On Vassiliev Invariants

Damian Lin

An essay submitted in fulfilment of
the requirements for the degree of
Master of Philosophy (Science)

Pure Mathematics
University of Sydney

THE UNIVERSITY OF
SYDNEY

July 30, 2025

Contents

Acknowledgements

Thanks to ...

Introduction

$\mathbb{T}$HE space of knots is the disconnected space of embeddings of $\mathbb{S}^1$ into $\mathbb{R}^3$, in which the connected components, the "rooms", are the knot types. Vassiliev studied the space of knots by looking at its "walls" of a specific type: those immersions of $\mathbb{S}^1$ into $\mathbb{R}^3$ which fail to be embeddings by having a single point of trasverse intersection. The addition of these walls makes the space of objects we study a connected space. Any two knots are connected by a path that passes through finitely many of the walls. The space of walls is also disconnected, but it can be connected by allowing paths to pass through finitely many "cornices" where two walls meet. These are immersions that fail to be embeddings by having two points of transverse intersection. The cornices are again disconnected but can be connected by "corners" and so on, with any immersions with m double points being connected by a finite number of immersions with $m + 1$. This produces the "stratification" of the space: a very schematic illustration is given below. Note that this picture doesn't properly capture the infinite-dimensional nature of the stratification. Nor some other missing details which will appear in Chapter 1.
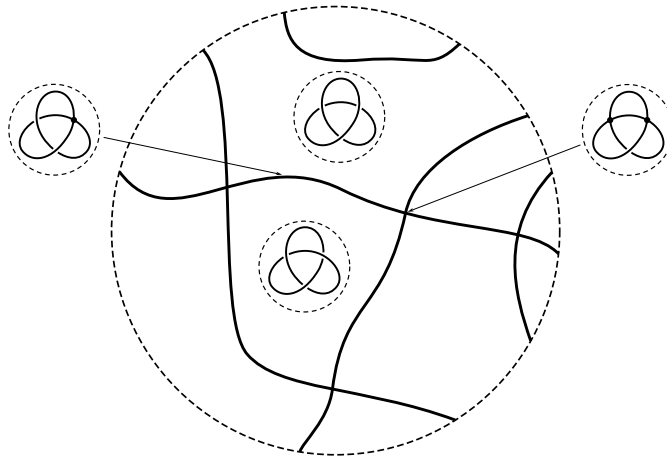
In [Vas90], Vassiliev makes a sequence of approximations of the cohomology ring of the space of knots, yielding a certain subring of the zeroth cohomology ring. Elements of this are locally constant functions on the space, so knot invariants. This subring of "Vassiliev" knot invariants can be computed on any knot by some procedure involving the homology group of the strata at a finite number of increasing depths.

Birman and Lin in [BL93] give an axiomatic definition of the Vassiliev invariants as those that respect the Vassiliev skein relation. It follows, as we will see in Chapter 1 that Vassiliev invariants can be described completely combinatorially by functions on chord diagrams obeying certain combinatorial rules.

By the work of Bar-Natan in [Bar95] the algebra of chord diagrams turns out to be equivalent to a different diagrammatic algebra, that of Jacobi diagrams, again up to a different set of combinatorial rules. This will be discussed in Chapter 3. This change of perspective introduces Lie theory in the following sense. A key relation in the algebra of Jacobi diagrams is a formal version of the relation in the universal enveloping algebra of a Lie algebra that the bracket is equal to the commutator. This is discussed in Chapter 4. A rigorous version of this statement is the work of Hinnich and Vaintrob [HV00] which constructs the algebra of Jacobi diagrams as the universal enveloping algebra object of some Lie algebra object in some tensor category.

A paragraph introducing welded knots: If knots have to do with the configuration space of some number of points, then welded knots have to do with the configuration space of some number of 'flying rings'. Arrowed Jacobi diagrams will need to be introduced. Kashiwara-Vergne may need to be mentioned.

In Chapter 6, we generalise the result that Jacobi diagrams are a universal enveloping algebra to directed(/welded/arrowed) Jacobi diagrams.

Some paragraphs talking about the rest of the Chapters.

