On Vassiliev Invariants

Damian Lin

An essay submitted in fulfilment of
the requirements for the degree of
Master of Philosophy (Science)

Pure Mathematics
University of Sydney

THE UNIVERSITY OF
SYDNEY

October 19, 2025

Contents

Acknowledgements

Thanks to ...

Introduction

THE introduction goes here.

1

Vassiliev invariants and chord diagrams

Something to maybe include somewhere:

The point of Vassiliev theory is to study the space of knots in the context of the singularities that lie between knots.

In view of this point, we spend Section 1.1 looking at a the stratification of the space of knots. This leads to a beautiful (and fruitful) classical analogy which we will explored in Section 1.2 and throughout this chapter.

In Section 1.3, with the context in mind, we introduce the main players in this theory.

VASSILIEV invariants are sophisticated to define in terms of the space of knots from the introduction, but the axiomatic definition of Birman-Lin [BL93] is much simpler. The definition also illustrates an analogy made by Bar-Natan [Bar95] in which Vassiliev invariants are "polynomial invariants". This is not meant in the sense that Vassiliev invariants take values in a polynomial ring (like say, the Jones polynomial), but rather that Vassiliev invariants have special properties not shared by all invariants, just as polynomial functions have special properties not shared by all functions.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m**-singular**.

Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular immersions with

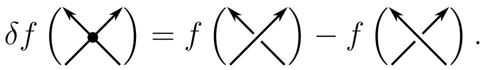
(a) triple points

(b) points with vanishing derivative

are excluded from the definition.

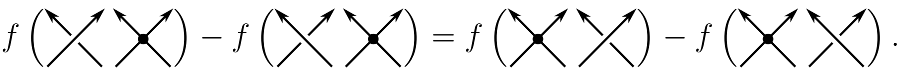
A singular knot with one double point is very close to two other knots. In one, the double point is replaced by a positive crosing, and in the other a negative crossing. If the conditions are right, we can extend a knot invariant to an invariant of singular knots by a procedure analagous to taking its derivative.

Definition 1.1.3 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



What are the conditions? For this to be a well-defined operation, it mustn't matter which double point we choose.

Definition 1.1.4 An invariant f of m-singular knots is **differentiable** if

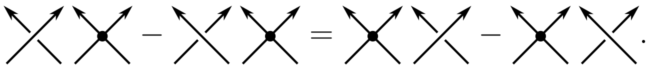


(DIFF)

If an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

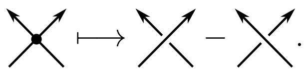
Definition 1.1.5 Define $\mathcal{K}_m^\bullet$ as the span of all m-singular knots, taken over $\mathbb{Q}$, modulo the following boundary relation (also known as a codifferentiability relation):



(DIFF*)

From now on, we will refer to elements $\mathcal{K}_m^\bullet$ as m-**singular knots**, i.e. the DIFF* relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m^\bullet \to \mathcal{K}_{m-1}^\bullet$ defined by

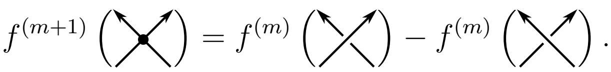


Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as an invariant of knots in $\mathcal{K}_m^\bullet$.

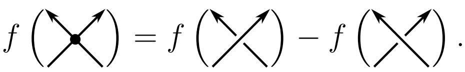
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

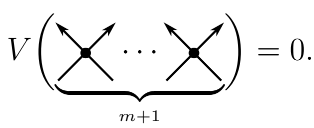


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation,



(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the most double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 In other words, Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

1.2 Integration and the fundamental theorem

To help see the bird's eye view, and following [Hut98], we phrase this in terms of an integration theory.

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group $\mathcal{O}_0$ is typically free abelian, and in our case this is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \operatorname{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behave like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

We wish to understand how to invert this process, namely:

Questions 1.2.2 (a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?

(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

(c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?

(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The following modules provide the tautological answers to the above questions in the general theory.

Definitions 1.2.3 (a) The **primary obstructions to integration** are the module

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the module

$$C\mathcal{O}_m = \mathcal{O}_m/\partial\mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the module

$$S\mathcal{O}_m = \ker\left(\partial_{m+1}\partial_m\right)/P\mathcal{O}_m,$$

and likewise the **order** k **obstructions to integration** are defined analagously.

(d) The **weights of integration** are the module

$$W\mathcal{O}_m = C\mathcal{O}_m/\pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the projection.

(e) The **finite type invariants** of order m are the module

$$FT\mathcal{O}_m = \ker\delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$ which provides good intuition for the general case.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the m-th and $(m{+}1)$-st strata such that the path only intersects the $(m{+}1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

To rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

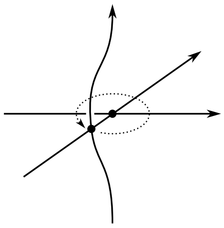
Due to the boundary relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

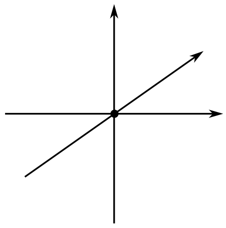
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a "path-integral" along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving "holes" in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

