EXAMPLE 13.3: Let Ω denote the annular domain $\left\{p=(x,y)\in R^2 \leqslant \|p\|_2 \leqslant 2\right\}$ Use MATLAB to create and plot

SOLUTION: We use a node deployment strategy that is based on that of Method 2 of part (a) of the last example, dist

(i) anglel = $\pi/2$, angle2 = $5\pi/6$, maxnodes = 500, r=0.25
(ii) anglel = $-\pi$, angle2 = $\pi/2$, maxnodes = 1500, r=0.25
(iii) angle $7\pi/6$, angle2 = $11\pi/6$, maxnodes = 1200, r=0.025
(*Triangulating General Convex Polygons*) (a) Write an M-file, call it `[x y tri ] =unipolytr i (xv, yv, maxnodes)`,
(b) Use your program to redo Exercise 1.
(c) Run your program, and plot the nodes and resulting triangulations to obtain triangulations for each of the following
(i) The rectangle with vertices $(\pm 1, \pm 10)$.
(ii) The triangle with vertices (0,0), (1,0), (0,8).
(iii) A regular octagon unit sidelength.
(iv) The septagon with vertices (0,0), (2,0), (16,1), (16,4), (13,5),(11,4), (1,3).
Suggestion: One way to view a convex polygon is that its set of points can be described as the intersection of all point
NOTE:(*Triangulating General Polygons*) Since any polygon can be decomposed into convex pieces, the program `unipolytri` of
Use the idea of the above note to redo Exercise for the Reader 13.4(b).
Use the idea of the above note to triangulate, using between 400 and 800 nodes, the decagon that has the following vert
Consider the symmetric (nonconvex) polygon consisting of the rectangle with vertices: $(\pm 1, -1), (\pm 1, 0)$ with two (left a
(a) Apply the method in the above note to triangulate this region by splitting it into the left and right halves (which ar
(b) Apply the method in the above note to triangulate this region by splitting it into the following three convex pieces:

The next three exercises will involve triangulations of the domains having domains with curved boundaries illustrated in
Let the elliptical region ω of Figure 13.20(a) have equation (for its boundary): $x^2 + 4y^2 - 4$. Use MATLAB to create an
(a) The nodes are more or less uniformly distributed with essentially a square grid (as in Method 1 of part (a) in the so
(b) The nodes are deployed on concentric ellipses of the same eccentricity as the boundary ellipse (cf. Method 2 of part
(c) The nodes are deployed in concentric ellipses (as in part (b)) but the density increases as we near the boundary (cf.
(d) The density of the nodes increases as we approach the interior point $(x, y) = (1, 0)$ and such that between 20 and 30
Let the region ω of Figure 13.20(b) be specified as follows: The square (outside) boundary has equations: $x - 0, 2, y - 0$
(a) The nodes are, more or less, uniformly distributed with essentially a square grid (cf. Method 1 of part (a) in the sol
(b) The density of the nodes increases as we near each of the two interior circle boundary portions and such that the sq
Let the region ω of Figure 13.20(c) be specified as follows: The outside circle has: center = (0, 0) and radius = 2; the in
(a) The nodes are more or less uniformly distributed with essentially a square grid (cf. Method 1 of part (a) in the solu
(b) The nodes are deployed on concentric circles (to the outer boundary circle) and more or less uniformly distributed.
(c) The density of the nodes increases as we near any of the four corner points on the inside square boundary and the o
Let ω denote the region of Figure 13.20(d). (a) Use MATLAB to plot an airfoil (the inside boundary of ω) by setting u
(b) The nodes are more or less uniformly distributed with essentially a square grid (cf. Method 1 of part (a) in the solu
(c) The density of the nodes increases as we near the airfoil (inside) part of the boundary and the outside rectangle will
Suggestions: To find appropriate x and y vectors for the foil, it is probably easiest to copy the figure down on graph p

[H] [width=0.9]6 Two triangulations of airfoils, (a) (left) A single component airfoil similar to that in Figure 13.20(d). T
NOTE:(Rectangular Elements) For domains whose boundaries are made up of only vertical and horizontal segments, rectangula

These functions reduce to linear functions on any of the four edges of the rectangle so that continuity is assured across b
For the domain in Figure 13.22(b), let the outer vertices be (1,1), (7,1), (7,3), (3,3), (3,6), (7,6), (7,8), and (1,8). Tessell
(a) Write down a formula for the basis function $\Theta_{(2,2)}$(x,y) corresponding to the interior node (2,2).
(b) Use MATLAB to draw a three-dimensional graph of this basis function.
(c) Repeat parts (a) and (b) for the basis function $\Theta_{(1,1)}$(x,y) corresponding to the interior node (1,1).
(d) Are these basis functions differentiable (smooth) across all edges of adjacent elements? (It was already pointed out t
Let the domain in Figure 13.22(b) have the vertices and tessellation of the last exercise. On this domain, consider the fo

(a) Use MATLAB to draw a three-dimensional graph of this function.
(b) Use MATLAB to draw a three-dimensional graph of the finite element interpolant to this function using the basis fu

where each term of the sum corresponds to a node of the tessellation.
(c) Create and plot a corresponding approximation to f(x,y) that arises from the triangularon of the domain using 60 tr
(d) Repeat part (b) except this time use squares of sidelength 1/4 in the tessellation. (So there will be 16 times as many
Suggestion: In parts (b) and (d), use the meshgri d command for each element and use the hol d on command.
The standard rectangular element has vertices $(\pm 1, \pm 1)$. (a) Show that the corresponding four local basis functions (viz

(The local basis function ρ_i corresponds to the vertex v_i, and they are written with the same orientation as the vertices
(b) Use MATLAB to draw three-dimensional graphs of each of these four local basis functions.