1

Vassiliev invariants and chord diagrams

$\mathbb{V}$ᴀssɪʟɪᴇᴠ invariants are sophisticated to define in terms of the space of knots from the introduction, but the axiomatic definition of Birman-Lin [BL93] is much simpler. The definition also illustrates an analogy first made by Bar-Natan [Bar95] in which Vassiliev invariants are "polynomial invariants". This is not meant in the sense that Vassiliev invariants take values in a polynomial ring (like say, the Jones polynomial), but rather that Vassiliev invariants have special properties not shared by all invariants, just as polynomial functions have special properties not shared by all functions.

1.1   Singular Knots

Definition 1.1.1   A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are double-points and transverse. When a singular knot has m such singularities, we call it m**-singular**.

Remark 1.1.2   Immersions with other types of singularities, are excluded from this definition, so the word "singular" in "singular knot" refers specifically to double point singularities. In particular immersions with

  (a) triple points

  (b) points with vanishing derivative

are excluded from the definition.

A singular knot with one double point is very close to two other knots, one where it's replaced by a positive crossing and one by a negative. If the conditions are right, we can extend a knot invariant to an invariant of singular knots by "taking its derivative".

Definition 1.1.3   The **derivative** δ of a differentiable m-singular knot invariant f is

$$\delta f \left(\vcenter{\hbox{\includegraphics{doublepoint}}} \right) = f \left(\vcenter{\hbox{\includegraphics{poscross}}} \right) - f \left(\vcenter{\hbox{\includegraphics{negcross}}} \right).$$

What are the conditions? For this to be a well-defined operation, it mustn't matter which double point we choose.

Definition 1.1.4 An invariant f of m-singular knots is **differentiable** if

$$f \left(\rotatebox{0}{\includegraphics{}} \right) - f \left(\rotatebox{0}{\includegraphics{}} \right) = f \left(\rotatebox{0}{\includegraphics{}} \right) - f \left(\rotatebox{0}{\includegraphics{}} \right). \tag{DIFF}$$

If an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points.

Rather than thinking about functions on knots satisfying certain relations, the modern version of this subject takes the philosophy of imposing relations on the objects directly.

Definition 1.1.5 Define $\mathcal{K}_m = \mathrm{span}(\{m\text{-singular knots}\})$ modulo the following boundary relation (also known as a codifferentiability relation):

$$\rotatebox{0}{\includegraphics{}} - \rotatebox{0}{\includegraphics{}} = \rotatebox{0}{\includegraphics{}} - \rotatebox{0}{\includegraphics{}}. \tag{DIFF*}$$

From now on, we will refer to elements $\mathcal{K}_m$ as m-**singular knots**, i.e. the DIFF* relation will be implicit in everything.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m \to \mathcal{K}_{m-1}$ defined by

$$\rotatebox{0}{\includegraphics{}} \longmapsto \rotatebox{0}{\includegraphics{}} - \rotatebox{0}{\includegraphics{}}.$$

Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as a invariant of knots in $\mathcal{K}_m$.

Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

$$f^{(m+1)} \left(\rotatebox{0}{\includegraphics{}} \right) = f^{(m)} \left(\rotatebox{0}{\includegraphics{}} \right) - f^{(m)} \left(\rotatebox{0}{\includegraphics{}} \right).$$

Often, we omit the superscript and write

$$f \left(\rotatebox{0}{\includegraphics{}} \right) = f \left(\rotatebox{0}{\includegraphics{}} \right) - f \left(\rotatebox{0}{\includegraphics{}} \right).$$

The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation, there is an integer m such that

$$V \left(\underbrace{\rotatebox{0}{\includegraphics{}} \cdots \rotatebox{0}{\includegraphics{}}}_{m+1} \right) = 0.$$

(b) The **order** of a Vassiliev invariant V it the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the highest number of double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 Vassiliev invariants of order m are those that vanish after $m + 1$ derivatives, just like degree m polynomials.

1.2   Integration Theory

To help see the bird's eye view, and following [Hut98], we phrase this in terms of an integration theory.

