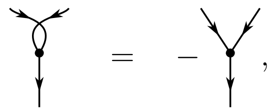
A tensor is a multilinear map $X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$, or equivalently (via the canonical isomorphism) an element of the vector space $X_1^* \otimes X_2^* \otimes \cdots X_n^* \otimes Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$. Such a tensor can be represented as a vertex with $m+n$ unbound directed edges: m incoming edges decorated by the corresponding vector spaces (in the example above, $X_1, \cdots X_m$), and n outgoing edges decorated by $Y_1, \cdots Y_n$. For example, the bracket in a Lie algebra $\mathfrak{g}$ is an element $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ expressed as



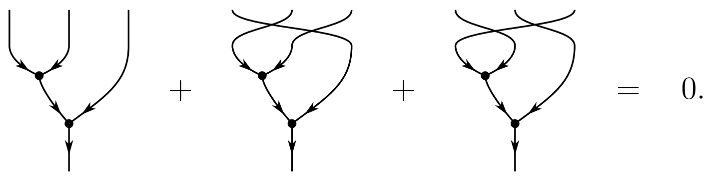
.

It will be obvious from each string diagram what tensor it represents, so we drop the edge labels.

Such a notation is useful because composition of tensors can be expressed graphically by connecting outgoing and incoming legs with the same decoration. In particular relations can therefore be expressed graphically. For example, the antisymmetry of the bracket $[y,x] = -[x,y]$ becomes


, and

the Jacobi relation $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ becomes

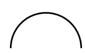


Looking at these relations in the tensor algebra $\mathcal{T}(\mathfrak{g})$, a hint at the Lie-theoretic structure emerges. The antisymmetry of the bracket, (drawn as a string diagram) looks like a directed version of `AS`. Similarly the string diagrammmatic Jacobi relation can be arranged into a directed version of `IHX`.

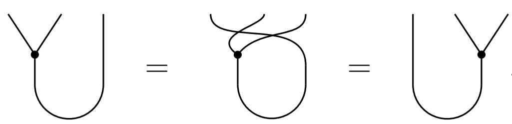
Furthermore, suppose $\mathfrak{g}$ is a metric Lie algebra so that it has an invariant, nondegenerate, bilinear form $\langle \cdot,\cdot \rangle \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. Being nondegenerate, it can be inverted to an element $c \in \mathfrak{g} \otimes \mathfrak{g}$ — this element is known as the Casimir element associated to $\mathfrak{g}$ (or simply the Casimir). These tensors can be diagrammatically represented as additional bivalent vertices

 and  .

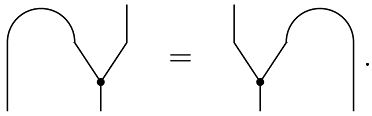
Recall that the bilinear form induces an isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$. Diagramatically, this can be used to change the arrow direction on any edge, allowing us to drop the edge arrows from the notation. Further, we also drop the dots in the metric and Casimir diagrams, so they look like

 and  .

Moreover, the invariance of the metric can be written as $\langle [x,y], z \rangle = \langle [y,z], x \rangle$ which can be represented graphically as cyclic invariance of the contraction of the bracket and the metric:

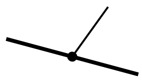

.

A similar relation holds for the Casimir in a metric Lie algebra, namely if the Casimir is $c = \sum_i e_i \otimes f_i$, then for $x \in \mathfrak{g}$, $\sum_i e_i \otimes [f_i, x] = \sum_i [x, e_i] \otimes f_i$. Diagramatically, we have



Just like the antisymmetry of the bracket and the Jacobi relation, these relations make sense when the diagrams are interpreted as parts of Jacobi diagrams. There is only one type of trivalent vertex in a Jacobi diagram, and since the edges around a vertex are only considered up to cyclic order, cyclic permutation of the order is "unseen on the level of Jacobi diagrams".

From the discussion above, we see that the internal structure of Jacobi diagrams (internal meaning not near the skeleton) is reflected in a metric Lie algebra. Let us try to find a similar reflection of the structure near the skeleton. A representation ρ of $\mathfrak{g}$ on a finite-dimensional vector space V can be written as a tensor $\rho \in \mathfrak{g}^* \otimes V^* \otimes V$. This takes a new kind of input and output, namely a $v \in V$ which we denote by a thicker line, like the skeleton of a Jacobi diagram.



We omit the arrows because they're unneccesary again — corresponding to the maps

$$f \otimes v \longmapsto f(v) \quad \text{and} \quad 1 \longmapsto \sum_i e_i \otimes e_i^*$$

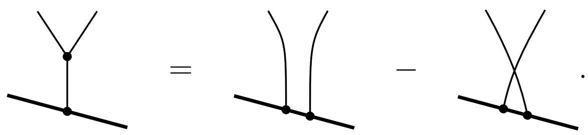
are diagrams

 and .

That the action of ρ on V be a Lie action is the equation

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

which diagramatically corresponds to



The following construction of Bar-Natan is the focus of his seminal article [Bar95]. It uses this diagrammatic calculus to produce weight systems from metric Lie algebras.

Construction 2.2.1 The construction takes a metric Lie algebra $\mathfrak{g}$, and produces a map $W_{\mathfrak{g}} : \mathcal{A} \to \mathcal{U}(\mathfrak{g})$ which is a $\mathcal{U}(\mathfrak{g})$-valued weight system. Given further a representation ρ of $\mathfrak{g}$ it produces a map $W_{\mathfrak{g}} : \mathcal{A} \to k$ which is a k-valued weight system. That is, given $\mathfrak{g}$ a metric Lie algebra and $J \in \mathcal{A}$, it produces an element of $\mathcal{U}(\mathfrak{g})$, and if also given a representation ρ it produces a scalar. We write v and u for the number of trivalent and univalent vertices respectively of J.

To each trivalent vertex of J, associate a copy of the tensor $[\cdot, \cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ (see Warning 2.2.2). For each edge between trivalent vertices, contract the corresponding tensors along the components corresponding to those half-edges. Where the signature of the components doesn't allow for contraction (when both components are covariant or both are contravariant, i.e. when there are two inputs or two outputs), contract one of them first with either the metric or the Casimir (whichever is allowed by its variance). The resulting tensor has u components (they may be co- or contra-variant).

Contract this tensor with a copy of the Casimir along all remaining covariant components. We define the result to be the tensor $T_{\mathfrak{g}}(J) \in \mathfrak{g}^{\otimes u}$. The linear order of its components must be a linear order that agrees with the cyclic order of the corresponding univalent vertices in J. Define $W_{\mathfrak{g}}(J)$ to be the projection of this tensor into $\mathcal{U}(\mathfrak{g})$, namely

$$W_{\mathfrak{g}}(J) = [T_{\mathfrak{g}}(J)] \in \mathcal{U}(\mathfrak{g}).$$

Constructing the k-valued weight system from the representation is as follows. A representation $\rho : \mathfrak{g} \to \mathrm{Hom}(V)$ of a Lie algebra extends uniquely to a representation of its universal enveloping algebra $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{Hom}(V)$. Define $W_{\mathfrak{g},\rho}$ as the trace of $W_{\mathfrak{g}}(J)$ with respect to this representation,

$$W_{\mathfrak{g},\rho}(J) = \mathrm{tr}(\rho(W_{\mathfrak{g}}(J))) \in \mathbb{Q}.$$

Warning 2.2.2 In Construction 2.2.1 when constructing $T_{\mathfrak{g}}(m)$, the tensor factors in the tensor corresponding to the bracket need to have the unusual cyclic order $(y^*, x^*, [x,y]_{\mathfrak{g}})$. This is because its projection into $\mathcal{U}(\mathfrak{g})$ should obey STU, and this is the cyclic order of the trivalent vertex in STU (it will become evident in the proof why we need this convention).

Example 2.2.3 Take the metric Lie algebra $(\mathfrak{sl}_2, \langle\cdot,\cdot\rangle)$ where $\mathfrak{sl}_2$ is defined by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h,$$

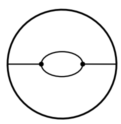
and the metric is

$$\langle h,h\rangle = 2, \quad \langle e,f\rangle = 1, \quad \langle f,e\rangle = 1.$$

Here we have chosen the normalisation of the metric that agrees with the trace in the adjoint representation, so

$$[\cdot,\cdot] = 2e^* \otimes h^* \otimes e - 2f^* \otimes h^* \otimes f + f^* \otimes e^* \otimes h.$$

We do the computations for

$J =$  .

Taking another copy $[\cdot,\cdot]'$ of the bracket tensor, one way to compute $W_{\mathfrak{sl}_2}(J)$ is to let $[\cdot,\cdot]$ take the left trivalent vertex, and associate the upward facing half-edge to the first component. Let $[\cdot,\cdot]'$ take the right trivalent vertex and associate the downward facing half-edge to the first component — the cyclic orders determine the rest. Then the computation is to take the contraction of $[\cdot,\cdot]$ along components 1 and 3 with $[\cdot,\cdot]'$ along components 3 and 1. This gives

$$2h^* \otimes h^* + e^* \otimes f^* + f^* \otimes e^*,$$

and contracting along each component with a Casimir to make them contravariant yields

$$T_{\mathfrak{sl}_2}(J) = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e.$$

Projecting this into $\mathcal{U}(\mathfrak{g})$ and writing it in the PBW-basis, we have

$$W_{\mathfrak{sl}_2}(J) = \frac{1}{2} h \otimes h - h + 2e \otimes f.$$

Finally, if we use the adjoint representation ad, defined by

$$\mathrm{ad}(h) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathrm{ad}(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{ad}(f) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we get

$$W_{\mathfrak{sl}_2,\mathrm{ad}}(J) = 3.$$

◊

Theorem 2.2.4 *In Construction 2.2.1, $W_\mathfrak{g}$ is a well-defined $\mathcal{U}(\mathfrak{g})$-valued weight system, and $W_{\mathfrak{g},\rho}$ is a well-defined k-valued weight system.*

Proof ([Bar95]) First we prove that the map $W_\mathfrak{g} : \mathcal{D} \to \mathcal{U}(\mathfrak{g})$ descends to a map from $\mathcal{A}$.

The map $T_\mathfrak{g} : \mathcal{D} \to \mathfrak{g}^{\otimes u}$ is invariant under `IHX` and `AS`. For `AS`, it follows from the anti-symmetry of $[\cdot,\cdot]$, and for `IHX` it follows from the Jacobi relation.

Furthermore, the map $\mathfrak{g}^{\otimes u} \to \mathcal{U}(\mathfrak{g})$ is invariant under `STU`. For if two chord diagrams differ by `STU`, on some univalent vertices associated with adjacent tensor factors y and x, the two sides of an `STU` map to

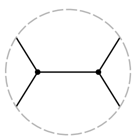
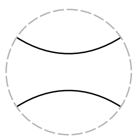
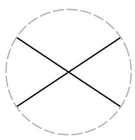
$$\cdots \otimes [x,y]_\mathfrak{g} \otimes \cdots \qquad \text{and} \qquad (\cdots \otimes x \otimes y \otimes \cdots) - (\cdots \otimes y \otimes x \otimes \cdots),$$

but the equality of these expressions is exactly the defining relation of $\mathcal{U}(\mathfrak{g})$. (Alternatively the well-definiedness of $W_\mathfrak{g}$ under `AS` and `IHX` also follows from this, because those relations are implied by `STU`.)

There is one other arbitrary choice that was made in the construction. The cyclic order of the univalent vertices of J induces a cyclic order on the components of the components of the tensor $T_\mathfrak{g} \in \mathfrak{g}^{\otimes u}$. However, in the construction, a linear order which respects that cyclic order was chosen. We must show that any choice of linear order respecting the cyclic order produces the same result. This is no problem for the well-definedness of $W_{\mathfrak{g},\rho}(J) = \mathrm{tr}(\rho(W_\mathfrak{g}(J)))$, as the trace is invariant under cyclic permutation. However for $W_\mathfrak{g}(J) = [T_\mathfrak{g}(J)] \in \mathcal{U}(\mathfrak{g})$ it remains to prove cyclic permutation invariance. In fact, a stronger statement is true. The diagram J itself is invariant under a cyclic permutation by Proposition 2.1.7, so of course this is true for $W_\mathfrak{g}(J)$. □

Let's look at a specific weight system for the Lie algebra $\mathfrak{sl}_2$ [Bar95; CV97].

Example 2.2.5 (Weight system for $\mathfrak{sl}_2$) We use the metric Lie algebra from Example 2.2.3. In [CV97], the following skein relation is derived:

$W_{\mathfrak{sl}_2}$  $= 2W_{\mathfrak{sl}_2}$  $- 2W_{\mathfrak{sl}_2}$ 

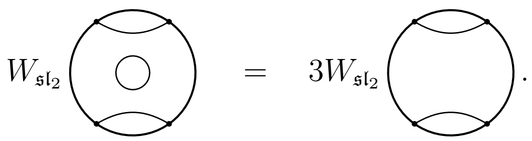
◊

Proof Compute both sides like in Construction 2.2.1, but without projecting into $\mathcal{U}(\mathfrak{g})$, leaving the result in $\mathfrak{sl}_2^{\otimes 4}$. Both give

$$\begin{aligned} &- h \otimes e \otimes h \otimes f + h \otimes e \otimes f \otimes h - h \otimes f \otimes h \otimes e + h \otimes f \otimes e \otimes h \\ &+ e \otimes h \otimes h \otimes f - e \otimes h \otimes f \otimes h + 2e \otimes f \otimes e \otimes f - 2e \otimes f \otimes f \otimes e \\ &+ f \otimes h \otimes h \otimes e - f \otimes h \otimes e \otimes h - 2f \otimes e \otimes e \otimes f + 2f \otimes e \otimes f \otimes e. \end{aligned}$$

□

Remark 2.2.6 When computing via the $\mathfrak{sl}_2$ skein relation above, it's possible to create a "bubble" (part of a diagram without any trivalent vertices). Since we are computing via contractions in the tensor algebra, this is to be interpreted as the contraction of the metric with the Casimir. In a finite-dimensional Lie algebra, this is just the dimension (so for $\mathfrak{sl}_2$, the factor 3). For example,



Question 2.2.7 In [CDM12, Remark 16.9] it is noted that the $\mathfrak{sl}_2$ skein relation is an analogue of the vector triple product rule for the cross product in $\mathbb{R}^3$. The relation has been further studied in [MS17] to determine a basis for $\mathcal{W}_{\mathfrak{sl}_2}$. There is also a cross product in $\mathbb{R}^7$, related to the exceptional Lie algebra $\mathfrak{g}_2$ — it also obeys a variant of the vector triple product rule. Is there a similar skein relation for $\mathcal{W}_{\mathfrak{g}_2}$? Are there other skein relations that the weight systems for the exceptional Lie algebras obey?

Construction 2.2.1 yields a way of extracting some information from $\mathcal{A}$ by plugging in a metric Lie algebra – doing so constructs some quotient of $\mathcal{A}$. This leads to the natural question of whether all of the information in $\mathcal{A}$ can be extracted by metric Lie algebras in the manner of the construction.

A computer enumeration of [Bar95] proves this for order $m \leq 9$:

m	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{W}_m$	1	1	2	3	6	10	19	33	60	104
$\dim(\mathcal{W}_{\text{Lie}})_m$	1	1	2	3	6	10	19	33	60	104

(here $\mathcal{W}_{\text{Lie}}$ denotes the dimension of the subspace of $\mathcal{W}_m$ spanned by weight systems coming from Construction 2.2.1). In other words

$$\mathcal{W}_{\text{Lie}} = \operatorname{span}\{W_{\mathfrak{g},\rho} \mid \mathfrak{g} \text{ a Lie algebra}, \rho \text{ a representation of } \mathfrak{g}\}.$$

In [Bar95], this is computed up to degree 9 only using the span of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ Lie algebras, providing a lower bound on $\dim(\mathcal{W}_{\text{Lie}})_m$. The upper bound comes from $\dim \mathcal{A}_m$. So in fact, up to degree 9, $\mathcal{W}_m$ is spanned even by only the $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ weight systems.

Conjecture 2.2.8 (Bar-Natan) *All weight systems are obtained as Lie algebra weight systems. In other words, the set $\{W_{\mathfrak{g},\rho} \mid \mathfrak{g}$ a Lie algebra, ρ a representation of $\mathfrak{g}\}$ spans $\mathcal{W}$.*

Indeed, the Lie action relation $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ as it was drawn graphically looks exactly like the STU relation in $\mathcal{A}$. However, looks turn out to be decieving and quite surprisingly this conjecture is false. The counterexample was found by Pierre Vogel [Vog97] in an attempt to answer the following related question:

Question 2.2.9 (Vogel) Is there some single universal Lie algebra object whose weight system spans W_{Lie}, the span of all Lie-algebraic weight systems?

2.3 Non-Lie algebraic weight systems

Conjecture 2.2.8 being false means that the STU relation describes more than the relation $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ for a representation ρ of a metric Lie algebra. In fact, the construction of metric Lie algebra weight systems seen in 2.2.1 is just one example of a more general construction introduced by Vogel and Vaintrob with the purpose of constructing weight systems coming from metric Lie super-algebras, and further generalisations of Lie algebras.

The most general type of objects these constructions apply to are 'Lie S-algebras' as coined by Vaintrob in [Vai94], but we will follow the more modern approach of [RW06; Rob01] and will refer to them as Lie algebra objects in a symmetric monoidal category.

Definition 2.3.1 A (**weak**) **monoidal category** is a category $\mathcal{C}$ equipped with a functor

$$\begin{array}{rccc} \otimes : & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C} \\ & (A,B) & \longmapsto & A\otimes B, \end{array}$$

a **unit** object $k \in \mathcal{C}$, and natural isomorphisms

$$\otimes\circ(\otimes\times\mathrm{id}) \longrightarrow \otimes\circ(\mathrm{id}\times\otimes), \quad k\otimes \_ \longrightarrow \mathrm{id} \quad \text{and} \quad \_\otimes k \longrightarrow \mathrm{id}$$

satisfying some relations known as the pentagon and triangle relations [Lei04, Sec. 1.2]. The natural isomorphisms give isomorphisms

$$(A\otimes B)\otimes C \cong A\otimes(B\otimes C), \qquad k\otimes A \cong A \cong A\otimes k$$

for every tuple of objects A, B and C in $\mathcal{C}$. If these isomorphisms are equalities, then $\mathcal{C}$ is a **strict** monoidal category.