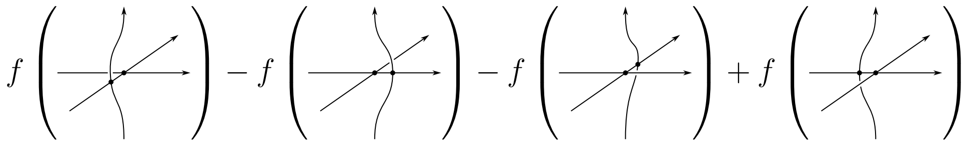
Definitions 1.2.5 (a) The following closed singular isotopy around a triple point:



around



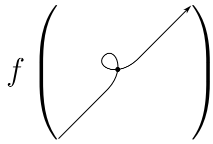
gives rise to the **topological four-term relation**



$= 0.$ (T4T)

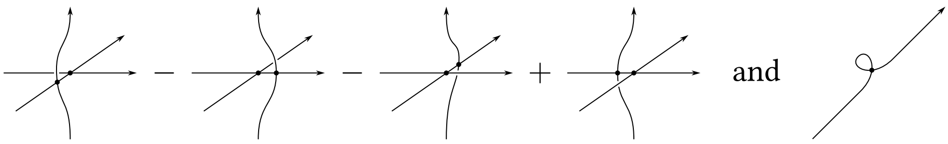
(b) The closed singular isotopy

around a point with a vanishing derivative gives rise to the **topological one-term relation**



$= 0.$ (T1T)

The necessity of the above relations for an invariant of m-singular knots to integrate is equivalent to the assertion that



are in $P\mathcal{K}^\bullet_m = \ker \partial_m$. Let us denote arbitrary such singular knots $\mathbf{T4T}$ and $\mathbf{T1T}$.

This raises the question of whether we have found all primary obstructions. Do $\mathbf{T4T}$ and $\mathbf{T1T}$ span $\ker \partial$?

Theorem 1.2.6 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $m-1$-singular knots if and only if it satisfies* T4T *and* T1T.

Proof (sketch) Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m-1$ double points, and additional worse singularities. The only events to consider are codimension $m+2$ (why $m+2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with DIFF relations, which we've said are implicit).

Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

In the order of Questions 1.2.2 and Definitions 1.2.3, we ought to talk next about the secondary and further obstructions to integration. However, let us postpone this until after we have discussed constants of integration, which will be a more natural place.

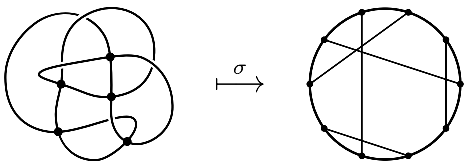
The constants of integration comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost. When integrating, values on these can be chosen freely.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}^\bullet_m = \mathcal{K}^\bullet_m/\partial\mathcal{K}^\bullet_{m+1}$. The difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. Thus, we see that constants of integration are singular knots which are blind to crossing changes, so the only data needed to describe them is the order in which the double points are traversed around the knot.

Definitions 1.2.7 (a) A **chord diagram** of order m is an element of $\mathcal{K}^\bullet_m/\partial\mathcal{K}^\bullet_{m+1}$. This is equivalent to an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}^\bullet_m$.

(b) The **chord diagram of an m-singular knot** k, denoted $\sigma(k)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of k. Traversing both k and the circle, label the points on the circle in the order in which the singular points of k are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(k)$.

Example 1.2.8



ZSUZSI READ TO HERE

Proposition 1.2.9 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}^*_m$ Then Q_0 is also an integral of P.

Proof The derivative of a function of chord diagrams is zero, as two knots which differ by a crossing change have the same chord diagram. Hence Q and Q_0 have the same derivative: P. □

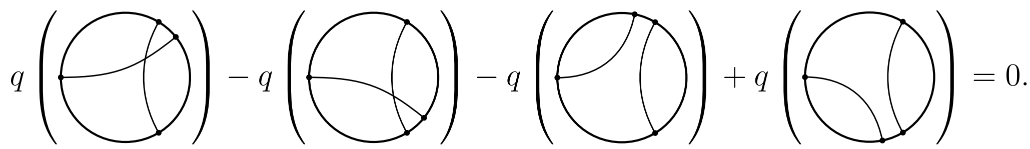
Remark 1.2.10 We defined a constant of integration as a set of objects (e.g. $c \in \mathcal{D}_m$) but perhaps it would have been more accurate to define it as a functional (e.g. $q \in \mathcal{D}^*_m$). After all the constant of integration on the real line is "$+\,C$" rather than the set $\{*\}$ with one object. There is no real risk of confusion, so let us be slightly loose and use the terminology for either.

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.11 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

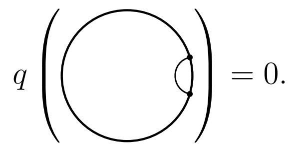
If we take the relations for a knot invariant to be integrable and project them into the space of chord diagrams, we get the following relations.

Definition 1.2.12 (a) The **four-term relation** is the relation



(4T)

(b) The **one-term relation** is the relation



(1T)

Remark 1.2.13 Just like T4T and T1T, 4T and 1T are not individual relations, but kinds of relations. To satisfy them, $q \in \mathcal{D}_m^*$ needs to satisfy them for all ways additional chords can be placed into the diagram to make a diagram of degree m, so long as they don't go between the close chord-ends.