Definition 1.2.1   An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group is $\mathcal{O}_0$ is typically free abelian, and in our case is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behaves like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

We wish to understand how to invert this process, namely:

Questions 1.2.2   (a) When does a functional in $\mathcal{O}_m^*$ "integrate" into a functional in $\mathcal{O}_{m-1}^*$?

  (b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?

  (c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

  (d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?

  (e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}_m^*$?

The following modules provide the tautological answers to the above questions in the general theory.

Definitions 1.2.3   (a) The **primary obstructions to integration** are the module

$$P\mathcal{O}_m = \ker \partial_m.$$

  (b) The **constants of integration** are the module

$$C\mathcal{O}_m = \mathcal{O}_m / \partial \mathcal{O}_{m+1}.$$

  (c) The **secondary obstructions to integration** are the module

$$S\mathcal{O}_m = \ker(\partial_{m+1}\partial_m)/P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.

(d) The **weights of integration** are the module

$$WO_m = CO_m / \pi(PO_m)$$

where $\pi : O_m \to CO_m$ is the projection.

(e) The **finite type invariants** of order m are the module

$$FTO_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m + 1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $O^\star = K^\star$ which should provide good intuition for the general case.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4   A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the mth and $(m+1)$st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m + 1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

To rephrase the definitions in section 1.1, in the integration theory $O^\star = K^\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m + 1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

Due to the boundary relation this is always well-defined.

Integration, is to construct from an invariant P of $(m + 1)$-singular knots an invariant Q of m-singular knot such that

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a "path-integral" along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving "holes" in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:
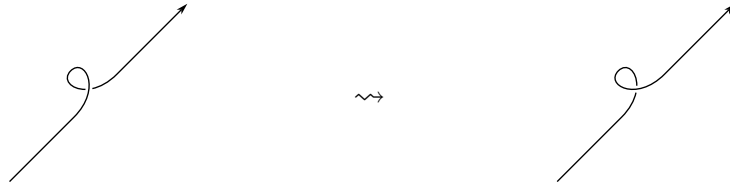
Definitions 1.2.5   (a) The following closed singular isotopy around a triple point:

around

gives rise to the **topological four-term relation**

$$f\left(\;\includegraphics{}\;\right) - f\left(\;\includegraphics{}\;\right) - f\left(\;\includegraphics{}\;\right) + f\left(\;\includegraphics{}\;\right) = 0. \quad \text{(T4T)}$$
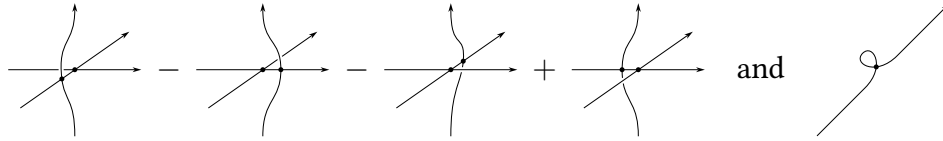
(b) The closed singular isotopy

around a point with a vanishing derivative gives rise to the **topological one-term relation**

$$f\left(\;\includegraphics{}\;\right) = 0. \quad \text{(T1T)}$$

The necessity of the above relations for an invariant of m-singular knots to integrate is equivalent to the assertion that

 and

are in $P\mathcal{K}_m = \ker \partial_m$. Let us denote arbitrary such singular knots **T4T** and **T1T**.

The question that arises is whether we have found all primary obstructions. Do **T4T** and **T1T** span $\ker \partial$?

Theorem 1.2.6 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $m - 1$-singular knots if and only if it satisfies* T4T *and* T1T.

> **Proof (sketch)** Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m - 1$ double points, and additional worse singularities. The only events to consider are codimension $m + 2$ (why $m + 2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with DIFF relations, which we've said are implicit).
>
> Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

In the order of Questions 1.2.2 and Definitions 1.2.3, we ought to talk next about the secondary and further obstructions to integration. However, let us postpone this until after we have discussed constants of integration, which will be a more natural place.