Remark 2.3.2 While we omit the details, we henceforth assume that these natural isomorphisms are equalities: $(A\otimes B)\otimes C = A\otimes(B\otimes C)$. This is acceptable by the MacLane coherence theorem for monoidal categories which says that every monoidal category is equivalent to a strict monoidal category. This also justifies why we omit the pentagon and triangle relations in the definition above. We refer the reader to [Lei04, Sec. 1.2] for details.

Definitions 2.3.3 (a) The **flip functor** is the functor

$$\begin{array}{rccc} \sigma : & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C}\times\mathcal{C} \\ & (A,B) & \longmapsto & (B,A). \end{array}$$

(b) A **symmetric monoidal category** is a monoidal category $\mathcal{C}$ equipped with a **symmetry natural isomorphism**

$$\tau : \otimes \longrightarrow \otimes\circ\sigma$$

satisfying the hexagon relation. The **hexagon relation** is the relation that the isomorphisms

$$\tau_{A,B} : A\otimes B \xrightarrow{\cong} B\otimes A$$

coming from the natural isomorphism τ obey

$$\tau_{A,B\otimes C} = (\mathrm{id}_B \otimes \tau_{A,C})\circ(\tau_{A,B}\otimes \mathrm{id}_C)$$

for every pair of objects A, and B in $\mathcal{C}$.

One might notice that the hexagon relation involves only three terms, rather than the six that the name suggests. This is because we have omited the reassociation natural isomorphisms that we assume are identities by Remark 2.3.2.

Examples 2.3.4 (a) Let **Vect** be the category of vector spaces and linear transformations. The symmetry natural isomorphism

$$\begin{array}{rccc} \tau: & X \times Y & \longrightarrow & Y \times X \\ & x \otimes y & \longmapsto & y \otimes x. \end{array}$$

gives **Vect** the structure of a symmetric monoidal category.

(b) A **super vector space** is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V with the decomposition

$$V = V_0 \oplus V_1.$$

The degree, $|x|$ of a homogeneous element $x \in V = V_0 \oplus V_1$ is $|x| = i \in \mathbb{Z}/2\mathbb{Z}$ where $x \in V_i$.

Let **sVect** denote the category of super vector spaces and degree preserving linear transformations. Defining the symmetry natural isomorphism on homogeneous elements $x, y \in V$ as

$$\begin{array}{rccc} \tau: & X \times Y & \longrightarrow & Y \times X \\ & x \otimes y & \longmapsto & (-1)^{|x||y|} y \otimes x. \end{array}$$

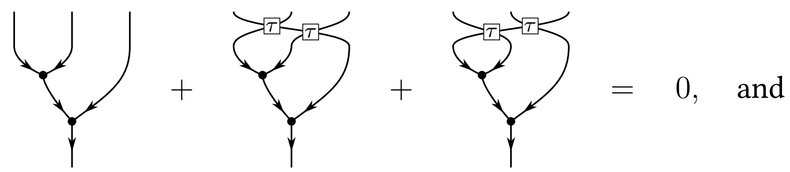
gives **sVect** the structure of a symmetric monoidal category. $\diamondsuit$

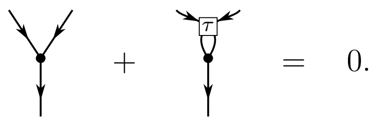
If (like in our examples) the monoidal category $\mathcal{C}$ is additive and the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear, then Lie algebra objects internal to $\mathcal{C}$ can be defined. These generalise Lie algebras in that they satisfy analogues of the Jacobi and antisymmetry identities involving the symmetry natural isomorphism τ. Of course, in the case $\mathcal{C} = \mathbf{Vect}$, τ is the trivial swap isomorphism, so the Lie algebra objects satisfy exactly the Jacobi and antisymmetry relations and are exactly the Lie algebras.

Definitions 2.3.5 (a) A **Lie algebra object** in $\mathcal{C}$ is an object L equipped with a bracket morphism $\beta : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ such that

$$(\beta \circ (\beta \otimes \mathrm{id})) \circ (1 + \tau_{123} + (\tau_{123})^2) = 0 \qquad \text{and} \qquad \beta + \beta \circ \tau = 0.$$

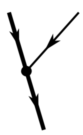
Graphically (in terms of the string diagrams of the previous section), this corresponds to



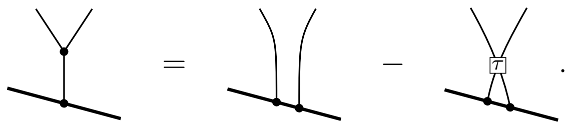


Here the trivalent vertex corresponds to the bracket morphism β.

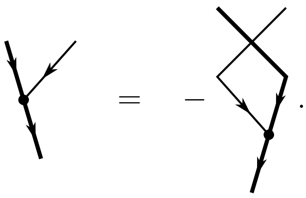
(b) A (right) **module** ρ **over a Lie algebra object** L **in** $\mathcal{C}$ is an object M with an action morphism $\rho : M \otimes L \to M$, diagramatically denoted



such that
$$\rho \circ (\mathrm{id} \otimes \beta) = \rho \circ (\rho \otimes \mathrm{id}) - \rho \circ (\rho \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau).$$
Here the thick line denotes M and this type of trivalent vertex denotes ρ. As a string diagram the equation becomes



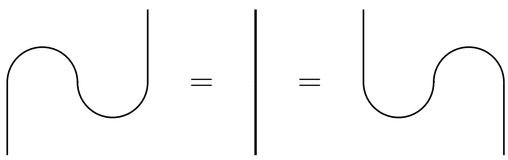
Note that any right module M can be given a natural left-module structure by
$$\bar{\rho} = -\rho \circ \tau$$
justifying the diagrammatic rule



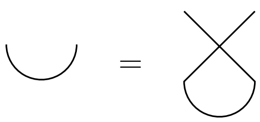
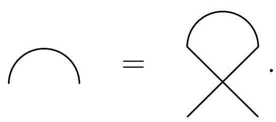
(c) A **metric Lie algebra object** L in $\mathcal{C}$ is a Lie algebra object in $\mathcal{C}$, further equipped with a metric morphism $m : L \otimes L \to k$ and Casimir morphism $c : k \to L \otimes L$ which abstract the metric and Casimir of Lie algebras in **Vect** by satisfying
$$(\mathrm{id} \otimes c) \circ (m \otimes \mathrm{id}) = \mathrm{id} = (c \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m)$$
and
$$m = \tau \circ m \quad \text{and} \quad c = c \circ \tau.$$
Diagramatically, these morphisms are

 and .

(like before these morphisms allow us to drop arrows on the thin lines). The former pair of relations is diagramatically



and the latter pair is

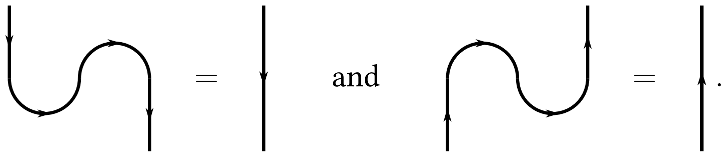
 and .

(d) A symmetric monoidal category $\mathcal{C}$ is **rigid** if for every object A there exists an object $A^\vee$ and a pair of morphisms $\iota : A^\vee \otimes A \to k$ and $\varepsilon : k \to A \otimes A^\vee$ — respectively called the **evaluation** and **coevaluation** — such that
$$(\mathrm{id}_A \otimes \varepsilon) \circ (\iota \otimes \mathrm{id}_A) = \mathrm{id}_A$$

and

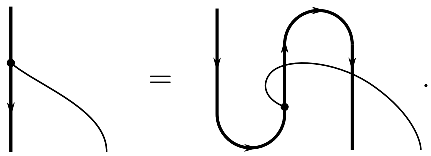
$$(\varepsilon \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_{A^\vee} \otimes \iota) = \mathrm{id}_{A^\vee}$$

Diagramatically, these are



An object in a monoidal category $\mathcal{C}$ is **rigid** if axioms above are satisfied just for that object without the whole category needing to be rigid.

For a rigid monoidal category $\mathcal{C}$, the dual of a (right) module may be turned into a (right) module by asserting that the module action commutes with the evaluation and coevaluation (that is evaluation and coevaluation are module maps). Diagramatically,



We will give concrete examples of Lie algebra objects in different symmetric monoidal categories later. For now, let's show that this data can be used to construct weight systems, generalising Construction 2.2.1.

Theorem 2.3.6 ([Vai94]) *Let $\mathcal{C}$ be a rigid, additive, symmetric monoidal category, L a metric Lie algebra in $\mathcal{C}$, and M a rigid module (a rigid object in the monoidal category of right modules over L). Then there is a weight system*

$$W_{L,M} : \mathcal{A} \longrightarrow \mathcal{C}(k, k).$$

If $\mathcal{C}$ is k-linear, then this is a k-valued weight system

$$W_{L,M} : \mathcal{A} \longrightarrow k.$$

The proof of this theorem is exactly the same as the proof of Construction 2.2.1, but considering everything in the generality of $\mathcal{C}$ rather than **Vect**. It is proved in [RW06].

The difference between this construction and Construction 2.2.1 is the treatment of the symmetry natural isomorphism τ which tells us what isomorphism to use when rearranging the tensor factors. The most obvious isomorphism would be the identity (as it was in the original construction) corresponding to when $\mathcal{C}$ is a strict symmetric (strict) monoidal category. However not every symmetric monoidal category is equivalent to a strict symmetric monoidal category. In other words, Lie algebra objects with non-trivial symmetry isomorphisms are necessary to pick up all the structure in $\mathcal{A}$.

Example 2.3.7 If we take $\mathcal{C} = \mathbf{sVect}$, the symmetric monoidal category of super vector spaces, then the Lie algebra objects are exactly the Lie superalgebras which we now define.

A **Lie superalgebra** $\mathfrak{g}$ is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading, equipped with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying two axioms. The **super symmetry** axiom is that for x, y homogeneous elements of $\mathfrak{g}_i$ and $\mathfrak{g}_j$ respectively,

$$[x, y] = -(-1)^{ij}[y, x].$$

The **super Jacobi identity**, that for homogeneous elements x, y and z in $\mathfrak{g}_i, \mathfrak{g}_j$ and $\mathfrak{g}_k$,

$$(-1)^{ik}[x,[y,z]] + (-1)^{ji}[y,[z,x]] + (-1)^{kj}[z,[x,y]] = 0.$$

Note that these identities are simply the axioms in Definition 2.3.5 (a), specialised to the category $\mathcal{C} = \mathbf{sVect}$.

The grading induces the splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the direct summand $\mathfrak{g}_0$ is known as the **even** part and the summand $\mathfrak{g}_1$ is known as the **odd** part. $\Diamond$

Shortly after Conjecture 2.2.8 was made, the following results were achieved based on constructions with the exceptional Lie superalgebra $\mathfrak{D}(2,1;\alpha)$ proving that metric Lie algebras alone do not see all the structure of $\mathcal{A}$.

Theorem 2.3.8 *There are primitive Jacobi diagrams of order 17 [Vog97] and 15 [Lie99] which vanish under all Lie algebra weight systems.*

Indeed (and this was known about shortly after but not published until significantly later) not even metric Lie superalgebras suffice.

Theorem 2.3.9 *Vogel's diagram of order 17 vanishes also on all Lie superalgebra weight systems [Vog11].*

Corollary 2.3.10 *The set of Lie (super)algebra weight systems does not span $\mathcal{W}$.*

In general, it is still unknown what exact level of generality one needs (what type of symmetric monoidal categories need to be considered) in order to generate all weight systems. We hereby provide a succinct review of the current state of the literature on the subject.

Roberts and Willerton in [Rob01] and [RW06] examine weight systems constructed from Lie algebra objects in the derived category of complex manifolds. Such weight systems are candidates for being able to detect knot orientation, which Lie algebra weight systems cannot [Bar95]. However, computing these weight systems is difficult, and to our knowledge, no computations exist in the literature.

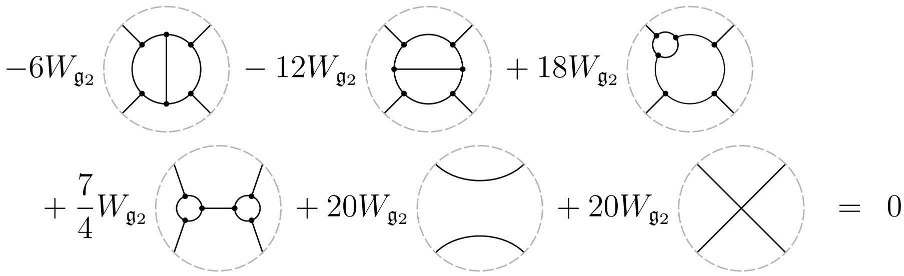
More recently Aizawa-Kimura [AK25] have conducted some preliminary investigations into the class of colour Lie algebras (also known as ϵ-Lie algebras). This class generalises the $\mathbb{Z}/2\mathbb{Z}$ grading on Lie superalgebras to a more general group. However, the example they present lies within the span of the $\mathfrak{sl}_2$ and $\mathfrak{gl}_{1|1}$ weight systems.

2.4 Some weight systems at the exceptional Lie algebras

The original motivation for the work of Vogel [Vog97; Vog11] (seen more explicity in [Vog99]) was to construct a universal object generalising all simple Lie algebras. Such an object would parametrise the relations in tensor powers of the adjoint representation of all simple Lie algebras, and its weight systems would span $\mathcal{W}$. For example, the relation for $W_{\mathfrak{sl}_2}$ given in Example 2.2.5 is entirely internal to the Lie algebra; it doesn't involve projecting to the universal enveloping algebra or choosing a representation (since the adjoint representation is canonical).

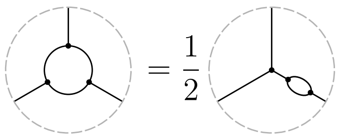
Recently, it seems there is renewed interest in Vogel's methods [KLS25; BM25; KMS25]. Inparticular, present implicitly in [Vog11] is a way to determine local relations internal to any simple Lie algebra in terms of two independent "Vogel parameters", and some complicated marked elements of $\mathcal{A}$ defined recursively. These were written out explicitly in the appendix of [KLS25], but the use of different normalisations of the Vogel parameters between this and [Vog11] makes computing explicit relations tedious. Moreover, the relations in Vogel's original work, are general enough to hold for any Lie superalgebra but when restricting to Lie algebras, the diagrammatic formulae can be simplified.

For example, [Vog11, Theorem 6.3(2)] gives the relation

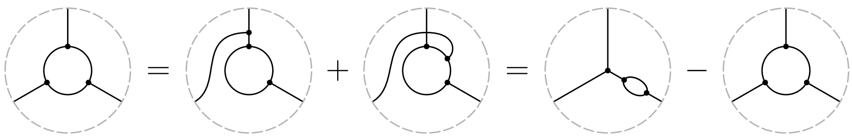


in $\mathcal{W}_{\mathfrak{g}_2}$. By some well-known identites internal to Jacobi diagrams, this relation can be simplified as follows.

Lemma 2.4.1 *In $\mathcal{A}$, the 'trivalent-bubble' vertex is proportional to the regular trivalent vertex with a bivalent bubble inserted on any one of the half-edges.*

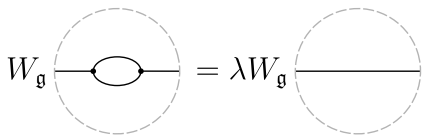


Proof Apply `IHX` to any pair of vertices, followed by three applications of `AS`.



□

Lemma 2.4.2 *In Lie algebraic weight systems, the bubble is proportional to the line.*

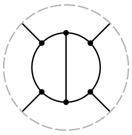
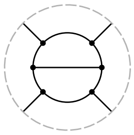
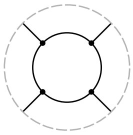


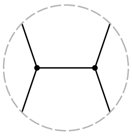
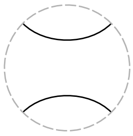
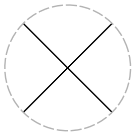
where $\lambda \in k$ is determined by the choice of metric.

This is proven in [CDM12, Lem. 6.15] by analysing the bracket tensor.

Remark 2.4.3 It would be interesting to have a purely categotical proof that doesn't refer to any specific representation, and uses the fact that Lie algebras are Lie algebra objects in the category **Vect**.

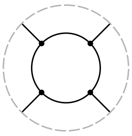
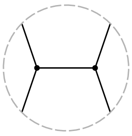
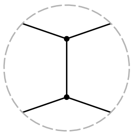
Simplifying Vogel's relation via these lemmas , gives the following relation. (We have used the metric in which $\lambda = 1/4$, i.e. our metric is 4 times $\langle \cdot, \cdot \rangle_K$, the Killing form.)

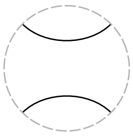
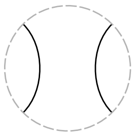
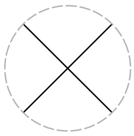
$-6W_{\mathfrak{g}_2}$  $-\ 12W_{\mathfrak{g}_2}$  $+\ 36W_{\mathfrak{g}_2}$ 

$+\ 7W_{\mathfrak{g}_2}$  $+\ 20W_{\mathfrak{g}_2}$  $-\ 20W_{\mathfrak{g}_2}$  $=\ 0$

This relation is true on a local level (like `IHX` and `AS` rather than `STU`). In-particular, it remains true after rotating all terms a quarter rotation clockwise. This symmetry is not obvious in the formula which suggests that perhaps the relation is a consequence of a similar, simpler relation.

Theorem 2.4.4 *For the Lie algebra $\mathfrak{g}_2$, the weight systems obey*

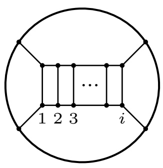
$W_{\mathfrak{g}_2}$  $= \frac{2}{3}W_{\mathfrak{g}_2}$  $+ \frac{2}{3}W_{\mathfrak{g}_2}$ 

$+\frac{5}{6}W_{\mathfrak{g}_2}$  $+ \frac{5}{6}W_{\mathfrak{g}_2}$  $+ \frac{5}{6}W_{\mathfrak{g}_2}$  .

Proof Compute the tensors corresponding to both sides. □

This is an analogue of a relation for $\mathfrak{sl}_3$ found by Yoshizumi and Kuga in [YK99] which was used recently by Yang in [Yan24] to compute the value of the $\mathfrak{sl}_3$ weight systems on the following infinite sequence of chord diagrams.