Proposition 1.2.14 *A weight system is characterised as a constant of integration that satisfies* 4T *and* 1T.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T and 1T. Since crossing changes are free, this is equivalent to satisfying the projection of T4T and T1T into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of on m-singular P is of the form

$$Q + q \circ \sigma.$$

Since integration is linear, to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that $S\mathcal{K}_m^\bullet$ vanishes. But this is a tautological statement of the general theory — it doesn't mean much if we don't know what $S\mathcal{K}_m^\bullet$ is.

Conjecture 1.2.15 *An invariant of m-singular knots satisfying* T4T *and* T1T *integrates m times into a genuine knot invariant.*

Remarks 1.2.16 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.6. The point is that it may not be possible to choose the integral to again satisfy T4T and T1T, which is what $S\mathcal{K}^\bullet$ measures.

(b) Computing $S\mathcal{K}_m^\bullet$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.

(c) This conjecture is proven in certain cases. It holds the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98].

The finite type invariants in $\mathcal{K}_\star^\bullet$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m+1$.

If we restrict Conjecture 1.2.15 to Vassiliev invariants, then we get the following.

Theorem 1.2.17 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* T4T, T1T *and the additional condition that $\delta^k v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant).*

There are various proofs of the fundamental theorem. They are listed in [BS97], and each proof is accompanied by a series of moral objections. To quote their introduction: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made". To summarise their philosophy:

Remark 1.2.18 We have the implication Conjecture 1.2.15 $\implies$ Theorem 1.2.17, and this is actually realised in the theory of braids. It is mysterious that the fate of the slightly stronger conjecture which comes from taking the natural topological approach to the fundamental theorem still remains unknown, and that there are grievances to be had with all known proofs.

In Section 1.5 we will look at equivalent formulation of the fundamental theorem.

1.3 The bialgebra of knots

Some planning notes for this chapter:

· Begin with the connected sum on plain knots (don't talk about modding by 1T 4T yet).

· This induces a connected sum on singular knots (define via commutative diagram).

· This nearly induces a conntected sum on chord diagrams... except need to mod out by 4T and 1T which we were gonna do anyway.

Speaking broadly, the aim of Vassiliev theory is to study the space of knots via the space of chord diagrams, using the information of the stratification of knots introduced in the first two sections. Knots and chord diagrams are not just vector spaces; they have some further structure. In this section, we synthesise all of this information into two algebraic structures.

Definitions 1.3.1 (a) The **space of knots**, denoted $\mathcal{K}$ is the vector space spanned (over $\mathbb{Q}$) by non-singular knots. Equivalently, $\mathcal{K} = \mathcal{K}_0^\bullet$.

(b) The space of knots is equipped with the **singular knot filtration**

$$\mathcal{K} \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

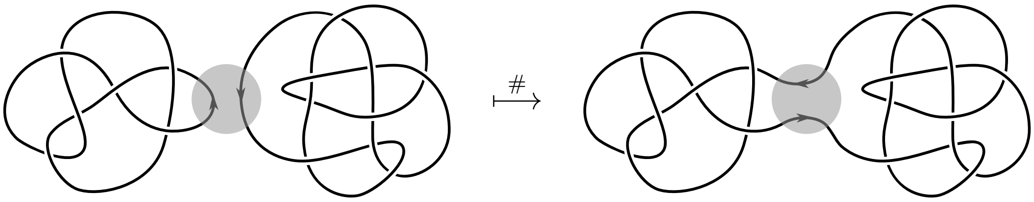
where the ith filtered component is the span of resolutions of singular knots with i double points, equivalently the image of δ^i; $\mathcal{K}_i = \delta^i(\mathcal{K}_i^\bullet)$.

Proposition 1.3.2 *The singular knot filtration is indeed a descending filtration of vector spaces.*

Proof A filtration of vector spaces is uninteresting. The only thing to check is that $\mathcal{K}_i \supset \mathcal{K}_j$ if $i < j$. If $k \in \mathcal{K}_j$, then $k = \delta^j(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}_j$. But then $k = \delta^j(k^\bullet) = \delta^i\delta^{j-i}(k^\bullet)$, so $k \in \delta^i(\mathcal{K}_i^\bullet)$. $\square$

The algebraic structure on knots comes from the following operation.

Definition 1.3.3 The **connected sum** of two knots k_1 and k_2 is the knot obtained by removing a small arc from each of k_1 and k_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

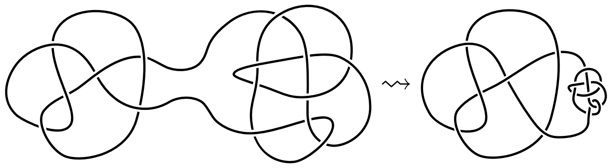


This definition is extended bilinearly to $\mathcal{K}$, i.e. linear combinations.

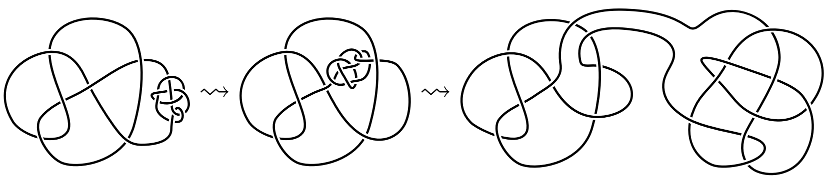
This is not a-priori well-defined. We have not specified where along either k_1 or k_2 the small arc is to be removed. However, by a classical knot-theoretic argument, the result is independent of the choice.

