Relativistic Quantum Field Theory

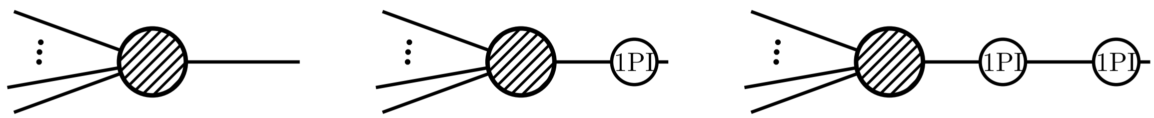
Physics 7651

Homework 9

Due: In class on Wednesday, November 2

1. **Counter terms on external legs** [5 points]

   In class we noted that one does not have to include diagrams with corrections on the external legs. Prove this statement by considering an arbitrary diagram and calculating the overall multiplicative factor associated with summing the series of diagrams formed by including 1PI corrections on a single external leg.



2. **Renormalization of the anharmonic oscillator in QM** [15 points]

   Consider an aharmonic oscillator specified by the following Lagrangian,

$$L = \frac{1}{2} Z \dot{q}^2 - \frac{1}{2} Z_\omega \omega^2 q^2 - Z_\lambda \lambda \omega^3 q^4$$

   We have set $m = 1$ so that λ is dimensionless.

   (a) Find the Hamiltonain H corresponding to L and write it as $H_{\text{free}} + H_{\text{int}}$ where $H_{\text{free}} = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2$ with $[Q, P] = i$.

   (b) Let $|0\rangle$ and $|1\rangle$ be the ground and first excited states of H_0. Let $|\Omega\rangle$ and $|I\rangle$ be the ground and first excited states of H. Define ω to be the excitation energy of H, $\omega \equiv E_I - E_\Omega$. We normalize the position operator Q by setting

$$\langle I|Q|\Omega\rangle = \langle 1|Q|0\rangle = (2\omega)^{-1/2}.$$

   Finally, to make things mathematically simpler, set $Z_\lambda = 1$. Write $Z = 1 + A$ and $Z_\omega = 1 + B$, where $A = \kappa_A \lambda + \mathcal{O}(\lambda^2)$ and $B = \kappa_B \lambda + \mathcal{O}(\lambda^2)$. Use perturbation theory to computer the $\mathcal{O}(\lambda)$ corrections to the unperturbed energy eigenvalues and eigenstates.

   (c) Find the numerical values of κ_A and κ_B that yield $\omega = E_I - E_\Omega$ and $\langle I|Q|\Omega\rangle = (2\omega)^{-1/2}$.

(d) Now think of the Lagrangian above as specifying a quantum field theory in $d = 1$ dimensions. Compute the $\mathcal{O}(\lambda)$ correction to the propagator. Fix κ_A and κ_B by requiring the propagator to have a pole at $k^2 = \omega^2$ with residue one. Do your results agree with those in (c)? Should they?

Hints: Write Q and P in terms of normalized creation and annihilation operators with $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. In part (d) use a Wick rotation to calculate the finite loop integral.

3. **Superficial divergences in ϕ^4** [10 points]

The **superficial degree of divergence** of a diagram D is the difference in the power of loop momenta in the numerator and the denominator. (This includes the integration measure so that it really gives the naively expected divergence for the expression.)

(a) ϕ^4 THEORY AND EVIL. (Special Halloween problem.) In ϕ^4 theory in four dimensions, use power counting to give the expression for the superficial degree of divergence D of a general diagram with E external legs, V vertices, I internal lines, and L loops. Give the relation between E, V, and I. Using this and the relation between L, V, and I, express D in terms of E and V.

(b) Given the above result, give a list of all amplitudes which are superficially divergent and give their superficial degree of divergence. Explain why some amplitudes which should be divergent according to (a) are actually not divergent. Specify which counter terms are required. Draw the leading order divergent diagram (including at least one vertex) for each counter term. *Hints: if you can't draw a diagram for an operator, there is usually a reason for this due to symmetry. Make sure that the leading order diagram you draw actually contributes to a given counter term—one counter term is subtle.*

(c) Redo (a) for ϕ^4 in general spactime dimension d. Briefly comment on implications for the superficial degree of divergences in this theory.