Definition 2.4.5 The ith **bookshelf diagram**, B_i, is the Jacobi diagram

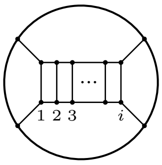
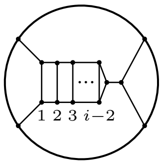
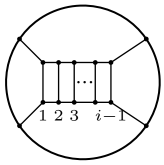


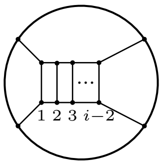
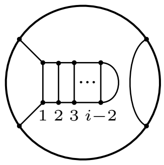
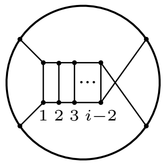
In [Yan24], a third-order recursion relation for the values of the $\mathfrak{sl}_3$ weight system on the bookshelf diagrams is found. We use the same idea with the new relation of Theorem 2.4.4 to compute their values under the $\mathfrak{g}_2$ weight system.

Theorem 2.4.6 *The values of the bookshelf diagram B_n are at most quadratic in the (quadratic) casimir element c of $\mathfrak{g}_2$. In particular,*

$$W_{\mathfrak{g}_2}(B_i) = c^2\left(\frac{1}{14}4^n + \frac{27}{112}(5/3)^n + \frac{11}{16}(-1)^n\right) + c\left(2^n + \frac{8}{3}(5/3)^n - \frac{11}{8}(-1)^n\right).$$

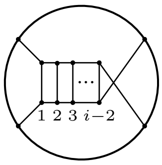
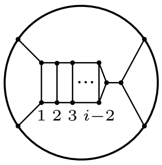
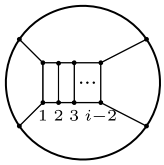
Proof Applying Theorem 2.4.4 to the rightmost box on a bookshelf diagram gives

$W_{\mathfrak{g}_2}$  $= \frac{2}{3}W_{\mathfrak{g}_2}$  $+\frac{2}{3}W_{\mathfrak{g}_2}$ 

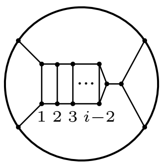
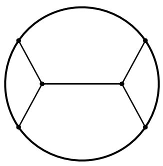
$+\frac{5}{6}W_{\mathfrak{g}_2}$  $+\frac{5}{6}W_{\mathfrak{g}_2}$  $+\frac{5}{6}W_{\mathfrak{g}_2}$  .

(2.4.6a)

Restricting ourselves to the final term of Equation 2.4.6a, an STU relation gives

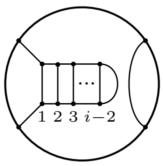
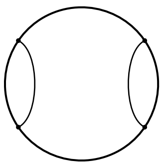
$W_{\mathfrak{g}_2}$  $= W_{\mathfrak{g}_2}$  $+ W_{\mathfrak{g}_2}$  .

Present here is a bookshelf diagram of order $i-2$ (the second term), and an additional term containing a bubble (the first term). Applying Lemma 2.4.1, we can remove the bubble from the first term, but doing so leaves a diagram with another bubble. After $i-2$ applications of the lemma we have

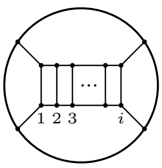
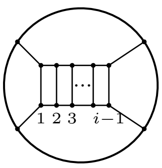
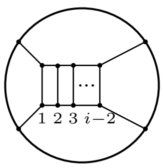
$W_{\mathfrak{g}_2}$  $= 2^{i-2}W_{\mathfrak{g}_2}$  $= 2^{i-2}4c^2.$

where the final equality comes a tabulation of low order chord diagrams in some Lie algebra representations in [CDM12, p. 181].

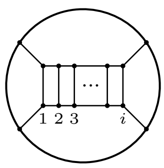
Now for the second-last term of Equation 2.4.6a. We apply Lemma 2.4.2 $i-2$ times, giving

$W_{\mathfrak{g}_2}$  $= 4^{i-2}W_{\mathfrak{g}_2}$  $= 4^{i-2}c^2.$

All in all, Equation 2.4.6a simplifies to

$W_{\mathfrak{g}_2}$  $= \frac{2}{3}W_{\mathfrak{g}_2}$  $+\frac{5}{3}W_{\mathfrak{g}_2}$  $+\frac{5}{6}4^{i-2}c^2+\frac{11}{6}2^{i-2}c$

so that setting

$g_i = W_{\mathfrak{g}_2}$ 

yields the second-order non-homogeneous recursion relation

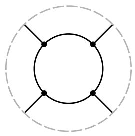
$$g_{i+2} = \frac{2}{3}g_{i+1} + \frac{5}{3}g_i + \frac{5}{6}4^i c^2 + \frac{11}{6}2^i c$$

for g_i which was fourth-order homogeneous form

$$g_{i+4} - \frac{20}{3}g_{i+3} + \frac{31}{3}g_{i+2} + \frac{14}{3}g_{i+1} - \frac{40}{3}g_i = 0.$$

To solve this we need the inital conditions g_1, g_2 and g_3. The first two come from the same tabulation in [CDM12, p. 181], whereas the third we must compute ourselves.

Computing in sagemath the tensor corresponding to the diagram



then projecting it to $\mathcal{U}(\mathfrak{g}_2)$ gives the value, but expressed in the PBW basis rather than as a polynomial in the Casimir, like the expressions we have for g_1 and g_2. By [CDM12, Theorem 6.1(3)], the value of any chord diagram under a Lie algebra weight system is a polynomial in the Casimir element and the higher order Casimirs elements of $\mathfrak{g}_2$. Since $\operatorname{rank}(\mathfrak{g}_2) = 2$, there is only one higher order Casimir to consider, and it is of degree 6. Therefore since g_3 is a degree 4 element of $Z(\mathfrak{g}_2)$, so it will be a polynomial in c of degree two or less, and the higher order Casimir element doesn't come into play. Evaluating $\rho(g_3)$ with ρ being the trivial representation, the seven-dimensional standard representation, and the fourteen-dimensional adjoint representation respectively gives the values

$$\begin{aligned}
\rho_{\text{tr.}}(c) &= 0 & \qquad \rho_{\text{tr.}}(g_3) &= 0\\
\rho_{\text{st.}}(c) &= 2 & \qquad \rho_{\text{st.}}(g_3) &= \frac{380}{9}\\
\rho_{\text{ad.}}(c) &= 4 & \qquad \rho_{\text{ad.}}(g_3) &= \frac{1120}{9}.
\end{aligned}$$

So by the Lagrange interpolation formula, we find that

$$g_3 = 5c^2 + \frac{100}{9}c.$$

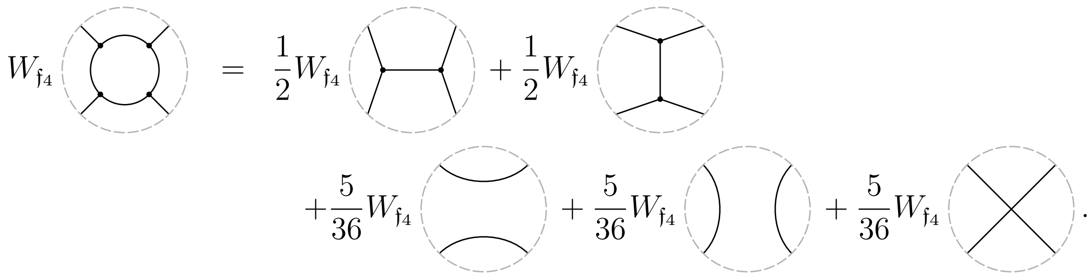
Solving the recurrence relation with these inital conditions yields the formula

$$g_i = c^2\left(\frac{1}{14}4^i + \frac{27}{112}(5/3)^i + \frac{11}{16}(-1)^i\right) + c\left(2^i + \frac{8}{3}(5/3)^i - \frac{11}{8}(-1)^i\right). \qquad \square$$

Remark 2.4.7 The result we used in the above proof from [CDM12, p. 181] first appears in the literature in the PhD thesis of A. Kaishev which we have been unable to source. Our computer program also verifies it.

The same computational method as in 2.4.4 can be used to find a relation in the $\mathfrak{f}_4$ weight system.

Theorem 2.4.8 *For the Lie algebra $\mathfrak{f}_4$, the weight systems obey*

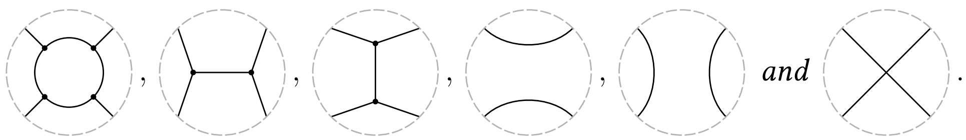


Proof Direct computation with sagemath. □

The dimension of $\mathfrak{g}_2$ is 14, and tensor contraction with sagemath is computable on the author's ThinkPad T14s with processor AMD Ryzen 7 PRO 5850U in a few seconds. One the other hand, $\dim \mathfrak{f}_4 = 52$, and calculation takes several hours on a server with two Intel Xeon E5-2667v3 CPUs. The main bottleneck is tensor contraction, and indeed for $\mathfrak{e}_6$ (which has dimension 78), the computation is not feasible on this hardware, at least not with the current algorithm. There is some hope that this can be done using more efficient tensor contraction algorithms than those offered by sagemath, but this has not yet been explored.

Finally, we note that for $\mathfrak{sl}_4$ the arguments given above do not work.

Theorem 2.4.9 *In the weight system corresponding to the Lie algebra $\mathfrak{sl}_4$, there is no linear relation between the diagrams*



Proof By computer verification with sagemath, the span of the corresponding tensors in $(\mathfrak{sl}_4)^{\otimes 4}$ is six-dimensional. □

2.5 A universal casimir Lie algebra object

We return to the universal Lie algebra of Vogel which was discussed in Section 2.3. For a long period, Vogel endevoured [Vog97; Vog11; Vog99] to construct this univeral Lie algebra, however attempts at its construction have so far failed [KMS25; BM25; KLS25].

There is, however, a theoretical construction of Hinnich–Vaintrob [HV00]: they construct a Lie algebra object in a symmetric monoidal category with a weight system which is isomorphic to $\mathcal{A}$ as a bialgebra. This guarantees that every Vassiliev invariant can be constructed from applying Construction 2.2.1 to some Lie algebra object in some symmetric monoidal category. However, it doesn't give any hints as to which symmetric monoidal category one needs to work in to construct a particular Vassiliev invariant. In this sense it is useful theoretically, but not computationally.

Furthermore, the this bialgebra is constructed as a generalised notion of a universal enveloping algebra, applied to a Lie algebra object is some symmetric monoidal category, giving credence to the statement hinted at by Construction 2.2.1 that "the algebra $\mathcal{A}$ behaves like the universal enveloping algebra of a Lie algebra".

In this section we review the Hinich–Vaintrob construction, and in Chapter 3, we will generalise it from Vassiliev invariants of classical knots to those of welded long knots.

A concept that is essential to the construction of Hinich–Vaintrob is that of an operad, which captures abstractly the general behaviour of associative operations and their iterated composition. Indeed, when operations on a set are composed by taking as one of their inputs the output of another operation, these compositions always obey certain rules. If the set is for example a vector space, the operations too form a vector space, and there is an action of the symmetric group S_n coming from the permutation of the inputs of the operation. An operad captures this concept of operations on some object of a category abstractly.

Warning 2.5.1 For all of the below, we assume the monoidal categories are k-linear, k a field of characteristic zero and the tensor product is bilinear (also known as tensor categories).

Definition 2.5.2 An **operad** $\mathcal{O}$ in the symmetric monoidal category **Set** consists of:

- a collection of sets, $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$, with $\mathcal{O}(n)$ known as the n-**ary operations**,
- an element $\mathrm{id} \in \mathcal{O}(1)$,
- an action of the symmetric group S_n on each $\mathcal{O}(n)$
- a series of **composition functions**: for each n and $k_1, k_2, \cdots, k_n \in \mathbb{N}$

$$\begin{array}{rccc} \circ_n : & \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \otimes \cdots \otimes \mathcal{O}(k_n) & \longrightarrow & \mathcal{O}(k_1 + k_2 + \cdots + k_n) \\ & \theta \otimes \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n & \longmapsto & \theta \circ (\theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n) \end{array}$$

  satisfying the following axioms.

(a) That for any $\theta \in \mathcal{O}(n)$, $\theta \circ_n (\mathrm{id}, \ldots, \mathrm{id}) = \theta = \mathrm{id} \circ_1 \theta$. (The **identity** axiom.)

(b) **Associativity of composition**. Let $\theta \in \mathcal{O}(n)$, and $\theta_1 \in \mathcal{O}(k_1)$, $\theta_2 \in \mathcal{O}(k_2)$, up to $\theta_n \in \mathcal{O}(k_n)$, and finally let $\theta_{1,1} \in \mathcal{O}(k_{1,1})$, $\theta_{1,2} \in \mathcal{O}(k_{1,2})$ through to $\theta_{k_n,1} \in \mathcal{O}(k_{k_n,1})$. The two ways of forming the composition of the three layers of operations: $\theta_{i,j}$ into θ_i into θ are equal. That is,

$$\theta \circ_n \Big(\theta_1 \circ_{k_1} (\theta_{1,1}, \ldots, \theta_{1,k_1}), \ldots, \theta_n \circ_{k_n} (\theta_{n,1}, \ldots, \theta_{n,k_n})\Big)$$

is equal to

$$\big(\theta \circ_n (\theta_1, \ldots, \theta_n)\big) \circ_{k_1+\cdots+k_n} (\theta_{1,1}, \ldots, \theta_{1,k_n}, \ldots, \theta_{n,1}, \ldots, \theta_{n,k_n}).$$

(c) An **equivariance** axiom, that for $\sigma \in S_n$, and all $\theta \in \mathcal{O}(n)$, and all $\theta_1, \ldots, \theta_n$,

$$(\theta\sigma) \circ_n (\theta_1, \ldots, \theta_n) = \big(\theta \circ_n (\theta_{\sigma^{-1}(1)}, \ldots, \theta_{\sigma^{-1}(n)})\big)\sigma_{\text{block}}$$

where $\sigma_{\text{block}} \in S_{k_1+\cdots+k_n}$ is the permutation that block-permutes blocks of size k_1 through to k_n.

(d) A further **equivariance** axiom, that for permutations $\sigma_i \in S_{k_i}$, and all $\theta \in \mathcal{O}(n)$, and all $\theta_1, \ldots, \theta_n$,

$$\theta \circ_n (\theta_1\sigma_1, \ldots, \theta_n\sigma_n) = \big(\theta \circ_n (\theta_1, \ldots, \theta_n)\big)(\sigma_1 \sharp \cdots \sharp \sigma_n)$$

where $(\sigma_1 \sharp \cdots \sharp \sigma_n) \in S_{k_1+\cdots+k_n}$ is the block-wise permutation of block i by σ_i.

(e) That the action of the symmetric group agrees with its action in the symmetric monoidal category **Set**.

Accordingly, operads can be considered over any symmetric monoidal category $\mathcal{C}$ by replacing **Set** with $\mathcal{C}$ in Definition 2.5.2 and adjusting sets to be objects of $\mathcal{C}$ and functions between sets to be morphisms between the corresponding objects.

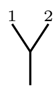
An algebra over an operad is an object in $\mathcal{C}$ whose operations obey the rules defined by $\mathcal{O}$.

Example 2.5.3 Recall that an operad is in particular a collection of vector spaces (or other objects of some symmetric monoidal category $\mathcal{C}$) $\mathcal{O}(n)$ for $n \in \mathbb{N}$, with a S_n action on each $\mathcal{O}(n)$. Such an object is called an S-module, and there is the category S-**mod** of S-modules whose morphisms are S-module morphisms: morphisms that commute with the action of S_n.

There is also a category **Oper** of operads. The morphisms from operads $\mathcal{O}$ to $\mathcal{P}$ are sequences of maps $\mathcal{O}(n) \to \mathcal{P}(n)$, $n \in \mathbb{N}$ that preserve the identity, preserve composition and commute with the action of S_n.

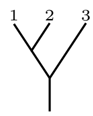
As such, there is a natural forgetful functor $\square : \mathbf{Oper} \to S\text{-}\mathbf{mod}$. It has a left-adjoint free operad functor $\Lambda : S\text{-}\mathbf{mod} \to \mathbf{Oper}$. For details of this construction we refer the reader to [Mar08].

We can build an S-module from a single generating operation. For example, let

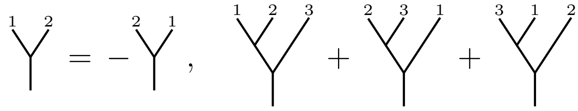
$\alpha =$ 

denote a generator operation with two inputs.

By abuse of notation we denote by α the S-module in which each $\mathcal{O}(n)$ is the set of iterated compositions of α. For example

 $\in \alpha(3).$

With this in mind, we can describe the **Lie operad** in a simple graphical presenation in terms of generators and relations:

$\mathcal{O}_{\text{Lie}} \quad = \quad \Lambda(\alpha) \quad /$  $= 0.$

◊

Recall that the goal of the Hinich–Vaintrob construction is to realise $\mathcal{A}$ as the universal enveloping algebra (or an appropriate generalisation thereof) of a Lie algebra object in some symmetric monoidal category. Indeed, $\mathcal{A}$ is a bialgebra, and Hinich–Vaintrob construct an object which is categorically a universal enveloping algebra, has a compatible coalgebra structure, and as a bialgebra is isomorphic to $\mathcal{A}$.

The first step is to create an category whose morphisms encode the internal structure of $\mathcal{A}$, i.e. the `AS` and `IHX` relations, but not yet with a choice of representation or the `STU` relations. An important feature of $\mathcal{A}$ that enabled the construction of Lie algebra weight systems was that contraction of tensors corresponding to the Lie bracket or metric/casimir elements was akin to joining parts of diagrams in $\mathcal{A}$. So a contraction-like operation would be a good start.

Indeed, composition in operads is reminiscent of contraction in a symmetric monoidal category, with the caveat that there are restictions on what can be contracted: only outputs with inputs. Furthermore, in an operad, every operation has exactly one output, and as a result there is no natural notion of tensor product in an operad.

A step towards describing the internal structure of $\mathcal{A}$ without specifying a Lie algebra is to allow multiple outputs, thereby considering not just operations with a single output, but with multiple outputs. Operads are abstractions of the operations on some object (say, A), i.e. morphisms $A^{\otimes n} \to A$. Slightly more generally, props are abstractions of all maps $A^{\otimes n} \to A^{\otimes m}$.

