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BOUNDARY NULL-CONTROLLABILITY OF SEMI-DISCRETE COUPLED PARABOLIC SYSTEMS IN SOME MULTI-DIMENSIONAL GEOMETRIES

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Abstract. The main goal of this paper is to investigate the controllability properties of semi-discrete in space coupled parabolic systems with less controls than equations, in dimension greater than 1. We are particularly interested in the boundary control case which is notably more intricate that the distributed control case, even though our analysis is more general.

The main assumption we make on the geometry and on the evolution equation itself is that it can be put into a tensorized form. In such a case, following [5] and using an adapted version of the Lebeau-Robbiano construction, we are able to prove controllability results for those semi-discrete systems (provided that the structure of the coupling terms satisfies some necessary Kalman condition) with uniform bounds on the controls.

To achieve this objective we actually propose an abstract result on ordinary differential equations with estimates on the control and the solution whose dependence upon the system parameters are carefully tracked. When applied to an ODE coming from the discretization in space of a parabolic system, we thus obtain uniform estimates with respect to the discretization parameters.

1. Introduction.

1.1. Motivating example.

In this paper, we are interested in null-controllability properties at any time $T > 0$ of coupled linear parabolic equations at the continuous level as well as at the semi-discrete in space level. In this introduction, we will focus

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on the following prototype system (which is in the so-called cascade form)

$$\begin{cases} \partial_t\alpha - \Delta\alpha = 0, & \text{in } (0,T)\times\Omega, \\ \partial_t\beta - \Delta\beta + \alpha = 0, & \text{in } (0,T)\times\Omega, \\ \alpha = 1_\Gamma v, & \text{on } (0,T)\times\partial\Omega, \\ \beta = 0, & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$), Γ is a non empty part of the boundary $\partial\Omega$, α and β are the two components of the system, and v is the boundary control we are looking for. The main difficulty in the analysis of the controllability of such system comes from the fact that we only have one boundary control v to drive the two components (α, β) to 0 at the final time. The coupling terms (here the term α in the equation for β) plays a key role in the problem and it can be seen as a kind of indirect control for the second component of the system. Note that this indirect controllability issue arises even if $\Gamma = \partial\Omega$.

Actually, it appears that the results for such systems may be quite different from the case of scalar equations or from the case of coupled systems with a distributed control. We refer for instance to the survey [3] for a review on that topic. In particular, it is explained in that reference that usual techniques based on Carleman estimates are useless on those problems. This is mainly because those techniques naturally give the controllability of the system when there is as many controls as components of the system (for (1) it would consist in another boundary control for β on Γ) and, in a second step, it is needed to prove that only one control is necessary. This is done, at the observability inequality level (see the discussion in Section 2.3) by removing one observation term thanks to the PDE itself. This last step cannot be done for boundary controls (or observations). This is why other approaches have to be developed.

Most of the results available up to now for such controllability problems are only proved in dimension $d = 1$ by using the so-called moments method. This is a quite powerful method but, unfortunately, restricted to autonomous problems in space dimension 1. In the multi-dimensional case, one of the more advanced result available in the literature is proved in [5], in the case where the geometry and the diffusion operator can be tensorized and this is also the case we shall consider in the present work. We also refer to [1] for results on similar multi-dimensional systems, yet under the geometric control condition which is not satisfied in the present study (see Figure 1).

