THEORETICAL ASTROPHYSICS II:
Polarized Radiative Transfer

HOMEWORK: Spectral Line Synthesis in LTE from given atmospheric model

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Contents

1 <u>INTRODUCTION</u>

The goal of this homework is to synthesize a spectral line profile in **Local Thermodynamic Equilibrium** from a given atmospheric model.

To do this, we are going to calculate level populations using the Boltzmann - and Saha equations, compute line opacity, source function and solve the radiative transfer equation.

The atmospheric model is provided by the **falc.dat**, which can be found in the repository of this report.

2 <u>SOLVE FOR THE IONIZATION STATE</u>

As a warm-up we are going to solve the Saha equation (only for Hydrogen) to compute the ionization state of Hydrogen as a function of height in the atmosphere. We will get the electron density and compare it with the one given in the model.

The Saha equation is given by

$$\frac{n_{j+1}n_e}{n_j} = \frac{2Z_{j+1}}{Z_j}\left(\frac{2\pi m_e k_B T}{h^2}\right)e^{\frac{E_j}{k_B T}} \tag{1}$$

Using the following constraints

$$n_{H^+} = n_e \qquad n_H + n_{H^+} + n_e = n_{tot} \qquad n_{tot} = \frac{p}{k_B T}$$

and defining the right hand side of equation (1) as

$$f(T,p) = \frac{2Z_{j+1}}{Z_j}\left(\frac{2\pi m_e k_B T}{h^2}\right)e^{\frac{E_j}{k_B T}}$$

we find

$$\frac{n_e^2}{n_{tot} - 2n_e} = f(T,p) \quad \Longleftrightarrow \quad n_e^2 + 2n_e^2 f(T,p) - n_{tot}f(T,p) = 0 \tag{2}$$

The equation is quadratic in n_e and the solution is given by the following expression

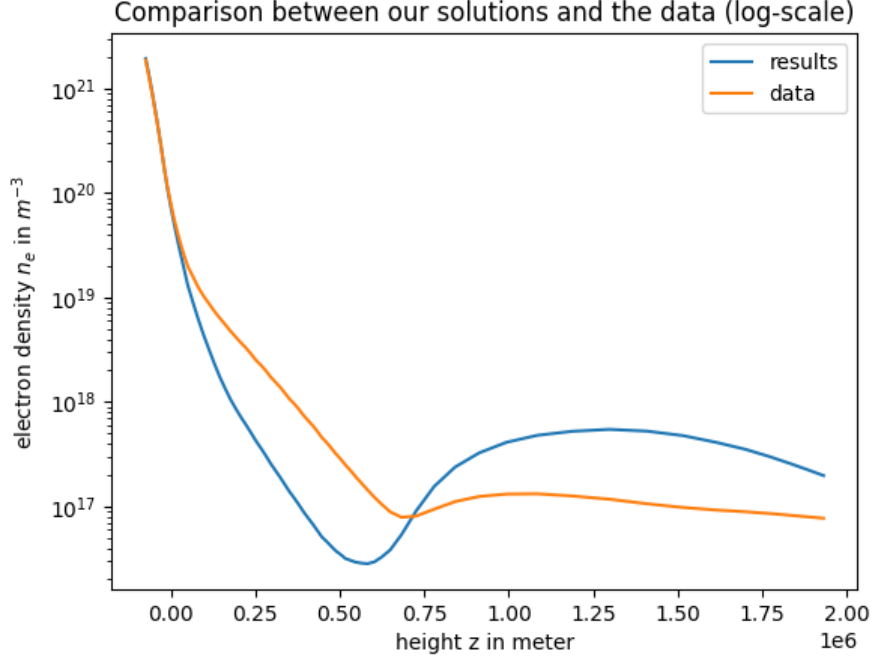$$n_e = -f(T,p) + \sqrt{f^2(T,p) + n_{tot}f(T,p)}$$

Figure 1: Comparison of electron density from the data with electron density calculated from our model

3   <u>ADDING ONE MORE SPECIES</u>

As we see, the error between our results and the data is getting bigger with the height. To fix this, we need to add more species that contribute electrons. The most important ones are Magnesium and Iron. We will substitute both of them with a proto-model for Magnesium only.