Definition 2.5.4 A **prop** is a symmetric monoidal category where every object is of the form $\mathbf{1}^{\otimes n}$ for some object $\mathbf{1}$. Equivalently, it's a symmetric monoidal category generated by a single object. Since props are categories, we use a boldface capital letter to denote them. We denote the monoidal unit by $\mathbf{0}$, the generating object by $\mathbf{1}$, and the object $\mathbf{1}^{\otimes n}$ by $\mathbf{n}$.

Definition 2.5.5 An **algebra** A **(in** $\mathcal{C}$**) over the prop** $\mathbf{P}$ is a symmetric monoidal functor $\alpha : \mathbf{P} \to \mathcal{C}$, where $\alpha(\mathbf{1}) = A$.

Remark 2.5.6 Under suitable definitions of morphisms of props and operads, there are the categories $\mathbf{Prop}$ and $\mathbf{Oper}$, and a pair of adjoint functors.

$$\begin{tikzcd} \mathbf{Oper} \arrow[d, shift right=2, "\mathbf{E}"] \\ \mathbf{Prop} \arrow[u, shift right=2, "\mathcal{F}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

The forgetful functor $\mathcal{F}$ takes a prop $\mathbf{P}$ to the operad whose operations are the $(\mathbf{m}, \mathbf{1})$-ary elements of the prop (i.e. morphisms with m inputs and one output):

$$\mathcal{F}(\mathbf{P})(m) = \mathbf{P}(\mathbf{m}, \mathbf{1}).$$

The free functor $\mathbf{E}$ maps an operad $\mathcal{O}$ to the prop whose $(\mathbf{m}, \mathbf{n})$-ary morphisms are tensor products of operations from the operad:

$$\mathbf{E}(\mathcal{O})(\mathbf{m}, \mathbf{n}) = \bigoplus_{f:[m]\to[n]} \bigotimes_{i=1}^{n} \mathcal{O}(|f^{-1}(i)|)$$

(here the outermost direct sum is over all functions f from $\{1, \dots, m\} \to \{1, \dots, n\}$). The prop $\mathbf{E}(\mathcal{O})$ is also referred to as the **envelope of the operad** $\mathcal{O}$.

Proposition 2.5.7 *The notions of algebra over a prop and algebra over an operad are equivalent in the sense that* $A \in \mathcal{C}$ *is an algebra over the operad* $\mathcal{O}$ *if and only if it is an algebra over the prop* $\mathbf{E}(\mathcal{O})$.

Now the main difference between contraction and partial composition in a prop is the marking of inputs and outputs. With metric Lie algebras, co- and contravariant indices could be contracted with the metric or casimir to invert their variance, but in a general prop there is no way to turn an input into an output.

Recall that the action of the symmetric groups S_n on an operad come from the symmetry of the symmetric monoidal category $\mathcal{C}$. But this only permutes inputs amongst themselves, rather than permuting inputs with outputs. In a certain kind of operad, called a cyclic operad, permutations of inputs with outputs can be effectively achieved, and the distinction between inputs and outputs is lost. This is achieved, for example by the metric in a metric Lie algebra which allows it to be identified with its dual and therefore the distinction between co- and contra- variant elements of the tensor algebra is lost.

Definition 2.5.8 A **cyclic operad** in $\mathcal{C}$ is an operad equipped with a cyclic structure. A **cyclic structure** is an action of each symmetric group S_{n+1} on $\mathcal{O}(n)$ which extends the S_n action on $\mathcal{O}(n)$ and satisfies the identity below.

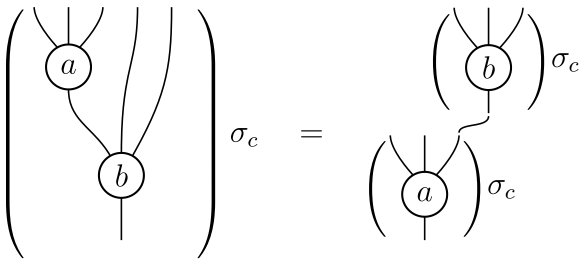
First we define what we mean by an extension of the S_n action. Formally, let S_n be the group of permutations of the set $\{1, \cdots, n\}$. The operad already has an action of S_n on each $\mathcal{O}(n)$ which agrees with the action of the symmetry τ of $\mathcal{C}$.

Let $S_n^+ \cong S_{n+1}$ be the group of permutations of $\{0, \cdots n\})$, and note that the subgroup of S_n^+ that fixes 0 is isomorphic to S_n. We say an action of S_n^+ extends the action of S_n if the restriction of the action to $S_n \subseteq S_n^+$ agrees with the action of S_n on $\mathcal{O}(n)$.

Let $\sigma_c \in S_{n+1}$ denote the cyclic permutation sending $0 \mapsto 1, 1 \mapsto 2, \cdots, n \mapsto 0$. To be a cyclic operad, the action of S_{n+1} must satisfy the identity

$$(a \circ_1 b)\sigma_c = (b\sigma_c) \circ_n (a\sigma_c)$$

for any $a \in \mathcal{O}(m), b \in \mathcal{O}(n)$. The axiom ensures that the action of S_{n+1} encodes that cyclically acting on a composition of operations is the same as cyclically acting and then composing. Graphically, this looks like the following.



We denote a cyclic operad with underlying operad $\mathcal{O}$ by $\mathcal{O}^{\mathrm{cyc}}$. This suffices because we do not consider operads with multiple cyclic structures.

The following definition generalises the definition of a metric Lie algebra (which is an algebra over the cyclic operad $\mathcal{O}_{\mathrm{Lie}}$) to an arbitrary cyclic operad.

Definition 2.5.9 For a cyclic operad $\mathcal{O}^{\mathrm{cyc}}$ in category $\mathcal{C}$, a **metric algebra** over $\mathcal{O}^{\mathrm{cyc}}$ is a pair (A, m). That is, a choice of object $A \in \mathcal{C}$ with together with a symmetric morphism $m : A \otimes A \to k$, such that (informally) using m to turn the output into an input and using its inverse c to turn the last input into an output implements the action of the cyclic operad. Formally:

- there exists a morphism $c : k \to A \otimes A$ such that the composition

  $$A \xrightarrow{c\otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes m} A$$

  is the identity (m is **invertible**).

- the composition

  $$\mathcal{O}(n) \otimes A^{\otimes n+1} \longrightarrow A \otimes A \xrightarrow{m} k$$

  is S_{n+1}-invariant (m is $\mathcal{O}$-**invariant**).

The conditions above can alternatively and more naturally be phrased in terms of props.

Definition 2.5.10 We define the **metric envelope of the cyclic operad** $\mathcal{O}^{\mathrm{cyc}}$, denoted $\mathbf{E}_m(\mathcal{O}^{\mathrm{cyc}})$ as the prop generated by the prop $\mathbf{E}(\mathcal{O}^{\mathrm{cyc}})$ along with two elements $m \in \mathbf{E}_m(\mathcal{O}^{\mathrm{cyc}})(\mathbf{2}, \mathbf{0})$ and $c \in \mathbf{E}_m(\mathcal{O}^{\mathrm{cyc}})(\mathbf{0}, \mathbf{2})$, satisfying the following conditions:

- The morphisms m and c are are symmetric and mutually inverse (in the sense of Definition 2.5.9).
- Let $\sigma_c \in S_{n+1}$ be the cyclic permutation of Definition 2.5.8. For each $f \in \mathcal{O}(n)$, the diagram

$$\begin{tikzcd} \mathbf{n} \arrow[r, "f\sigma_c \otimes \mathrm{id}"] \arrow[d, "c\otimes\mathrm{id}"'] & \mathbf{1} \\ \mathbf{n+2} \arrow[r, "\mathrm{id}\otimes f\otimes\mathrm{id}"'] & \mathbf{3} \arrow[u, "\mathrm{id}\otimes m"'] \end{tikzcd}$$

  commutes.

In other words, in $\mathbf{E}_m(\mathcal{O}^{\mathrm{cyc}})$, the action of σ_c corresponds to the cyclic permutation of inputs and outputs by means of the metric and the casimir.

Example 2.5.11 Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{E}_m(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}})$ are exactly the metric Lie algebra objects in the category $\mathcal{C}$. $\lozenge$

This generalises the definition of metric Lie algebra to an arbitrary symmetric monoidal category. Indeed any prop for metric algebras over a cyclic operad in any symmetric monoidal category gives rise to a weight system. This also permits the following slight *further* generalisation of the type of objects that give rise to weight systems, which is necessary for the universal construction of Hinnich and Vaintrob.

Definition 2.5.12 The **casimir envelope of the cyclic operad** $\mathcal{O}^{\mathrm{cyc}}$, denoted $\mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}})$ is the prop generated by the prop $\mathbf{E}(\mathcal{O})$ along with the element $c \in \mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}})(\mathbf{0}, \mathbf{2})$ such that:

- The morphism c is symmetric.
- For each $f \in \mathcal{O}(n)$, the diagram

$$\begin{tikzcd} \mathbf{n-1} \arrow[r, "c\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes c"'] & \mathbf{n+1} \arrow[d, "\mathrm{id}\otimes f"] \\ \mathbf{n+1} \arrow[r, "f\sigma_c\otimes\mathrm{id}"'] & \mathbf{2} \end{tikzcd}$$

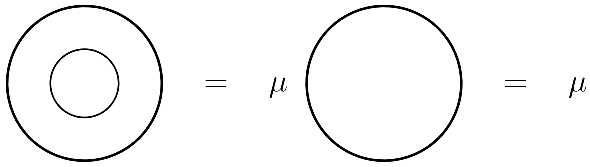
  commutes.

Example 2.5.13 Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}})$ are exactly the casimir Lie algebra objects in the category $\mathcal{C}$. $\lozenge$

This generalises the notion of a Lie algebra object with a casimir element (but not necessarily with a metric). We call such an object a casimr Lie algebra object.

The reason we did not see an analagous definition earlier in this chapter is the following. In the case of finite-dimensional Lie algebras, every casimir Lie algebra is a metric Lie algebra. This is because in finite-dimensional vector spaces, a casimir, being an invariant symmetric map $c : k \to A \otimes A$ is always invertible, and its inverse is a metric. So, any rigid casimir Lie algebra object is a metric one. For example, finite-dimensional casimir Lie algebras, superalgebras, etc.

The reason to consider casimir Lie algebra objects as opposed to just metric Lie algebra objects to make the universal construction is the following. Consider specifying a metric Lie algebra object $\mathfrak{g}$ in some symmetric monoidal category $\mathcal{C}$. Then the weight system $\mathcal{W}_{\mathfrak{g}}$ has a relation of the form



for some scalar μ. Here the first diagram is formed by pairing the casimir c with the metric m. This was discussed in Remark 2.2.6. For example if $\mathcal{C} = \mathbf{Vect}$, this is the dimension of the Lie algebra, or if $\mathcal{C} = \mathbf{sVect}$ it is the superdimension. This relation is just one example of a relation between diagrams with no univalent vertices to scalars.

In $\mathcal{A}$ there are no such relations. In fact, there are no chord diagrams that have only trivalent vertices and no univalent vertices. So, metric Lie algebras are the wrong objects to consider if we want to find a bialgebra isomorphic to $\mathcal{A}$. Instead, algebras having only casimirs should be considered.

Definition 2.5.14 The **universal Casimir Lie algebra object** is the generating object $\mathbf{1}$ in the symmetric monoidal category $\mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}})$. We denote this as $\mathbf{1}_c \in \mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}})$.

This object $\mathbf{1}_c \in \mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}}$ is the object to which Hinich–Vaintrob apply the following construction which generalises the notion of a universal enveloping algebra. As in the construction of the classical universal enveloping algebra, this proceeds via constructing a generalisation of the tensor algebra.

Definitions 2.5.15 The **global section** is the functor

$$\begin{array}{rccc} \Gamma: & \mathcal{C} & \longrightarrow & \mathbf{Vect} \\ & X & \longmapsto & \mathcal{C}(k, X) \end{array}$$

where k is the tensor unit in $\mathcal{C}$.

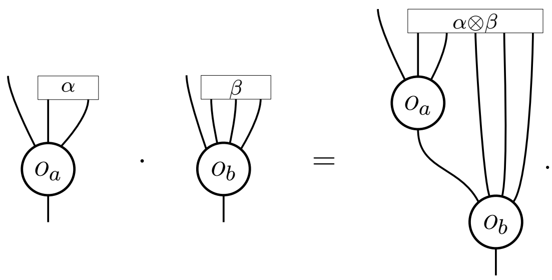
Definitions 2.5.16 The **external tensor algebra of the algebra** A **over the operad** $\mathcal{O}$ is the algebra

$$T(\mathcal{O}, A) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(n+1) \otimes_{S_n} \Gamma(A^{\otimes n}).$$

Let $o_a \otimes_{S_n} \alpha \in \mathcal{O}(n+1) \otimes_{S_n} \Gamma(A^{\otimes n})$, and $o_b \otimes_{S_n} \beta \in \mathcal{O}(m+1) \otimes_{S_n} \Gamma(A^{\otimes m})$. Their product is defined by the following operad composition

$$(o_a \otimes_{S_n} \alpha) \cdot (o_b \otimes_{S_n} \beta) = (o_a \circ_1 o_b) \otimes_{S_n} (\alpha \otimes \beta).$$

Diagramatically this composition looks like



Recall that since A is an $\mathcal{O}$-algebra it comes equipped with maps

$$\mathcal{O}(k) \otimes A^{\otimes k} \longrightarrow A.$$

For example, in $\mathcal{O}_{\text{Lie}}$, setting $o \in \mathcal{O}(k)$, induces the map $A^k \to A$ implementing iterated composition of the bracket $[\cdot,\cdot]_A$ according to the binary tree o. Taking tensor powers of these maps induces the maps

$$\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \otimes A^{\otimes \sum_i m_i} \longrightarrow A^{\otimes n}$$

and applying the global section functor, we have maps

$$\eta_{(m_1,\ldots,m_n)} : \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \otimes \Gamma(A^{\otimes \sum_i m_i}) \longrightarrow \Gamma(A^{\otimes n}).$$

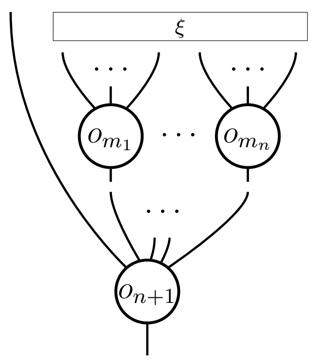
Definitions 2.5.17 The **external enveloping algebra** $U(\mathcal{O},A)$ **of the algebra** A **over the operad** $\mathcal{O}$ is the quotient of $T(\mathcal{O},A)$ by the weakest equivalence relation making the following diagram commute for all $n, m_1, \ldots, m_n \in \mathbb{N}$.

$$\begin{tikzcd}
\mathcal{O}(1+n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}(m_i) \otimes \Gamma(A^{\otimes \sum_i m_i}) \arrow[r, "\mathrm{id}\otimes\eta_{(m_1,\ldots,m_n)}"] \arrow[d, "\circ_2\otimes\cdots\otimes\circ_{n+1}"'] & \mathcal{O}(1+n) \otimes \Gamma(A^{\otimes n}) \arrow[d, hook] \\
\mathcal{O}(1+(\sum_i m_i)) \otimes \Gamma(A^{\otimes \sum_i m_i}) \arrow[r, hook] & T(\mathcal{O},A)
\end{tikzcd}$$

It is easy to get lost in notation here. Let us represent an element

$$o_{n+1} \otimes \bigotimes_{i=1}^{n} o(m_i) \otimes \xi \in \mathcal{O}(1+n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}(m_i) \otimes \Gamma(A^{\otimes \sum_i m_i})$$

pictorially as



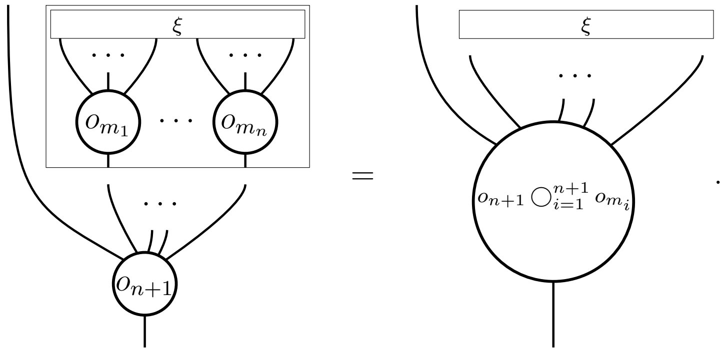
$$\in \quad \mathcal{O}(1+n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}(m_i) \otimes \Gamma(A^{\otimes \sum_i m_i}).$$

There are two ways to produce an element of the external tensor algeba $T(\mathcal{O},A)$ here.