Proposition 1.3.4 *The connected sum* $\# : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ *forms a well-defined operation. It does not matter where along either knot the small arc was removed, the results are ambient-isotopic.*

Proof We exhibit an ambient isotopy starting at $k_1 \# k_2$ where the small arc is removed from k_1 as in the example above. The part of the connected sum coming from k_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of k_1.



Then, k_2 is then isotoped along k_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along k_1, and the same argument with the roles of k_1 and k_2 reversed completes the proof. $\square$

Proposition 1.3.5 *The connected sum makes* $(\mathcal{K}, \#)$ *into a filtered algebra. That is, the connected sum respects the filtration,*

$$K_i \# K_j \subset K_{i+j}.$$

Proof Indeed, the connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. The question is whether the connected sum respects the filtration.

If $k \otimes \ell \in \mathcal{K}_i \otimes \mathcal{K}_j$, then there are $k_\bullet$ and $\ell_\bullet$ in $\mathcal{K}_i^\bullet$ and $\mathcal{K}_j^\bullet$ that resolve to k and ℓ, respectively. Similarly, the 'connected sum' $k_\bullet \# \ell_\bullet$ resolves by δ^{i+j} to $k \# \ell$, which is therefore in $\delta^{i+j}(\mathcal{K}_{i+j})$.

Here, 'connected sum' is enclosed in inverted commas due to the following technicality. Connected sums of singular knots with singular knots were not part of Definition 1.3.3. Even if we ensure that the small arcs removed from a singular knot do not contain a singular point, still, this is ill-defined. The ambiguity is that the resulting singular knot may depend on from which side of the singular point the arc was removed i.e. the repositioning argument in the proof of Proposition 1.3.4 fails due to the presence of singular points. After taking the resolution under δ^{i+j} however, the repositioning argument now works again, so any of the choices of connected sum in $k_\bullet \# \ell_\bullet$ produce a singular knot which resolves to $k \# \ell$. □

Say something intelligent to introduce this definition.

I guess it's something like: We assert that knots are (semi)group-like. This must be somehow related to (some?) chord diagrams being like exponentials (primitives?).

Definition 1.3.6 The coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is defined on knots k as

$$\Delta(k) = k \otimes k$$

and extended bilinearly to $\mathcal{K}$.

Proposition 1.3.7 *The connected sum and coproduct make* $(\mathcal{K}, \#, \Delta)$ *into a bialgebra.*

Proof A bialgeba is a coalgebra that is also an algebra with compatible product and coproduct. That $(\mathcal{K}, \Delta)$ forms a coalgebra is trivial (the counit is the augmentation map denoted ε). And we have already seen that $(\mathcal{K}, \#)$ is an algebra. So it remains only to check the compatibility conditions. We check that the product and coproduct are compatible:

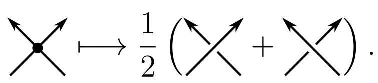
$$\begin{aligned} \Delta(k \# \ell) &= k \# \ell \otimes k \# \ell \\ &= (k \otimes k) \# (\ell \otimes \ell) \\ &= \Delta(k) \# \Delta(\ell) \end{aligned}$$

and the rest are trivial. □

Proposition 1.3.8 *With the singular knot filtration,* $\mathcal{K}$ *is a filtered bialgebra. That is, the coproduct* Δ *also respects the filtration,*

$$\Delta(\mathcal{K}_j) \subset \sum_{i=0}^{j} \mathcal{K}_i \otimes \mathcal{K}_{j-i}.$$

We give a proof, due to Willerton [Wil96] which follows directly from his Lemma 1.3.10. The lemma gives a formula for the coproduct of an element of $\mathcal{K}_n$ in terms of a singular knot that resolves to it by looking at the 2^n ways of resolving some of singular points in one cofactor and the rest in the other. If I is a subset of the singular points of a singular knot, let δ^I be the operator that resolves the singular points I. Let μ^I be the operator that averages singular points in I, where averaging a singular point is sending



Remark 1.3.9 There is one technicality. To make the above definitions of μ^I and δ^I, we must forget the `DIFF`* relation on $\mathcal{K}^\bullet_m$ (or work in the appropriate lift) as we wish to resolve with respect to specific double points. This provides no mathematical difficulty, so we chose not to do the reader the disservice of altering the notation. But for Lemma 1.3.10, its proof and the proof of Proposition 1.3.8, let $\mathcal{K}^\bullet_m$ not contain the quotient by the `DIFF`* relation.

Lemma 1.3.10 (Willerton) *Suppose $k^\bullet \in \mathcal{K}^\bullet_m$, and let S denote the set of singular points of $k^\bullet$.*

$$\Delta(\delta^S(k^\bullet)) = \sum_{I\subset S} \mu^{\bar{I}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}}(k^\bullet)$$

where $\bar{I} = S \setminus I$.