(b) The triangulation of part (a) was rather uniform and had 1795 nodes. In this part we try to work with a smaller nu

We now move on to describe the FEM for the general case of the BVP (10):

Under the assumptions indicated in the theoretical discussion earlier in this section, this BVP can be shown to be equiv

Minimize the functional:

over the following set of admissible functions:

Note that the class of admissible functions requires only the Dirichlet boundary conditions (on Γ_1). The Robin bounda

Analogous to the one-dimensional method presented in Section 10.5, the FEM will solve a corresponding finite-dimensio
The basis functions corresponding to nodes on the boundary portion Γ_1 will have their coefficients determined by the D

FEM FOR THE BVP (10)-GENERAL CASE:

Step #1: Decompose the domain into elements, and represent the set of nodes and elements using matri

Step #2: Use the Dirichlet BCs $u(x, y) = g(x, y)$ on Γ_1 to determine the coefficients of the Dirichlet bound

Step #3: Assemble the $n \times n$ stiffness matrix A and load vector b needed to determine the remaining co

Step #4: Solve the stiffness equation $Ac = b$, and obtain the FEM solution

As before, the coefficients $c_1, c_2, \cdots, c_n$ will eventually be determined as the solution vector $c = [c_1 c_2 \cdots c_n]'$ of a linear s

and

In these formulas the integrals over Γ_2 are with respect to arclength (i.e., positively oriented line integrals). These can b

The assembly process can be coded much like the way we did it for Example 13.7. The only new feature here is the pres

from the definition of line integrals. Such integrals could be done with MATLAB's `quad` (or `quadl`)-but in a general FE

This formula, known as the **Newton-Coates formula** with three equally spaced points, is exact for polynomials up to

EXERCISE FOR THE READER 13.13: (a) Write an M-file `lineint = bdyintapprox(fun, tri, redges)` that works

In our next example, we will solve a BVP over an odd-shaped region. The problem is carefully constructed so that the e

EXAMPLE 13.8: Use the finite element method to solve the following mixed BVP over the parabolically shaped dom

(a) Use first a triangulation with between 300 and 500 nodes that are more or less uniformly distributed. Compare with

*Appendix A: Introduction to MATLAB's Symbolic Toolbox **A.l: WHAT ARE SYMBOLIC COMPUTATI**

This appendix is meant as a quick reference for occasions in which exact mathematical calculations or manipulation
Computing the (formula) for the derivative or antiderivative of a function
Simplifying or combining algebraic expressions
Computing a definite integral exactly and expressing the answer in terms of known functions and constants such as π, e
Finding analytical solutions of differential equations (if possible)
Solving algebraic or matrix equations exactly (if possible)
Such exact arithmetic computations are known collectively as **symbolic computations**. MATLAB is unable to perfor

There are also circumstances where the precision of MATLAB's floating point arithmetic is not good enough for a given
The remainder of this appendix will present a brief survey of some of the functionality and features of the Symbolic Too

A.2: ANALYTICAL MANIPULATIONS AND CALCULATIONS

To begin a symbolic calculation, we need to declare the relevant variables as symbolic. To declare x, y as symbolic varia

```
>> syms x y
```

Let's now do a few algebraic manipulations. The basic algebra manipulation commands that MAPLE has are as follows

>> $p2 = (x + 2 * y)\hat{}2;, p4 = (x + 2 * y)\hat{}4;$
`>> expand(p2) %Multiplies out the binomial product.`
`->ans =` $x\hat{}2 + 4 * x * y + 4 * y\hat{}2$
`>> expand(p4)`
`->` $ans = x\hat{}4 + 8 * x\hat{}3 * y + 24 * x\hat{}2 * y\hat{}2 + 32 * x * y\hat{}3 + 16 * y\hat{}4$
`>> pretty(ans) %Puts the answer in a prettier form.`

```
>> 4  3  2 2  3  4
x + 8 x y + 24 x y + 32 x y +16y
```

In general, for any sort of analytical expression exp, the command expand (exp) will use known analytical identities to t

`>> prett y (expand (tan` $(x + 2 * y)$ `))->`

To clean up (simplify) any sort of analytical expression (involving powers, radicals, trig functions, exponential functions
>> $h = x6 - x\hat{}5 - 12 * x^4 - 2 * x\hat{}3 + 41 * x\hat{}2 + 51 * x + 18;$

```
>> pretty(factor(h))
->   2     3
(x+2(x-3)(x+1)
```

This function will also factor positive integers into primes. This brings up an important point. MATLAB also has a fun

`>> factor(3^101-1)`
??? Error using ==¿ factor
The maximum value of n allowed is 2^32
`>> factor(sym(3^101-1)) %declaring the integer input as symbolic`
`%brings forth the MAPLE version this command.`
`>>ans =` (2)^110*(43)*(47)*(89)*(6622026029)

Whereas the Student Version of MATLAB includes access to many of the Symbolic Toolbox commands that one might