On Vassiliev Invariants

Damian Lin

An essay submitted in fulfilment of
the requirements for the degree of
Master of Philosophy (Science)

Pure Mathematics
University of Sydney

December 13, 2025

Contents

Acknowledgements

Thanks to ...

Introduction

The introduction goes here.

1

Vassiliev invariants and chord diagrams

Something to maybe include somewhere:

The point of Vassiliev theory is to study the space of knots in the context of the singularities that lie between knots.

In view of this point, we spend Section 1.1 looking at a the stratification of the space of knots. This leads to a beautiful (and fruitful) classical analogy which we will explored in Section 1.2 and throughout this chapter.

In Section 1.3, with the context in mind, we introduce the main players in this theory.

POLYNOMIAL functions are a special type of function. They are related in a natural way to the derivative, they are defined by a finite amount of combinatorial data, and they can be used to approximate any continuous function.

In the same way, the Vassiliev invairants are a special type of functions. And this analogy, first made by Dror Barn-Natan in [Bar95] is by no means superficial. As we will come to see, Vassiliev invariants enjoy analogues of the first two properties above, and conjeturally also the third.

In this chapter we give a version of the introductory theory of Vassiliev invariants in which the analogy above is made as explicitly as possible. This also leads a natural interpretation of the defining relations of the algebra $\mathcal{A}$ which is the fundamental object of study in the field.

1.1 Singular knots

Definition 1.1.1 A **singular knot** is an immersion of S^1 into $\mathbb{R}^3$ which fails to be an embedding at finitely many singularities, and where the singularities are all double-points of transverse intersection. When a singular knot has m such singularities, we call it m**-singular**.

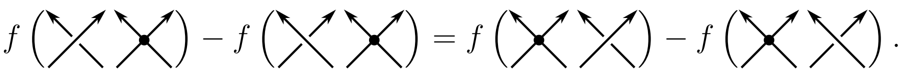
Remark 1.1.2 Immersions with other types of singularities, are excluded from this definition, so the word “singular” in “singular knot” refers specifically to double point singularities. In particular immersions with

(a) triple points
(b) points with vanishing derivative
are excluded from the definition.

A singular knot with one double point is very close to two other knots. In one, the double point is replaced by a positive crosing, and in the other a negative crossing.

Just as we have notions of ambient isotopy for knots, and knot invairants, we can have m-singular isotopy and m-singular knot invariants.

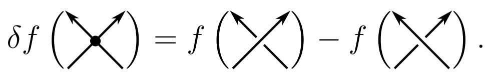
Definition 1.1.3 An invariant f of m-singular knots is **differentiable** if



(DIFF)

If an m-singular knot invariant is differentiable, we can extend it to an invariant of $(m+1)$-singular knots by a procedure analagous to taking its derivative.

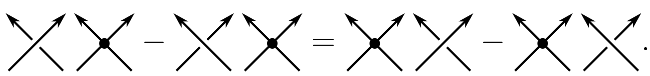
Definition 1.1.4 The **derivative** δ of a differentiable m-singular knot invariant f is an $(m+1)$-singular knot invariant



A regular knot invariant (which is an invariant of 0-singular knots) satisfies this condition vaccuously so is differentiable and its derivative is an invariant of 1-singular knots. Furthermore, if an invariant of m-singular knots is differentiable, so is its derivative, so it can be extended to any number of double points. In-particular, regular knot invariants have derivatives of any order.

Rather than thinking about functions on knots satisfying certain relations, the modern view of this subject takes the philosophy of imposing relations on the objects directly.

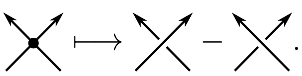
Definition 1.1.5 Define $\mathcal{K}_m^\bullet$ as the span of all m-singular knots, taken over $\mathbb{Q}$, modulo the following boundary relation (also known as a codifferentiability relation):



(DIFF*)

From now on, we will refer to elements $\mathcal{K}_m^\bullet$ as m-**singular knots**, i.e. the DIFF* relation will be implicitly assumed.

Definition 1.1.6 The **boundary** operation is the map $\partial : \mathcal{K}_m^\bullet \to \mathcal{K}_{m-1}^\bullet$ defined by

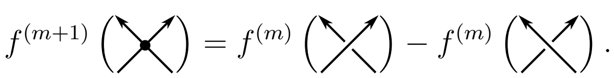


Remark 1.1.7 The derivative operation and the DIFF relation are dual to the boundary operation and the DIFF* relation. For example, a differentiable invariant of knots is the same as an invariant of knots in $\mathcal{K}_m^\bullet$.

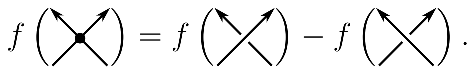
Any knot invariant, f can be extended to an invariant $f^{(m)}$ of m-singular knots by the Vassiliev skein relation

$$f^{(0)} = f$$

and

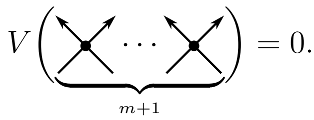


Often, we omit the superscript and write



The Vassiliev skein relation extends a knot via its derivative, or chooses a value on $(m+1)$-singular knots to agree with the difference of values on its boundary.

Definitions 1.1.8 (a) A knot invariant V is a **Vassiliev invariant** of order (or type) m if when extended to singular knots via the Vassiliev skein relation,



(b) The **order** of a Vassiliev invariant V is the highest m such that V is a Vassiliev invariant of order m. (That is, the order of a Vassiliev invariant is the most double points a knot K can have without $V(K)$ having to vanish).

Remark 1.1.9 In other words, Vassiliev invariants of order m are those that vanish after $m+1$ derivatives, just like degree m polynomials.

1.2 The stratification of the space of knots and integration

To help see the bird's eye view we phrase the analogy between Vassiliev invariants and polynomials in terms of an integration theory, following [Hut98].

Definition 1.2.1 An **integration theory** $(\mathcal{O}_\star, \partial_\star)$ is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. In case we need to refer to a specific map, let ∂_m denote the map ∂ whose domain is $\mathcal{O}_m$. Note that we do not assume $\partial^2 = 0$.

The group $\mathcal{O}_0$ is typically free abelian, and in our case this is the primary object we want to study. The groups $\mathcal{O}_m$ are also typically free abelian groups, and can often be thought of as m-singular objects of some kind. The map ∂ takes an m-singular object x to some combination of $(m-1)$-singular objects near x.

By fixing an abelian group G and setting $\mathcal{O}_m^* = \mathrm{Hom}(\mathcal{O}_m, G)$, we get the sequence

$$\cdots \xleftarrow{\delta} \mathcal{O}_m^* \xleftarrow{\delta} \mathcal{O}_{m-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathcal{O}_1^* \xleftarrow{\delta} \mathcal{O}_0^*$$

where δ_m is the transpose of ∂_m. The maps δ behave like derivatives: $\delta(f)$ for $f \in \mathcal{O}_m^*$ defines f on $\mathcal{O}_{m+1}^*$ as some combination of its values on "close" m-singular objects.

Questions 1.2.2 We wish to understand how to invert this process, namely:

(a) When does a functional in $\mathcal{O}_m^*$ "integrate" to a functional in $\mathcal{O}_{m-1}^*$?
(b) Is the integral of a functional in $\mathcal{O}_m^*$ uniquely defined, or are there choices to be made?
(c) When does such a functional integrate multiple times, in-particular when does it integrate m times into a functional in $\mathcal{O}_0^*$, (i.e. a function on the non-singular objects)?

(d) If there are choices to be made in integration, do they affect whether the new functional is integrable again?
(e) Which functions on the non-singular objects $\mathcal{O}_0$ are obtained by m consecutive integrations of functionals in $\mathcal{O}^*_m$?

The answers to the above questions are given precisely by the following modules

Definitions 1.2.3 (a) The **primary obstructions to integration** are the module

$$P\mathcal{O}_m = \ker \partial_m.$$

(b) The **constants of integration** are the module

$$C\mathcal{O}_m = \mathcal{O}_m / \partial\mathcal{O}_{m+1}.$$

(c) The **secondary obstructions to integration** are the module

$$S\mathcal{O}_m = \ker (\partial_{m+1}\partial_m)/P\mathcal{O}_m,$$

and likewise the **order k obstructions to integration** are defined analagously.
(d) The **weights of integration** are the module

$$W\mathcal{O}_m = C\mathcal{O}_m/\pi(P\mathcal{O}_m)$$

where $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ is the projection.
(e) The **finite type invariants** of order m are the module

$$FT\mathcal{O}_m = \ker \delta^{m+1},$$

where δ^{m+1} denotes $m+1$ applications of δ with appropriate indices, ending with δ_m.

It may not be entirely obvious how these definitons provide answers to the questions above. In the rest of this section we will see the truth of this in the case $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$.

Recall from the picture from the introduction that m-singular knots are components of the stratification of the space of knots of codimension m.

Definition 1.2.4 A **singular isotopy**, $\Psi(t)$ of m-singular knots is a path in the union of the m-th and $(m+1)$-st strata such that the path only intersects the $(m+1)$st stratum transversally and finitely many times. The intersections $\{\Psi_s : 1 \leq s \leq r\}$ of the path with the $(m+1)$st stratum are called the **singularities** of the singular isotopy. The **signs** $\varepsilon_s = \varepsilon(s) : \{s\} \to \{\pm 1\}$ of the singularities give the signs of the corresponding intersection.

Figure: singular isotopy.

To rephrase the definitions in Section 1.1, in the integration theory $\mathcal{O}_\star = \mathcal{K}^\bullet_\star$, differentiation constructs from an invariant Q of m-singular knots, an invariant Q' of $(m+1)$-singular knots such that along singular isotopies from k_0 to k_1,

$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s Q'(\Psi_s).$$

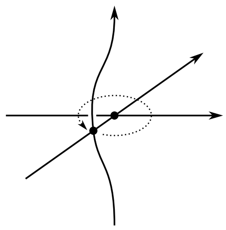
Due to the boundary relation this is always well-defined.

Integration is to construct from an invariant P of $(m+1)$-singular knots an invariant Q of m-singular knot such that

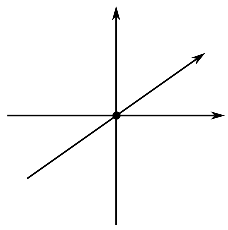
$$Q(k_0) - Q(k_1) = \sum_{s=1}^{r} \varepsilon_s P(\Psi_s),$$

in which case we write $P = Q'$. This is like a “path-integral” along a singular isotopy. For this to be well-defined, P needs to be path-independent. Equivalently, all integrals along closed paths (where $k_0 = k_1$) must vanish. In particular, recall from Remark 1.1.2 that triple points and points with vanishing derivative are excluded from all levels of the stratification, leaving “holes” in the strata. The vanishing of integrals along singular isotopies around such holes give rise to the following relations, and satisfying these are necessary conditions for P to integrate:

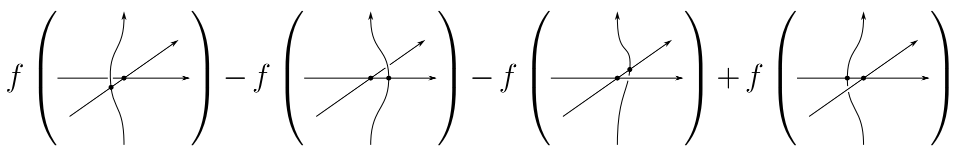
Definitions 1.2.5 (a) The following closed singular isotopy around a triple point:



around

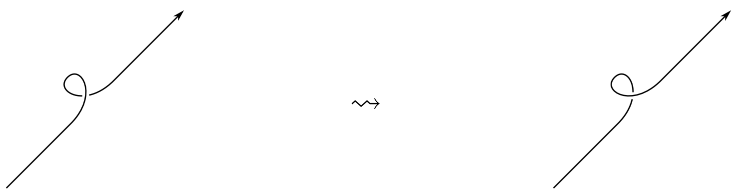


gives rise to the **topological four-term relation**

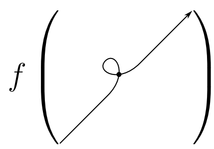


$= 0.$ (T4T*)

(b) The closed singular isotopy

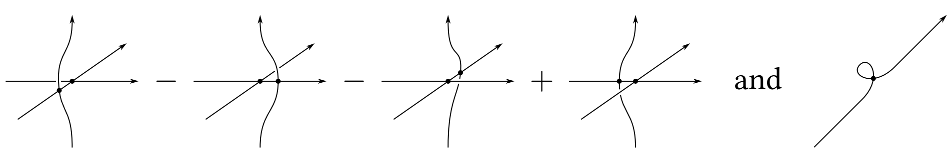


around a point with a vanishing derivative gives rise to the **topological one-term relation**



$= 0.$ (T1T*)

So, combinations of m-singular knots of the form



are in $P\mathcal{K}^\bullet_m = \ker \partial_m$. Any nonvanishing functionals on them can't integrate. But do combinations of these forms span the primary obstructions? Are they the only possible reasons that that f doesn't integrate?

Theorem 1.2.6 (Stanford [Sta96]) *An invariant f of m-singular knots integrates to an invariant of $(m-1)$-singular knots if and only if it satisfies* `T4T`* *and* `T1T`*.

Proof (sketch) Let $\gamma = \Psi(t)$ be a singular isotopy and let $\Phi(\gamma)$ be the path integral along it. Construct a homotopy from γ to the constant singular isotopy in the stratification of knots with at least $m-1$ double points, and additional worse singularities. The only events to consider are codimension $m+2$ (why $m+2$? is it to do with homotopy being "2"-dimensional?) events. They all change the $\Phi(\gamma)$ by $f(\mathbf{T4T})$ or $f(\mathbf{T1T})$ (or similar with `DIFF` relations, which we've said are implicit).

Once the homotopy is complete, we are done as $\Phi(\gamma_{\text{const}}) = 0$. On a down-to-earth level, why does this imply f can be integrated?

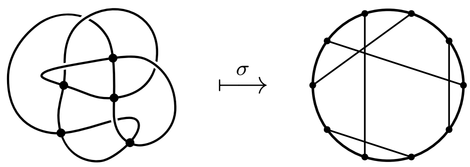
Continuing with Questions 1.2.2 and Definitions 1.2.3, the constants of integration comprise the information that is lost when differentiating. In the case of knots, given an invariant f of m-singular knots, differentiating defines an invariant δf on $(m+1)$-singular knots as the difference of f on two "neighbouring" m-singular knots. This way, the individual value of f on either of these m-singular knots is lost, and when integrating, a choice has to be made.

Recall from Definition 1.2.3 (b) that $C\mathcal{K}^\bullet_m = \mathcal{K}^\bullet_m / \partial\mathcal{K}^\bullet_{m+1}$. What do the classes of this quotient look like? Well since the difference of an m-singular knot and the same m-singular knot with one crossing changed is the image of some $(m+1)$-singular knot under ∂. Thus, we see that the data describing the constants of integration are singular knots which are blind to crossing changes, so the only data needed to describe them is the order in which the double points are traversed around the knot.

Definitions 1.2.7 (a) A **chord diagram** of order m is an element of $\mathcal{K}^\bullet_m / \partial\mathcal{K}^\bullet_{m+1}$. This is equivalent to an oriented circle with a distinguished set of m pairs of points, considered up to orientation-preserving diffeomorphism of the circle. In figures, chords are drawn between each pair of points. The vector space spanned by chord diagrams of order m is denoted $\mathcal{D}_m$, so $\mathcal{D}_m = C\mathcal{K}^\bullet_m$.

(b) The **chord diagram of an m-singular knot** k, denoted $\sigma(k)$ is the chord diagram formed by the following process. Place $2m$ points on an oriented circle, two for each singular point of k. Traversing both k and the circle, label the points on the circle in the order in which the singular points of k are traversed. Each label is given twice, pairing up the $2m$ points, forming $\sigma(k)$.

Example 1.2.8



◊

Proposition 1.2.9 *Suppose P is an integrable m-singular invariant with integral Q. Let Q_0 differ from Q by a function on chord diagrams, that is*

$$Q_0(k) = Q(k) + q(\sigma(k))$$

for $q \in \mathcal{D}^*_m$. Then Q_0 is also an integral of P.