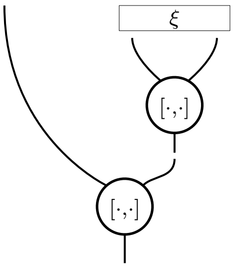
The first is to use the $\mathcal{O}$ algebra structure of A to apply the map

$$\eta_{(m_1,\ldots,m_n)} : \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \otimes \Gamma(A^{\otimes \sum_i m_i}) \longrightarrow \Gamma(A^{\otimes n})$$

to $o_{m_1} \otimes \cdots \otimes o_{m_n} \otimes \xi$. The second is to compose the operations o_{m_i} into the $(i+1)$st components of the operation o_{n+1}. The definition asserts that these are equal in $U(\mathcal{O}, A)$. Pictorially,

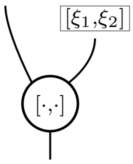


Example 2.5.18 Let $\mathfrak{g}$ be a Lie algebra (in **Vect**). We examine the relation in the external enveloping algebra $U(\mathcal{O}_{\text{Lie}}, \mathfrak{g})$ coming from making the diagram of Definition 2.5.17 commute in the case $n = 1, m_1 = 2$ on the object



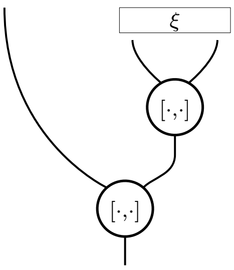
$$\in \quad \mathcal{O}_{\text{Lie}}(2) \otimes \mathcal{O}_{\text{Lie}}(2) \otimes \Gamma(\mathfrak{g}^{\otimes 2})$$

Let us write $\xi = \xi(1) = \xi_1 \otimes \xi_2$. Applying the Lie action map first gives



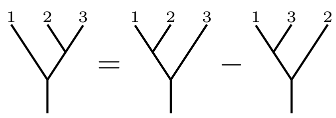
$$\in \quad \mathcal{O}_{\text{Lie}}(2) \otimes \Gamma(\mathfrak{g}).$$

On the other hand, applying the operadic composition first gives

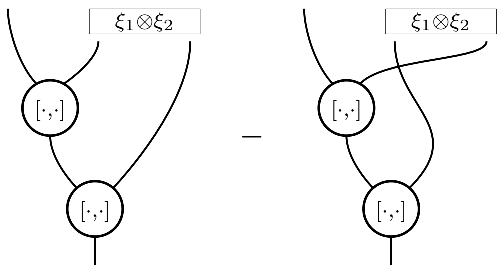


$$\in \quad \mathcal{O}_{\text{Lie}}(3) \otimes \Gamma(\mathfrak{g}^{\otimes 2})$$

and by the relations in $\mathcal{O}_{\text{Lie}}$ from Example 2.5.3,

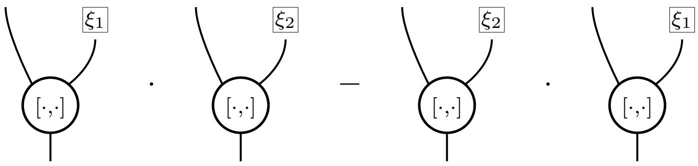


this is equal to



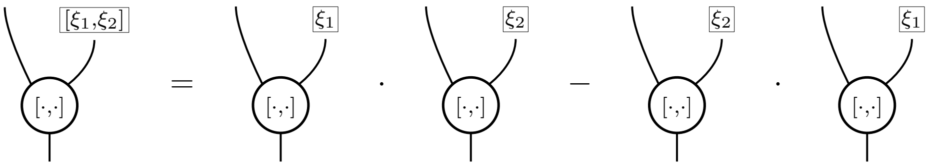
$$\in \quad \mathcal{O}_{\text{Lie}}(3) \otimes \Gamma(\mathfrak{g}^{\otimes 2}).$$

Recalling the definition of the multiplication in Definition 2.5.16, each of these terms is a product in $T(\mathcal{O}_{\mathrm{Lie}}, \mathfrak{g})$, so we have



$$\in \quad \mathcal{O}_{\mathrm{Lie}}(3) \otimes \Gamma(\mathfrak{g}^{\otimes 2}).$$

In total, we have the relation



in $U(\mathcal{O}_{\mathrm{Lie}}, \mathfrak{g})$. Alternatively, written as a formula, we have

$$[\xi_1, \xi_2] = \xi_1 \cdot \xi_2 - \xi_2 - \xi_1$$

which is exactly the defining relation of a classical universal enveloping algebra. $\Diamond$

Theorem 2.5.19 ([HV00]) *The algebra $U(\mathcal{O}_{Lie}, \mathbf{1}_c)$ is isomorphic as an algebra to $\mathcal{A}$.*

We fill out some details left out of the proof in [HV00].

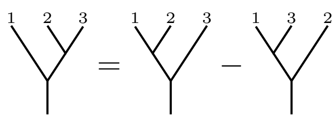
Lemma 2.5.20 *Every element of $\mathcal{O}_{Lie}(n)$, can be written in the form $L\pi$ where $L \in \mathcal{O}_{Lie}(n)$ is the left-skewed binary tree*

$$L = [\cdot,\cdot] \circ_1 \cdots \circ_1 [\cdot,\cdot] \circ_1 [\cdot,\cdot]$$

and $\pi \in kS_n$.

Proof Recall from the definition of an operad, there is a S_n-action on $\mathcal{O}_{\mathrm{Lie}}(n)$. We proceed by induction.

Firstly, the claim is true in $\mathcal{O}_{\mathrm{Lie}}(2)$, as a single bracket is left-skewed. The claim is also true in $\mathcal{O}_{\mathrm{Lie}}(3)$ because of the relations in $\mathcal{O}_{\mathrm{Lie}}$



or where $\sigma = (1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2) \in S_3$

$$[\cdot,\cdot] \circ_2 [\cdot,\cdot] = ([\cdot,\cdot] \circ_1 [\cdot,\cdot])(1 - \sigma).$$

Recall from Example 2.5.3 the definition of the Lie operad as the free operad $\Lambda([\cdot,\cdot])$ on a single generating binary operation $[\cdot,\cdot]$ modulo some relations. Let $O \in \Lambda([\cdot,\cdot])$. From the equivariance axioms of the operad, if there is some composition into the second component in O, then $(1-\sigma) \in kS_3$ determines some $\tau \in kS_n$ such that in $\mathcal{O}_{\mathrm{Lie}}$, $O = O'\tau$ and the composition in O' is into the first component. Iterated application of this fact completes the proof. $\square$

Proof of Theorem 2.5.19 Let $\mathbf{n}_c$ denote $(\mathbf{1}_c)^{\otimes n}$. For the rest of this proof we denote $\mathbf{E}_c(\mathcal{O}^{\mathrm{cyc}}_{\mathrm{Lie}})$ simply by $\mathbf{E}_c$.

The algebra $U(\mathcal{O}_{\mathrm{Lie}}, \mathbf{1}_c)$ is an associative algebra in $\mathbf{Vect}$ whose objects are elements of

$$\begin{aligned} T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{1}_c) &= \bigoplus_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Lie}}(n+1) \otimes_{S_n} \Gamma(\mathbf{n}_c) \\ &= \bigoplus_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Lie}}(n+1) \otimes_{S_n} \mathbf{E}_c(\mathbf{0}, \mathbf{n}) \end{aligned}$$

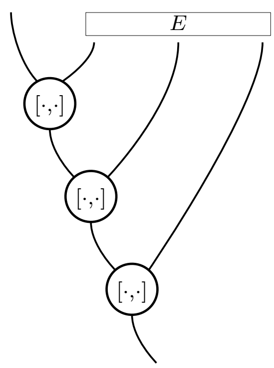
Lemma 2.5.20 induces (vector space) isomorphisms for $n \in \mathbb{N}$

$$\mathcal{O}_{\mathrm{Lie}}(n+1) \otimes_{S_n} \mathbf{E}_c(\mathbf{0}, \mathbf{n}) \cong \mathbf{E}_c(\mathbf{0}, \mathbf{n})$$

and therefore an vector space isomorphism

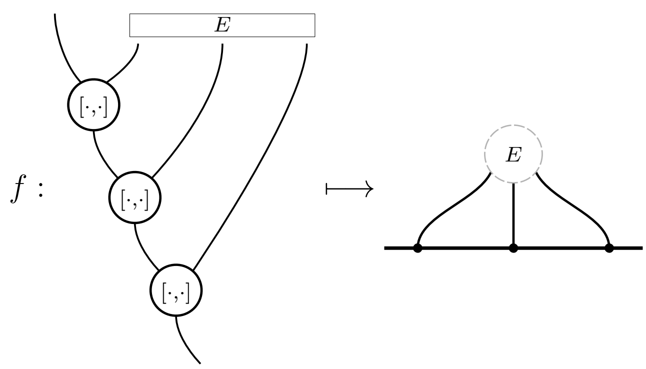
$$T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{1}_c) \cong \bigoplus_{n\in\mathbb{N}} \mathbf{E}_c(\mathbf{0}, \mathbf{n}).$$

Explicitly, $L\pi \otimes_{S_n} E_0 \cong L \otimes_{S_n} E_0\pi \mapsto E_0\pi$, and write $E = E_0\pi$. In words, a homogeneous element of $T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{1}_c)$ in degree n can be written as composition of left-skewed brackets tensored with a unique morphism E of $\mathbf{E}_c(\mathbf{0}, \mathbf{n})$.



A morphism $E \in \mathbf{E}_c(\mathbf{0}, \mathbf{n})$ is associated to a diagram in $\mathcal{A}$; choose a presentation for E in the symmetric monoidal category $\mathbf{E}_c(\mathbf{0}, \mathbf{n})$, this determines a unitrivalent graph with a cyclic order on the internal vertices, and a linear order on the univalent vertices induced by the order $\mathbf{1} < \mathbf{2} < \cdots < \mathbf{n}$. By Proposition 2.1.7, this map factors through the natural projection from linear orders on univalent vertices to cyclic orders. This therefore defines a map $f : T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{1}_c) \to \mathcal{A}$.

Informally, f is the map which takes an element of the tensor algebra, and interprets it as a linear Jacobi diagram where the skeleton is the path from the leftmost leaf to the root.



We need to show that f factors through the projection to $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$. All of the relations of Definition 2.5.17 follow from the relation in Example 2.5.18, so it suffices to prove that the map factors through that relation. This is just the STU in $\mathcal{A}$.

The map f is surjective because every Jacobi diagram can be presented by a graph whose edges do not have local minima, informally, the casimir and the bracket suffice to draw any unitrivalent graph. It is injective because STU is a generating relation for $\mathcal{A}$ (Proposition 2.1.5) and its preimages in $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ are related by the the relation in Example 2.5.18. This proves the algebra isomorphism. □

We turn to the coalgebra structure of $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ and the bialgebra isomorphism to $\mathcal{A}$. Recall from Proposition 2.1.8 that the coproduct structure on $\mathcal{A}$ is given by the formula

$$\Delta(A) = \sum_{C \subset S} A_C \otimes A_{\overline{C}}$$

where S is the set of connected components of A (not considering the skeleton), $\overline{C} = S \setminus C$, and A_C is the Jacobi diagram consisting of only the subset C of connected components.

Since in the map $f : U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c) \to \mathcal{A}$, the path from the leftmost leaf to the root becomes the skeleton, the correct notion of primitive in $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ is connected components, excluding the path from the leftmost leaf to the root.

Definition 2.5.21 The coproduct Δ on $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ is defined by pulling the coproduct on $\mathcal{A}$ back through the algebra isomorphism of Theorem 2.5.19.

Remark 2.5.22 In [HV00] a operad-theoretic characterisation of the primitive elements is given in terms of modular operads. Recalling from Corollary 2.1.9 that by the formula for the coproduct, the primitive elements are exactly the connected components, this also gives a definition of the coproduct without reference to connected components.

With this definition, we obtain as a corollary an upgrade of the isomorphism of Theorem 2.5.19 to the level of bialgebras.

Theorem 2.5.23 ([HV00]) *The bialgebra $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ is isomorphic as a bialgebra to $\mathcal{A}$.*

Proof Recall that in the map $f : U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c) \to \mathcal{A}$ of Theorem 2.5.19, the path from the leftmost leaf to the root of an element of $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ becomes the skeleton of a chord diagram in $\mathcal{A}$. Therefore connected components in $\mathcal{A}$ not considering the skeleton map to connected components in $U(\mathcal{O}_{\text{Lie}}, \mathbf{1}_c)$ not considering the path from the leftmost leaf to the root. As such, the formula for the coproduct in both spaces is the same. □

Theorem 2.5.23 is a formalisation of the statement that "$\mathcal{A}$ behaves like the universal enveloping algebra of a Lie algebra". In the next chapter we look at a generalisation of this statement in the context of welded knots.

3

Welded knots, $I\mathfrak{g}$ and arrow diagrams

TAKEN generally, knot theory is the study of codimension two embedded submanifolds. The theory of welded knots, or w-knots is a diagrammatic generalisation of knot theory that relates to knotted tori in $\mathbb{R}^4$, and it has its own flavour of Vassiliev invariants. Generalisations of the Vassiliev invariants to objects other than knotted circles in $\mathbb{R}^3$ take the name **finite type invariants**.

On one hand, welded knots are more complicated topological objects, relating to ribbon embeddings of tori in four dimensions. On the other hand however their finite type invariant theory is much more manageable than the classical case, so studying the simpler, welded version of $\mathcal{A}$ may shed light on how to study the algebra $\mathcal{A}$, as well as being interesting in its own right.

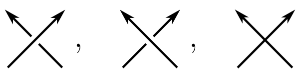
3.1 Welded knots

A good and thorough exposition of the theory of welded knots is [BD16; BD17]. Their presentation is in terms of virtual knots and circuit algebras.

The definition of classical knots is first and foremost topological: they are ambient isotopy classes of embeddings of oriented circles into $\mathbb{R}^3$. Due to the Reidemeister theorem, we may instead work with equivalence classes of planar knot diagrams under the Reidemeister moves.

With welded knots, the definition is the other way around. We will define them diagrammatically as equivalence classes of planar knot diagrams under an extended set of Reidemeister moves.

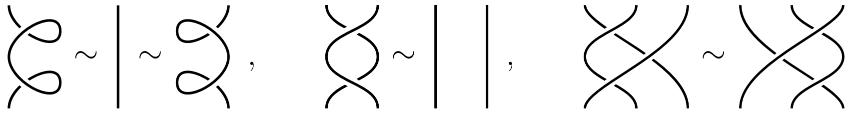
Definition 3.1.1 A **welded knot** is an immersed curve in $\mathbb{R}^3$, where each four-valent planar vertex is decorated as one of the following types, respecting the orientation of the planar curve.



The third crossing type is called a **virtual crossing**.

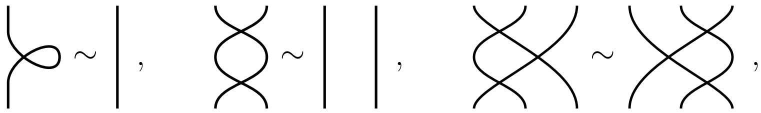
The equivalence relations are planar isotopy and:

(a) The **(framed) Reidemeister moves**

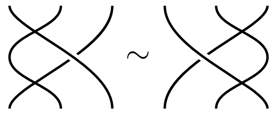


(R1$^{\mathtt{f}}$, R2, R3)

(b) The **virtual Reidemeister moves**

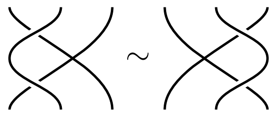


(VR1, VR2, VR3)



(VR4)

(c) The **overcrossings commute** relation



(OC)

Remarks 3.1.2 (a) There should technically be two variants of R1$^{\mathtt{f}}$, another in which all crossings are changed, and the same is true for VR4. This is even before orientation is taken into account: all of the moves are true for each choice of orientation of each strand.

(b) The moves VR1, VR2 and VR3 govern how virtual crossings interact with each other. The move VR4 governs how virtual crossings interact with classical crossings. It implies that strands consisting of only virtual crossings may be freely rerouted around classical crossings. The interpretation of this fact can be found in [Kau99].

(c) In the OC relation, the order of the two overcrossings along the middle strand changes, hence the name. The version of OC in which the crossings are changed is called **undercrossings commute** and is *not* an imposed relation: undercrossings do *not* commute for welded knots. Indeed, allowing undercrossings to commute would trivialise welded knots. For an explanation of why this lack of symmetry is to be expected in terms of the tube map, the curious reader is referred to [BD16, Sec. 2.2].

There is also a topological interpretation of welded knots, but it is not known exactly how well the topological object represents the diagrammatic one: welded knots are ambient isotopy classes of ribbon-embedded tori in $\mathbb{R}^4$, up to a global orientation reversal move, and potentially some further moves [Sat00]. The map from welded knots to ribbon embedded tori is known as the tube map. We do not discuss the details here, instead referring the reader to [Aud16]. For the purposes of this thesis, we define finite type invariants of welded knots from their diagrammatic definition, remaining confident that welded knots topologically correspond to something close to ribbon-knotted tori modulo a global orientation reversal move in $\mathbb{R}^4$.

In fact, we can even avoid the trouble of the global orientation reversal move by passing to welded long knots. Since we have not yet defined classical long knots, we start with that. The word “long” indicates that the role of the circle in the definition is replaced by a line.

Definition 3.1.3 A (classical) **long knot** an embedding of the line $k : \mathbb{R} \hookrightarrow \mathbb{R}^3$, with the condition that $k(t) = (t, 0, 0)$ for all large $|t|$.

For the finite type theory, the algebra $\mathcal{A}$ of Jacobi diagrams (and chord diagrams), is replaced with an algebra defined similarly, but with the the circular skeleton replaced by a line. The relations are the same (either `STU` or `4T`), but instead of the univalent vertices having a cyclic order they have a linear order. The notation $\mathcal{A}(|)$ is used for this algebra, in which the skeleton is drawn as an argument.

There was no need to define long knots earlier because in the classical case, long knots and knots are in bijection with each other and the theory (including the finite type theory) is the same for both long and regular knots. Recall Proposition 2.1.7 as evidence of this fact. In the welded case however the long theory has the advantage of eliminating the global orientation reversal move.

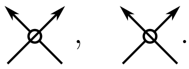
Definition 3.1.4 A **welded long knot** is a welded knot, but instead of a planar curve it is a planar immersion $\ell : \mathbb{R} \to \mathbb{R}^2$ such that $\ell(t) = (t, 0)$ for all large $|t|$.

For why this eliminates the global orientation reversal we refer the reader again to [Sat00] and also [BD17, Corrigendum].

3.2 Arrow diagrams

The finite type theory of welded long knots is defined similarly to that of classical long knots.

Definition 3.2.1 A **singular welded long knot** is a welded long knot whose crossings can also be of the following types

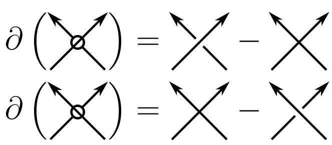


They are known as **positive semi-virtual crossings** and **negative semi-virtual crossings**, and they correspond to topological singularities of ribbon-knotted tori.

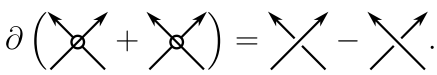
The set of m-singular welded long knots forms an algebra with the product given by connected sum given by concatenation and coproduct $\Delta(k) = k \otimes k$. This is called the **algebra of m-singular welded long knots**, $\mathcal{K}^{\bullet}_{w,m}(|)$.

The algebra of 0-singular welded long knots, or just the algebra of welded long knots $\mathcal{K}^{\bullet}_{w,0}(|) = \mathcal{K}_w(|)$ is filtered by the images of succesive powers of ∂, and the filtration is compatible with the bialgebra structure.

Just like in the classical case, the notions of invariants of m-singular welded long knots, their differentiability, and the derivative and boundary operators transfer over the the welded case. However now the derivative and boundary depend on the type of singularity.