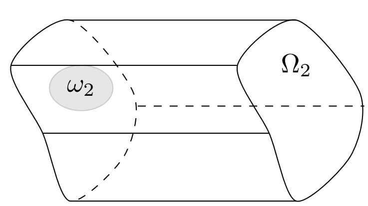


FIGURE 1. Typical geometric situation

In this work, the geometric assumption we shall consider for problem (1), is as follows: $\Omega = \Omega_1 \times \Omega_2$ with $\Omega_1 = (0, a)$ and $\Omega_2 \subset \mathbb{R}^{d-1}$, and $\Gamma = \{0\} \times \omega_2$ (see

Figure 1). Then, we can rewrite the system as follows

$$\begin{cases} \partial_t \alpha - \partial_{x_1}^2 \alpha - \Delta_2 \alpha = 0, & \text{in } (0,T) \times \Omega_1 \times \Omega_2, \\ \partial_t \beta - \partial_{x_1}^2 \beta - \Delta_2 \beta + \alpha = 0, & \text{in } (0,T) \times \Omega_1 \times \Omega_2, \\ \alpha = \beta = 0, & \text{on } (0,T) \times \Omega_1 \times \partial\Omega_2, \\ \alpha = 1_{\{0\}\times\omega_2} v, & \text{on } (0,T) \times \{0,a\} \times \Omega_2, \\ \beta = 0, & \text{on } (0,T) \times \{0,a\} \times \Omega_2, \end{cases} \tag{2}$$

where Δ_2 is the $(d-1)$-dimensional Laplace operator in Ω_2. The fact that the diffusion operator is split into two parts, each of them acting on different sets of variables, is crucial in the analysis. That is the reason why we shall adopt a tensor product formalism that consists essentially in identifying $L^2(\Omega)$ to $L^2(\Omega_1)\widehat{\otimes}L^2(\Omega_2)$ (see Remark 3.1 for a definition of $\widehat{\otimes}$) and in writing the two equations above in the following equivalent form

$$\begin{cases} \partial_t \alpha + (-\partial_{x_1}^2) \otimes \mathcal{I}\, \alpha + \mathcal{I} \otimes (-\Delta_2)\, \alpha = 0, & \text{in } (0,T), \\ \partial_t \beta + (-\partial_{x_1}^2) \otimes \mathcal{I}\, \beta + \mathcal{I} \otimes (-\Delta_2)\, \beta + \alpha = 0, & \text{in } (0,T), \end{cases} \tag{3}$$

where the same symbol $\mathcal{I}$ is used for the identity operator in $L^2(\Omega_1)$ and $L^2(\Omega_2)$. All the necessary notations and properties concerning tensor products will be recalled in Section 3.1.

By exploiting this tensor product structure, even though the tensor product formalism was not explicitly used, it was proved in [5], that the null-controllability of (3) holds at any time $T > 0$.

1.2. **Passing to the discrete world.** We are now interested in semi-discrete versions of the controllability result for (1) mentioned just before. To simplify the presentation in this introduction, we assume that $d = 2$, that $\Omega_1 = \Omega_2 = (0,1)$ and that the computation grid is made of $N \times N$ uniformly distributed points $(ih, jh)_{1\leq i,j\leq N}$ with $h = \frac{1}{N+1}$. The semi-discrete system we consider is obtained by the finite difference method and reads

$$\begin{cases} \partial_t \alpha_{i,j} + \dfrac{4\alpha_{i,j} - \alpha_{i-1,j} - \alpha_{i+1,j} - \alpha_{i,j-1} - \alpha_{i,j+1}}{h^2} = 0, & \forall 1 \leq i,j \leq N, \\ \partial_t \beta_{i,j} + \dfrac{4\beta_{i,j} - \beta_{i-1,j} - \beta_{i+1,j} - \beta_{i,j-1} - \beta_{i,j+1}}{h^2} + \alpha_{i,j} = 0, & \forall 1 \leq i,j \leq N, \\ \alpha_{i,0} = \beta_{i,0} = \alpha_{i,N+1} = \beta_{i,N+1} = 0, & \forall 1 \leq i \leq N, \\ \alpha_{0,j} = v_j 1_{\omega_2}(jh), & \forall 1 \leq j \leq N, \\ \alpha_{N+1,j} = \beta_{0,j} = \beta_{N+1,j} = 0, & \forall 1 \leq j \leq N. \end{cases} \tag{4}$$

The grid geometry is essentially the one described in Figure 2, where the control $v = (v_j)_j$ is only appearing in the first equation of the system (the one for $\alpha_{i,j}$) and only on the boundary points represented by the symbol ■ corresponding to the subdomain ω_2.