Proof The derivative of an n-singular invariant that factors through σ is zero: crossing changes do not change the chord diagram, so when a crossing change occurs, the function does not change. Hence Q and Q_0 have the same derivative: P. □

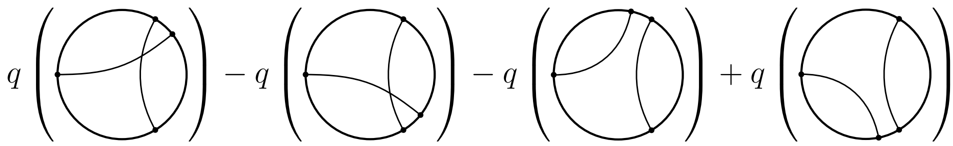
Remark 1.2.10 We defined a constant of integration as a set of objects (e.g. $c \in \mathcal{D}_m$) but perhaps it would have been more accurate to define it as a functional (e.g. $q \in \mathcal{D}_m^*$). After all the constant of integration on the real line is "$+\ C$" rather than the set $\{*\}$ with one object. There is no real risk of confusion, so let us be slightly loose and use the terminology for either.

Since we are trying to integrate more than once, we might wish to know which constants of integration are themselves integrable.

Definition 1.2.11 A **weight system** is an integrable $w \in \mathcal{D}_m^*$.

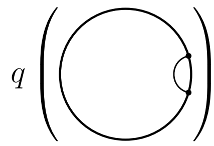
If we take the relations for a knot invariant to be integrable and project them into the space of chord diagrams, we get the following relations.

Definitions 1.2.12 (a) The **four-term relation** is the relation



$= 0.$ (4T*)

(b) The **one-term relation** is the relation



$= 0.$ (1T*)

Just like T4T* and T1T*, 4T* and 1T* are not individual relations, but classes of relations. There's one 4T* relation for all ways of placing other chords in-between the three far-apart chord ends, in all of the four diagrams. There's a 1T* relation for all ways of placing chord diagrams between either of the two far-apart chord ends. In other words, any chord diagram with an 'isolated chord' that doesn't cross any other chords in the diagram is killed by a 1T* relation.

Proposition 1.2.13 *A weight system is characterised as a constant of integration that satisfies* 4T* *and* 1T*.

Proof A weight system defines an m-singular invariant that is also invariant under crossing change. To integrate it must satisfy 4T* and 1T*. Since crossing changes are free, this is equivalent to satisfying the projection of T4T* and T1T* into chord diagrams. □

We return now to the secondary (and higher) obstructions. A general integral of an m-singular P is of the form

$$Q + q \circ \sigma.$$

Since integration is linear, to be integrable again, both terms need to be integrable. The latter we have just seen as the condition that q is a weight system. A sufficient condition for the former to be integrable is that $S\mathcal{K}_m^\bullet$ vanishes. But this is a tautological statement of the general theory — it doesn't mean much if we don't know what $S\mathcal{K}_m^\bullet$ is.

Conjecture 1.2.14 *An invariant of m-singular knots satisfying* T4T* *and* T1T* *integrates m times into a genuine knot invariant.*

Remarks 1.2.15 (a) At first glance, this conjecture looks like it follows from Theorem 1.2.6. The point is that it may not be possible to choose the integral to again satisfy T4T* and T1T*, which is what $S\mathcal{K}^\bullet$ measures.

(b) Computing $S\mathcal{K}^\bullet_m$ is dual to computing $\ker \partial_{m+1}\partial_m / \ker \partial_m$ (we saw a similar thing with the primary obstructions). Computing $\ker \partial^2$ is the hard part — it's not too hard to find some elements, but whether they form a spanning set is open.

(c) This conjecture is proven in certain cases. It holds in the integration theory for braids [Hut98], and in a certain sense it's "half"-proven for knots [Wil98].

The finite type invariants in $\mathcal{K}^\bullet_\star$ are simply the Vassiliev invariants, as checked by a simple comparison between Definitions 1.1.8 and 1.2.3. In other words, Vassiliev invariants of order m are those which vanish on parts of the strata at and above some depth $m+1$.

If we restrict Conjecture 1.2.14 to Vassiliev invariants, then we get the following.

Theorem 1.2.16 (Fundamental theorem of Vassiliev invariants) *Let v be an invariant of m-singular knots satsifying* T4T*, T1T* *and the additional condition that $\delta v = 0$. Then v integrates m times into a genuine knot invariant (which is a Vassiliev invariant of order m).*

There are various proofs of the fundamental theorem. They are listed in [BS97], and each proof is accompanied by a series of moral objections. In the words of Bar-Natan: "Always the method is indirect and very complicated, and/or some a-priori unnatural choices have to be made".

Remark 1.2.17 We have the implication Conjecture 1.2.14 $\implies$ Theorem 1.2.16, and this is actually realised in the theory of braids. It is mysterious that the fate of the slightly stronger conjecture which comes from taking the natural topological approach to the fundamental theorem still remains unknown, and that there are grievances to be had with all known proofs of the theorem.

In Section 1.5 we will look at equivalent formulation of the fundamental theorem.

1.3 Knots and Vassiliev invariants

Speaking broadly, the aim of Vassiliev theory is to study the space of knots via the space of chord diagrams, using information from the stratification of knots introduced in the first two sections. But this space is not just a vector space; it has some further structure which we wish to incorporate. In this section, we synthesise all of this information into one algebraic structure.

Definitions 1.3.1 (a) The **space of knots**, denoted $\mathcal{K}$, is the vector space spanned, over $\mathbb{Q}$, by non-singular knots. Equivalently, $\mathcal{K} = \mathcal{K}^\bullet_0$.

(b) The space of knots is equipped with the **singular knot filtration**

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

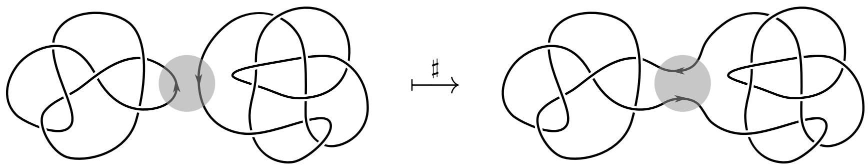
where the ith filtered component $\mathcal{K}_i$ is the span of resolutions of singular knots with i double points, equivalently the image of δ^i, that is, $\mathcal{K}_i = \delta^i(\mathcal{K}^\bullet_i)$.

Proposition 1.3.2 *The singular knot filtration is indeed a descending filtration of vector spaces.*

Proof This being a filtration of vector spaces, the only thing to check is that if $i < j$, then $\mathcal{K}_i \supset \mathcal{K}_j$. If $k \in \mathcal{K}_j$, then $k = \delta^j(k^\bullet)$ for some $k^\bullet$ in $\mathcal{K}_j$. But then $k = \delta^j(k^\bullet) = \delta^i\delta^{j-i}(k^\bullet)$, so $k \in \delta^i(\mathcal{K}_i^\bullet)$. □

The algebraic structure on knots comes from the following operation.

Definition 1.3.3 The **connected sum** of two knots k_1 and k_2 is the knot obtained by removing a small arc from each of k_1 and k_2, then connecting the two embedded intervals into a single knot in an orientation-preserving way.

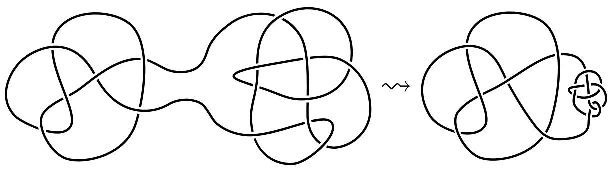


This definition is extended bilinearly to $\mathcal{K}$, i.e., to linear combinations of knots.

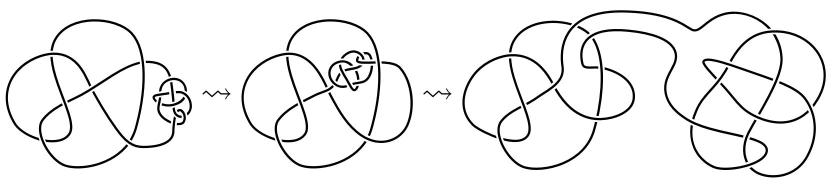
The connected sum of knots is not a-priori well-defined, as we have not specified where along either k_1 or k_2 the small arc is to be removed. However, by a classical knot-theoretic argument, the result is independent of this choice.

Proposition 1.3.4 *The connected sum $\sharp : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ operations is well-defined. It does not matter where along either knot the small arc was removed, the results are ambient-isotopic.*

Proof We exhibit an ambient isotopy starting at $k_1 \sharp k_2$ where the small arc is removed from k_1 as in the example above. The part of the connected sum coming from k_2 is shrunk by ambient isotopy. Since it can be shrunk arbitrarily small, let it be shrunk to lie within a small tubular neighbourhood of k_1.



Then, k_2 is then isotoped along k_1, reenlarged and isotoped back to its original position.



The above argument works for any choice of small arc removed along k_1, and the same argument with the roles of k_1 and k_2 reversed completes the proof. □

Proposition 1.3.5 *The connected sum respects the descending filtration:*

$$K_i \sharp K_j \subset K_{i+j}.$$

That is, the connected sum makes $(\mathcal{K}, \sharp)$ into an ascending filtered algebra.

Proof Indeed, the connected sum being a well-defined operation makes $\mathcal{K}$ into an algebra. The question is whether the connected sum respects the filtration.

If $k \otimes \ell \in \mathcal{K}_i \otimes \mathcal{K}_j$, then there are $k_\bullet$ and $\ell_\bullet$ in $\mathcal{K}_i^\bullet$ and $\mathcal{K}_j^\bullet$ that resolve to k and ℓ, respectively. Similarly, the 'connected sum' $k_\bullet \sharp \ell_\bullet$ resolves by δ^{i+j} to $k \sharp \ell$, which is therefore in $\delta^{i+j}(\mathcal{K}_{i+j})$.

Here, 'connected sum' is enclosed in inverted commas due to the following technicality. Connected sums of singular knots with singular knots were not part of Definition 1.3.3. Even if we ensure that the small arcs removed from a singular knot do not contain a singular point, still, this is ill-defined. The ambiguity is that the resulting singular knot may depend on from which side of the singular point the arc was removed i.e. the repositioning argument in the proof of Proposition 1.3.4 fails due to the presence of singular points. After taking the resolution under δ^{i+j} however, the repositioning argument now works again, so any of the choices of connected sum in $k_\bullet \sharp \ell_\bullet$ produce a singular knot which resolves to $k \sharp \ell$. □

Say something intelligent to introduce this definition.

I guess it's something like: We assert that knots are (semi)group-like. This must be somehow related to (some?) chord diagrams being like exponentials (primitives?).

Definition 1.3.6 The coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is defined on knots k as

$$\Delta(k) = k \otimes k$$

and extended bilinearly to $\mathcal{K}$.

Proposition 1.3.7 *The connected sum and coproduct are compatible. That is,* $(\mathcal{K}, \sharp, \Delta)$ *forms a bialgebra.*

Proof A bialgeba is a coalgebra that is also an algebra with compatible product and coproduct. That $(\mathcal{K}, \Delta)$ forms a coalgebra is trivial (the counit is the augmentation map denoted ε). And we have already seen that $(\mathcal{K}, \sharp)$ is an algebra. So it remains only to check the compatibility conditions. We check that the product and coproduct are compatible:

$$\begin{aligned}\Delta(k \sharp \ell) &= k \sharp \ell \otimes k \sharp \ell \\ &= (k \otimes k) \sharp^{\otimes 2} (\ell \otimes \ell) \\ &= \Delta(k) \sharp^{\otimes 2} \Delta(\ell)\end{aligned}$$

where $\sharp^{\otimes 2}$ denotes the component-wise tensor product on $\mathcal{K} \otimes \mathcal{K}$.

Checking the unit and counit is trivial. □

So far, $\mathcal{K}$ is a bialgebra whose product respects the filtration. The same is true of the coproduct.

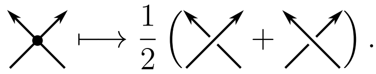
Proposition 1.3.8 *The coproduct* Δ *also respects the filtration,*

$$\Delta(\mathcal{K}_j) \subset \sum_{i=0}^{j} \mathcal{K}_i \otimes \mathcal{K}_{j-i}.$$

That is, with the singular knot filtration, $\mathcal{K}$ *is a filtered bialgebra.*

We give a proof, due to Willerton which follows directly from Lemma of 1.3.10 of [Wil96]. The lemma is a formula for the coproduct of an element of $k \in \mathcal{K}_m$ that comes from some $k^\bullet \in \mathcal{K}^\bullet_m$. The formula is terms of the 2^m ways of resolving some of singular points in one cofactor and the rest in the other, but first we need some notation.

If I is a subset of the singular points of a singular knot, let δ^I be the operator that resolves the singular points I. Let μ^I be the operator that averages singular points in I, where averaging a singular point is sending



Remark 1.3.9 There is one technicality. To make the above definitions of μ^I and δ^I, we must forget the `DIFF`* relation on $\mathcal{K}^\bullet_m$ (or work in the appropriate lift) as we wish to resolve with respect to specific double points. This provides no mathematical difficulty, so we chose not to do the reader the disservice of altering the notation. But for Lemma 1.3.10, its proof and the proof of Proposition 1.3.8, let $\mathcal{K}^\bullet_m$ not contain the quotient by the `DIFF`* relation.

Lemma 1.3.10 (Willerton) *Suppose $k^\bullet \in \mathcal{K}^\bullet_m$, and let S denote the set of singular points of $k^\bullet$.*

$$\Delta(\delta^S(k^\bullet)) = \sum_{I \subset S} \mu^{\bar{I}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}}(k^\bullet)$$

where $\bar{I} = S \smallsetminus I$.

Proof We proceed by induction on m. In the base case of $m = 0$, $S = \emptyset$, and $k^\bullet = k$ is a genuine knot, so

$$\begin{aligned}\Delta(\delta^0(k)) &= \Delta(k)\\ &= k \otimes k\\ &= \sum_{I\subset\emptyset} \mu^{\bar{I}}\delta^I(k) \otimes \mu^I\delta^{\bar{I}}(k).\end{aligned}$$

The inductive step is as follows. Let $k^\bullet \in \mathcal{K}^\bullet_{m+1}$. Let J denote all singular points of $k^\bullet$, and $x \in J$ denote a specific singular point. Furthermore, let $k^{\bullet+}$ (resp. $k^{\bullet-}$) denote the m-singular knots obtained from $k^\bullet$ when x is replaced by a positive (resp. negative) crossing, so that $\delta^{\{x\}}(k^\bullet) = k^{\bullet+} - k^{\bullet-}$. We examine

$$\sum_{I\subset J} \mu^{\bar{I}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}}(k^\bullet).$$

Decomposing the sum based on whether $x \in I$ yields

$$\sum_{x\in I\subset J} \mu^{\bar{I}}\delta^{I\smallsetminus\{x\}}(k^\bullet) \otimes \mu^{I\smallsetminus\{x\}}\delta^{\bar{I}}(k^\bullet) + \sum_{x\notin I\subset J} \mu^{\bar{I}\smallsetminus\{x\}}\delta^I(k^\bullet) \otimes \mu^I\delta^{\bar{I}\smallsetminus\{x\}}(k^\bullet),$$

then resolving either δ or μ on x,

$$\begin{aligned}&\frac{1}{2}\sum_{x\in I\subset J} \mu^{\bar{I}}\delta^{I\smallsetminus\{x\}}(k^{\bullet+} - k^{\bullet-}) \otimes \mu^{I\smallsetminus\{x\}}\delta^{\bar{I}}(k^{\bullet+} + k^{\bullet-})\\ +&\frac{1}{2}\sum_{x\notin I\subset J} \mu^{\bar{I}\smallsetminus\{x\}}\delta^I(k^{\bullet+} + k^{\bullet-}) \otimes \mu^I\delta^{\bar{I}\smallsetminus\{x\}}(k^{\bullet+} - k^{\bullet-}).\end{aligned}$$