Proof We proceed by induction on m. In the base case of $m = 0$, $S = \emptyset$, and $k^\bullet = k$ is a genuine knot, so

$$\begin{aligned}\Delta(\delta^0(k)) &= \Delta(k)\\ &= k \otimes k\\ &= \sum_{I\subset\emptyset} \mu^{\bar{I}}\delta^I(k) \otimes \mu^I\delta^{\bar{I}}(k).\end{aligned}$$

The inductive step is as follows. Let $k^\bullet \in \mathcal{K}^\bullet_{m+1}$. Let J denote all singular points of $k^\bullet$, and $x \in J$ denote a specific singular point. Furthermore, let $k^{\bullet+}$ (resp. $k^{\bullet-}$) denote the m-singular knots obtained from $k^\bullet$ when x is replaced by a positive (resp. negative) crossing, so that $\delta^{\{x\}}(k^\bullet) = k^{\bullet+} - k^{\bullet-}$. We examine

$$\sum_{I\subset J} \mu^{\bar{I}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}}(k^\bullet).$$

Decomposing the sum based on whether $x \in I$ yields

$$\sum_{x\in I\subset J} \mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^\bullet) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^\bullet) + \sum_{x\notin I\subset J} \mu^{\bar{I}\setminus\{x\}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^\bullet),$$

then resolving either δ or μ on x,

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J} \mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^{\bullet+} - k^{\bullet-}) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^{\bullet+} + k^{\bullet-})\\ +&\frac{1}{2}\sum_{x\notin I\subset J} \mu^{\bar{I}\setminus\{x\}}\delta^I(k^{\bullet+} + k^{\bullet-}) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^{\bullet+} - k^{\bullet-}).\end{aligned}$$

Expanding, yields the cumbersome,

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J}\Big(\mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^{\bullet+}) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^{\bullet+}) + \mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^{\bullet+}) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^{\bullet-})\\ &\qquad - \mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^{\bullet-}) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^{\bullet+}) - \mu^{\bar{I}}\delta^{I\setminus\{x\}}(k^{\bullet-}) \otimes \mu^{I\setminus\{x\}}\delta^{\bar{I}}(k^{\bullet-})\Big)\\ +&\frac{1}{2}\sum_{x\notin I\subset J}\Big(\mu^{\bar{I}\setminus\{x\}}\delta^I(k^{\bullet+}) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^{\bullet+}) - \mu^{\bar{I}\setminus\{x\}}\delta^I(k^{\bullet+}) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^{\bullet-})\\ &\qquad + \mu^{\bar{I}\setminus\{x\}}\delta^I(k^{\bullet-}) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^{\bullet+}) - \mu^{\bar{I}\setminus\{x\}}\delta^I(k^{\bullet-}) \otimes \mu^I\delta^{\bar{I}\setminus\{x\}}(k^{\bullet-})\Big)\end{aligned}$$

but since $\{I \mid I \subset J, x \notin I\}$ is in bijection with $\{I \smallsetminus \{x\} \mid I \subset J, x \in I\}$, in each of the above sums, the corresponding terms have the same indices. Hence, the first and last terms in each sum combine, and the second and third terms cancel out to give

$$\sum_{x \in I \subset J} \mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}(k^{\bullet +}) \otimes \mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}(k^{\bullet -}) \otimes \mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}(k^{\bullet -}).$$

Since neither $\mu^{\bar{I}} \delta^{I \smallsetminus \{x\}}$ or $\mu^{I \smallsetminus \{x\}} \delta^{\bar{I}}$ are with respect to x, this can be written

$$\sum_{I \subset J \smallsetminus \{x\}} \mu^{\bar{I}} \delta^{I}(k^{\bullet +}) \otimes \mu^{I} \delta^{\bar{I}}(k^{\bullet +}) - \mu^{\bar{I}} \delta^{I}(k^{\bullet -}) \otimes \mu^{I} \delta^{\bar{I}}(k^{\bullet -})$$

which by the inductive hypothesis is

$$\begin{aligned} \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet +})) - \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet -})) &= \Delta(\delta^{J \smallsetminus \{x\}}(k^{\bullet +}) - \delta^{J \smallsetminus \{x\}}(k^{\bullet -})) \\ &= \Delta(\delta^{J}(k^{\bullet})). \end{aligned}$$

□

Proof of Proposition 1.3.8 The operators δ^I and $\mu^{\bar{I}}$ commute since they are evaluating different singular points. Let I be an arbitrary subset of S, and let $|I| = i$ and $|S| = j$, then the left cofactor is in $\mathcal{K}_i$ and the right in $\mathcal{K}_{j-i}$. □

Remark 1.3.11 Not all knot invariants respect the singular knot filtration, but the point of the Vassiliev invariants is that they're the ones that are natural with respect to the singular knot filtration.

To illustrate this point, let's look at the dual filtered bialgebra of $\mathcal{K}$.

Definition 1.3.12 The **filtered bialgebra of Vassiliev invariants** is the set of Vassiliev invariants, with the (increasing) filtration given by the degree of the Vassiliev invariant, the product given by multiplication pointwise

$$V_1 V_2(k) = V_1(k) V_2(k),$$

and the coproduct η given by

$$\eta(V)(k_1 \otimes k_2) = V(k_1 \# k_2).$$

Proposition 1.3.13 *The filtered bialgebraic dual of the (descending) filtered bialgebra $\mathcal{K}$ is the (ascending) filtered bialgebra of Vassiliev invariants.*

Proof Put a proof here.