Since we now have more species, we get more equation which we have to solve:

$$\frac{n_{H+}n_e}{n_H} = \frac{2Z_{H+}}{Z_H}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{\frac{3}{2}} e^{-\frac{13.6eV}{k_B T}} = f_1(T,p)$$

$$\frac{n_{MgII}n_e}{n_{MgI}} = \frac{2Z_{MgII}}{Z_{MgI}}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{\frac{3}{2}} e^{-\frac{7.646eV}{k_B T}} = f_2(T,p)$$

$$n_H + n_{H+} + n_{MgI} + n_{MgII} = n_{tot} = \frac{p}{k_B T}$$

Abundance of Magnesium: $n_{MgI} + n_{MgII} = 7.6 \times 10^{-5} n_{tot}$

$$n_e = n_{H+} + n_{MgII}$$

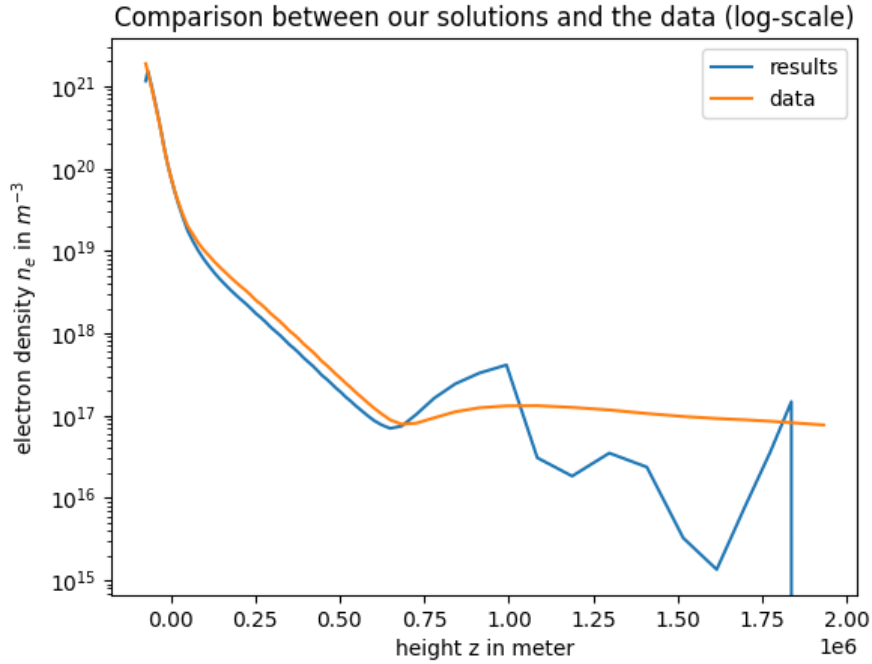for the partition functions we use: $Z_{MgI} = 2$, $Z_{MgII} = 1$.

Figure 2: Comparison of electron density from the data with electron density calculated from our model after adding more species.

4 CALCULATE LEVEL POPULATIONS

We are going to synthesize the Magnesium MgIb2 line at 517.2nm. For this we only need the energies of the level and their statistical weights given by $g = 2J + 1$. In the case of the Magnesium line g is given by $g = 2 \cdot 1 + 1 = 3$.