Expanding, yields the cumbersome formula

$$\begin{aligned}
&\frac{1}{2}\sum_{x\in I\subset J}\Big(\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet+})+\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet-})\\
&\qquad\qquad-\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet+})-\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet-})\Big)\\
+&\frac{1}{2}\sum_{x\notin I\subset J}\Big(\mu^{\bar I\smallsetminus\{x\}}\delta^{I}(k^{\bullet+})\otimes\mu^{I}\delta^{\bar I\smallsetminus\{x\}}(k^{\bullet+})-\mu^{\bar I\smallsetminus\{x\}}\delta^{I}(k^{\bullet+})\otimes\mu^{I}\delta^{\bar I\smallsetminus\{x\}}(k^{\bullet-})\\
&\qquad\qquad+\mu^{\bar I\smallsetminus\{x\}}\delta^{I}(k^{\bullet-})\otimes\mu^{I}\delta^{\bar I\smallsetminus\{x\}}(k^{\bullet+})-\mu^{\bar I\smallsetminus\{x\}}\delta^{I}(k^{\bullet-})\otimes\mu^{I}\delta^{\bar I\smallsetminus\{x\}}(k^{\bullet-})\Big).
\end{aligned}$$

Since $\{I \mid I\subset J, x\notin I\}$ is equal to $\{I\smallsetminus\{x\}\mid I\subset J, x\in I\}$, in each of the above sums, the corresponding terms have the same indices. Hence, the first and last terms in each sum combine, and the second and third terms cancel out to give

$$\sum_{x\in I\subset J}\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet+})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet+})-\mu^{\bar I}\delta^{I\smallsetminus\{x\}}(k^{\bullet-})\otimes\mu^{I\smallsetminus\{x\}}\delta^{\bar I}(k^{\bullet-}).$$

Since neither $\mu^{\bar I}\delta^{I\smallsetminus\{x\}}$ or $\mu^{I\smallsetminus\{x\}}\delta^{\bar I}$ are with respect to x, this can be written

$$\sum_{I\subset J\smallsetminus\{x\}}\mu^{\bar I}\delta^{I}(k^{\bullet+})\otimes\mu^{I}\delta^{\bar I}(k^{\bullet+})-\mu^{\bar I}\delta^{I}(k^{\bullet-})\otimes\mu^{I}\delta^{\bar I}(k^{\bullet-})$$

which by the inductive hypothesis is

$$\begin{aligned}
\Delta(\delta^{J\smallsetminus\{x\}}(k^{\bullet+}))-\Delta(\delta^{J\smallsetminus\{x\}}(k^{\bullet-}))&=\Delta(\delta^{J\smallsetminus\{x\}}(k^{\bullet+})-\delta^{J\smallsetminus\{x\}}(k^{\bullet-}))\\
&=\Delta(\delta^{J}(k^{\bullet})).
\end{aligned}$$

□

Proof of Proposition 1.3.8 The operators δ^I and $\mu^{\bar I}$ commute since they are evaluating different singular points. Let I be an arbitrary subset of S, and let $|I|=i$ and $|S|=j$, then the left cofactor is in $\mathcal{K}_i$ and the right in $\mathcal{K}_{j-i}$. □

Not all knot invariants respect the singular knot filtration, as we will see. The point of the Vassiliev invariants is that they're the ones that are natural with respect to the singular knot filtration. Indeed, the Vassiliev invariants are obtained naturally from $\mathcal{K}$ via the following construction.

The dual filtered bialgebra construction makes a filtered bialgebra whose mth filtered component is the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$. The result is the Vassiliev invariants, along with a product transpose to the coproduct in $\mathcal{K}$ and coproduct transpose to the product in $\mathcal{K}$.

Definition 1.3.11 The **filtered bialgebra of Vassiliev invariants**, denoted $\mathcal{V}$, is the vector space of Vassiliev invariants with an ascending filtration by degree

$$\mathcal{V}_0\subset\mathcal{V}_1\subset\mathcal{V}_2\subset\cdots,\qquad \mathcal{V}=\bigcup_{m=0}^{\infty}\mathcal{V}_m.$$

The product is given by pointwise multiplication

$$V_1\cdot V_2(k)=V_1(k)V_2(k),$$

and the coproduct, η, is given by

$$\eta(V)(k_1\otimes k_2)=V(k_1\,\sharp\,k_2).$$

Proposition 1.3.12 *The filtered bialgebraic dual of the descending filtered bialgebra of singular knots $\mathcal{K}$ is the ascending filtered bialgebra of Vassiliev invariants $\mathcal{V}$.*

We don't prove this textbook fact, but let us sketch the main points. Indeed, the set of functionals in $\mathcal{K}^*$ that vanish on $\mathcal{K}_{m+1}$ is $\mathcal{V}_n$.

Pulling $V_1 \otimes V_2$ back along Δ,

$$(V_1 \otimes V_2) \circ \Delta : k \mapsto V_1(k)V_2(k),$$

recovers the formula for the product $V_1 \cdot V_2$ in $\mathcal{V}$. Similarly, pulling V back along $\sharp$ is

$$V \circ \sharp(k_1 \otimes k_2) \mapsto V(k_1 \sharp k_2)$$

recovers the formula for the coproduct, η.

Here we rely on the fact that $(\mathcal{K}\otimes\mathcal{K})^*$ is canoncially isomorphic to $\mathcal{K}^*\otimes\mathcal{K}^*$, which follows from $\mathcal{K}$ being finite type (finite-dimensional in each filtered component).

We didn't prove that the product and coproduct in $\mathcal{V}$ respect the ascending filtration. However the filtered algebra dual of a decreasing filtered bialgebra is always an increasing filtered bialgebra (for the proof, as well as the full details of the dual filtered bialgebra construction, the reader is invited to consult [CDM12, Appendix A.2.4]).

Remark 1.3.13 Alternatively, this can be proved directly in $\mathcal{V}$. Proving that $\sharp$ respects the filtration on $\mathcal{K}$ was easy, and is just as easy in the dual case. However, the proof that Δ respects the filtration on $\mathcal{K}$ was cumbersome, and so is its dual. But it is worth looking into how it can be understood by a continuation of the polynomial analogy due to Willerton [Wil96].

The generalised Leibniz theorem of multivariable calculus says that if f and $g : \mathbb{R}^m \to \mathbb{R}$ are differentiable, then (in similar derivative notation to as above)

$$\frac{\partial^{|I|}(fg)}{\partial_{x_I}} = \sum_{J\subset\{1,\cdots,i\}} \frac{\partial^{|J|}f}{\partial_{x_J}} \cdot \frac{\partial^{|\bar{J}|}g}{\partial_{x_{\bar{J}}}}.$$

This says that the derivative of a product of f and g with respect to some variables is the sum of every way of taking some of those derivatives with respect to f and some with respect to g.

A kind of dual theorem follows from this. For $c \in \mathbb{R}$, if given that c comes from some f and $g : \mathbb{R}^m \to \mathbb{R}$ by taking their derivatives with respect to all the variables and evaluating all remaining variables in the result, then we get a cofactorisation for c in $\mathbb{R} \otimes \mathbb{R}$: i.e.

$$c = \left.\frac{\partial^{|I|}(fg)}{\partial_{x_I}}\right|_{\{x_I=a_I\}} \text{ implies } \left(\sum_{J\subset\{1,\cdots,i\}} \left.\frac{\partial^{|J|}f}{\partial_{x_J}}\right|_{\{x_J=a_{\bar{J}}\}} \otimes \left.\frac{\partial^{|\bar{J}|}g}{\partial_{x_{\bar{J}}}}\right|_{\{x_J=a_J\}} \right) \xrightarrow{\mu} c,$$

where $\mu : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is multiplication.

As it turns out, this theorem is pretty useless in the multivariable calculus case. The filtration on $\mathbb{R}$ coming from being the derivative of some function evaluated at some point is trivial, and so every $c \in \mathbb{R}$ comes from some such f and g, and it's easy to construct such f and g.

But the dual theorem in the case of knots is exactly Willerton's Lemma 1.3.10, where the averaging map plays the role of evaluation. Furthermore, the knot version of the generealised Leibniz theorem [Wil96] is that if $k^\bullet \in \mathcal{K}_m^\bullet$ and S the set of singular points of $k^\bullet$, then

$$(V_1 \cdot V_2)(\delta^S(k^\bullet)) = \sum_{I\subset S} V_1(\mu^{\bar{I}}\delta^I(k^\bullet)) \otimes V_2(\mu^I\delta^{\bar{I}}(k^\bullet)).$$

It follows directly from this that if V_1 is of type m and V_2 of type n, then $V_1 \cdot V_2$ is of type $m+n$: for if $k^\bullet \in \mathcal{K}^\bullet_{m+1}$, then either $|I| > m$ or $|\hat{I}| > n$, so in each summand, one of the cofactors is a Vassiliev invariant being evaluated above its order, so zero. Hence $(V_1 \cdot V_2)(\delta^S(k^\bullet)) = 0$.

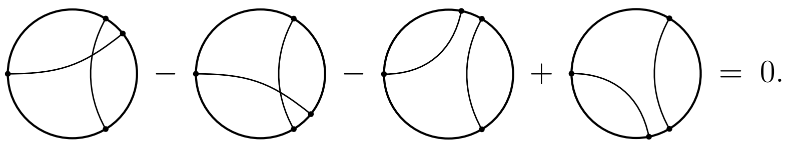
The analogy is that the multivariable calculus analogue of this argument is a proof that polynomials are filtered by degree.

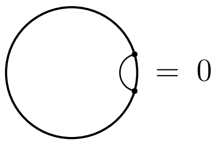
Example 1.3.14 In view of Remark above about non-Vassiliev invariants, let's define a knot invariant. (Define f_{3_1}.)

1.4 Chord diagrams and weight systems

This bialgebra strucutre on knots is closely related to a similar bialgebra structure on chord diagrams. Knots are complicated and chord diagrams are much simpler, and the general idea is to study the former via the latter.

In Section 1.2 we saw how functions on chord diagrams specify Vassiliev invariants, so long as the functions satisfy 4T* and 1T*. We can instead encode this directly into the algebra of chord diagrams by the following relations.


(4T)


(1T)

Definition 1.4.1 We define $\mathcal{A}_m$, the **space of chord diagrams** of degree m as

$$\mathcal{A}_m = \mathcal{D}_m / \texttt{4T}, \texttt{1T},$$

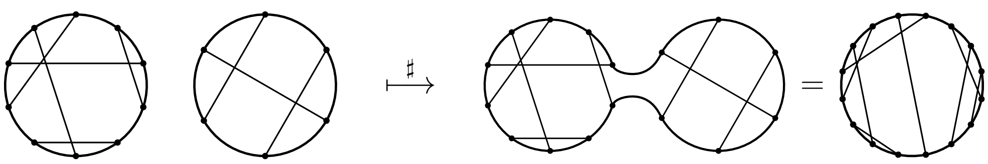
and $\mathcal{A}$, the **space of chord diagrams** as

$$\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m.$$

Warning 1.4.2 Both elements of $\mathcal{A}$ and $\mathcal{D}$ are known as chord diagrams. From now on when we say "a chord diagram", we mean an element of $\mathcal{A}$ unless otherwise specified.

The algebra $\mathcal{A}$ has multiplication and coproduct operations that mirror those in $\mathcal{K}$.

Definition 1.4.3 The **connected sum of two chord diagrams** A_1 and A_2 is the chord diagram obtained by cutting the two circles of A_1 and A_2 and connecting the two intervals in an orientation-preserving way.



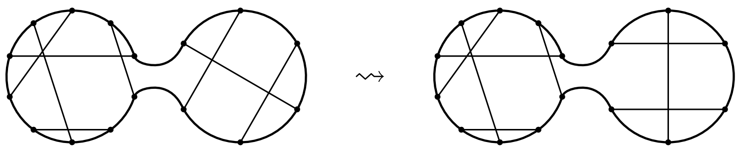
The definition is extended bilinearly to elements of $\mathcal{A}$.

Again, this is not, a-priori, a well-defined operation, as the location of the cut on each circle was not specified. Indeed in the algebra $\mathcal{D}$ this is ill-defined. However the 4T relation in $\mathcal{A}$ takes care of this.

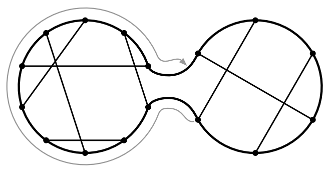
Proposition 1.4.4 *The connected sum operation $\sharp : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is well-defined.*

Proof We will prove that the connected sums, given any two choices of connection locations, are equal modulo 4T.

Let us denote the first chord diagram as a_1 and the second as a_2. Without loss of generality, it suffices to prove that without change in the connection location of a_1, we can change the connection location on a_2. Indeed it suffices to prove that we can rotate a_2 by one 'click', like so:

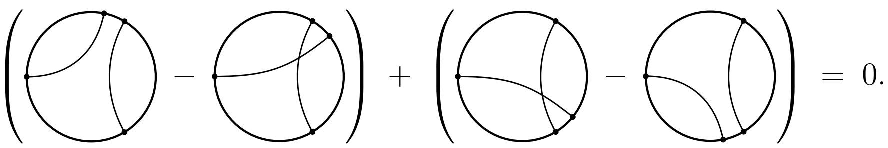


This is equivalent to sliding a single chord endpoint on the second diagram all the way through the first diagram, along the path of the grey arrow.



Which we show can be achieved by a series of 4T relations.

We can rewrite 4T as



A sliding move of our special chosen endpoint of a_2 over an endpoint of some chord of a_1 is achieved by subtracting the first two terms of the rearranged 4T. But every chord of a_1 is encountered twice in the path. In the other instance it is encountered, the sliding is achieved by subtracting the remaining two terms of 4T. So, the two connected sums $a_1 \sharp a_2$ differ by a sum of 4T relations, completing the proof. □

Proposition 1.4.5 *The connected sum operation makes $\mathcal{A}$ into a graded algebra.*

Proof No chords are lost during the connected sum: the 4T relation is homogenous with respect to degree, and the connected sum of a chord diagram containing a 1T chord relation still contains a 1T chord. So, the connected sum of a chord diagram of order i and a chord diagram of order j is a chord diagram of order $i + j$. □

Just as there is a connected sum operation in $\mathcal{A}$ reminiscent to that in $\mathcal{K}$, there is a coproduct too.

Definition 1.4.6 The **coproduct of a chord diagram** A is the sum of ways of partitioning its chords between two subdiagrams. Specifically, if S is the set of chords of A, and $J \subset S$, let $\widehat{J} = S \setminus J$. Denote by A_J the chord diagram A but with only the chords in $J \subset S$, and the rest deleted. Then

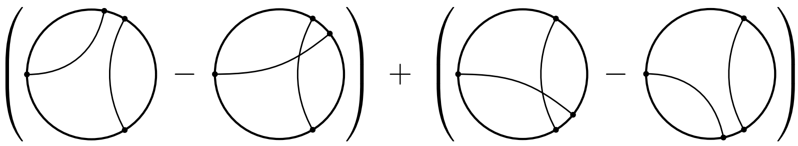
$$\Delta(A) = \sum_{J \subset S} A_J \otimes A_{\overline{J}}.$$

Proposition 1.4.7 *The coproduct Δ is well-defined in $\mathcal{A}$, and makes $\mathcal{A}$ into a graded bialgebra.*

Proof We need to check: that the coproduct factors through the quoitents, that the coproduct respects the grading, and that the compatibility condition holds.

That Δ factors through 1T is easy: an isolated chord in A remains isolated and appears in one cofactor of every term of $\Delta(A)$.

Also, Δ factors through 4T. Suppose that $K = A_1 - A_2 + A_3 - A_4$ is some combination of chord diagrams to be killed by 4T. This means that K looks like

$K =$ 

where there may be other chords O that the above diagrams have in common, as well as those shown. Note that there is one moving chord in the above diagram and one stationary chord. Let us label these m and s. Take the same partition $J \sqcup \overline{J}$ of S for all of the four chord diagrams at once, and write as the resulting coproduct $\Delta(A_i) = C_i \otimes D_i$. Suppose without loss of generality that m was partitioned into the C_i's. Then $D_1 = D_2 = D_3 = D_4$, so this term of the coproduct factors as $(C_1 - C_2 + C_3 - C_4) \otimes D_1$ and either:

- s was also partitioned into the C_i's, and the relation remains a 4T, or
- s was partitioned into the D_i's, and so $C_1 = C_2$ and $C_3 = C_4$,

and in either case, that term of the coproduct is killed.