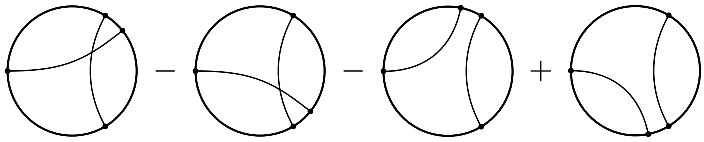
Example 1.3.14 In view of Remark 1.3.11, let's define a knot invariant. (Define f_{3_1}.)

Remark 1.3.15 Let's continue the polynomial analogy with an observation due to Willerton in [Wil96]. Taking the dual statement to Lemma 1.3.10, we get a theorem analogous to the Leibniz theorem for functions of multiple variables. The statement that Vassiliev invariants are a filtered algebra then follows from the definition of Vassiliev invariants and the lemma. The same argument in multivariable calculus uses the Leibniz theorem to prove that polynomials are a filtered vector space.

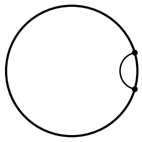
1.4 The Hopf algebra of chord diagrams

This bialgebra strucutre on knots is heavily related to a Hopf algebra structure on chord diagrams. The general idea is to study the former via the latter.

In Section 1.2 we saw how functions on chord diagrams specify Vassiliev invariants, so long as the functions satisfy 4T and 1T. We can instead encode this directly into the algebra of chord diagrams by the following relations.



$$= 0. \tag{4T*}$$



$$= 0 \tag{1T*}$$

Definition 1.4.1 We define $\mathcal{A}_m$, the **space of chord diagrams** of degree m as

$$\mathcal{A}_m = \mathcal{D}_m/\texttt{4T}^*, \texttt{1T}^*,$$

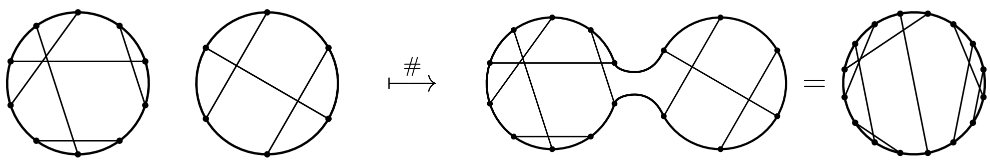
and $\mathcal{A}$, the **space of chord diagrams** as

$$\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m.$$

Warning 1.4.2 Both elements of $\mathcal{A}$ and $\mathcal{D}$ are known as chord diagrams. From now on when we say "a chord diagram", we mean an element of $\mathcal{A}$ unless otherwise specified.

The algebra $\mathcal{A}$ has multiplication and coproduct operations that mirror those in $\mathcal{K}$.

Definition 1.4.3 The **connected sum** of two chord diagrams A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



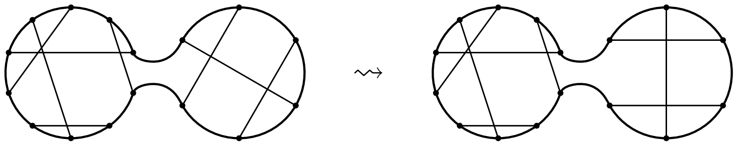
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed in the algebra $\mathcal{D}$ this is ill-defined. However the 4T* relation in $\mathcal{A}$ takes care of this.

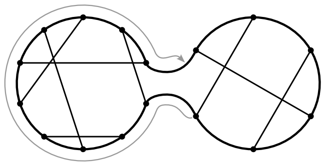
Proposition 1.4.4 *The connected sum operation $\# : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is well-defined.*

Proof We will prove that the connected sums, given any two choices of connection locations, are equal modulo 4T*.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one 'click', like so:

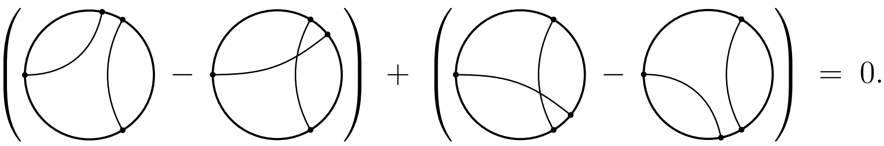


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow.



Which we show can be achieved by a series of 4T* relations.

We can rewrite 4T* as



A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T*. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T*. So, the two connected sums $a_1 \# a_2$ differ by a sum of 4T* relations, completing the proof. □

Proposition 1.4.5 *The connected sum operation makes $\mathcal{A}$ into a graded algebra.*

Proof The connected sum of a chord diagram of order i and a chord diagram of order j is a chord diagram of order $i + j$.

The coproduct on chord diagrams (will this be relevant if we use the other coproduct later anyway?). And the antipode I suppose.

Proposition 1.4.6 *The graded Hopf-algebraic dual of $\mathcal{A}$ is the the Hopf algebra $\mathcal{W}$ of weight systems.*

Proof We have constructued it as so. A map $f : \mathcal{A} \to \mathbb{F}$ satisfies relations that make it a weight system.