Again, the derivative δ is defined on m-singular invariants of welded long knots as the adjoint to ∂. As a consequence of these definitons, we can express the positive-to-negative crossing change via a sum of semivirtuals

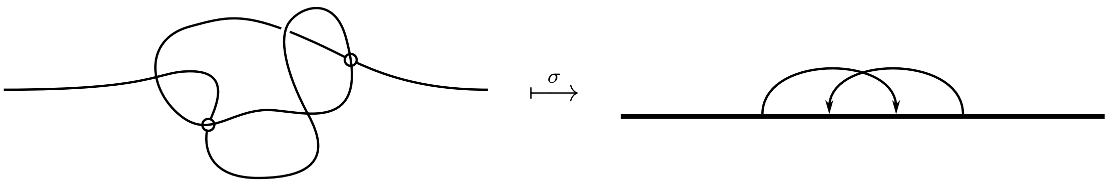


Long arrow diagrams are defined from singular welded knots just as chord diagrams are defined from singular knots.

Definitions 3.2.2 (a) A **long arrow diagram** of order m is an oriented line (the **skeleton**) with a distinguished set of m ordered pairs of points, considered up to orientation-preserving diffeomorphism of the line. The set of all long arrow diagrams forms an algebra $\mathcal{D}_w(|)$. The product is induced by the natural connected sum operation on long knots, and the coproduct is given by formula as in the classical case (Definition 1.4.6), but with single arrows considered primitive. This is called the **bialgebra of long arrow diagrams**, and it is also graded by degree, where single arrows are of degree one.

(b) The **long arrow diagram of an m-singular welded long knot**, denoted $\sigma(k)$ is the arrow diagram formed as follows. Traverse the welded long knot, and whenever a singularity is encountered, place a point on the skeleton. If the hollow part of the singularity is traversed, place a tail, and if the filled part is traversed place a head. Points on the skeleton corresponding to the same singularity are paired, and the order of the pair is tail first, then head.

Example 3.2.3



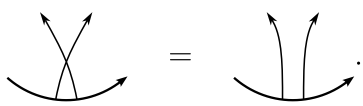
$\Diamond$

Just as in the classical case, there are natural relations to consider on arrow diagrams induced by welded equivalence. Namely, there are directed versions of the `4T`, and another relation called the `TC` relation.

A directed `4T` equation holds for each choice of orientation of the two chords involved in the classical `4T` relation. We skip the definitions of the directed `4T`; like for classical knots, we prefer to work with unitrivalent diagrams and directed versions of `STU`, `AS` and `IHX` which we will soon define, and from which (just like the classical case) the directed `4T` relation can easily be deduced.

As for the `TC` relation, it is a consequence of the `OC` relation, and the germ of the vast differences between the welded and classical finite type theories.

Definition 3.2.4 The **TC relation** (tails commute relation)



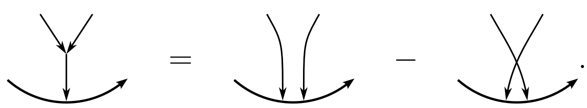
(`TC`)

Definition 3.2.5 A **long arrow Jacobi diagram** is an oriented unitrivalent graph where every trivalent vertex has at least one vertex oriented inward, and at least one oriented outwards, with the following additional data:

- at each trivalent vertex, a cyclic order of the incident edges,
- a fixed linear order on the univalent vertices,

modulo:

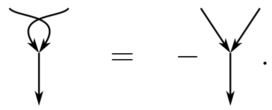
(a) Any **directed STU** relations. For example,



(`STU`)

Relations hold for both choices of vertex type (two-heads-one-tail, one-head-two-tails) and all three choices of orientation of the trivalent vertex. Note that even though the skeleton is the line rather than the circle, we still draw it curved to distinguish it.

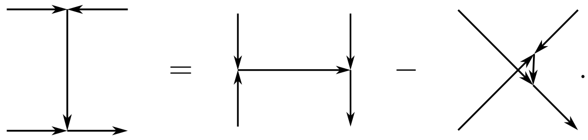
(b) Any **directed AS** relation. For example,



(AS)

Again, for any choice of vertex type and rotation.

(c) Any **directed IHX** relations. For example,



(IHX)

These hold for any compatible choice of both vertices in the diagrams.

(d) The TC relations, as given in Definition 3.2.4.

The algebra of long arrow Jacobi diagrams is denoted $\mathcal{J}_w(|)$, and as in the classical case (since directed STU, TC imply directed 4T) we also write it as $\mathcal{A}_w(|)$.

The main difference between $\mathcal{J}_w(|) = \mathcal{A}_w(|)$ and $\mathcal{J}(|) = \mathcal{A}(|)$ is the tails commute relation which drastically reduces the complexity of the welded verson of the algebra, as evident in the following important proposition about the structure of $\mathcal{A}_w(|)$.

Proposition 3.2.6 (Two-in-one-out rule) *In $\mathcal{A}_w(|)$, any diagram containing a trivalent vertex with one edge oriented inwards and two edges oriented outward vanishes.*

Proof Take a diagram with a one-in-two-out vertex. Apply STU's to all other vertices of the arrow diagram. The result is a sum of arrow diagrams, each containing a trivalent one-in-two-out vertex. We show that each such arrow diagram vanishes. Since there is only one trivalent vertex in each diagram, the one-in-two-out vertex's incoming edge has its tail on the skeleton. Therefore applying the relevant STU relation equates the diagram with a difference of two diagrams that differ only by the order of placement of two tails on the skeleton. By TC, the difference is zero. □

Resultingly, only two versions of STU, one of AS and one of IHX remain nontrivial relations in $\mathcal{A}_w(|)$.

Warning 3.2.7 From this point it is clear when talking about welded objects we mean long welded objects. So, we write $\mathcal{A}_w(|)$ as $\mathcal{A}_w$, and don't necessarily specify that they are long.

The algebras $\mathcal{K}_w$ and $\mathcal{A}_w$ are also coalgebras, with coproducts that mirror the classical case.

Definitions 3.2.8 (a) The coproduct Δ on $k \in \mathcal{K}_w$ is given by the formula

$$\Delta(k) = k \otimes k.$$

(b) The coproduct Δ on $A \in \mathcal{A}_w$ is given by the formula

$$\Delta(A) = \sum_{C \subset S} A_C \otimes A_{\overline{C}}$$

where S is the set of connected components (not considering the skeleton) of A, and $\overline{C} = S \setminus C$.

We omit the proofs that these turn the $\mathcal{A}_w$ and $\mathcal{K}_w$ into bialgebras because they mirror the proofs in the classical case.

The existence of a finite type invariant Z for welded knots establishes a welded version of the fundamental theorem.

Theorem 3.2.9 ([BD16]) *There exists a universal Vassiliev invariant $Z_w : \mathcal{K}_w \to \mathcal{A}_w$.*

Unlike the involved construction for the Kontsevich integral Z, the construction of Z_w is relatively simple. The `TC` relation does all of the heavy lifting. A proof can be found in [BD16].

Corollary 3.2.10 (Fundamental theorem of Vassiliev invariants of welded knots) *We have the graded bialgebra isomorphism*

$$\mathcal{A}_w \cong \operatorname{gr} \mathcal{K}_w.$$

Equivalently,

$$\mathcal{V}_w \cong \operatorname{gr} \mathcal{W}_w.$$

3.3 Welded weight systems and Lie algebras

There is a variant of Construction 2.2.1 for welded knots, taking a Lie algebra and constructing a weight system. The difference is that vertices in string diagrams now have incoming edges ("heads") and outgoing edges ("tails"), and diagramatically, edges can only be contracted in a way that agrees with the half-edges' orientations. Recall that this restriction was not imposed in the classical case, and accomodating this was the reason to restrict to metric Lie algebras. Hence, in the welded case we can consider any Lie algebra, without the metric restriction.

Construction 3.3.1 Given a Lie algebra $\mathfrak{g}$, we obtain a map $W_\mathfrak{g} : \mathcal{A}_w \to \mathcal{U}(I\mathfrak{g})$.

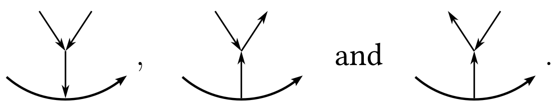
Apply Construction 2.2.1, but when associating to each trivalent vertex a copy of the tensor $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, also ensure that the inward oriented half-edges are associated to g^* components, and the outward oriented half-edges to the $\mathfrak{g}$ component. In addition to the preservation of cyclic orders (recall Warning 2.2.2), this determines a bijection between the half-edges of each trivalent vertex and the components of the associated tensor.

For each edge between trivalent vertices, contract the corresponding tensors along the components corresponding to those half-edges. The signature of the components will always allow for the contraction from the fact that connected half-edges are always oriented coherently.

The result is a tensor product in copies of $\mathfrak{g}$ and $\mathfrak{g}^*$, which can be interpreted as an element of $(I\mathfrak{g})^{\otimes u}$ where u is the number of univalent vertices of the diagram in $\mathcal{A}_w$.

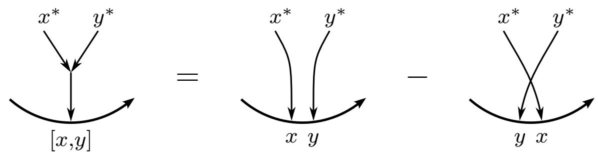
Note an imporant difference to the classical case. There are now two types of univalent vertices that connect to the external line: heads and tails. Heads correspond to elements of $\mathfrak{g}$ and tails to elements of $\mathfrak{g}^*$. Therefore, the role of $\mathcal{U}(\mathfrak{g})$ in the classical case must be played here by some other associative algebra whose basis elements are words in $\mathfrak{g} \sqcup \mathfrak{g}^*$.

Analysing the relations involving the external line should tell us what kind of object replaces $\mathcal{U}(\mathfrak{g})$. From the two-in-one-out rule, there are three non-trivial cases to examine:



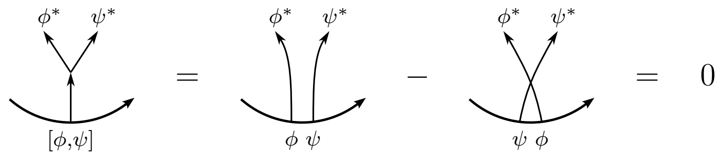
In all three cases, they obey commutator-like relations (the versions of STU), so the correct associative algebra target for the weight system is still the universal enveloping algebra of some Lie algebra. However since the connections to the external line can now be either heads or tails, representing elements of $\mathfrak{g}$ and $\mathfrak{g}^*$, the Lie algebra as a vector space is $\mathfrak{g}^* \oplus \mathfrak{g}$.

Interpreting tails and heads in these STU relations as elements of $\mathfrak{g}^* \oplus 0$ or $0 \oplus \mathfrak{g}$ rather than $\mathfrak{g}^*$ or $\mathfrak{g}$ determines the bracket on $\mathfrak{g}^* \oplus \mathfrak{g}$. For example, the STU with two heads on the external line implies that the bracket on $0 \oplus \mathfrak{g}$ should be inherited from the bracket on $\mathfrak{g}$.



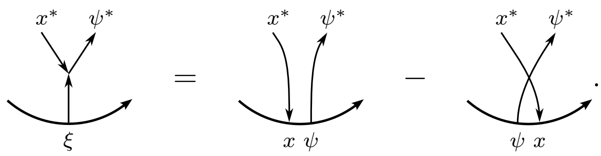
So, $[0 \oplus x, 0 \oplus y] = 0 \oplus [x, y]_{\mathfrak{g}}$.

To determine the bracket on $\mathfrak{g}^* \oplus 0$, we need to examine one of the trivial cases not shown above. Indeed, the tails commute relation neccecitates that the bracket on $\mathfrak{g}^* \oplus 0$ be the zero-bracket.



This gives $[\phi \oplus 0, \psi \oplus 0] = 0 \oplus 0$.

The bracket structure is known on the $0 \oplus \mathfrak{g}$ and $\mathfrak{g}^* \oplus 0$ direct summands, so the total Lie algebra structure will be a semidirect product $\mathfrak{g}^* \rtimes \mathfrak{g}$, and what remains is to specify an action of $\mathfrak{g}$ on $\mathfrak{g}^*$. With a bit of work, the remaining 'mixed' STU relations determine this. One such relation is



Let us determine the functional ξ in terms of x and ψ. There is only one type of nonzero trivalent vertex in $\mathcal{A}_w$, so this is the same tensor as $0 \oplus \mathfrak{g}$ case, just viewed from a different component. That tensor is $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ so ξ satisfies the equation $[\cdot,\cdot] = x^* \otimes \xi \otimes \phi^*$ whereby

$$[x, \xi^*] = \phi^* \quad \text{so} \quad \phi([x, \xi^*]) = 1.$$

Therefore, $\xi \in \mathfrak{g}^*$ is the functional

$$\xi : t \longmapsto \phi([x, t]).$$

This is also known as the **coadjoint action**, $\xi = \mathrm{ad}_x^*(\phi)$ because it is the functional adjoint to $\mathrm{ad}_x(\phi)$. Writing this in terms of the bracket on $\mathfrak{g}^* \otimes \mathfrak{g}$,

$$[0 \oplus x, \psi \oplus 0] = \mathrm{ad}_x^*(\psi) \oplus 0.$$

Similarly, the AS relation gives $[\phi \oplus 0, 0 \oplus y] = -\mathrm{ad}_y^*(\phi) \oplus 0$. Hence, the total Lie algebra structure on $\mathfrak{g}^* \oplus \mathfrak{g}$ is given by the Lie algebra $I\mathfrak{g}$ defined as follows.

Definition 3.3.2 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{g}^*$ denote the vector space $\mathfrak{g}^*$ with the structure of an abelian Lie algebra. Then let $I\mathfrak{g}$ denote the Lie algebra $\mathfrak{g}^* \rtimes \mathfrak{g}$ with the bracket

$$[\phi \oplus x, \psi \oplus y] = \mathrm{ad}_x^*(\psi) - \mathrm{ad}_y^*(\phi) \oplus [x, y].$$

Remark 3.3.3 This construction coincides with the Drinfeld double [Dri88] of the Lie bialgebra $\mathfrak{g}$, in the case where $\mathfrak{g}$ is co-commutative.

Proposition 3.3.4 ([BD16]) *Construction 3.3.1 gives a well-defined algebra homomorphism* $\mathcal{A}_w \to \mathcal{U}(I\mathfrak{g})$.

Proof The relations in $\mathcal{A}_w$ are generated by STU and TC. We have shown that these relations are also true in $\mathcal{U}(I\mathfrak{g})$. □

3.4 The universal welded weight system

The aim of this section is to prove a Hinnich-Vaintrob style statement for $\mathcal{A}_w$. That is, to construct a Lie algebra object in a category, whose universal enveloping algebra is isomorphic to $\mathcal{A}_w$.

Recall that in the classical case, the Lie algebra object was in the category $\mathbf{E}_c(\mathcal{O}_{\text{Lie}})$, since its morphisms and composition can be used to implement contraction (the connection of trivalent vertices) in $\mathcal{A}$ which resembles the structure of $\mathfrak{g}$. In $\mathcal{A}_w$, the contraction is similar, and as seen from Section 3.3 but instead resembles the structure of $I(\mathfrak{g})$; trivalent vertices can only be contracted along half-edges whose orientations agree. Recall from Construction 2.2.1 that it was the addition of the Casimir element which casued the ambiguation of elements $\mathfrak{g}$ with $\mathfrak{g}^*$, allowing directionless contraction, and indeed $I(\mathfrak{g})$ is not necessarily a Casimir Lie algebra. Thus, the corresponding category in the welded case should be able to distinguish elements of $\mathfrak{g}$ and $\mathfrak{g}^*$. This can be achieved with a type of category called a 2-coloured prop.

Definitions 3.4.1 An n-**coloured prop** is a symmetric monoidal category generated by n objects.

The definition of an algebra over a prop (Definition 2.5.5) extends naturally to coloured props as well.

Since the morphisms in the welded case should reflect the structure of $I(\mathfrak{g})$, and the bracket on $I(\mathfrak{g})$ decomposes depending on whether $x \in I(\mathfrak{g})$ is in $\mathfrak{g}$ or $\mathfrak{g}^*$, the category ought to have two objects, $\mathbf{1}$ and $\mathbf{1}^\vee$ and be linear. Further, instead of the contraction being implemented through composition and the metric and Casimir elements as in $\mathbf{E}_c$, the objects $\mathbf{1}$ and $\mathbf{1}^\vee$ should behave like duals, so recalling Definition 2.3.5 (d), should be rigid with evaluation and coevaluation morphisms $\iota : \mathbf{1}^\vee \otimes \mathbf{1} \to k$ and $\varepsilon : k \to \mathbf{1} \otimes \mathbf{1}^\vee$. The contraction in the corresponding category for the welded case should be implemented via composition and these evaluation and coevaluation morphisms.

In view of these comments, the following category $\mathbf{T}_\iota$ is analogous to the metric envelope of the Lie operad $\mathbf{E}_m$, but includes as objects both tensor powers of $\mathfrak{g}$ but also tensor powers of $\mathfrak{g}^*$ and its morphisms describe the structure of $I\mathfrak{g}$ rather than $\mathfrak{g}$.

Definition 3.4.2 We define a symmetric monoidal category $\mathbf{T}_\iota$. It is a k-linear category and it is generated additively and monoidally by the objects $\mathbf{1}$ and $\mathbf{1}^\vee$. Denote the objects $\mathbf{1}^{\otimes n}$ and $(\mathbf{1}^\vee)^{\otimes n}$ as $\mathbf{n}$ and $\mathbf{n}^\vee$ respectively, as well as $(\mathbf{1}^\vee)^{\otimes n} \otimes \mathbf{1}^{\otimes k}$ as $\mathbf{n}^\vee + \mathbf{k}$.

The generating morphisms of $\mathbf{T}_\iota$ are given by

$$\mathbf{T}_\iota(\mathbf{k}, \mathbf{n}) = \bigoplus_{f:[k]\to[n]} \bigotimes_{i=1}^{n} \mathcal{O}_{\text{Lie}}(|f^{-1}(i)|),$$

as well as evaluation and coevaluation morphisms $\iota : \mathbf{1}^\vee \otimes \mathbf{1} \to k$ and $\varepsilon : k \to \mathbf{1} \otimes \mathbf{1}^\vee$. These morphisms satisfy the relations induced by $\mathcal{O}_{\text{Lie}}$ and the relations

$$(\mathrm{id}_{\mathbf{1}} \otimes \varepsilon) \circ (\iota \otimes \mathrm{id}_{\mathbf{1}}) = \mathrm{id}_{\mathbf{1}}$$

and

$$(\varepsilon \otimes \mathrm{id}_{\mathbf{1}^\vee}) \circ (\mathrm{id}_{\mathbf{1}^\vee} \otimes \iota) = \mathrm{id}_{\mathbf{1}^\vee}$$

(that is, $\mathbf{1}$ and $\mathbf{1}^\vee$ are rigid with coevaluation and evaluation maps ι and ε respectively).