The coproduct clearly satisfies

$$\Delta(\mathcal{A}_m) \subset \bigoplus_{i+j=m} \mathcal{A}_i \otimes \mathcal{A}_j = (\mathcal{A} \otimes \mathcal{A})_m,$$

so it is graded.

The compatibility condition holds. If A has chord set S and B has chord set T, then

$$\begin{aligned}\Delta(A) \sharp^{\otimes 2} \Delta(B) &= \left(\sum_{J' \subset S} A_{J'} \otimes A_{\overline{J'}}\right) \sharp^{\otimes 2} \left(\sum_{J'' \subset T} B_{J''} \otimes B_{\overline{J''}}\right) \\ &= \sum_{J \subset S \sqcup T} (A \sharp B)_J \otimes (A \sharp B)_{\overline{J}} \\ &= \Delta(A \sharp B).\end{aligned}$$

□

We have shown that $\mathcal{A}$ is a graded bialgebra of finite type. In fact $\mathcal{A}$ is an even more specific structure.

Definition 1.4.8 A **connected, commutative, cocommutative** graded bialgebra of finite type is a graded bialgebra, A, of finite type for which

- The unit $k \to A_0$ is an isomorphism (**connectedness**)
- The product is commutative, $m \circ \tau = m$
- The coproduct is cocommutative, $\tau \circ \Delta = \Delta$

Proposition 1.4.9 *The bialgebra $\mathcal{A}$ is a connected, commutative, cocommutative graded bialgebra of finite type.*

Proof The connetedness isomorphism is given by ...
We have already shown it is commutative.
The coproduct is clearly cocommutative. □

Connected, commutative, cocommutative graded bialgebras of finite type are very rigid structures. In-particular, a classical strucutral theorem applies, and such a bialgebra can be understood in terms of its primitive elements.

Definition 1.4.10 An element x is **primitive** in a coalgebra (so in-particular in a bialgebra) with coproduct Δ if it satisfies

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

The set of primitive elements of a bialgebra A is denoted $\mathcal{P}(A)$, and if A is graded, then let $\mathcal{P}_n(A)$ denote the set of primitive elements of degree n.

Theorem 1.4.11 (Milnor-Moore) *Let H be a connected, commutative, cocommutative bialgebra of finite type, over a field of characteristic zero. Then, as an algebra, H is isomorphic to the symmetric algebra of $\mathcal{P}(H)$. In other words, H is a polynomial algebra in $\mathcal{P}(H)$.*

We refer to [Les24] or [CDM12] for a proof. This fact leads to important consequences about the structure of $\mathcal{A}$ and its relation to Lie algebras, which will the subject of Chapter 2.

Definition 1.4.12 The weight systems (Definition 1.2.11), denoted $\mathcal{W}$, form a graded bialgebra, the **graded bialgebra of weight systems** with grading given by degree m, product given by pointwise multiplication

$$W_1 \cdot W_2(a) = W_1(a)W_2(a),$$

and coproduct, η given by

$$\eta(W)(a_1 \otimes a_2) = W(a_1 \sharp a_2).$$

In fact, every graded object is a filtered object with the naturally induced filtration. Considering a graded object as such allows us to take its dual filtered bialgebra (which we refer to as its dual graded bialgebra in this case).

Proposition 1.4.13 *The graded bialgebra $\mathcal{W}$ is the dual graded bialgebra of the graded bialgebra $\mathcal{A}$.*

1.5 The fundamental theorem of Vassiliev invariants

In Section 1.2, we gave the fundamental theorem of Vassiliev invariants. The point of this theorem is that is establishes a particular relationship between the algebras of the previous two chapters, $\mathcal{K}$ and $\mathcal{A}$ (or equivalently, between $\mathcal{W}$ and $\mathcal{V}$). But admittedly, in the form of Theorem 1.2.16, it's not a-priori obvious why this is the case. Here we give a restatement of that theorem that makes the relationship explicit.

Definition 1.5.1 The **associated graded** algebra of a filtered algebra A is the algebra formed by the direct sum of the succesive quotients of the filtered components of A. For an algebra with a descending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m+1},$$

and for an algebra with an ascending filtration,

$$\operatorname{gr} A = \bigoplus_{m=0}^{\infty} A_m / A_{m-1},$$

where $A_{-1} = \{0\}$.

Theorem 1.5.2 (Fundamental theorem) *The algebra of weight systems is isomorphic to the associated graded algebra of the algebra of Vassiliev invariants, $\mathcal{W} \cong \operatorname{gr} \mathcal{V}$, or on the level of graded components,*

$$\bigoplus_{m=0}^{\infty} \mathcal{W}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{V}_m / \mathcal{V}_{m-1}.$$

Equivalently, this can be stated in the dual setting as follows. The algebra of chord diagrams is isomorphic to the associated graded algebra of the algebra of knots, $\mathcal{A} \cong \operatorname{gr} \mathcal{K}$, or on the level of graded components,

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m \cong \bigoplus_{m=0}^{\infty} \mathcal{K}_m / \mathcal{K}_{m+1}.$$

The equivalence of this version of the theorem with the version (Theorem 1.2.16) given in Section 1.2 will be proven at the end of this section.

We can break the fundamental theorem up into two parts.

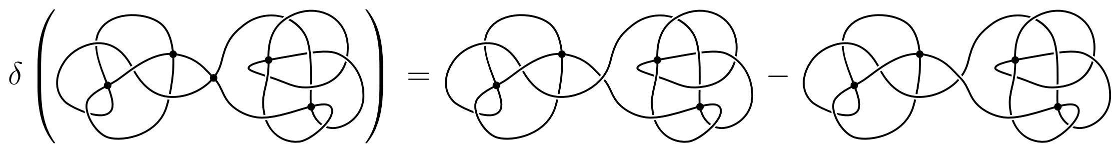
Vassiliev	Every Vassiliev invariant modulo Vassiliev invariants of higher order gives a Weight system, so a map $$\mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{W}_m.$$	Every chord diagram gives an element of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$, so a map $$\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}.$$
Kontsevich	Every weight system gives a Vassiliev invariant modulo Vassiliev invariants of higher order, so a map $$\mathcal{W}_m \to \mathcal{V}_m/\mathcal{V}_{m-1}$$ which is inverse to the above.	Every equivalence class of $\mathcal{K}_m$ modulo $\mathcal{K}_{m+1}$ gives a chord diagram, so a map $$\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m.$$ which is inverse to the above.

The point of the part of this theorem due to Vassiliev is that the relations in $\mathcal{A}$ are compatible with the relations of $\mathcal{K}_n/\mathcal{K}_{n+1}$. Indeed, $\mathcal{A}$ was constructed in this way, and much of that work was already done in Section 1.2, and the following proof involves nothing new.

Proof of Theorem 1.5.2 (Vassiliev) For the map $\mathcal{A}_m \to \mathcal{K}_m/\mathcal{K}_{m+1}$ we take the following. Given $a \in \mathcal{A}_m$, take a singular knot $k^\bullet \in \mathcal{K}^\bullet_m$ whose chord diagram is a, then resolving it to an element of $k \in \mathcal{K}_m$, and projecting that into the quotient $\mathcal{K}_m/\mathcal{K}_{m+1}$.

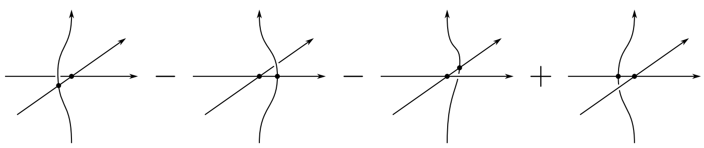
This is well-defined since any other $k^{\bullet\prime}$ also has chord diagram a. Any two m-singular knots with the same chord diagram differ by some crossing changes, in particular this applies to $k^\bullet$ and $k^{\bullet\prime}$. But if $k^\bullet$ and $k^{\bullet\prime}$ differ by a crossing change, then $\delta^m(k^\bullet)$ and $\delta^m(k^{\bullet\prime})$ differ by an element of $\mathcal{K}_{m+1}$, so $[\delta^m(k^\bullet)] = [\delta^m(k^{\bullet\prime})]$. This argument that any $k^\bullet \in \mathcal{K}^\bullet_m$ can be chosen to represent $[\delta^m(k^\bullet)]$as long as it has chord diagram a will be used again below, and we refer to it as the crossing-change argument.

Recalling that $\mathcal{A}_m = \mathcal{D}_m/\mathtt{1T}, \mathtt{4T}$, we need to show that the map factors through the quotient. Indeed 1T is in the kernel: a type 1T chord diagram is sent to a singular knot with a singular point that is passed through twice in a row when the knot is traversed. The resolution, by the crossing-change argument, can be chosen to be a difference of two knots that are isotopic except around the 1T double point:



$= 0.$

Similarly, 4T is also in the kernel. A combination of chord diagrams appearing in a 4T relation is sent to a combination of singular knots which by the crossing-change argument can all be chosen to be identical except near a small region, where



but after resolving around the singular point that they don't have in common this becomes

figure of the 8 terms that cancel out.

The isomorphism respects the algebra structure as $a_1 \sharp a_2$ is sent to $[k_{a_1\sharp a_2}] \in \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ that comes from the resolution of a singular knot $k^\bullet_{a_1\sharp a_2}$ with chord diagram $a_1 \sharp a_2$. But likewise a_1 and a_2 map to $[k_{a_1}]$ and $[k_{a_2}]$ which are resolutions of singular knots $k^\bullet_{a_1}$ and $k^\bullet_{a_2}$. The induced operation from the connect sum $\mathcal{K}_m/\mathcal{K}_{m+1} \otimes \mathcal{K}_n/\mathcal{K}_{n+1} \to \mathcal{K}_{m+n}/\mathcal{K}_{m+n+1}$ takes these to $[k_{a_1} \sharp k_{a_2}]$, which comes from the resolution of $k^\bullet_{a_1} \sharp k^\bullet_{a_2}$. These singular knots may not a-priori be the same, but they have the same chord diagram, so by the crossing-change argument they can be chosen to be the same. Hence their resolutions are the same.

Using similar arguments, from the formula of Willerton's Lemma, the isomorphism can be shown to respect the bialgebra structure.

The dual version of the statement follows from the regular version, but direct proofs are illustrative and we return to them later. □

For now, let's turn to the Vassiliev part of the fundamental theorem, with the following definition.

The part of Theorem 1.5.2 due to Maxim Kontsevich [Kon93] is much more involved. Kontsevich constructed an integral invariant which proves the fundamental theorem, as well as containing the information of all Vassiliev invariants at the same time. We will not give a detailed exposition of Kontsevich's invariant here, but they abound in the literature, for example [BS97; CD05; CDM12] in order of increasing level of detail. Rather we will boil the Kontsevich integral down to a single universal property.

Definition 1.5.3 The **completion** of a filtered algebra A is the filtered algebra

$$\widehat{A} = \varprojlim_{m\to\infty} A_m/A_{m+1}.$$

In other words this is the degree-completion of the graded algebra $\operatorname{gr} A$. Note that $\widehat{A}$ is only a filtered algebra, not a graded one, as its elements can have infinitely many terms non-zero, so it doesn't decompose as a direct sum.

Definition 1.5.4 A **universal Vassiliev invariant** is a knot invariant $Z : \mathcal{K} \to \widehat{\mathcal{A}}$ with the following property. If $k \in \mathcal{K}_m$ is a linear combination of knots with $k = \delta^m(k^\bullet)$ and $k^\bullet$ has chord diagram $a \in \mathcal{A}_m$, then

$$Z(k) = a + \text{higher degree terms.}$$

Remark 1.5.5 Another equivalent way of defining a universal Vassiliev invariant is as follows. If f is a descending-filtration-respecting map $f : A \to B$, then define the associated graded map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ that sends $a_m + A_{m+1} \mapsto f(a_m) + B_{m+1}$. In other words, a graded map coming from the filtered map f that forgets information about higher degrees. A universal Vassiliev invariant is a map $Z : \mathcal{K} \to X$ whose associated graded $\operatorname{gr} Z : \operatorname{gr} \mathcal{K} \to \operatorname{gr} X$ is an isomorphism.

The Kontsevich integral is such a map with $X = \widehat{\mathcal{A}}$. In particular, note that $\operatorname{gr} \widehat{\mathcal{A}} = \mathcal{A}$ (because the direct sum implicit in gr means $a \in \operatorname{gr} \widehat{\mathcal{A}}$ cannot have infinitely many non-zero terms). So since Z is $\mathcal{K} \to \widehat{\mathcal{A}}$ and satisfies this property, then $\operatorname{gr} \mathcal{K} \cong \mathcal{A}$.

Theorem 1.5.6 (Kontsevich Integral) *There exists a universal Vassiliev invariant, denoted $Z(k)$, called the Kontsevich integral.*

Proof of Theorem 1.5.2 (Kontsevich) Take the map $k \in \mathcal{K}_m \to \mathcal{A}_m$ coming from killing the higher degree terms in the Kontsevich integral, and taking the lowest order non-zero chord diagram. This factors through the quotient to a map $\mathcal{K}_m/\mathcal{K}_{m+1} \to \mathcal{A}_m$ since by Theorem 1.5.6, any additional $k' \in \mathcal{K}_{m+1}$ contributes only higher degree terms, which get killed. It is easy to see that the two maps are inverses. □

Again, it's worth looking at the proof in the dual setting too.

Lemma 1.5.7 *Post-composing the Kontsevich integral with a weight system of order m,*

$$k \longmapsto W(Z(k))$$

(or more precisely, the following composition)

$$\mathcal{K} \xrightarrow{Z} \widehat{\mathcal{A}} \xrightarrow{\pi_m} \mathcal{A}_m \xrightarrow{W} \mathbb{Q}$$

gives a Vassiliev invariant of order m.

Proof The map $W \circ Z$ is clearly an invariant, since Z is an invariant. It's Vassiliev since on linear combinations $k \in \mathcal{K}_{m+1}$, $Z(k)$ has nothing in degree m, so composing with a weight system of degree m gives zero. $\square$

Proof of Theorem 1.5.2 (dual) The map defined by Lemma 1.5.7, $\mathcal{W}_m \to \mathcal{V}_m$ is injective, as only the zero weight system gives the zero invariant.

However, it is not surjective. We show that the map, written as

$$\mathcal{W}_m \xrightarrow{Z^*} \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}$$

forces a choice of Vassiliev invariant of degree $m-1$. Being as explicit as possible, let

$$\Omega(k) = \begin{cases} k \in \mathcal{K}_m & (W \circ Z)(k) \\ k \notin \mathcal{K}_m & 0 \end{cases}, \quad \text{and} \quad \Theta(k) = \begin{cases} k \in \mathcal{K}_m & 0 \\ k \notin \mathcal{K}_m & (W \circ Z)(k) \end{cases},$$

then $W \circ Z = \Omega + \Theta$, where Ω recovers the weight system W, when $k \in \mathcal{K}_m$, and Θ is some finite type invariant of order $m-1$, determined by W.

In essence, we have found that the cokernel of Z^* is $\mathcal{V}_{m-1}$, so we get the desired isomorphism $\mathcal{W}_m \cong \mathcal{V}_m/\mathcal{V}_{m-1}$, at least on the level of vector spaces. $\square$

The natural question arises, what was this summand Θ? Fixing $n < m$ and a knot $k \in \mathcal{K}_n$ with chord diagram a_k, it has Kontsevich integral

$$Z(k) = a_k + \begin{matrix}\text{terms of} \\ \text{order } (n+1)\end{matrix} + \cdots + \begin{matrix}\text{terms of} \\ \text{order } m\end{matrix} + \cdots .$$

Applying the projection π_m, all that remains are some chord diagrams of order m, with some coefficients depending on the intricacies of the Kontsevich integral for that particular knot. Composing with the weight system, this is a $\mathbb{Q}$-valued Vassiliev invariant of order $m-1$ determined by W.

The name 'universal Vassiliev invariant' we gave to the Kontsevich integrals and invariants of its kind is indeed justified. Every Vassiliev invariant can be obtained through the Kontsevich integral.

