Relativistic Quantum Field Theory

Physics 7651

Homework 7

Due: In class on Wednesday, October 19

1. **Some cross section calculations** [5 points]

   Here we will the theory in Problem 1 of Homework 6 to experimental observables. Recall that the Lagrangian of that theory was

   $$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 + |\partial\psi|^2 - M^2|\psi|^2 - g\phi\psi^\dagger\psi.$$

   (a) Compute, to leading order in g, the center-of-mass differential cross section and total cross section for the elastic scattering of $\psi\psi^\dagger$.

   (b) Do the same for $\psi\psi^\dagger$ annihilating into $\phi\phi$. *Hint:* you already calculated the amplitude in Homework 6. Be careful not to double-count final states.

   (c) Do the same for $\psi\phi \to \psi\phi$. *Hint:* you calculated the amplitude in Homework 6.

2. **Kinematics of a simple theory** [10 points]

   (a) Work out the three-body phase space $D_{(3)}$ for massless final states. Explicitly give the integration range for $dE_1\, dE_2$. Draw the Dalitz plot for this process.

   (b) Let A, B, C, and D be four real scalar fields with Lagrangian

   $$\mathcal{L} = \frac{1}{2}\left[(\partial A)^2 - m^2A^2 + (\partial B)^2 + (\partial C)^2 + (\partial D)^2\right] + gABCD.$$

   Compute, to the lowest non-vanishing order in g, the total decay width of the A. How would the answer have differed if the interaction were instead gAB^3?

3. **Particle decay in a thermal universe** [5 points]

   Consider a theory in which a heavy real scalar φ can decay into ligher real scalars ϕ. The particular interaction is not relevant. Thus far we have studied such a decay with the implicit assumption that the universe was empty of ϕ particles before the decay. Sometimes (e.g. in cosmology) we would like to compute the decay of a φ not in an empty universe, but one that was already filled with a thermal distribution of ϕ particles.

This is not hard to do if you treat the final state particles as non-interacting and assume that there are no other φ type particles in the initial distribution of particles[1].

Show that in this case the only change in the formalism presented in class is that the phase space density D has an additional multiplicative factor $f(E/k_BT)$ for reach final state ϕ, where E is the ϕ energy, k_B is Boltzmann's constant, and f is a function that you must find.

Hints: (1) For any system in thermal equilibrium, the probability of finding a system in its n^{th} energy eigenstate is proportional to $\exp(-E_n/k_BT)$. (2) For a single harmonic oscillator, $\langle n|a^\dagger|n-1\rangle = \sqrt{n}$.

4. **Introduction to scalar QED** [10 points]

   We have referred to the conserved charge of a complex scalar field ψ as its electric charge. In this problem we will see how this is done by introducing a photon, A_μ which couples to this charge. The photon is a vector particle (you can think of it as the quantized vector potential), which is something we haven't seen before. We'll meet the photon later in this course and delve further into gauge theories next semester; for now we'll just get our feet wet.

   One might like to couple the photon directly to the Noether current as $\mathcal{L}_I = -j^\mu A^\mu$, but this is not gauge invariant. The correct Lagrangian for scalar QED is obtained through minimal coupling and is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\psi)^\dagger D^\mu\psi - m^2|\psi|^2, \qquad\qquad D_\mu = \partial_\mu + ieA_\mu,$$

   where the F^2 term is the photon kinetic term and e is the electric coupling.

   (a) Write out the Feynman rule for the coupling between a charged scalar, its antiparticle, and the μ^{th} component of the photon assuming all momenta flowing into the vetex. Draw a charged particle as a line with an arrow and the photon as a wiggly line. *Hint*: this is a derivative coupling. You learned how to deal with this in Problem 4 of HW6 so that you can 'read off' the Feynman rule from the interaction Lagrangian[2].

   (b) The propagator for a photon in Feynman gauge is

$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}.$$

   Calculate the differential cross section $d\sigma/d\cos\theta$ for elastic $\psi\psi^\dagger$ scattering.

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[1] This frequently happens in cosmology: the φ may be heavy and produced in thermal equilibrium at an early epoch when the temperature was very high. The expansion of the universe rapidly brings these particles out of equilibrium ('freeze out') and reduces their density to a negligible value. They then decay in an environment consisting of a hot gas of the much less massive ϕ particles.

[2] See Preskill 4.33 for a discussion of subtleties. If you are particularly conerned about formalities, you may simply take $H_I = -L_I$ and assume that ∂_0 commutes with time ordering.