1.5 The Kontsevich integral

In Section 1.2, we gave the fundamental theorem of Vassiliev invariants. The point of this theorem is that is establishes a particular relationship between the algebras of the previous two chapters, $\mathcal{K}$ and $\mathcal{A}$ (or equivalently, between $\mathcal{W}$ and $\mathcal{V}$). But admittedly, in the form of Theorem 1.2.17, it's not a-priori obvious why this is the case. Here we give a restatement of that theorem that makes the relationship explicit.

Definition 1.5.1 The **associated graded** algebra of a filtered algebra A is the algebra formed by the direct sum of the succesive quotient by the filtered components of A. For an algebra with a descending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m+1},$$

and for an algebra with an ascending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m-1},$$

where $A_{-1} = \{0\}$.

Theorem 1.5.2 (Fundamental theorem) *The algebra of weight systems is isomorphic to the associated graded algebra of the algebra of Vassiliev invariants, $\mathcal{W} \cong \operatorname{gr} \mathcal{V}$, or on the level of graded components,*

$$\bigoplus_{m=0}^{\infty} \mathcal{W}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{V}_m / \mathcal{V}_{m-1}.$$

Equivalently, it can be stated in the dual setting as follows. The algebra of chord diagrams is isomorphic to the associated graded algebra of the algebra of knots, $\mathcal{A} \cong \operatorname{gr} \mathcal{K}$, or on the level of graded components,

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{K}_m / \mathcal{K}_{m+1}.$$

We can break the fundamental theorem up into two parts.

Vassiliev	Every Vassiliev invariant modulo Vassiliev invariants of higher order gives a Weight system, so a map $$\mathcal{V}_m / \mathcal{V}_{m-1} \to \mathcal{W}_m.$$	Every chord diagram gives an element of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$, so a map $$\mathcal{A}_m \to \mathcal{K}_m / \mathcal{K}_{m+1}.$$
Kontsevich	Every weight system gives a Vassiliev invariant modulo Vassiliev invariants of higher order, so a map $$\mathcal{W}_m \to \mathcal{V}_m / \mathcal{V}_{m-1}$$ which is inverse to the above.	Every equivalence class of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$ gives a chord diagram, so a map $$\mathcal{K}_m / \mathcal{K}_{m+1} \to \mathcal{A}_m.$$ which is inverse to the above.

It is actually only the second part of this theorem, due to Kontesevich that is equivalent to Theorem 1.2.17. The first part, due to Vassiliev, is additional (so in reality, this 'reformulation' is stronger), but considered "easy" relative to the second part. If the goal is to describe $\mathcal{K}$ by $\mathcal{A}$, it states that the relations in $\mathcal{A}$, 4T* and 1T* are truly compatible with $\mathcal{K}_n/\mathcal{K}_{n+1}$. Indeed this was the point of Section 1.2, so most of the work is already done.

Proof of Theorem 1.5.2 (Vassiliev) We start by defining the map $a \in \mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$. The map is given by choosing a singular knot $k^\bullet \in \mathcal{K}_m^\bullet$ whose chord diagram is a, then resolving it to an element of $k \in \mathcal{K}_m$, and projecting that into the quotient $\mathcal{K}_m/\mathcal{K}_{m+1}$. This is well-defined since any other $k^{\bullet\prime}$ also has chord diagram a. So by allowing crossing changes in $\mathcal{K}_m^\bullet$, $k^\bullet$ and $k^{\bullet\prime}$ may be made equivalent. In the presence of n singular points, the δ-images of singular knots in $\mathcal{K}_n^\bullet$ are in the δ-image of $\mathcal{K}_{n+1}^\bullet$, so equivalent in the quotient.

Insert relevant figures.

Recalling that $\mathcal{A}_m = \mathcal{D}_m/1\mathrm{T}^*, 4\mathrm{T}^*$, we need to show that the map is well-defined with respect to those relations. Indeed 1T* is in the kernel, as any singular knot with a 1T* singular point resolves to a difference of two of the same knots when that singular point is resolved.

4T* is also in the kernel as it comes from a closed singular isotopy in the strata. When resolving such a closed isotopy, things always cancel out.

Insert figures and make an argument about the dual case.

□

Let's turn to the other part of the fundamental theorem, with the following definition.

Definition 1.5.3 The **completion** of a filtered algebra A is the filtered algebra

$$\widehat{A} = \varprojlim_{m\to\infty} A_m/A_{m+1},$$

and in the cases we care about, this will be equal to the degree-completion of $\operatorname{gr} A$, though I am not sure if that is true in general.

It is only filtered and not graded since infinitely many terms are non-zero and this is a technical restriction on direct sums.

The part of Theorem 1.5.2 due to Maxim Kontsevich [Kon93] is much more involved. Konsevich constructed an integral invariant which proves the fundamental theorem, as well as containing the information of all Vassiliev invariants at the same time. We will not give a detailed exposition of Kontsevich's invariant here, but they abound in the literature, for example [BS97; CD05; CDM12] in order of increasing level of detail. Rather we will boil the Kontsevich integral down to the following key property.