Theorem 1.5.8 *If Z is a universal Vassiliev invariant, then every Vassiliev invariant factors through Z.*

Proof Let $V \in \mathcal{V}_m$. Following the proof above, we can project V to $\mathcal{V}_m/\mathcal{V}_{m-1}$ to get a weight system W_m. Subtracting $W_m \circ Z$ leaves a Vassiliev invariant of one less degree. In other words, via the isomorphism

$$\begin{aligned} \mathcal{V}_m &\cong \mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0 \\ &\cong \mathcal{W}_m \oplus \mathcal{W}_{m-1} \oplus \cdots \oplus \mathcal{W}_1 \oplus \mathcal{W}_0, \end{aligned}$$

V can be written as a sequence of weight systems of degree from 1 to m such that V factors through the Kontsevich integral

$$V = \sum_{i=0}^{m} (W_m \circ Z) = \left(\bigoplus_{i=0}^{m} W_m \right) \circ Z.$$

$\square$

Corollary 1.5.9 *A universal Vassiliev invariant (in-particular, the Kontsevich integral Z) is exactly as strong as the set of Vassiliev invariants.*

Definition 1.5.10 Taking the projection $\mathcal{V}_m \to \mathcal{V}_m/\mathcal{V}_{m+1} \cong \mathcal{W}_m$ yields a weight system. The **canonical Vassiliev invariants** are those Vassiliev invariants whose weight systems W recover them completely via $W \circ Z$.

In other words, not all bases of $\mathcal{V}_m$ are created equal. The canonical Vassiliev invariants are those that are homogenous with respect to the splitting of $\mathcal{V}_m$

$$\mathcal{V}_m/\mathcal{V}_{m-1} \oplus \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \oplus \cdots \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_0$$

induced by the Kontsevich integral. Canonical Vassiliev invariants were first defined in [BG96] and used to prove the Melvin-Morton-Rozansky conjecture relating the coefficients of Alexander polynomial of a knot to those of the coloured Jones polynomials. This is a good example of how the theory of Vassiliev invariants is useful for probing the structure of knots.

Question 1.5.11 (Bar-Natan–Garoufalidis) There are a few known from-the-bottom-up constructions of universal Vassiliev invariants of knots. However they are all equivalent to, or conjecturally equivalent to the Kontsevich integral. Yet the Kontsevich integral is not the only invariant that can satisfy the defining degree property of a universal Vassiliev invariant.

Is there a reason why the Kontsevich integral, or equivalently, this splitting appears to be canonical?

Finally, we return to the equivalence of the two fundamental theorems.

Proof (Equivalence of Theorems 1.2.16 and 1.5.2) For the forward direction, suppose Theorem 1.2.16 holds. This states that every invariant $v^\bullet$ of m-singular knots satisfying `T4T`* and `T1T`* and further that $\delta v^\bullet = 0$, then $v^\bullet$ integrates to an invariant v of 0-singular knots.

First we prove that the Kontsevich part of the fundamental theorem holds. Let W be a weight system of order m. Then W defines an invariant $v_W^\bullet$ of m-singular knots by $v_W^\bullet(k) = W(\sigma(k))$. The derivative of $v_W^\bullet$ is zero. Indeed,

$$\delta v_W^\bullet(k) = W(\sigma(k^+)) - W(\sigma(k^-))$$

for some knots k^+ and k^- that differ by crossing changes, but σ is invariant under crossing changes. Also, since W is a weight system it satisfies `4T`* and `1T`*, and so W' satisfies `T4T`* and `T1T`*. Therefore the conditions of Theorem 1.2.16 and integrates into a Vassiliev invariant.

The Vassiliev part of the theorem is independent of the original version and was proven separately in Section 1.2.

Now for the reverse direction, suppose Theorem 1.5.2 holds. Take a m-singular knot invariant $v^\bullet$ satisfying `4T`* and `1T`*, and that $\delta v^\bullet = 0$. This defines a weight system $W_{v^\bullet}$, which by the Kontsevich part of Theorem 1.5.2 gives an invariant class in $\mathcal{V}_m/\mathcal{V}_{m-1}$. Explicitly, this invariant is $W_{v^\bullet} \circ Z$. But by definition of the derivative of an m-singular invariant,

$$\delta^m(W_{v^\bullet} \circ Z)(k^\bullet) = (W_{v^\bullet} \circ Z)(\delta^m k^\bullet)$$

which by the definition of a universal Vassiliev invariant is just $v^\bullet$.

Thus, $W_{v^\bullet} \circ Z$ is the mth derivative of $v^\bullet$, so $v^\bullet$ integrates into a Vassiliev invariant of order m, completing the proof. $\square$

2

Lie theory and Jacobi diagrams

The fundamental theorem of Vassiliev invariants states that the bialgebra of Vassiliev invariants can be broken up into nice combinatiorial weight systems. So to understand $\mathcal{V}$ it suffices to understand $\mathcal{W}$, or equivalently its dual $\mathcal{A}$. There is a hint that the structure of $\mathcal{A}$ may relate to Lie algebras.

2.1 Jacobi diagrams, AS, STU and IHX

This side of the story reframes the bialgebra $\mathcal{A}$ as an isomorphic bialgebra known as the algebra of Jacobi diagrams to illuminate the Lie theory connections.

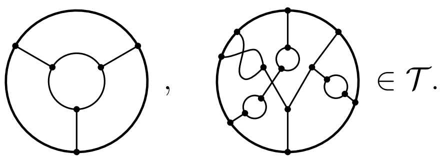
Definition 2.1.1 A **unitrivalent diagram** is a unitrivalent graph (with loops and multiple edges allowed) with the following additional data:

- each trivalent vertex has a fixed cyclic order of incident edge-connections,
- the set of univalent vertices has a fixed cyclic order.

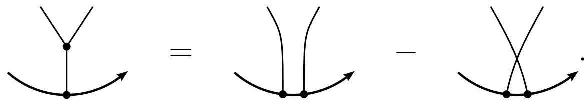
The vector space of unitrivalent diagrams is denoted $\mathcal{T}$.

When drawing unitrivalent diagrams, there are two notation conventions. Firstly, the fixed cyclic order of the univalent edges of is specified by drawing them connected to a circle (the cyclic order is induced by traversing the circle anticlockwise). Secondly, all the trivalent vertices are taken with the anticlockwise cyclic ordering unless an arrow around that vertex indicates otherwise.

In particular, from the first point, all chord diagrams are unitrivalent diagrams with only univalent vertices (the chord ends). Further examples of unitrivalent diagrams would be



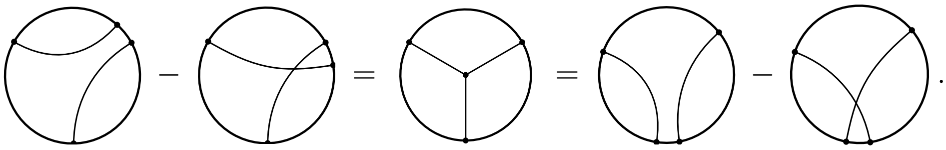
Definition 2.1.2 The **STU relation** is the relation



(STU)

As usual, this is not an individual relation but a type of relations, true in any diagrams that are identical except for the subdiagrams being as shown.

Note that the for the chord diagrams inside the algebra of Jacobi diagrams, the STU relations imply the 4T relations, as

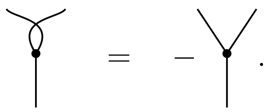


Definition 2.1.3 The algebra $\mathcal{J}$ of Jacobi diagrams is the vector space $\mathcal{T}/\mathtt{STU}$, with the product $\sharp$ defined the same way as it was for chord diagrams.

This is well-defined. The proof of Proposition 1.3.4 showed that the product $\sharp$ being well-defined on $\mathcal{A}$ was a consequence of the 4T relations, which are implied by the STU relations. From the STU relations, we may deduce the following other relations which hold in $\mathcal{J}$.

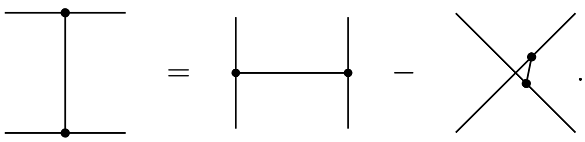
Proposition 2.1.4 *The following relations are consequences of the* STU *relation in* $\mathcal{J}$*:*

(a) The ***AS relation*** *(antisymmetry relation),*



(AS)

(b) The ***IHX relation****,*



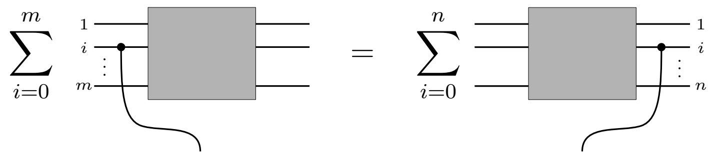
(IHX)

Proof (a) Take two diagrams which differ only by AS at one (trivalent) vertex. If the vertex at which the AS relation resides is adjacent to a univalent vertex (i.e. touches the outer circle), then this is immediate from applying STU to both diagrams at that vertex.

If the vertex is not immediately adjacent to a univalent vertex, then it is some d vertices 'in the way'. By applying STU to those vertices yields a sum of 2^d diagrams, all identical except for differing by AS, now on a vertex adjacent to a univalent vertex.

(b) A similar argument applies. If one of the two vertices of the IHX is adjacent to the circle, then the result is a direct consequence of an STU on each of the vertices. Otherwise, some STUs may be required first. □

Lemma 2.1.5 (Generalised IHX) *The following holds in* $\mathcal{J}$ *for any subgraph consisting of trivalent vertices that can be inserted into the grey box.*



The result is standard, see Chapter 5.2 of [CDM12] for a proof.

We have already spoiled the surprise that in the end, $\mathcal{A}$ and $\mathcal{J}$ will be isomorphic as bialgebras. In fact, as algebras, this is clearly true so far, as $\mathcal{J}$ is just a change of basis from $\mathcal{A}$. Since $\mathcal{A}$ spans $\mathcal{J}$, we can attempt to lift the coproduct from $\mathcal{A}$ directly onto $\mathcal{J}$.

Proposition 2.1.6 *The coproduct Δ on $J \in \mathcal{J}$ defined by taking a Jacobi diagram, representing it as a chord diagram via* STU*, taking the coproduct in $\mathcal{A}$, then interpreting the result as a Jacobi diagram via the inclusion of $\mathcal{A}$ into $\mathcal{J}$, is also given by the following formula.*

$$\Delta(J) = \sum_{C \subset S} J_C \otimes J_{\overline{C}},$$

where S is the set of connected components of J, and $\overline{C} = S \setminus C$.

Proof Note that this has the same symbolic form as the coproduct in $\mathcal{A}$ given in Definition 1.4.6, but with chords replaced by connected components of Jacobi diagrams. However, when working in $\mathcal{A} \subset \mathcal{J}$, there are only univalent vertices, so the connected components are exactly the chords. Since $\mathcal{A}$ forms a basis for $\mathcal{J}$, and the formula is linear, it extends to all of $\mathcal{J}$. □

Corollary 2.1.7 *The primitive elements $\mathcal{P}(\mathcal{A})$ are the connected Jacobi diagrams.*

Corollary 2.1.8 *The bialgebras $\mathcal{A}$ and $\mathcal{J}$ are isomorphic.*

Warning 2.1.9 Justified by this isomorphism, we write $\mathcal{J} = \mathcal{A}$, and $\mathcal{A}$ is the preferred choice of notation for both chord diagrams and Jacobi diagrams.

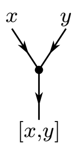
2.2 Lie algebra weight systems

Similar diagrammatic relations to STU, AS and IHX satisfied by $\mathcal{A}$ are appear also in the context of a graphical notation for multilinear maps, a fact which we may exploit to probe $\mathcal{A}$. Before seeing how, let us review this graphical notation following [Thu00; RW06].

Remark 2.2.1 This diagrammatic calculus is well-known but it goes by many names: string diagram calculus, Penrose calculus, tensor calculus, diagrammatic calculus for tensors, etc. We call them string diagrams.

A tensor is a multilinear map $X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$, or equivalently (via the classical canonical isomorphism) an element of the vector space $X_1^* \otimes X_2^* \otimes \cdots X_n^* \otimes Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$.

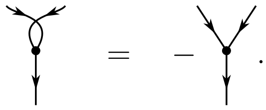
A tensor can be represented as a vertex with $m+n$ unbound directed edges; m incoming edged decorated by the corresponding vector spaces (in the example above, $X_1, \cdots$), and n outgoing edges decordated by $Y_1, \cdots$. For example, the bracket in a Lie algebra $\mathfrak{g}$ is an element $[\cdot,\cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, expressed as



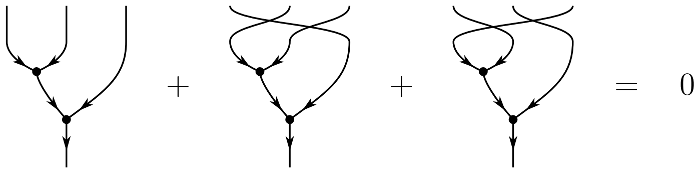
.

In fact, since the relation if true for any elements $x, y \in \mathfrak{g}$, the labels on the edges can be dropped.

Such a notation is useful because composition of tensors can be expressed graphically by connecting outgoing and incoming legs with the same decoration. Relations can therefore be expressed graphically, for example, the antisymmetry of the bracket $[y, x] = -[x, y]$ becomes



the Jacobi relation $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ becomes



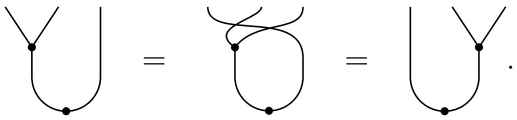
Looking at the relations these relations in the tensor algebra $\mathcal{T}(\mathfrak{g})$, the first solid evidence of Lie-theoretic structure in this story emerges. The antisymmetry of the bracket, drawn as a string diagram looks like a directed version of `AS`. Similarly the string diagram Jacobi relation can be arranged into a directed version of `IHX`.

Furthermore, suppose $\mathfrak{g}$ is a metric Lie algebra. Then it has an invariant, nondegenerate, bilinear form $\langle \cdot, \cdot \rangle \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. Being nondegenerate, it can be inverted to an element $c \in \mathfrak{g} \otimes \mathfrak{g}$. We have additional bivalent vertices

 and  .

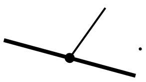
These can be used to change the arrow direction on any edge, allowing us to drop the edge arrows from the notation.

Moreover, the invariance of the metric can be written as $\langle [x, y], z \rangle = \langle [y, z], x \rangle$ which graphically can be represented as cyclic invariance of the contraction of the bracket and the metric



A similar argument works for the casimir.

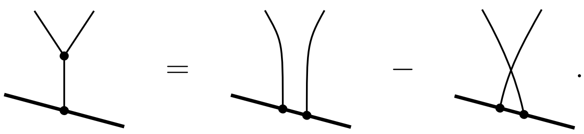
A representation of $\mathfrak{g}$ into a finite-dimensional vector space V takes the form $\rho \in \mathfrak{g}^* \otimes V^* \otimes V$. This takes a new kind of input an output, namely a $v \in V$ which we denote by a thick line at a shallow angle



The identity the action is a Lie action,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

is graphically



Again, arrows are unnecessary on the thick edges corresponding to inputs and outputs of V, as cup and cap vertices similar to the metric and casimir for $\mathfrak{g}$ are given by the maps

$$f \otimes v \longmapsto f(v) \quad \text{and} \quad 1 \longmapsto \sum_i e_i \otimes e_i^*.$$

The famous construction of Bar-Natan [Bar95] uses this diagrammatic calculus to take metric Lie algebras and produce weight systems.

Theorem 2.2.2 *The construction below which takes a lie algebra $\mathfrak{g}$ produces a well-defined weight system $W_{\mathfrak{g}}$ valued in $\mathcal{U}(\mathfrak{g})$. Furthermore if a representation, ρ of $\mathfrak{g}$ is given, then a $\mathbb{C}$-valued weight system is produced.*

Construction 2.2.3 Given a Lie algebra $\mathfrak{g}$, the vertex m with signature $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ is associated to a genuine element of the tensor algebra $T_{\mathfrak{g}}(m) \in \mathfrak{g}^{\otimes 3}$. For each trivalent vertex of the Jacobi diagram $J \in \mathcal{A}$, take $T_{\mathfrak{g}}(m)$, and take all of their tensor products, in arbitrary order. There being v trivalent vertices, this lies in $\mathfrak{g}^{\otimes 3v}$.

