Relativistic Quantum Field Theory

Physics 7651

Homework 4

Due: In class on Wednesday, Sept. 28

1. **The Spin-Statistics Theorem and CPT** [5 points]

   (a) **Can a scalar field describe fermions?**. Unlike bosons, fermions obey *anti-commutation* relations. Compare the quantization of a free scalar field using canonical commutation ($-$) versus anti-commutation ($+$) relations,
   $$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}\right]_{\mp} = \delta^{(3)}\left(\mathbf{k} - \mathbf{k}'\right).$$
   Calculate the (anti-)commutatator of the field for spatial separations $(x - x')^2 = -r^2 < 0$. What can you conclude about the choice of using commutators or anti-commutators for a scalar field based on the requirement of causality?

   (b) **CPT theorem for complex scalars**. Argue that for an arbitrary *interacting* Lagrangian theory of complex scalar fields ψ_i and real scalar fields ϕ_a that CPT is a good symmetry.

2. **An explicit form for the parity operator** [5 points] A scalar field ϕ transforms under parity as
   $$\phi(\mathbf{x}, t) \to P\phi(\mathbf{x}, t)P^{-1} = \eta_P \phi(-\mathbf{x}, t), \qquad (*)$$
   where $\eta = \pm 1$ is the **intrinsic parity** of the field. In this problem we will demonstrate an explicit expression for the parity operator P in terms of creation and annihilation operators. Define the operator exponentiations
   $$P_1 = \exp\left[-i\frac{\pi}{2}\sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\right] \qquad P_2 = \exp\left[i\frac{\pi}{2}\eta_P \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}\right].$$
   Prove that
   $$P_1 a_{\mathbf{k}} P_1^{-1} = -i a_{\mathbf{k}} \qquad P_2 a_{\mathbf{k}} P_2^{-1} = i\eta_P a_{-\mathbf{k}}$$
   and show that the operator $P = P_1 P_2$ is unitary and satisfies $(*)$, i.e. $P_1 P_2$ gives an explicit expression for the parity operator. *Hint*: Use the fact that for any operators A and B,
   $$e^{i\alpha A} B e^{-i\alpha A} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} B_n \qquad \text{where } B_0 = B, \quad B_n = [A, B_{n-1}].$$

3. **C, P, and T for the Schrödinger field** [10 points] The charge-conjugation C, parity P, and time-reversal T operators act on a complex scalar field $\psi(\mathbf{x},t)$ via

$$\begin{aligned} C\psi(\mathbf{x},t)C^{-1} &= \psi^*(\mathbf{x},t), \\ P\psi(\mathbf{x},t)P^{-1} &= \psi(-\mathbf{x},t), \\ T\psi(\mathbf{x},t)T^{-1} &= \psi(\mathbf{x},-t), \end{aligned}$$

where C and P are unitary while T is anti-unitary. Recall the Schödinger field theory that you quantized in Problem 4 of Homework 3. In this non-relativistic theory, only two of these three operators exist—which two? Define these two operators in terms of their action on creation and annihilation operators.

4. **Coherent states** [10 points] Coherent states have many applications across physics. In this problem we will work out some of their properties. It is sufficient to consider a single harmonic oscillator; the generalization to a free field (*many* oscillators) is trivial. Let $H = \frac{1}{2}(p^2 + q^2)$ and—as usual—define creation and annihilation operators,

$$a = \frac{q + ip}{\sqrt{2}} \qquad\qquad a^\dagger = \frac{q - ip}{\sqrt{2}}.$$

Define the coherent state $|z\rangle$ by $|z\rangle = Ne^{za^\dagger}|0\rangle$, where z is a complex number and N is a real positive constant such that $\langle z|z\rangle = 1$.

(a) Find N and compute $\langle z|z'\rangle$.

(b) Show that $|z\rangle$ is an eigenstate of a and find its eigenvalue. (*Hint*: a is not Hermitian so that its eigenstates may be non-orthogonal with complex eigenvalues.)

(c) The set of all coherent states for all values of z is not only complete, but it's *over*-complete: the energy eigenstates can be constructed by taking successive derivatives at $z = 0$, so the coherent states with z in some small real interval around the origin are already enough. Despite this, show that there is a completeness relation,

$$1 = \alpha \int d\mathrm{Re}z\, d\mathrm{Im}z\; e^{-\beta z^* z}|z\rangle\langle z|$$

and find the real constants α and β. Hint: evaluate this equation between arbitrary coherent states.

(d) Show that for any polynomial in two variables, $F(p,q)$,

$$\langle z| \;:\; F(p,q) \;:\; |z\rangle = F(\bar{p}, \bar{q}),$$

and find the real numbers $\bar{p}$ and $\bar{q}$ in terms of z.

(e) In the q-representation, the statement that $|z\rangle$ is an eigenstate of a with known eigenvalue is a first-order diffential equation for $\langle q|z\rangle$, which is the position-space wave-function of $|z\rangle$. Solve this equation and find the wave function. For simplicity, don't bother with overall normalization factors.