In terms of string diagrams, the string diagrams that describe $\mathbf{T}_\iota$ are oriented, so they have heads and tails.

Again however, there is the problem that $\mathbf{T}_\iota$ also describes arrow diagrams with no univalent vertices. Recall that the motivation behind Definition 2.5.12 was to exclude the metric morphism as not to describe chord diagrams with no univalent vertices. The motivation for the following definition is similar.

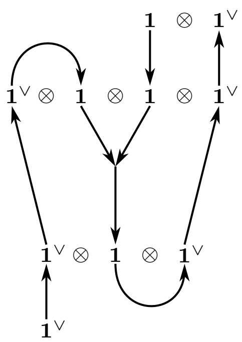
Definition 3.4.3 The symmetric monoidal category $\mathbf{T}_\varepsilon$ is the symmetric monoidal subcategory of $\mathbf{T}_\iota$ generated by the morphisms β and ε (but not ι) as well as the morphisms

$$a_\ell = (\mathrm{id}_{\mathbf{1}^\vee} \otimes \iota) \circ (\mathrm{id}_{\mathbf{1}^\vee} \otimes \beta \otimes \mathrm{id}_{\mathbf{1}^\vee}) \circ (\varepsilon \otimes \mathrm{id}_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}^\vee})$$

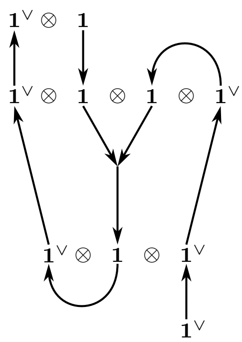
and

$$a_r = ((\iota \circ \tau) \otimes \mathrm{id}_{\mathbf{1}^\vee}) \circ (\mathrm{id}_{\mathbf{1}^\vee} \otimes \beta \otimes \mathrm{id}_{\mathbf{1}^\vee}) \circ (\mathrm{id}_{\mathbf{1}^\vee} \otimes \mathrm{id}_{\mathbf{1}} \otimes (\varepsilon \circ \tau))$$

which in terms of string diagrams are



and



respectively. (These morphisms a_ℓ and a_r give the "bracket" between $\mathbf{1}$ and $\mathbf{1}^\vee$ in the sense of $I\mathfrak{g}$.)

Denote $\mathbf{I} = \mathbf{1}^\vee \oplus \mathbf{1} \in \mathbf{T}_\varepsilon$. We will define a morphism $y : \mathbf{I} \otimes \mathbf{I} \to \mathbf{I}$ that makes $\mathbf{I}$ into a Lie algebra object in $\mathbf{T}_\varepsilon$. Note that by the k-linearity of $\mathbf{T}_\varepsilon$, a map of sigature $\mathbf{I} \otimes \mathbf{I} \to \mathbf{I}$ is determined by eight maps of similar signature between objects $\mathbf{1}^\vee$ or $\mathbf{1}$. Indeed expanding $\mathbf{I} = \mathbf{1}^\vee \oplus \mathbf{1}$, we get by the distributivity of $\otimes$ with $\oplus$ that

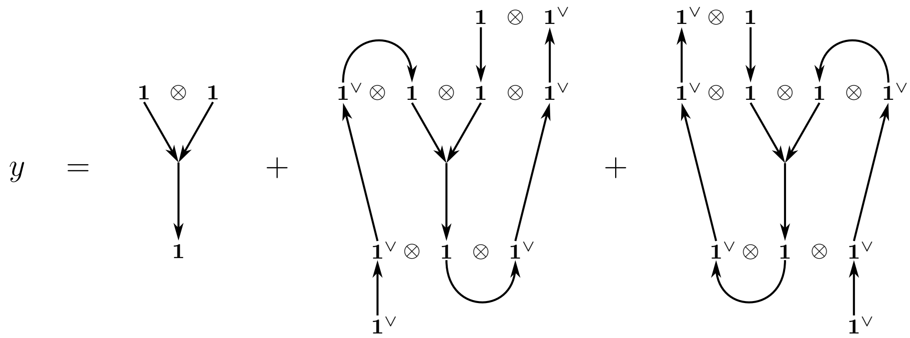
$$y : (\mathbf{1}^\vee \otimes \mathbf{1}^\vee) \oplus (\mathbf{1}^\vee \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}^\vee) \oplus (\mathbf{1} \otimes \mathbf{1}) \longrightarrow \mathbf{1}^\vee \oplus \mathbf{1}.$$

Theorem 3.4.4 *Let $y : \mathbf{I} \otimes \mathbf{I} \to \mathbf{I}$ be the morphism*

$$y = \beta \oplus a_\ell \oplus a_r \oplus 0.$$

Then $\mathbf{I}$ is a Lie algebra object of $\mathbf{T}_\varepsilon$ with morphism y.

Proof We have



(where here the trivalent vertex is β). There are two statements to prove:

(a) The generalised antisymmetry of $(\mathbf{I}, y)$, namely

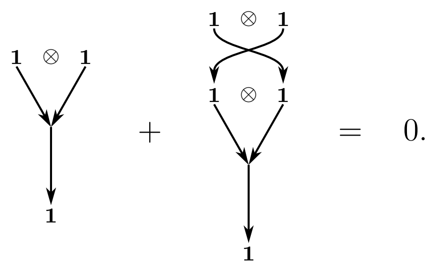
$$y + y \circ \tau = 0$$

(b) The generalised Jacobi identity of $(\mathbf{I}, y)$, namely

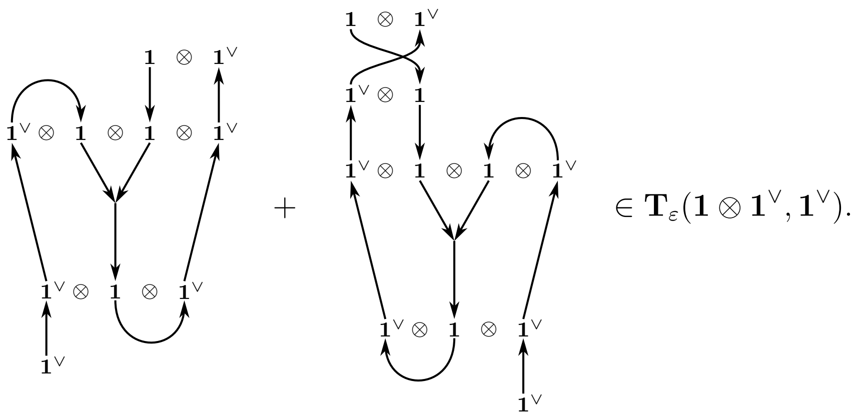
$$(y \circ (y \otimes \mathrm{id}_{\mathbf{I}})) \circ (1 + \tau_{123} + \tau_{132}) = 0.$$

For the proof of (a), observe that $y + y \circ \tau$ contains six terms: $\beta + \beta \circ \tau \in \mathbf{T}_\varepsilon(\mathbf{1} \otimes \mathbf{1}, \mathbf{1})$, $a_\ell + a_r \circ \tau \in \mathbf{T}_\varepsilon(\mathbf{1} \otimes \mathbf{1}^\vee, \mathbf{1}^\vee)$ and $a_r + a_\ell \circ \tau \in \mathbf{T}_\varepsilon(\mathbf{1}^\vee \otimes \mathbf{1}, \mathbf{1})$. It suffices to check that each of these components are zero.

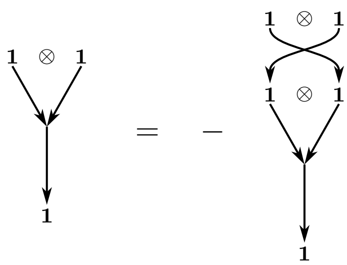
In the component $\mathbf{T}_\varepsilon(\mathbf{1} \otimes \mathbf{1}, \mathbf{1})$, we have $\beta + \beta \circ \tau = 0$. This relation is simply the antisymmetry of the bracket β (with which $\mathbf{1}$ is a Lie algebra object in $\mathbf{T}_\varepsilon$) — in terms of string diagrams this is



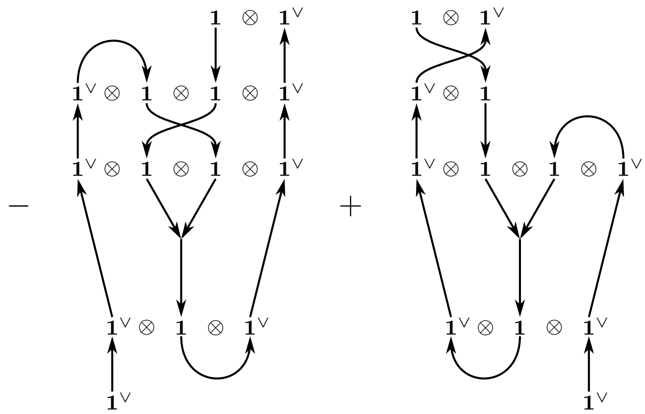
We also have $a_\ell + a_r \circ \tau \in \mathbf{T}_\varepsilon(\mathbf{1} \otimes \mathbf{1}^\vee, \mathbf{1}^\vee)$. Diagramatically this is



Substituting

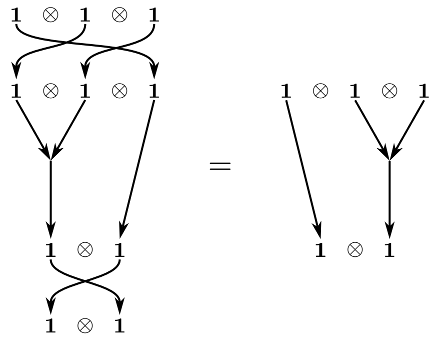


into the first term, we get



$$\in \mathbf{T}_\varepsilon(\mathbf{1}\otimes\mathbf{1}^\vee, \mathbf{1}^\vee).$$

Indeed, the two diagrams are equal and therefore their difference is zero. This follows from the relation



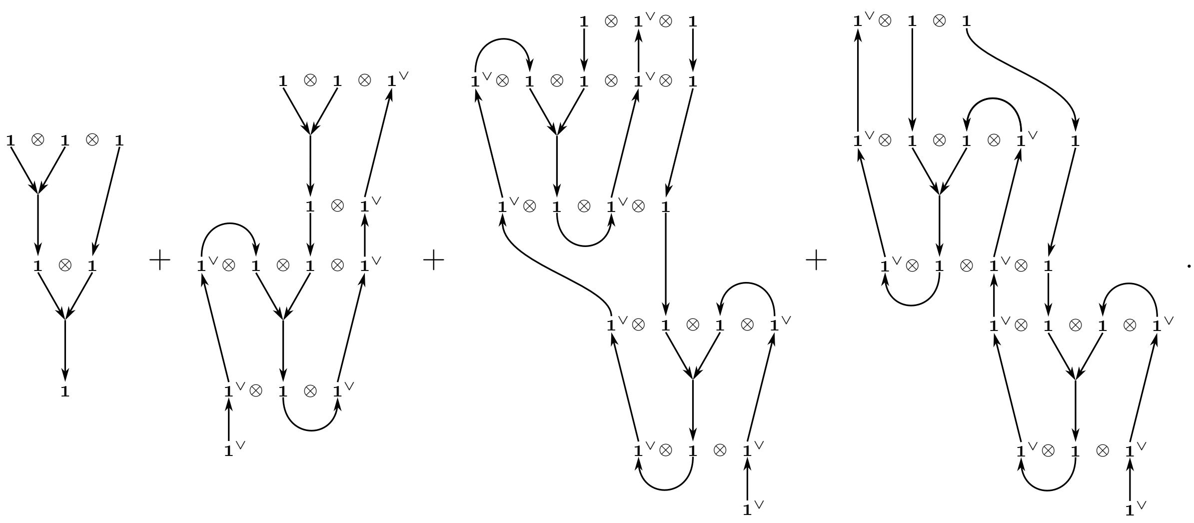
which is a consequence of the fact that morphisms in $\mathbf{T}_\varepsilon$ are direct sums of elements of $\mathcal{O}_{\mathrm{Lie}}$ (Definition 3.4.2), and the equivariance axioms of the Lie operad in Definition 2.5.2.

The same argument proves that $a_r + a_\ell \circ \tau = 0 \in \mathbf{T}_\varepsilon(\mathbf{1}^\vee \otimes \mathbf{1}, \mathbf{1})$, completing the proof of (a).

Continuing on to the proof of (b), we first compute $y \circ (y \otimes \mathrm{id}_{\mathbf{I}})$. There are nine terms to consider, but some vanish due to being in different components of the direct sum — only those terms where the output object of the first morphism matches the input morphism of the second morphism are non-zero. For example, $a_r \circ (\beta \otimes \mathrm{id}_{\mathbf{I}}) = 0$ is one of the vanishing terms. Of the nine terms, only four fail to vanish for this reason. Namely,

$$y \circ (y \otimes \mathrm{id}_{\mathbf{I}}) = \beta \circ (\beta \otimes \mathrm{id}_{\mathbf{I}}) + \beta \circ (a_\ell \otimes \mathrm{id}_{\mathbf{I}}) + a_\ell \circ (a_r \otimes \mathrm{id}_{\mathbf{I}}) + a_r \circ (a_r \otimes \mathrm{id}_{\mathbf{I}}).$$

The string-diagrammatic version of this expression is



It remains to show precomposing this expression with $(1+\tau_{123}+\tau_{132})$ gives the zero morphism from $\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}\to\mathbf{I}$.

Indeed, for the first term we have

$$\beta\circ(\beta\otimes\mathrm{id}_{\mathbf{I}})\circ(1+\tau_{123}+\tau_{132})=0$$

from the fact that $\mathbf{1}$ is a Lie algebra object with morphism β.

Note here that if $\mathbf{x},\mathbf{y},\mathbf{z}\in\{\mathbf{1},\mathbf{1}^\vee\}$ and $f:\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\to\mathbf{1}^\vee$ then $f\circ(1+\tau_{123}+\tau_{132})$ has components in $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\to\mathbf{1}^\vee$, $\mathbf{y}\otimes\mathbf{z}\otimes\mathbf{x}\to\mathbf{1}^\vee$ and $\mathbf{z}\otimes\mathbf{x}\otimes\mathbf{y}\to\mathbf{1}^\vee$. Therefore, the analysis with the remaining three terms is not as simple, as the domains are not cyclically invariant tensor products in $\{\mathbf{1},\mathbf{1}^\vee\}$.

It remains to show that

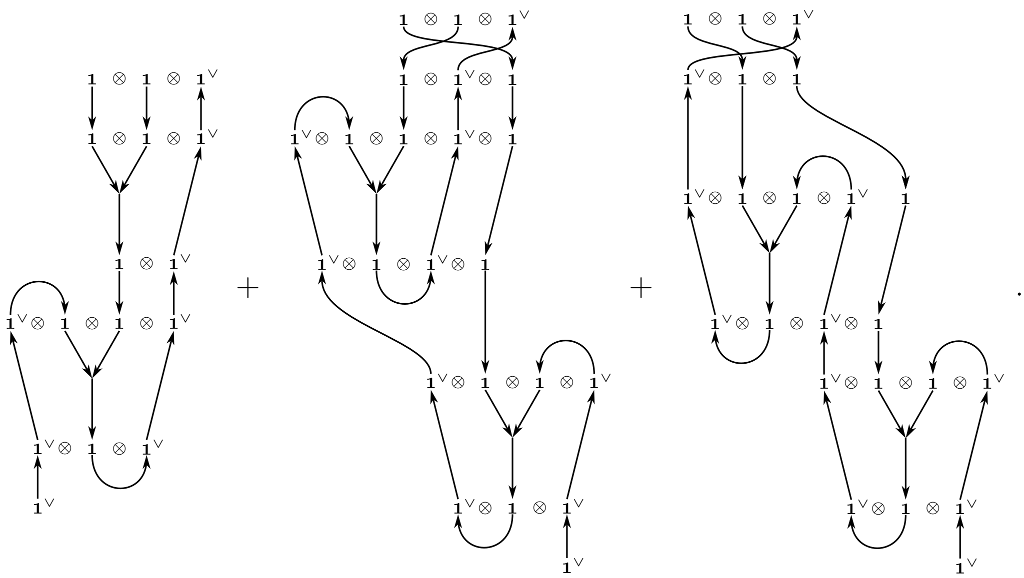
$$(\beta\circ(a_\ell\otimes\mathrm{id}_{\mathbf{I}})+a_\ell\circ(a_r\otimes\mathrm{id}_{\mathbf{I}})+a_r\circ(a_r\otimes\mathrm{id}_{\mathbf{I}}))\circ(1+\tau_{123}+\tau_{132})$$

vanishes. This morphism in $\mathbf{T}_\varepsilon(\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I},\mathbf{I})$ only has components in $\mathbf{T}_\varepsilon(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}^\vee,\mathbf{1}^\vee)$, $\mathbf{T}_\varepsilon(\mathbf{1}\otimes\mathbf{1}^\vee\otimes\mathbf{1},\mathbf{1}^\vee)$ and $\mathbf{T}_\varepsilon(\mathbf{1}^\vee\otimes\mathbf{1}\otimes\mathbf{1},\mathbf{1}^\vee)$ so it suffices to check these components.