For each edge between the trivalent vertices in the Jacobi diagram, contract the corresponding variables in $\mathfrak{g}^{\otimes 3v}$. The result lies in $\mathfrak{g}^{\otimes 3v-2e}$ where e is the number of edges between trivalent vertices.

The number of univalent vertices matches the number of remaining uncontracted trivalent vertices, $u = 3v - 2e$. In-particular the edges between univalent and trivalent vertices, along with the cyclic order on univalent vertices determines a cyclic order on the remaining trivalent vertices. So, a cyclic order on the factors of $\mathfrak{g}^{\otimes 3v-2e} = \mathfrak{g}^{\otimes u}$. Permute the factors into an order which respects the induced cyclic order (there are many — choose one). Projecting from the tensor algebra of $\mathfrak{g}$ into its universal enveloping algebra, we obtain an element $W_{\mathfrak{g}}(J) \in \mathcal{U}(\mathfrak{g})$.

A representation $\rho : \mathfrak{g} \to \mathrm{Hom}(V)$ of $\mathfrak{g}$ extends uniquely to a representation $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{Hom}(V)$ of $\mathcal{U}(\mathfrak{g})$. Use this representation to take the trace of $W_{\mathfrak{g}}(J)$. The result is an element of the ground field $W_{\mathfrak{g},\rho}(J) = \mathrm{tr}(\rho(T_{\mathfrak{g}}(J))) \in \mathbb{C}$.

Warning 2.2.4 When constructing $T_{\mathfrak{g}}(m)$, the tensor factors in the tensor corresponding to the bracket need to have the unusual cyclic order $(y, x, [x, y]_{\mathfrak{g}})$. This is because its projection into $\mathcal{U}(\mathfrak{g})$ should obey STU, and this is the cyclic order of the trivalent vertex in STU.

Hence, we have the functions $W_{\mathfrak{g}} : \mathcal{A} \to \mathcal{U}(\mathfrak{g})$ and $W_{\mathfrak{g},\rho} : \mathcal{A} \to \mathbb{C}$ which we claim are weight systems.

Proof The contruction creates a string diagram which is well-defined modulo certain diagrammatic relations as seen at the start of this section such as the cyclic invariance of the contraction of the metric and bracket. By [Vai94, Proposition 3.1.1], string diagrams modulo these relations are exactly encoded by unitrivalent graphs with a cyclic order on trivalent vertices, and a linear order on univalent vertices.

Explicitly have diagrams for all the relations, involving the symmetry natural transformation, so that we can later say that this could not be the on-the-nose identity.

Thus, to prove that the map $W_{\mathfrak{g}} : \mathcal{A} \to \mathcal{U}(\mathfrak{g})$ is well-defined it suffices to show that the string diagrams obey the STU relations when interpreted as elements of $\mathcal{U}(\mathfrak{g})$. If two chord diagrams differ by STU, on some univalent vertices associated with adjacent tensor factors y and x, the construction gives

$$\cdots \otimes [x, y]_{\mathfrak{g}} \otimes \cdots \qquad \text{and} \qquad (\cdots \otimes x \otimes y \otimes \cdots) - (\cdots \otimes y \otimes x \otimes \cdots),$$

but this equality is exactly the quotient in the $\mathcal{U}(\mathfrak{g})$.

Indeed `AS` and `IHX` follow from `STU` but they are also easy to see directly. `AS` follows from the skew-symmetry of the vertex m, being built out of the bracket, and `IHX` follows from the Jacobi identity in $\mathfrak{g}$.

Finally, a Jacobi diagram's univalent vertices only have a cyclic order. But in the construction a true order respecting that cyclic order was chosen. We should show that any order respecting the cyclic order leads to the same result. It suffices to show that the tensor is cyclically symmetric. Indeed, a stronger statement is true: anything satisfying the `STU` relation is cyclically symmetric. Since it is stronger, let us prove it instead for $\mathcal{A}$.

It is equivalent to prove that any Jacobi diagram on a long line is cyclically symmetric. Examine the operation of taking the univalent vertex (which we will here call legs) in the first position and moving it to the last position, reducing by one the positions of all other legs. This generates the cyclic group on the number of univalent vertices, so it is enough to show this operation preserves the Jacobi diagram. The `STU` relation gives the cost of commuting the first leg past the ith leg: the same diagram with the first leg attached to the ith leg. Thus cost commuting the first leg to the very last place is the sum over the other legs of attaching it to those legs. We will show that this cost is zero.

We split this sum by the connected components of the ith leg. On components that don't contain the first leg, this is the sum, over legs, of the same diagram, with the first leg attached to the ith leg. This vanishes by the generalised `IHX`. On the component that contains the first leg, generalised `IHX` produces a diagram with a loop at the top, which is killed by `AS`.

That $W_{\mathfrak{g},\rho}$ is a genuine weight system into $\mathbb{C}$ follows from that a representation on $\mathfrak{g}$ extends uniquely to $\mathcal{U}(\mathfrak{g})$. □

Confirm: This obeys all the relations of $\mathcal{A}$, and more. Does this make it a quotient of $\mathcal{A}$?

Let's look at a specific weight system for the Lie algebra $\mathfrak{sl}_2$ [Bar95; CV97].

Example 2.2.5 (Weight system for $\mathfrak{sl}_2$) We choose as our Lie algebra $\mathfrak{sl}_2$, generated by h, e and f with commutators

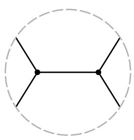
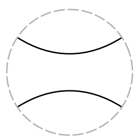
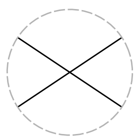
$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h.$$

The unique invariant bilinear form up to scalar multiple is

$$\langle h,h\rangle = 2, \quad \langle e,f\rangle = 1, \quad \langle f,e\rangle = 1,$$

(those pairings not dipsplayed are zero). Here we have chosen the normalisation of the metric that agrees with the trace in the adjoint representation.

In [CV97], the following skein relation is derived:

$W_{\mathfrak{sl}_2}$  $= 2W_{\mathfrak{sl}_2}$  $- 2W_{\mathfrak{sl}_2}$ 

Proof Compute both sides by contracting and permuting tensors for the bracket, and casimir/metric in $\mathfrak{sl}_2$. Both give

$$\begin{aligned} &-h\otimes e\otimes h\otimes f + h\otimes e\otimes f\otimes h - h\otimes f\otimes h\otimes e + h\otimes f\otimes e\otimes h \\ &+ e\otimes h\otimes h\otimes f - e\otimes h\otimes f\otimes h + 2e\otimes f\otimes e\otimes f - 2e\otimes f\otimes f\otimes e \\ &+ f\otimes h\otimes h\otimes e - f\otimes h\otimes e\otimes h - 2f\otimes e\otimes e\otimes f + 2f\otimes e\otimes f\otimes e. \end{aligned}$$ □

In fact, this skein relation is an analogue of the vector triple product rule for the cross product in $\mathbb{R}^3$, which is related to the Lie algebra $\mathfrak{sl}_2$ [CDM12]. It has been further studied in [MS17]. There is also a cross product in $\mathbb{R}^7$, related to the exceptional Lie algebra $\mathfrak{g}_2$. However it does not obey the vector triple product rule.

Remark 2.2.6 When computing via the $\mathfrak{sl}_2$ skein relation above, it's possible to create a "bubble" (part of a diagram without any trivalent vertices). Since we are computing via contractions in the tensor algebra, this is to be interpreted as the contraction of the metric with the casimir. In a finite-dimensional Lie algebra, this is just the dimension, so for $\mathfrak{sl}_2$, the factor 3.

Question 2.2.7 Are there similar skein relations that the weight systems for the exceptional Lie algebras obey?

Bar-Natan's construction yields a way of extracting some information from $\mathcal{A}$ by plugging in a metric Lie algebra – doing so constructs some quotient of $\mathcal{A}$. This naturally begs the question whether every all the information in $\mathcal{A}$ can be extracted with some metric Lie algebra.

Computer enumeration of chord diagrams [Bar95] prove this for order $m \leq 9$, the order to which the dimensions $\dim \mathcal{A}_m = \dim \mathcal{W}_m$ were known:

m	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104
$\dim(\operatorname{span}(W_{\mathfrak{sl}_n, \mathfrak{gl}_n}))$	1	1	2	3	6	10	19	33	60	104

Here $\operatorname{span}(W_{\{\mathfrak{g},\rho\}})$ denotes the dimension of the subspace of $\mathcal{W}_m$ spanned by weight systems coming from Costruction 2.2.3 on representations of classical Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$.

Conjecture 2.2.8 (Bar-Natan) *All weight systems are obtained as Lie algebra weight systems. In other words the set $\{W_{\mathfrak{g},\rho} \mid \mathfrak{g}$ a Lie algebra, ρ a representation of $\mathfrak{g}\}$ spans $\mathcal{W}$.*

Indeed the Lie action relation $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ as it was drawn graphically looks exactly like the STU relation in $\mathcal{A}$. However, looks can be decieving and quite surprisingly this conjecture is false. The counterexample was found by Pierre Vogel [Vog97] in an attempt to answer the following related question:

Question 2.2.9 (Vogel) Is there some single universal Lie algebra object whose weight system spans W_{Lie}, the span of all Lie-algebraic weight systems?

We will look at examples of more general weight systems in Section 2.3.

2.3 Non-Lie algebraic weight systems

Conjecture 2.2.8 being false implies that STU is more general than $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$. The same is true for IHX and AS compared to their strictly Lie-theoretic counterparts. In fact, Construction 2.2.3 is just one example of a more general construction introduced by Vogel and Vaintrob to construct weight systems coming from metric Lie super-algebras.

The most general type of objects these constructions apply to are called by Vaintrob [Vai94] 'Lie S-algebras', but we will follow the more modern approach of [RW06; Rob01] and they will be known as Lie algebra objects in a symmetric monoidal category.

Definition 2.3.1 A **(weak) monoidal category** is a category $\mathcal{C}$ equipped with a functor

$$\begin{array}{rccc} \otimes : & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C} \\ & (A,B) & \longmapsto & A\otimes B, \end{array}$$

an **unit** object $k\in\mathcal{C}$, and natural isomorphisms

$$\otimes\circ(\otimes\times\mathrm{id})\longrightarrow\otimes\circ(\mathrm{id}\times\otimes)\qquad\text{and}\qquad\otimes\longrightarrow\mathrm{id}$$

satisfying some relations known as the pentagon and triangle relations. The natural isomorphisms give isomorphisms

$$(A\otimes B)\otimes C\cong A\otimes(B\otimes C),\qquad k\otimes A\cong A\cong A\otimes k$$

for every tuple of objects A, B and C in $\mathcal{C}$. If these isomorphisms are equalities, then $\mathcal{C}$ is a **strict** monoidal category.

Remark 2.3.2 Omitting the details, we assume that these natural isomorphisms are equalities, for example that $(A\otimes B)\otimes C = A\otimes(B\otimes C)$. This is acceptable by the coherence theorem for monoidal categories which says that every monoidal category is equivalent to a strict monoidal category, and it's why we omit the pentagon and triangle relations in the definition above. We refer to [Lei04] for details.

Definitions 2.3.3 (a) The **flip functor** is the functor

$$\begin{array}{rccc} \sigma : & \mathcal{C}\times\mathcal{C} & \longrightarrow & \mathcal{C}\times\mathcal{C} \\ & (A,B) & \longmapsto & (B,A). \end{array}$$

(b) A **symmetric monoidal category** is a monoidal category $\mathcal{C}$ equipped with a **symmetry natural isomorphism** τ

$$\otimes\longrightarrow\otimes\circ\sigma$$

satisfying the hexagon relation. The **hexagon relation** is the relation that the isomorphisms

$$\tau_{A,B}: A\otimes B\xrightarrow{\cong} B\otimes A$$

coming from the natural isomorphism τ obey

$$\tau_{A,B\otimes C} = (\mathrm{id}_B\otimes\tau_{A,A})\circ(\tau_{A,B}\otimes\mathrm{id}_C)$$

for every pair of objects A, and B in $\mathcal{C}$.

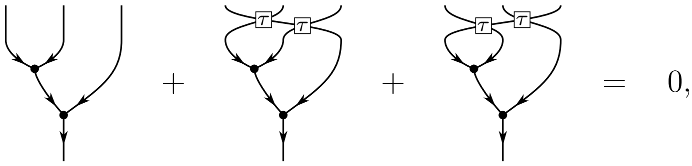
The hexagon relations have six terms instead of three when the associators omitted due to Remark 2.3.2 are reintroduced.

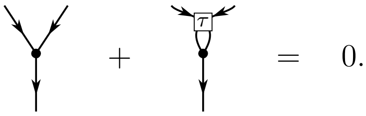
If the tensor category is additionally additive, we can define the following.

Definitions 2.3.4 (a) A **Lie algebra object** in an additive symmetric tensor category $\mathcal{C}$ is an object L equipped with a bracket morphism β such that

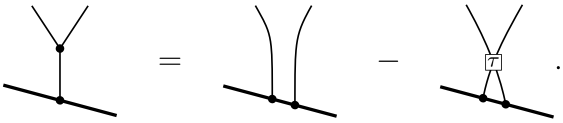
$$(\beta\circ(\beta\otimes\mathrm{id}))\circ(1+\tau_{123}+(\tau_{123})^2)=0\qquad\text{and}\qquad\beta+\beta\circ\tau=0.$$

Graphically, in terms of the string diagrams of the previous section, this is





(b) A **representation of a Lie algebra object** L in $\mathcal{C}$ into an object V in $\mathcal{C}$ is a morphism $\rho : L \otimes V \to V$, such that



(c) A **metric Lie algebra object** L in $\mathcal{C}$ is a Lie algebra object in $\mathcal{C}$, further equipped with the following modules over L...

We will give various concrete examples of Lie algebra objects in different symmetric tensor categories later. But for now, let's show that this data can still be used to construct weight systems, generalising Construction 2.2.3.

Theorem 2.3.5 ([Vai94]) *Let $\mathcal{C}$ be a rigid, additive, symmetric monoidal category, L a metric Lie algebra in $\mathcal{C}$, and M a dualisable representation η. Then, there is a weight system*

$$W_{L,\eta} : \mathcal{A} \longrightarrow \mathcal{C}(k,k) \cong k.$$

Proof Similar to the proof of the construction... □

The difference between this construction and Bar-Natan's original one is the treatment of the symmetry natural isomorphism τ. This tells us which isomorphism to use when rearranging the tensor factors. The most obvious isomorphism would be the identity, as it was in the original construction, corresponding to when $\mathcal{C}$ is a strict symmetric (strict) monoidal category. However unlike for general monoidal categories, not every symmetric monoidal category is equivalent to a strict symmetric monoidal category. At a down-to-earth level, Lie algebra objects with non-trivial symmetry isomorphisms are necessary to pick up all the structure in $\mathcal{A}$.

Example 2.3.6 If we take $\mathcal{C} = \mathbf{sVect}$, the symmetric monoidal category of super vector spaces, the Lie algebra objects are the following.

A **Lie superalgebra** $\mathfrak{g}$ is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading, equipped with a bracket $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ satisfying some axioms to follow. The grading induces the splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the direct summand $\mathfrak{g}_0$ is known as the **even** part and the summand $\mathfrak{g}_1$ is known as the **odd** part.

The axioms a Lie superalgebra must satisfy are the following analogues of the usual lie algebra axioms. Let x, y, z be homogeneous elements, so in $\mathfrak{g}_i, \mathfrak{g}_j$, and $\mathfrak{g}_k$ respectively, then **super symmetry** axiom is

$$[x,y] = -(-1)^{ij}[y,x],$$

and the **super Jacobi identity** is the axiom

$$(-1)^{ik}[x,[y,z]] + (-1)^{ji}[y,[z,x]] + (-1)^{kj}[z,[x,y]] = 0. \qquad \diamondsuit$$

Shortly after Conjecture 2.2.8 was made, the following results were achieved based on constructions with the exceptional Lie superalgebra $\mathfrak{D}(2,1;\alpha)$.