To calculate level population of each level we are going to use the Boltzmann equation

$$n_{j,i} = \frac{n_j g_i}{Z_i} e^{\frac{E_i}{k_B T}} \tag{3}$$

In our case we get the two equations

$$n_l = \frac{3 n_{MgI}}{2} e^{-\frac{2.7115919 eV}{k_B T}} \qquad\qquad n_u = \frac{3 n_{MgI}}{2} e^{-\frac{5.1078270 eV}{k_B T}}$$
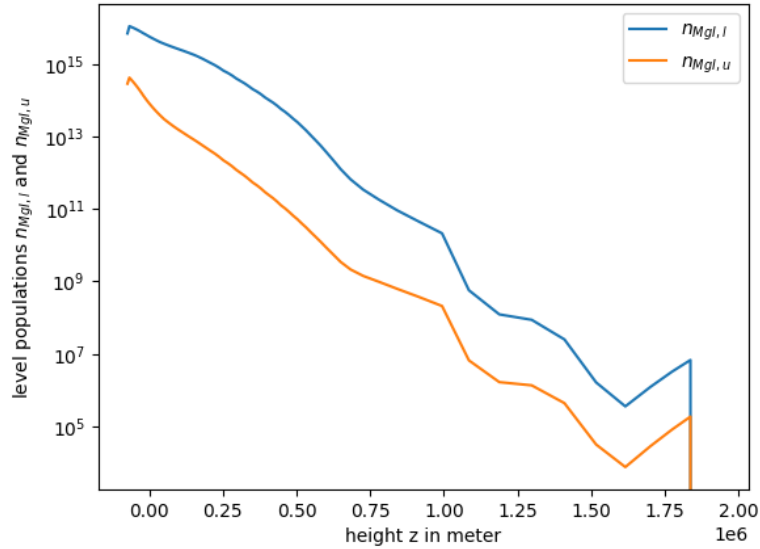

Figure 3: Comparison between our results for the lower level population and the upper level population of Magnesium.

4

5 LINE OPACITY AND SOURCE FUNCTION

We assume a LTE, in this case the source function is just the Planck function and we focus on the opacity given by:

$$\chi_\nu = (n_l B_{lu} - n_u B_{ul}) \frac{h\nu}{4\pi} \phi(\nu)$$

with: $n_l = n_{MgI,l}$, $n_u = n_{MgI,u}$, Einstein coefficients B_{lu} and B_{ul} and the line profile $\phi(\nu)$.

The task is now to calculate the Einstein coefficients. To do this we need to use a third Einstein coefficient, which we get from NIST website.

In our case this coefficient is given by: $A_{ul} = 3.46 \times 10^{-5} \frac{1}{s}$ For B_{ul} we use: $B_{ul} = \frac{c^2}{2h\nu^3} A_{ul}$ and for $B_{lu} = \frac{g_u}{g_l} B_{ul} = B_{ul}$

6 CALCULATION OF THE LINE PROFILE

Now we pick a wavelength range ($\lambda_0 = 517.2nm$, $\lambda \in [517.0nm, 517.4nm]$). We are going to calculate the line profile at each depth (2D-array depth vs. wavelength). For this we will work through the following steps:

1. Calculate the Doppler width from the magnesium atomic mass and temperature at each depth.

2. Calculate the damping frequency (constant value for A_{ul}).

3. Convert the wavelength grid to dimensionless frequency offset $u = (\nu - \nu_0)/\Delta\nu_D$ and convert damping to $a = \Gamma/(4\pi\Delta\nu_D)$.

4. Calculate the Voigt profile $\phi(\nu)$ at each depth and wavelength point.

Finally we are able to calculate the line opacity everywhere, at each depth and wavelength/frequency point: $\chi_\nu = (n_l B_{lu} - n_u B_{ul})\frac{h\nu}{4\pi}\phi(\nu)$
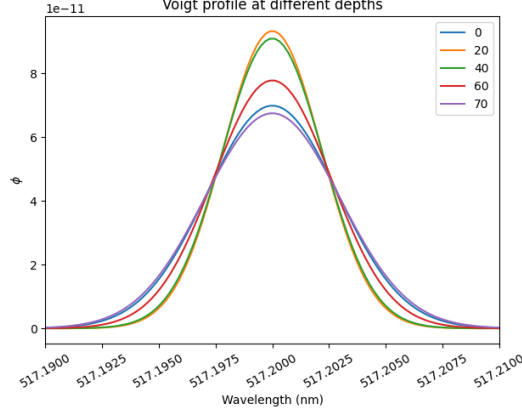