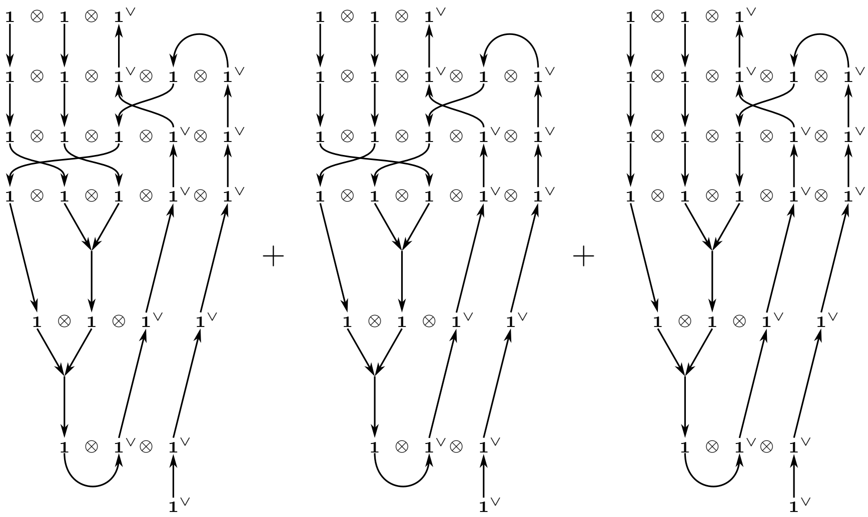
We provide the calculation in the $\mathbf{T}_\varepsilon(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}^\vee,\mathbf{1}^\vee)$ component — the proof that the other components follow from similar arguments. Indeed in $\mathbf{T}_\varepsilon(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}^\vee,\mathbf{1}^\vee)$ we have the contributions

$$\beta\circ(a_\ell\otimes\mathrm{id}_{\mathbf{I}})\circ 1+a_\ell\circ(a_r\otimes\mathrm{id}_{\mathbf{I}})\circ\tau_{132}+a_r\circ(a_r\otimes\mathrm{id}_{\mathbf{I}})\circ\tau_{123}.$$

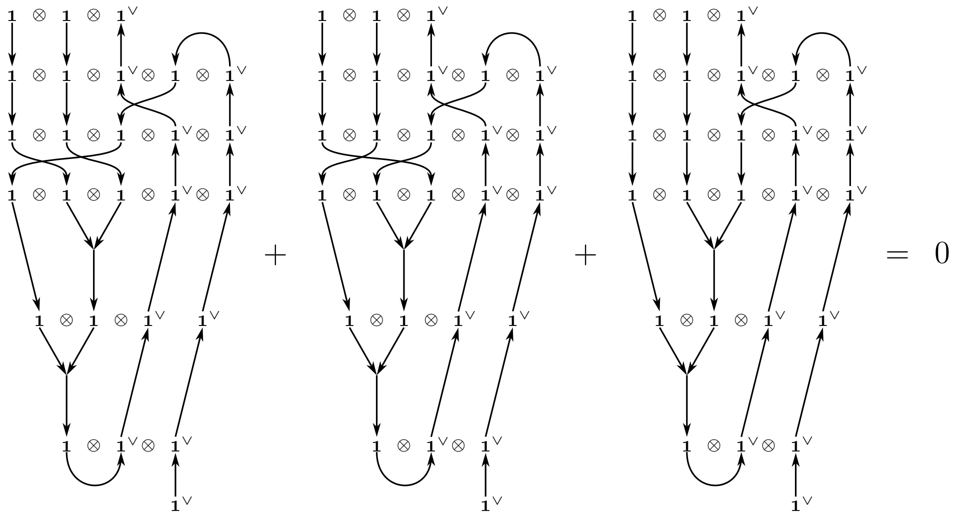
Diagramatically, that is



Simplifying these terms via the rigidity relations of Definiton 2.3.5 (d) as well as the antisymmetry of the bracket β, we arrive at



(specifically, the first and third terms are simplified only by the rigidity relations, while the second term is simplified by the rigidity relations and two applications of the antisymmetry of β). Finally, β is a Lie algebra object in $\mathbf{T}_\varepsilon$, and since the three terms in this expression factor through the generalised Jacobi identity for β, we have



completing the proof. $\square$

We therefore obtain the following description of the algebra $\mathcal{A}_w$.

Theorem 3.4.5 *Let $\mathbf{I}$ denote the Lie algebra object $\mathbf{1} \oplus \mathbf{1}^\vee$ (with the morphism y) in the two-coloured prop $\mathbf{T}_\varepsilon$. The algerba $U(\mathcal{O}_{\mathrm{Lie}}, \mathbf{I})$ is isomorphic to $\mathcal{A}_w$ as a filtered algebra.*

With the appropriate definitions in place, the proof follows the proof of Theorem 2.5.19.

Proof We denote objects $\mathbf{I}^{\otimes n}$ in $\mathbf{T}_\varepsilon$ as $\mathbf{nI}$. Recall the definition of $T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{I})$ as

$$\begin{aligned} T(\mathcal{O}_{\mathrm{Lie}}, \mathbf{I}) &= \bigoplus_{n \in \mathbb{N}} \mathcal{O}_{\mathrm{Lie}}(n+1) \otimes_{S_n} \Gamma(\mathbf{nI}) \\ &= \bigoplus_{n \in \mathbb{N}} \mathcal{O}_{\mathrm{Lie}}(n+1) \otimes_{S_n} \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{nI}). \end{aligned}$$

By Lemma 2.5.20, from the S_n-action on $\mathcal{O}_{\text{Lie}}$, any element $t \in T(\mathcal{O}_{\text{Lie}}, \mathbf{I})$ may be uniquely written as

$$t = \bigoplus_{n \in \mathbb{N}} L_{n+1} \otimes_{S_n} A_n$$

with L_{n+1} the left-skewed bracket in $\mathcal{O}_{\text{Lie}}(n+1)$ and $A_n \in \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{nI})$. This defines a vector space isomorphism

$$\begin{array}{ccc} T(\mathcal{O}_{\text{Lie}}, \mathbf{I}) & \longrightarrow & \bigoplus_{n \in \mathbb{N}} \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{nI}) \\ t & \longmapsto & \bigoplus_{n \in \mathbb{N}} A_n. \end{array}$$

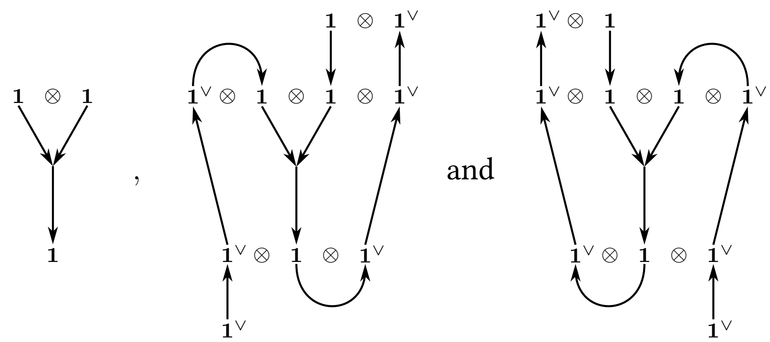
We define the map $f : T(\mathcal{O}_{\text{Lie}}, \mathbf{I}) \to \mathcal{A}_w$ as the above isomorphism composed with sending each A_n to the arrow diagram in $\mathcal{A}_w$ that it presents.

Recall that by Definition 2.5.17, the associative algebra $U(\mathcal{O}, \mathbf{I})$ is a quotient of $T(\mathcal{O}, \mathbf{I})$ by the weakest relations that assert the following diagram commutes for all $n, m_1, \ldots, m_n \in \mathbb{N}$.

$$\begin{tikzcd} \mathcal{O}_{\text{Lie}}(1+n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}_{\text{Lie}}(m_i) \otimes \mathbf{T}_\varepsilon(\mathbf{0}, \sum_i \mathbf{m}_i) \arrow[r, "\mathrm{id}\otimes\eta_{(m_1,\ldots,m_n)}"] \arrow[d, "\circ_2\otimes\cdots\otimes\circ_{n+1}"'] & \mathcal{O}_{\text{Lie}}(1+n) \otimes \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{n}) \arrow[d, hook] \\ \mathcal{O}_{\text{Lie}}(1 + (\sum_i m_i)) \otimes \mathbf{T}_\varepsilon(\mathbf{0}, \sum_i \mathbf{m}_i) \arrow[r, hook] & T(\mathcal{O}_{\text{Lie}}, A) \end{tikzcd}$$

Just like in the classical case of Example 2.5.18, the relations in the algebra $\mathcal{A}_w$ can be recovered from this diagram. Indeed, in the special case $n = 1, m_1 = 2$, the diagram is an equality in $\mathcal{O}_{\text{Lie}}(2) \otimes \mathcal{O}_{\text{Lie}}(2) \otimes \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{2I})$. Specifying the morphism $y \otimes y \otimes X$ and leaving as arbitrary $X \in \mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{2I})$, this reduces to the directed `STU` and `TC` relations. The map f therefore factors through the quotient to $U(\mathcal{O}_{\text{Lie}}, \mathbf{I})$.

The map f is clearly surjective as every arrow diagram which can be presented by a graph whose edges do not have local minima. The arrow diagram can therefore be presented with only coevaluation maps (caps), and trivalent two-in-one-out vertices. A morphism in $\mathbf{T}_\varepsilon(\mathbf{0}, \mathbf{nI})$ is built by presenting the two-in-one-out vertices by morphisms



and caps by the coevaluation morphism ε.

The map is injective because we have shown that $U(\mathcal{O}_{\text{Lie}}, \mathbf{I})$ has the directed `STU` relations and the `TC` relation. Just like in the classical case, the `AS` and `IHX` relations follow from `STU`, so any equalities in $\mathcal{A}$ are also observed in $U(\mathcal{O}_{\text{Lie}}, \mathbf{I})$. $\square$

Just like the classical case, we define a bialgebra structure on $U(\mathcal{O}_{\text{Lie}}, \mathbf{I})$ via the algebra isomorphism with A_w.

Definition 3.4.6 The coproduct Δ on $U(\mathcal{O}_{\mathrm{Lie}}, \mathbf{I})$ is defined by pulling the coproduct on A_w back via the isomorphism of Theorem 3.4.5.

We obtain as a corollary the following Hinich–Vaintrob theorem for the bialgebra $\mathcal{A}_w$.

Theorem 3.4.7 *The bialgerba $U(\mathcal{O}_{Lie}, \mathbf{I})$ is isomorphic to $\mathcal{A}_w$ as a filtered bialgebra.*

Recall that the raison d'etre of Theorem 2.5.23 was to formalise the informal statement that "the bialgebra $\mathcal{A}$ of chord diagrams behaves like the universal enveloping algebra of a Lie algebra". Interpreting Theorem 3.4.7 in the same manner, its statement becomes that "the bialgebra $\mathcal{A}_w$ or arrow diagrams behaves like the universal enveloping algebra $\mathcal{U}(I\mathfrak{g})$ of a cocommutative Drinfeld double $I\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$".

Remark 3.4.8 For classical weight systems Theorem 2.5.19 proves that every weight system (so every functional on $\mathcal{A}$) comes from applying Construction 2.2.1 to some Lie algebra object in some symmetric monoidal category. However, for welded weight systems, this is also achieved in as a corollary of [BD16, Thm. 3.26] which states that the universal finite type invariant of welded knots Z_w is equivalent to the Alexander polynomial. Indeed, recall that the Alexander polynomial factors through the weight system $W_{\mathfrak{gl}_{1|1}}$, and since every welded weight system factors through Z_w, they all factor through $W_{\mathfrak{gl}_{1|1}}$. As such, all welded weight systems arise via Construction 2.2.1 from Lie superalgebras, specifically $\mathfrak{gl}_{1|1}$. Theorem 3.4.5 gives an alternative, categorical proof of this fact.

References

[AK25] Naruhiko Aizawa and Daichi Kimura. "Universal weight systems from a minimal $\mathbb{Z}_2^2$-graded lie algebra". In: *Journal of Knot Theory and Its Ramifications* 34.10 (2025).

[Aud16] Benjamin Audoux. "On the welded Tube map". In: *Knot Theory and Its Applications*. Vol. 670. American Mathematical Society, 2016.

[Bar95] Dror Bar-Natan. "On the Vassiliev knot invariants". In: *Topology* 34.2 (1995), pp. 423–472.

[BD16] Dror Bar-Natan and Zsuzsanna Dancso. "Finite-type invariants of w-knotted objects, I: w-knots and the Alexander polynomial". In: *Algebraic & Geometric Topology* 16.2 (2016), pp. 1063–1133.

[BD17] Dror Bar-Natan and Zsuzsanna Dancso. "Finite type invariants of w-knotted objects II: tangles, foams and the Kashiwara–Vergne problem". English. In: *Mathematische Annalen* 367.3-4 (Apr. 2017), pp. 1517–1586.

[BG96] Dror Bar-Natan and Stavros Garoufalidis. "On the Melvin–Morton–Rozansky conjecture". In: *Inventiones mathematicae* 125.1 (May 1996), pp. 103–133.

[BL93] Joan S. Birman and Xiao-Song Lin. "Knot polynomials and Vassiliev's invariants". In: *Inventiones Mathematicae* 111.2 (1993), pp. 225–270.

[BM25] Liudmila Bishler and Andrei Mironov. "On refined Vogel's universality". In: *Physics Letters B* 867 (2025), p. 139596. ISSN: 0370-2693.

[BS97] Dror Bar-Natan and Alexander Stoimenow. *The Fundamental Theorem of Vassiliev Invariants*. 1997. arXiv: q-alg/9702009.

[CD05] Sergei Chmutov and Sergei Duzhin. *The Kontsevich integral*. 2005. arXiv: math/0501040.

[CDM12] Sergei Chmutov, Sergei Duzhin, and Jacob Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

[CV97] Sergei Chmutov and Alexander Varchenko. "Remarks on the vassiliev knot invariants coming from $\mathfrak{sl}_2$". In: *Topology* 36.1 (1997), pp. 153–178.

[Dea94] John Dean. "Many classical knot invariants are not Vassiliev invariants". In: *Journal of Knot Theory and Its Ramifications* 03.01 (1994), pp. 7–10.

[Dri88] V. G. Drinfeld. "Quantum groups". In: *Journal of Soviet Mathematics* 41.2 (1988), pp. 898–915.

[Hut98] Michael Hutchings. "Integration of Singular Braid Invariants and Graph Cohomology". In: *Transactions of the American Mathematical Society* 350.5 (1998), pp. 1791–1809.

[HV00] Vladimir Hinich and Arkady Vaintrob. "Cyclic operads and algebra of chord diagrams". In: *Selecta Mathematica, New Series* 8 (May 2000).

[Kau99] Louis H. Kauffman. "Virtual Knot Theory". In: *European Journal of Combinatorics* 20.7 (1999), pp. 663–691. ISSN: 0195-6698.

[KLS25] Dmitry Khudoteplov, Elena Lanina, and Alexey Sleptsov. *Construction of Lie algebra weight system kernel via Vogel algebra*. 2025. arXiv: 2411.14417.

[KMS25] Dmitry Khudoteplov, A. Morozov, and A. Sleptsov. "Can Yang-Baxter imply Lie algebra?" In: *Physics Letters B* 867 (2025), p. 139586. ISSN: 0370-2693.

[Kon93] Maxim Kontsevich. “Vassiliev’s knot invariants”. In: *Advances in Soviet Mathematics* 16(2) (1993), pp. 137–150.

[Lei04] Tom Leinster. *Higher Operads, Higher Categories.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[Les24] Christine Lescop. *Invariants of Links and 3-Manifolds from Graph Configuration.* European Mathematical Society Press, 2024.

[Lie99] Jens Lieberum. “On Vassiliev invariants not coming from semisimple Lie algebras”. In: *Journal of Knot Theory and Its Ramifications* 08.05 (1999), pp. 659–666.

[Mar08] Martin Markl. “Operads and PROPs”. In: ed. by M. Hazewinkel. Vol. 5. Handbook of Algebra. North-Holland, 2008, pp. 87–140.

[MS17] Jean-Baptiste Meilhan and Sakie Suzuki. “Riordan trees and the homotopy $\mathfrak{sl}_2$ weight system”. In: *Journal of Pure and Applied Algebra* 221.3 (2017), pp. 691–706.

[Rob01] Justin Roberts. *Rozansky-Witten theory.* 2001. arXiv: math/0112209.

[RW06] Justin Roberts and Simon Willerton. “On the Rozansky-Witten weight systems”. In: *Algebraic & Geometric Topology* 10 (2006), pp. 1455–1519.

[Sat00] Shin Satoh. “Virtual knot presentation of ribbon torus-knots”. In: *Journal of Knot Theory and Its Ramifications* 09.04 (2000), pp. 531–542.

[Sta96] Ted Stanford. “Finite-type invariants of knots, links, and graphs”. In: *Topology* 35.4 (1996), pp. 1027–1050.

[Thu00] Dylan P. Thurston. *Wheeling: A diagrammatic analogue of the Duflo isomorphism.* PhD thesis. 2000. arXiv: math/0006083.

[Tra94] Rolland Trapp. “Twist sequences and Vassiliev invariants”. In: *Random Knotting and Linking.* Series on Knots and Everything. 1994, pp. 159–173.

[Vai94] Arkady Vaintrob. “Vassiliev knot invariants and Lie S-algebras”. In: *Mathematical Research Letters* 1 (1994), pp. 579–595.

[Vas90] V. A. Vassiliev. In: *Cohomology of knot spaces.* Dec. 1990, pp. 23–70.

[Vas92] V. A. Vassiliev. *Complements of discriminants of smooth maps : topology and applications.* Translations of mathematical monographs, v. 98. Providence, R.I: American Mathematical Society, 1992.

[Vog11] Pierre Vogel. “Algebraic structures on modules of diagrams”. In: *Journal of Pure and Applied Algebra* 215.6 (2011), pp. 1292–1339.

[Vog97] Pierre Vogel. “Algebraic structures on modules of diagrams”. An early preprint version of [Vog11]. 1997. URL: https://webusers.imj-prg.fr/~pierre.vogel/.

[Vog99] Pierre Vogel. “Vassiliev theory and the universal Lie algebra”. Unpublished preprint. 1999. URL: https://webusers.imj-prg.fr/~pierre.vogel/.

[Wil96] Simon Willerton. “Vassiliev invariants and the Hopf algebra of chord diagrams”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 119.1 (1996), pp. 55–65.

[Wil98a] Simon Willerton. “A Combinatorial Half-Integration from Weight System to Vassiliev Knot Invariant”. In: *Journal of Knot Theory and Its Ramifications* 07.04 (1998), pp. 519–526.

[Wil98b] Simon Willerton. “Vassiliev invariants as polynomials”. In: vol. 42. 1998, pp. 457–463.

[Yan24] Zhuoke Yang. “On Values of $\mathfrak{sl}_3$ Weight System on Chord Diagrams whose Intersection Graph is Complete Bipartite”. In: *Moscow Mathematical Journal* 24 (2024), pp. 125–140.

[YK99] Shutaro Yoshizumi and Ken’ichi Kuga. “Some formulas in $\mathfrak{sl}_3$ weight systems”. In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 75.7 (1999), pp. 134–136.