Theorem 2.3.7 *There are primitive Jacobi diagrams of order at least 17 [Vog97] and at least 15 [Lie99] which vanish on all Lie algebra weight systems.*

Furthermore, as was known about shortly after but not published until significantly later,

Theorem 2.3.8 *Vogel's diagram of order 17 vanishes also on all Lie superalgebra weight systems [Vog11].*

Corollary 2.3.9 *The set of Lie (super)algebra weight systems does not span $\mathcal{W}$.*

In general, it is still unknown to what exact level of generality one needs to go (what type of symmetric monoidal categories need to be considered) in order to generate all weight systems. We hereby provide a succinct review the current state of the literature on the subject.

Roberts and Willerton in [Rob01] and [RW06] examine weight systems constructructed from Lie algebra objects in the derived category of complex manifolds. Such weight systems are candidates for being able to detect knot orientation, which Lie algebra weight systems cannot [Bar95]. However, computing these weight systems is difficult, and to our knowledge, no computations exist in the literature.

More recently Aizawa-Kimura [AK25], have conducted some preliminary investigations into the class of colour Lie algebras (also known as ϵ-Lie algebras). This class generalises the $\mathbb{Z}/2\mathbb{Z}$ grading on Lie superalgebras to a more general type of group and sign rule. The example they present lies within the span of the $\mathfrak{sl}_2$ and $\mathfrak{gl}_{1|1}$ weight systems.

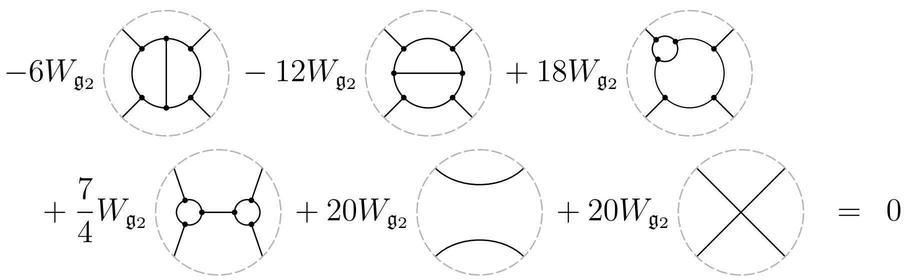
2.4 Some weight systems at the exceptional Lie algebras

The original motivation for the work of Vogel [Vog97; Vog11] and written more explicity in [Vog99] was to construct a universal object generalising all simple Lie algebras, whose weight systems span $\mathcal{W}$. **Insert a summary of the Vogel universality story.**

The relation for $W_{\mathfrak{sl}_2}$ given above is entirely internal to the lie algebra; it doesn't involve projecting to the universal enveloping algebra or choosing a representation. In terms of Vogel universality, this is known as 'lying in the adjoint sector'.

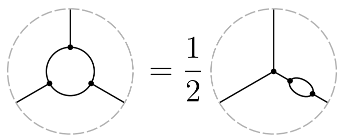
Present implicitly in [Vog11] is a way to determine, local relations internal to any simple lie algebra in terms of its parameters α, β and γ, and some complicated 'marked' elements of $\mathcal{A}$ defined recursively. These were written out explicitly in the recent preprint [KLS25, Appendix]. The use of different normalisations of the parameters α, β and γ in these two sources makes computing explicit relations tedious. Furthermore, the relations are general enough to hold for any Lie superalgebra, but can be simplified in the specific case of a Lie algebra.

Theorem 6.3(2) of [Vog11] gives a relation in $W_{\mathfrak{g}_2}$.

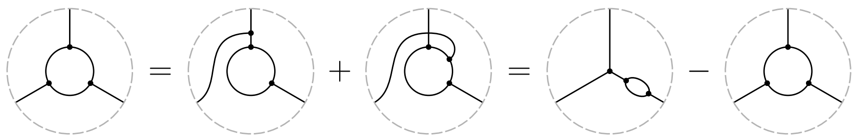


However, by some well-known identies internal to Jabobi diagrams, these diagrams can be simplified.

Lemma 2.4.1 *In $\mathcal{A}$ the 'trivalent-bubble' vertex is proprtional to the regular trivalent vertex with a bivalent bubble inserted on any one of the half-edges.*

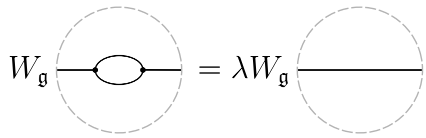


Proof Apply `IHX` to any pair of vertices, then three applications of `AS`.



□

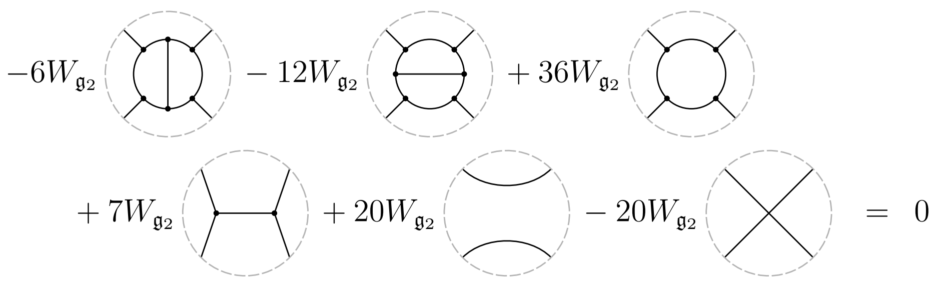
Lemma 2.4.2 *In Lie algebraic weight systems, the bubble is proportional to the line.*



Proof It's in CDM but it's unsatisfying. If it can be proved for Lie algebras it should be able to be proved in a way where it's obvious it doesn't work for Lie superalgebras, with without referring to the dual basis and its properties.

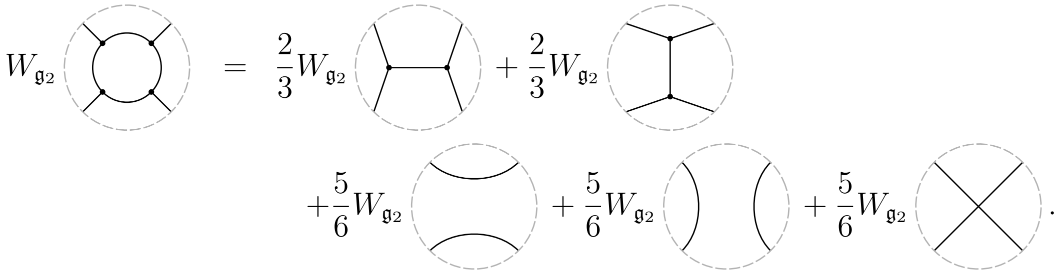
Simplifying Vogel's relation via these lemmas (we use the metric in which $\lambda = 1/4$), gives the following relation.

Proposition 2.4.3



This relation is true for any diagram into which each term is inserted. In-particular, it remains true after rotating all terms a quarter rotation clockwise. However it's not obvious that this is true, and various other relations have to be applied to turn the rotated version into the original relation. This suggests that perhaps the relation is a consequence of a similar relation.

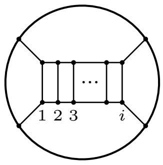
Theorem 2.4.4 *In the algebra $\mathfrak{g}_2$, the weight systems obey*



Proof Compute the tensors corresponding to both sides. □

This is an analogue of a relation for $\mathfrak{sl}_3$ found by Yoshizumi and Kuga in [YK99]. This was used recently by Yang in [Yan24] to compute the value of the $\mathfrak{sl}_3$ weight systems on the following infinite sequence of chord diagrams.

Definition 2.4.5 The ith **bookshelf diagram**, B_i, is the Jacobi diagram

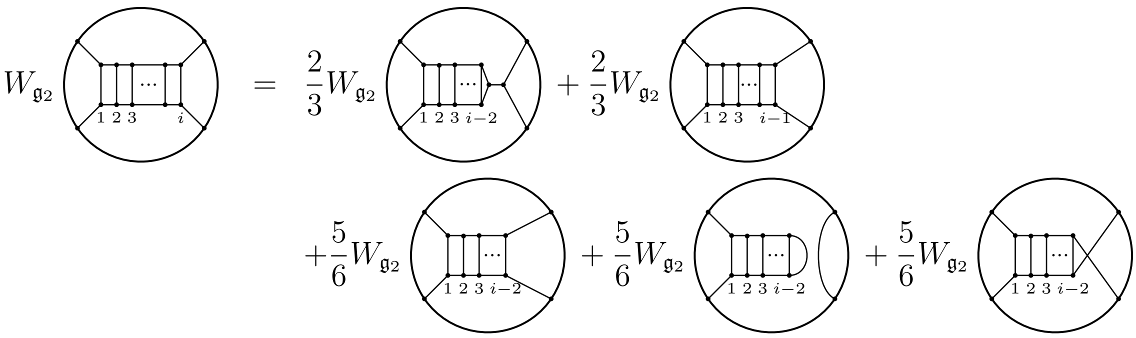


In [Yan24], a third-order recursion relation for the values of the $\mathfrak{sl}_3$ weight system on the bookshelf diagrams is found. We use the same idea with the new relation of Theorem 2.4.4 to compute their values under the $\mathfrak{g}_2$ weight system.

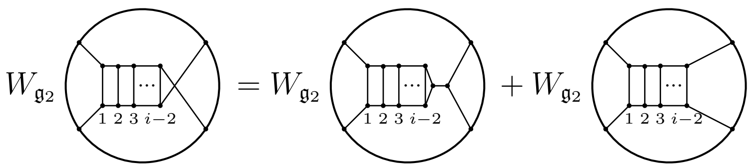
Theorem 2.4.6 *The values of the bookshelf diagram B_i are at most quadratic in the (quadratic) casimir element c of $\mathfrak{g}_2$. In particular,*

$$W_{\mathfrak{g}_2}(B_i) = c^2\left(\frac{1}{14}4^n + \frac{27}{112}(5/3)^n + \frac{11}{16}(-1)^n\right) + c\left(2^n + \frac{8}{3}(5/3)^n - \frac{11}{8}(-1)^n\right).$$

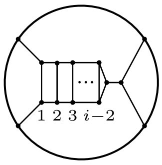
Proof Apply Theorem 2.4.4, to the rightmost box on a bookshelf diagram.

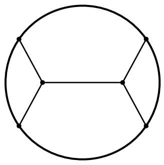


Applying STU to the final term gives

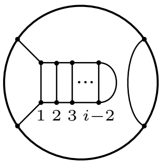


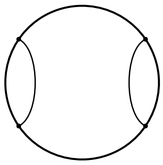
By Lemma 2.4.1, we have

$W_{\mathfrak{g}_2}$ 

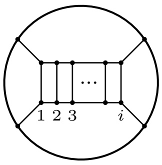
$= 2^{i-2} W_{\mathfrak{g}_2}$  $= 2^{i-2} 4c^2$

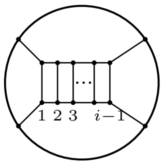
This calculation of the $\mathfrak{g}_2$ weight system on this diagram first appears in the literature in the PhD thesis of A. Kaishev which we have been unable to source, however it also appears in [CDM12, p. 181]. Our computer program also verifies it. By Lemma 2.4.2,

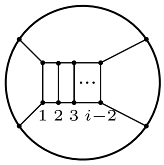
$W_{\mathfrak{g}_2}$ 

$= 4^{i-2} W_{\mathfrak{g}_2}$  $= 4^{i-2} c^2$

All in all,

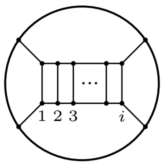
$W_{\mathfrak{g}_2}$ 

$= \frac{2}{3} W_{\mathfrak{g}_2}$ 

$+ \frac{5}{3} W_{\mathfrak{g}_2}$ 

$+ \frac{5}{6} 4^{i-2} c^2 + \frac{11}{6} 2^{i-2} c$

so setting

$g_i = W_{\mathfrak{g}_2}$ 

we have the second-order non-homogeneous recursion relation for g_i

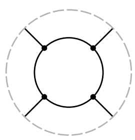
$$g_{i+2} = \frac{2}{3} g_{i+1} + \frac{5}{3} g_i + \frac{5}{6} 4^i c^2 + \frac{11}{6} 2^i c.$$

Its fourth-order homogeneous form is

$$g_{i+4} - \frac{20}{3} g_{i+3} + \frac{31}{3} g_{i+2} + \frac{14}{3} g_{i+1} - \frac{40}{3} g_i = 0.$$

To solve this we need the inital conditions g_1, g_2 and g_3. The first two come from the same computations of Kaishev [CDM12, p. 181], whereas the third we have to compute ourselves.

Computing in sagemath the tensor corresponding to the diagram



then projecting it to $\mathcal{U}(\mathfrak{g}_2)$ gives the value, but expressed in the PBW basis rather than as a polynomial in the casimir. There are two generating casimirs of $\mathfrak{g}_2$. The other is of degree 6, but g_3 is a degree 4 element of $Z(\mathfrak{g}_2)$, so it will be a polynomial in c of degree two or less. Evaluating

$\rho(g_3)$ with ρ being the trivial representation, the seven-dimensional standard representation, and the fourteen-dimensional adjoint representation gives the following values.

$$\begin{aligned} \rho_{\text{tr.}}(c) &= 0 & \qquad \rho_{\text{tr.}}(g_3) &= 0 \\ \rho_{\text{st.}}(c) &= 2 & \qquad \rho_{\text{st.}}(g_3) &= \frac{380}{9} \\ \rho_{\text{ad.}}(c) &= 4 & \qquad \rho_{\text{ad.}}(g_3) &= \frac{1120}{9} \end{aligned}$$

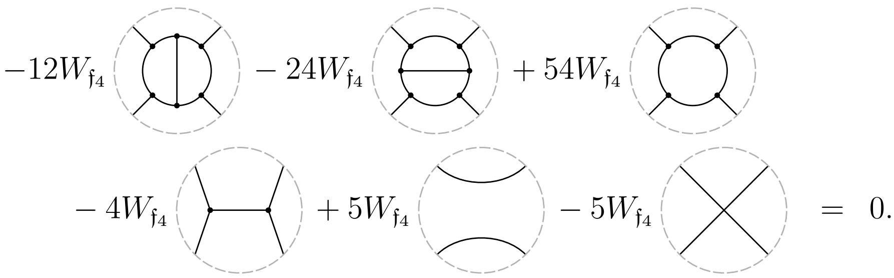
So by the Lagrange interpolation formula,

$$g_3 = 5c^2 + \frac{100}{9}c.$$

Solving the recurrence relation with these inital conditions yields the formula

$$g_i = c^2\left(\frac{1}{14}4^i + \frac{27}{112}(5/3)^i + \frac{11}{16}(-1)^i\right) + c\left(2^i + \frac{8}{3}(5/3)^i - \frac{11}{8}(-1)^i\right). \qquad \square$$

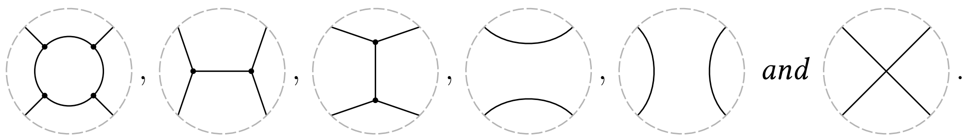
We use the same method to compute the a relation in the weight system $W_{\mathfrak{g}}$ for the exceptional Lie algebra $\mathfrak{g} = \mathfrak{f}_4$.



The dimension of $\mathfrak{g}_2$ is 14, and tensor contraction with sagemath takes about ten seconds on a [computer specs]. One the other hand, $\dim \mathfrak{f}_4 = 52$, and calculation takes several hours on [computer specs escona].

Finally, we note that for $\mathfrak{sl}_4$ the arguments given above do not work.

Theorem 2.4.7 *In the weight system corresponding to the Lie algebra $\mathfrak{sl}_4$, there is no linear relation between the diagrams*



Proof By computer verification with sagemath, the span of the corresponding tensors in $(\mathfrak{sl}_4)^{\otimes 4}$ is six-dimensional. $\square$

2.5 A universal metric Lie algebra object

There is, however, a theoretical construction of Hinnich-Vaintrob [HV00]. They construct a Lie algebra object in a symmetric tensor category with a "weight system" (in a certain sense) which is isomorphic to $\mathcal{A}$. In Chapter 3, we will examine an object, analagous to the space of welded knots, rather than knots.