The constants of integration comprise the information that is lost when integrating. In the case of knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two neighbouring m-singular knots, but the value of the invariant on either of these

m-singular knots is lost. When integrating, values on these can be chosen freely. Recall from Definition 1.2.3 (b) that $C\mathcal{K}_m = \mathcal{K}_m/\partial\mathcal{K}_{m+1}$. The difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. We see that constants of integration are singular knots which are blind to crossing changes, so the only data needed to describe them is the order in which the double points are traversed around the knot.

Definitions 1.2.7   (a) A **chord diagram** of order m is an element of $\mathcal{K}_m/\partial\mathcal{K}_{m+1}$. This is equivalent to an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. Chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}_m$.

   (b) The **chord diagram of an m-singular knot** K, denoted $\pi(K)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of K. Traversing both K and the circle, label the points on the circle in the order in which the singular points of K are traversed. Each label is given twice, pairing up the $2m$ points, forming $\pi(K)$.

Example 1.2.8
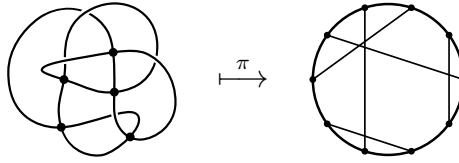
Proposition 1.2.9   *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\pi(k))$$

for $q \in \mathcal{D}_m^*$ Then Q_0 is also an integral of P.

Proof   The derivative of a function of chord diagrams is zero, as two knots which differ by a crossing change have the same chord diagram. Hence Q and Q_0 have the same derivative: P.   □

Remark 1.2.10   We defined a constant of integration as a set of objects (e.g. $c \in \mathcal{D}_m$) but perhaps it would have been more accurate to define it as a functional (e.g. $q \in \mathcal{D}_m^*$). After all the constant of integration on the real line is "$+ C$" rather than the set $\{*\}$ with one object. There is no real risk of confusion, so let us be slightly loose and use the terminology for either.

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.11   A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

If we take the relations for a knot invariant to be integrable and project them into the space of chord diagrams, we get the following relations.

Definition 1.2.12   (a) The **four-term relation** is the relation

$$4T = 0 \tag{4T}$$

(b) The **one-term relation** is the relation

$$1T = 0 \tag{1T}$$

Proposition 1.2.13 *A weight system is characterised as a constant of integration that satisfies* 4T *and* 1T.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T and 1T. Since crossing changes are free, this is equivalent to satisfying the projection of T4T and T1T into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of on m-singular P is of the form

$$Q + q \circ \pi.$$

Since integration is linear, to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that $S\mathcal{K}_m$ vanishes. But this is a tautological statement of the general theory — it doesn't mean much when we don't know what $S\mathcal{K}_m$ is.

Conjecture 1.2.14 *An invariant of m-singular knots satisfying* T4T *and* T1T *integrates m times into a genuine knot invariant.*

Remarks 1.2.15 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.6. The point is that it may not be possible to choose the integral to again satisfy T4T and T1T, which is what $S\mathcal{K}$ measures.

(b) Computing $S\mathcal{K}_m$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.

(c) This conjecture is proven in certain cases. It holds the integration theory for braids [Hut98]. In a certain sense it's "half"-proven for knots [Wil98]. In Chapter 2 we extend this to "a little more than half".

Theorem 1.2.16 (Fundamental theorem of Vassiliev invariants) *Let f be an invariant of m-singular knots satsifying* T4T, T1T *and the additional condition that $\delta f = 0$. Then f integrates m times into a genuine knot invariant.*

Corollary 1.2.17 *The fundamental theorem of Vassiliev invariants holds even if the additional condition is replaced by $\delta^k f = 0$ for some k.*

1.3   Third subsection

2

Combinatoral integration of weight systems

3

Lie theory and Jacobi diagrams

3.1 First subsection

3.2   Second section

3.3   Third subsection

4

Jacobi diagrams as a universal enveloping algebra

5

Welded knots and arrow diagrams

5.1   First subsection

5.2   Second section

5.3   Third subsection

6

Arrowed Jacobi diagrams as a universal enveloping algebra

6.1   First subsection

6.2   Second section

6.3   Third subsection

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