Theorem 1.5.4 (Kontsevich) *There is an invariant of knots, $Z : \mathcal{K} \to \widehat{\mathcal{A}}$, called the Kontsevich integral, with the following property. If $k \in \mathcal{K}_m$ is a linear combination of knots with $k = \delta^m(k^\bullet)$ and $k^\bullet$ has chord diagram $a \in \mathcal{A}_m$, then*

$$Z(k) = a + \textit{higher degree terms.}$$

Proof of Theorem 1.5.2 (Kontsevich) Take the map $k \in \mathcal{K}_m \to \mathcal{A}_m$ coming from killing the higher degree terms in the Kontsevich integral, and taking the lowest order non-zero chord diagram. This factors through the quotient to a map $\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m$ since by Theorem 1.5.4, any additional $k' \in \mathcal{K}_{m+1}$ contributes only higher degree terms, which get killed. It is easy to see that the two maps are inverses.

Make argument about the dual case, which is taking the kontsevich integral then composing with a weight system gives a Vassiliev invariant.

⋆ ⋆ ⋆

Definition 1.5.5 There is a category of filtered algebras and filtration-respecting maps, and there is a category of graded algebras and grading-respecting maps. The **associated graded** map of a filtration-respecting map $f : A \to B$ is the grading-respecting map $\mathrm{gr}\, f : \mathrm{gr}\, A \to \mathrm{gr}\, B$ defined by taking a representative a_m to $f(a_m)$ then projecting onto B_m/B_{m+1},

$$\mathrm{gr}\, f : a_m + A_{m+1} \mapsto f(a_m) + B_{m+1}.$$

This makes gr a functor between these two categories.

Part about one direction of the f.t. being easy. Talk about why the other isn't (also how in the dual case it makes it hard to define a map knots to chord diagrams).

The fundamental theorem of Vassiliev invariants allows us to establish a relationship between knots and chord diagrams, or equivalently, Vassiliev invariants and weight systems. It's not a-priori obvious how Theorem 1.2.17 does this, but is apparent in the following equivalent reformulation.

The part due to Kontsevich is very involved. Kontsevich defined a knot invariant called the Kontsevich integral which is an incredibly fascinating object. We skip many of the details.

Remark 1.5.6 It is important to note that a-priori, we have no well-defined map $\mathcal{K} \to \mathcal{A}$ that respects the singular knot filtration. We do have such a map for singular knots $\mathcal{K}^\bullet \to \mathcal{A}$, but to extend this to $\mathcal{K}$, we need to be sure that a linear combination in $\mathcal{K}$ coming from resolving a singular knot can only be done in one way, or at least by singular knots with only one type of chord diagram. This fact is provided by the Kontsevich integral.

⋆

Definition 1.5.7 A universal Vassiliev invariant is a map

$$U : \mathcal{K} \to X$$

to some space X, such that any Vassiliev invariant can be factored through U. For any Vassiliev invariant v, there is a linear functional f on X such that $v = f \circ U$.

Remark 1.5.8 Definition 1.5.7 is also known to some as a (homomorphic) expansion, for example [BD16], and related to the concept of a formality map.

Theorem 1.5.9 (Universal Vassiliev Invariant)

2

Lie theory and Jacobi diagrams

2.1 First subsection

2.2 Second section

2.3 Third subsection

3

Jacobi diagrams as a universal enveloping algebra

3.1 First subsection

3.2 Second section

3.3 Third subsection

4

Welded knots and arrow diagrams

4.1 First subsection

4.2 Second section

4.3 Third subsection

5

Arrowed Jacobi diagrams as a universal enveloping algebra

5.1 First subsection

5.2 Second section

5.3 Third subsection

References

[Bar95] Dror Bar-Natan. "On the Vassiliev knot invariants". In: *Topology* 34.2 (1995), pp. 423–472.

[BD16] Dror Bar-Natan and Zsuzsanna Dancso. "Finite-type invariants of w-knotted objects, I: w-knots and the Alexander polynomial". In: *Algebraic & Geometric Topology* 16.2 (2016), pp. 1063–1133.

[BL93] Joan S. Birman and Xiao-Song Lin. "Knot polynomials and Vassiliev's invariants". In: *Inventiones Mathematicae* 111.2 (1993), pp. 225–270.

[BS97] Dror Bar-Natan and Alexander Stoimenow. *The Fundamental Theorem of Vassiliev Invariants*. 1997. arXiv: q-alg/9702009.

[CD05] Sergei Chmutov and Sergei Duzhin. *The Kontsevich integral*. 2005. arXiv: math/0501040.

[CDM12] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

[Hut98] Michael Hutchings. "Integration of Singular Braid Invariants and Graph Cohomology". In: *Transactions of the American Mathematical Society* 350.5 (1998), pp. 1791–1809.

[Kon93] Maxim Kontsevich. "Vassiliev's knot invariants". In: *Advances in Soviet Mathematics* 16(2) (1993), pp. 137–150.

[Sta96] Ted Stanford. "Finite-type invariants of knots, links, and graphs". In: *Topology* 35.4 (1996), pp. 1027–1050.

[Wil96] Simon Willerton. "Vassiliev invariants and the Hopf algebra of chord diagrams". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 119.1 (1996), pp. 55–65.

[Wil98] Simon Willerton. "A Combinatorial Half-Integration from Weight System to Vassiliev Knot Invariant". In: *Journal of Knot Theory and Its Ramifications* 07.04 (1998), pp. 519–526.