Note, however that the objects in this category are not sets, but simply objects, so the result is useful theoretically rather than computationally.

Their construction uses the language of operads and props, which we review. For all of the below, we assume categories are linear.

Some operations on a set can always be composed by taking as input the output of another operation. These compositions alyways obey certain rules. If the set is a vector space, the operations too form a vector space. An operad captures this concept of operations on some object on a category abstractly.

Definition 2.5.1 An **operad** $\mathcal{O}$ in monoidal category $\mathcal{C}$ consists of:

- a collection of objects in $\mathcal{C}$, $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$, with $\mathcal{O}(n)$ known as the n**-ary operations**
- an element $\mathrm{id} \in \mathcal{O}(1)$.
- an action of the symmetric group S_n on each $\mathcal{O}(n)$
- a series of composition morphisms: for each n and $k_1, k_2, \cdots, k_n \in \mathbb{N}$

$$\begin{aligned} \circ_n : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \otimes \cdots \otimes \mathcal{O}(k_n) &\longrightarrow \mathcal{O}(k_1 + k_2 + \cdots + k_n) \\ \theta \otimes \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n &\longmapsto \theta \circ (\theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n) \end{aligned}$$

satisfying the following axioms: [Do I want all of these??]

- Associativity – but not of the operations, of composition of the operations
- Equivariance – permuting the inputs of θ, permuting the morphisms input to θ, and inversely block-permuting their inputs does nothing.
- The action of the symmetric group agrees with its action in $\mathcal{C}$.

Definition 2.5.2 An **algebra** A **over the operad** $\mathcal{O}$ **(in** $\mathcal{C}$**)** is an object of $\mathcal{C}$ satisfying certain natural compatibility conditions.

In essence, an algebra over an operad is an object in $\mathcal{C}$ whose operations obey the rules defined by $\mathcal{O}$.

Example 2.5.3 Nobody has ever showed me a definition of the Lie operad. I suppose it's just a quotient of the rooted binary trees operad. Also add an example of at least one other operad (maybe associative or commutative).

Composition in operads is something like contraction in a symmetric monoidal category, but with restictions on what can be contracted: only outputs and inputs. From two operations, one can take their tensor product, but this doesn't fit into the notion of an operad, as it has two outputs.

A step towards allowing general contractions is to allow multiple outputs, thereby considering not just operations but, for example in $\mathcal{C} = \mathbf{Vect}$, general multilinear functions. Operads were abstractions of the operations on some object (say, A), i.e. morphisms $A^{\otimes n} \to A$. Props will be abstractions of all maps $A^{\otimes n} \to A^{\otimes m}$.

Definition 2.5.4 A **prop** is a symmetric monoidal category where every object is of the form $A^{\otimes n}$ for some object A.

Definition 2.5.5 An **algebra** A **over the prop P (in** $\mathcal{C}$**)** is tensor functor $\alpha : \mathbf{P} \to \mathcal{C}$.

Since this is a tensor-functor from a tensor category generated by a single object, this is equivalent to a choice of object in $\mathcal{C}$ such that [...figure out what...].

Remark 2.5.6 There is an adjunction Prop $\to$ Oper. The operad to prop direction looks at all the morphisms $(n \to 1)$ in the prop, the other direction takes arbitrary tensor products of things in the operad.

Proposition 2.5.7 *The notions of algebra over a prop and algebra over an operad are equivalent. $A \in \mathcal{C}$ is an algebra over the operad $\mathcal{O}$ if and only if it is an algebra over the prop $\mathbf{P}(\mathcal{O})$.*

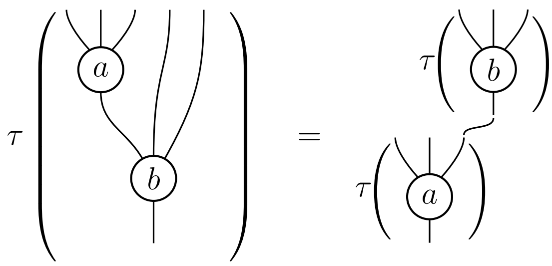
Now the only difference between contraction and what is governed by our prop is the inputs vs outputs. Remember the action of the symmetric group comes from the symmetry of the symmetric monoidal category $\mathcal{C}$. But this only permutes inputs and outputs among themselves. We want a way to permute inputs with outputs. This cannot actually be done in an operad, but if the operad is of a certain kind, the same effect can be achieved in terms of how the operad behaves within the the prop it generates.

Definition 2.5.8 A **cyclic operad** in $\mathcal{C}$ is a pair $(\mathcal{O}, \tau_+)$ where τ_+ is an action of each symmetric group S_{n+1} on $\mathcal{O}(n)$, extending the action coming from the symmetry τ of $\mathcal{C}$, and satisfying the axiom

$$(a \circ_1 b)\tau_+ = (b\tau_+) \circ_n (a\tau_+)$$

for any $a \in \mathcal{O}(m)$, $b \in \mathcal{O}(n)$. Here to specify the action of S_{n+1}, the action of $S_n \subset S_{n+1}$ is predetermined by the action of the symmetry τ.

This has the following graphical interpretation.



We denote an arbitrary cyclic operad $\mathcal{O}_{\tau_+}$ to reflect that whether an operad is cyclic depends both on $\mathcal{O}$ and the choice of action extending the action τ of $\mathcal{C}$ on $\mathcal{O}$.

The definition of a metric Lie algebra involved axioms on both the bracket, but also about the symmetry on the bracket and the form.

Definition 2.5.9 For a cyclic operad $(\mathcal{O}, \tau_+)$ in category $\mathcal{C}$, a **metric algebra** over $(\mathcal{O}, \tau_+)$ is a pair (A, b). That is, a choice of object $A \in \mathcal{C}$ with together with a symmetric morphism $m : A \otimes A \to k$, such that:

- there exists a morphism $c : k \to A \otimes A$ such that the composition

$$A \xrightarrow{c \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes m} A$$

  is the identity (m is **invertible**).

- the composition

$$\mathcal{O}(n) \otimes A^{\otimes n+1} \longrightarrow A \otimes A \xrightarrow{m} k$$

  is S_{n+1}-invariant (m is $\mathcal{O}$-**invariant**).

(REALLY? Surely it just has to agree with the S_n-action. What about swapping!)

The conditions above can alternatively and more naturally be phrased in terms of props.

Definition 2.5.10 We define the **prop for metric algebras over the cyclic operad** $(\mathcal{O}, \tau_+)$, denoted $\mathbf{P}_m(\mathcal{O}, \tau_+)$ as the prop generated by the prop $\mathbf{P}(\mathcal{O})$ along with two elements $m \in \mathbf{P}_m(\mathcal{O}, \tau_+)(2, 0)$ and $c \in \mathbf{P}_m(\mathcal{O}, \tau_+)(0, 2)$, satisfying the following conditions:

- The morphisms b and c are are symmetric and mutually inverse (in the sense of Definition 2.5.9).

- For each $f \in \mathcal{O}(n)$, the composition

$$A^{\otimes n} \xrightarrow{c \otimes \mathrm{id}} A^{\otimes n+2} \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} A^{\otimes 3} \xrightarrow{\mathrm{id} \otimes b} A$$

  is equal to $f\tau_+$.

In other words, in $\mathbf{P}_m(\mathcal{O}, \tau_+)$, the action of τ_+ corresponds to the cyclic permutation of inputs and outputs by means of the metric and the casimir.

Proposition 2.5.11 *Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{P}_m(Lie, \tau_+)$ are exactly the metric Lie algebra objects in the category $\mathcal{C}$.*

Proof Kind of by definition

This generalises the definition of metric Lie algebra to an arbitrary symmetric monoidal category. Indeed any prop for metric algebras over a cyclic operad in any symmetric monoidal category gives rise to a weight system. But furthermore, this permits the following slight *further* generalisation of the type of objects that give rise to weight systems, which is necessary for the universal construction of Hinnich and Vaintrob.

Definition 2.5.12 The **prop for casimir algebras over the cyclic operad** $(\mathcal{O}, \tau_+)$, denoted $\mathbf{P}_c(\mathcal{O}, \tau_+)$ is the prop generated by the prop $\mathbf{P}(\mathcal{O})$ along with the element $c \in \mathbf{P}_c(\mathcal{O}, \tau_+)$ such that:

- The morphism c is symmetric.

- For each $f \in \mathcal{O}(n)$, the diagram

$$\begin{tikzcd} A^{\otimes n-1} \arrow[r, "c\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes c"'] & A^{\otimes n+1} \arrow[d, "\mathrm{id}\otimes f"] \\ A^{\otimes n+1} \arrow[r, "f\tau\otimes \mathrm{id}"'] & A^{\otimes 2} \end{tikzcd}$$

  commutes.

Proposition 2.5.13 *Let $(\mathcal{C}, \tau)$ be a symmetric monoidal category. Then the algebras over the prop $\mathbf{P}_c(Lie, \tau_+)$ are exactly the casimir Lie algebra objects in the category $\mathcal{C}$.*

Proof Kind of by definition

This generalises the notion of a Lie algebra (object) with a casimir element (but not necessarily with a metric). We call such algebra (object) a casimr Lie algebra (object).

The reason we did not see an analagous definition earlier in this chapter is the following. In the case of finite-dimensional Lie algebras, every casimir Lie algebra is a metric Lie algebra. This is because in finite-dimensional vector spaces, a casimir, being an invariant symmetric map $c : k \to A \otimes A$ is always invertible, and its inverse is a metric. So, any finite-dimensional casimir Lie algebra (superalgebra, etc.) is a metric one.

The reason we need to consider casimir Lie algebra objects as opposed to just metric Lie algebra objects to make the universal construction is the following.

Consider the tensor category of a specific finite-dimensional (casimir) hence metric Lie algebra object in a specific symmetric tensor category. In $\mathcal{C}(k, k)$ we have the object formed by the contraction of the tensor with the metric. Since this is finite-dimensional, this is just the trace, therefore the dimension of the algebra. So, in the tensor category we have the relation $m \circ c = \dim \mathfrak{g}$. If we are trying to find an algebra which contains all the relations true in the tensor category of any metric Lie algebra object, then we arrive at a problem. All the relations of this form are contradictory.

Two obvious choices present themselves. Removing all relations, therefore allowing symbolic objects of the form $circle$, we construct an algebra bigger that $\mathcal{A}$. The other option is to not allow any circles by instead generalising the notion of casimir Lie algebra objects, (or something – this could use some thought)

Definition 2.5.14 The object $\mathbb{L}_m \in \mathbf{P}(Lie)$ Hard question: what do I put as the symmetry action here? Usually the symmetry is determined by the category. But here we defined the category as generated by the prop $\mathbf{P}(\mathcal{O})$ with an additional element. Does this all go round in circles?

Theorem 2.5.15 (Hinnich-Vaintrob) *Under [certain mild conditions] on $\mathcal{O}$, $\mathcal{C}$ and A, there exists an algebra $U(\mathcal{O}, A)$, the quotient of the external tensor algebra, obeying some important relation analogous to the defining property of the external enveloping algebra of a Lie algebra, but general to any operad.*

Theorem 2.5.16 (Hinnich-Vaintrob) $U(\mathcal{O} = Lie, A = 1 = \mathbb{L}_c \in \mathbf{P}_c(Lie))$ *is isomorphic as a Hopf algebra to* $\mathcal{A}$*.*

Proof We appeal to Hinnich and Vaintob a lot. □

In the next chapter we look at a generalisation of this statement in the context of welded knots.

3

Welded knots, isometry Lie algebras and arrow diagrams

3.1 Welded knots

3.2 Arrow diagrams

3.3 A universal welded weight system

References

[AK25] Naruhiko Aizawa and Daichi Kimura. "Universal weight systems from a minimal $\mathbb{Z}_2^2$-graded lie algebra". In: *Journal of Knot Theory and Its Ramifications* 34.10 (2025).

[Bar95] Dror Bar-Natan. "On the Vassiliev knot invariants". In: *Topology* 34.2 (1995), pp. 423–472.

[BG96] Dror Bar-Natan and Stavros Garoufalidis. "On the Melvin–Morton–Rozansky conjecture". In: *Inventiones mathematicae* 125.1 (May 1996), pp. 103–133.

[BS97] Dror Bar-Natan and Alexander Stoimenow. *The Fundamental Theorem of Vassiliev Invariants*. 1997. arXiv: `q-alg/9702009`.

[CD05] Sergei Chmutov and Sergei Duzhin. *The Kontsevich integral*. 2005. arXiv: `math/0501040`.

[CDM12] Sergei Chmutov, Sergei Duzhin, and Jacob Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

[CV97] Sergei Chmutov and Alexander Varchenko. "Remarks on the vassiliev knot invariants coming from $\mathfrak{sl}_2$". In: *Topology* 36.1 (1997), pp. 153–178.

[Hut98] Michael Hutchings. "Integration of Singular Braid Invariants and Graph Cohomology". In: *Transactions of the American Mathematical Society* 350.5 (1998), pp. 1791–1809.

[HV00] Vladimir Hinich and Arkady Vaintrob. "Cyclic operads and algebra of chord diagrams". In: *Selecta Mathematica, New Series* 8 (May 2000).

[KLS25] Dmitry Khudoteplov, Elena Lanina, and Alexey Sleptsov. *Construction of Lie algebra weight system kernel via Vogel algebra*. 2025. arXiv: `2411.14417`.

[Kon93] Maxim Kontsevich. "Vassiliev's knot invariants". In: *Advances in Soviet Mathematics* 16(2) (1993), pp. 137–150.

[Lei04] Tom Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[Les24] Christine Lescop. *Invariants of Links and 3-Manifolds from Graph Configuration*. European Mathematical Society Press, 2024.

[Lie99] Jens Lieberum. "On Vassiliev invariants not coming from semisimple Lie algebras". In: *Journal of Knot Theory and Its Ramifications* 08.05 (1999), pp. 659–666.

[MS17] Jean-Baptiste Meilhan and Sakie Suzuki. "Riordan trees and the homotopy $\mathfrak{sl}_2$ weight system". In: *Journal of Pure and Applied Algebra* 221.3 (2017), pp. 691–706.

[Rob01] Justin Roberts. *Rozansky-Witten theory*. 2001. arXiv: `math/0112209`.

[RW06] Justin Roberts and Simon Willerton. "On the Rozansky-Witten weight systems". In: *Algebraic & Geometric Topology* 10 (2006), pp. 1455–1519.

[Sta96] Ted Stanford. "Finite-type invariants of knots, links, and graphs". In: *Topology* 35.4 (1996), pp. 1027–1050.

[Thu00] Dylan P. Thurston. *Wheeling: A diagrammatic analogue of the Duflo isomorphism*. PhD thesis. 2000. arXiv: `math/0006083`.

[Vai94] Arkady Vaintrob. "Vassiliev knot invariants and Lie S-algebras". In: *Mathematical Research Letters* 1 (1994), pp. 579–595.

[Vog11] Pierre Vogel. “Algebraic structures on modules of diagrams”. In: *Journal of Pure and Applied Algebra* 215.6 (2011), pp. 1292–1339.

[Vog97] Pierre Vogel. “Algebraic structures on modules of diagrams”. An early preprint version of [Vog11]. 1997. URL: https://webusers.imj-prg.fr/~pierre.vogel/.

[Vog99] Pierre Vogel. “Vassiliev theory and the universal Lie algebra”. Unpublished preprint. 1999. URL: https://webusers.imj-prg.fr/~pierre.vogel/.

[Wil96] Simon Willerton. “Vassiliev invariants and the Hopf algebra of chord diagrams”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 119.1 (1996), pp. 55–65.

[Wil98] Simon Willerton. “A Combinatorial Half-Integration from Weight System to Vassiliev Knot Invariant”. In: *Journal of Knot Theory and Its Ramifications* 07.04 (1998), pp. 519–526.

[Yan24] Zhuoke Yang. “On Values of $\mathfrak{sl}_3$ Weight System on Chord Diagrams whose Intersection Graph is Complete Bipartite”. In: *Moscow Mathematical Journal* 24 (2024), pp. 125–140.

[YK99] Shutaro Yoshizumi and Ken’ichi Kuga. “Some formulas in $\mathfrak{sl}_3$ weight systems”. In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 75.7 (1999), pp. 134–136.