Figure 4: Implemented Voigt profile at different depths.
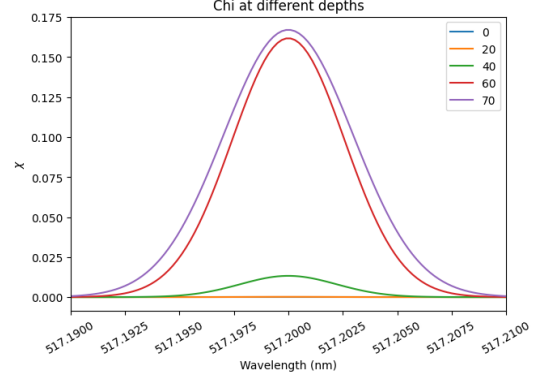

Figure 5: Line opacity χ at different depths.

7 TRANSFER EQUATION

We finally have: $\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$

Now we use:

$$I(\tau_1) = I(\tau_2)e^{-(\tau_2-\tau_1)} + \int_{\tau_1}^{\tau_2} S(\tau)e^{-(\tau-\tau_1)}d\tau \qquad (4)$$

We start at the bottom and set: $I_\nu(\tau_{\nu,max}) = B_\nu(T(\tau_{\nu,max}))$

And then step by step calculate I_ν at each depth, until we reached the top of the atmosphere.

To move step by step, we use:

$$I(\tau_i) = I(\tau_{i+1})e^{-(\tau_{i+1}-\tau_i)} + S(\tau_i)(1 - e^{-(\tau_{i+1}-\tau_i)}) \qquad (5)$$
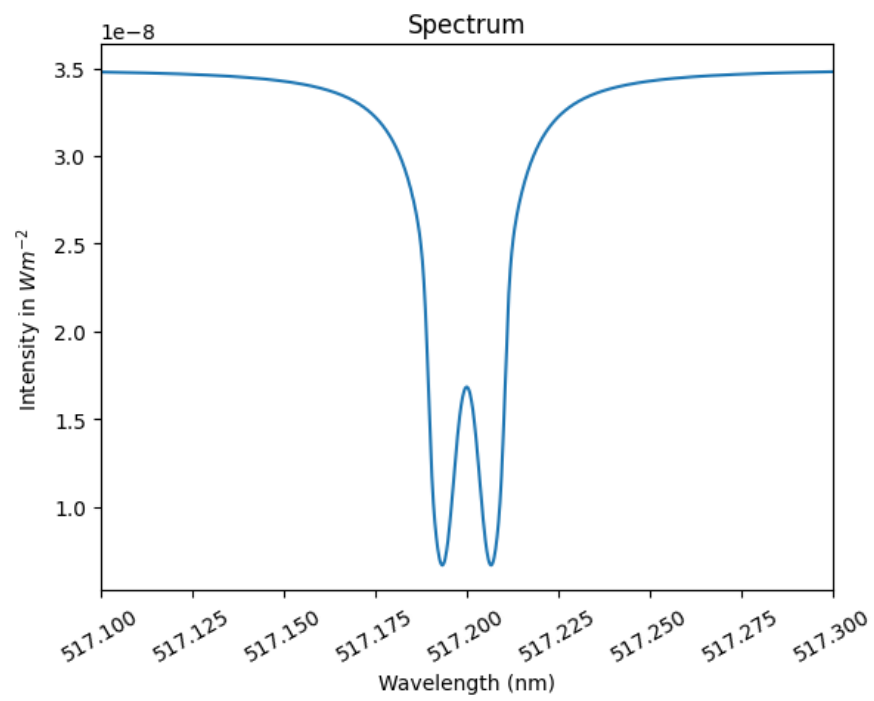
Figure 6: Final result: Intesity of our Line.