At each time t, both components $\alpha_h = (\alpha_{i,j})_{i,j} \in \mathbb{R}^{N\times N}$ and $\beta_h = (\beta_{i,j})_{i,j} \in \mathbb{R}^{N\times N}$ of the system are now considered as elements of the tensor product $\mathbb{R}^N \otimes \mathbb{R}^N$ and we observe that the five-point discrete Laplace operator can be written as the tensor product $\mathrm{A}_h \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_h$ with the usual definition of the three-point discrete

Laplace matrix

$$\mathrm{A}_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

We finally end up with the following equivalent form of our semi-discrete system

$$\begin{cases} \partial_t \alpha_h + (\mathrm{A}_h \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_h)\alpha_h & = \mathrm{B}_h v_h, \\ \partial_t \beta_h + (\mathrm{A}_h \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_h)\beta_h + \alpha_h = 0, \end{cases} \tag{5}$$

where B_h is a matrix that accounts for the influence of the control v_h in the system through the boundary conditions in (4), the precise definition of which will be given in Section 4.1. This is the semi-discrete version of (3).

For this particular system, the main result of this paper is the following (see Section 4.1 for the precise statement and definition of the norms involved).

Theorem. *There exist $C > 0$ and $h_0 > 0$ such that for any $h < h_0$, any time $T > 0$ and any initial data $\alpha_h^0, \beta_h^0 \in \mathbb{R}^N \otimes \mathbb{R}^N$, (with $h = \frac{1}{N+1}$) there exists a control $v_h \in L^2(0, T, \mathbb{R}^N)$ such that*

$$[\![v_h]\!]_{L^2(0,T,\mathbb{R}^N)} \leq C e^{\frac{C}{T}} (\|\alpha_h^0\| + \|\beta_h^0\|),$$

and the associated solution to (5) satisfies

$$\|\alpha_h(T)\| + \|\beta_h(T)\| \leq C e^{-C/h^2} e^{\frac{C}{T}} (\|\alpha_h^0\| + \|\beta_h^0\|).$$

It is well known, see [7, 19], that we cannot expect in general to achieve exactly $\alpha_h(T) = \beta_h(T) = 0$ since the semi-discrete system may be not even approximately controllable. In this sense, achieving exponentially small targets with respect to h is an optimal result.

Our aim will be to provide similar results for more general semi-discrete systems. That is the reason why, in order to formulate them more conveniently for any number of coupled equations and to ease the reading of the proofs, we shall actually gather the two components $(\alpha_h, \beta_h) \in (\mathbb{R}^N \otimes \mathbb{R}^N)^2$ into a single unknown $y_h \in \mathbb{R}^N \otimes \mathbb{R}^N \otimes \mathbb{R}^n$, with $n = 2$ in the present case, in such a way that the considered system (5) will finally be written in the compact form

$$\partial_t y_h + (\mathrm{A}_h \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_h) \otimes \mathsf{I}\, y_h + \mathrm{I} \otimes \mathrm{I} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y_h = \mathrm{B}_h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_h.$$

In the sequel of this paper we shall not explicitly mention the subscript h for the notation of quantities related to the discretization process but taking into account the fact that the considered spaces, norms and operators are grid-dependent is a central point in the analysis.

1.3. **Main results and outline of the paper.** Considering the previous discussion we shall analyze in this paper the controllability of parabolic systems of n components and m controls of the following tensorized form

$$\partial_t y + \mathcal{A}_1 \otimes \mathcal{I} \otimes \mathsf{I}\, y + \mathcal{I} \otimes \mathcal{A}_2 \otimes \mathsf{I}\, y + \mathcal{I} \otimes \mathcal{I} \otimes \mathsf{C} y = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathsf{B} v, \quad \text{in } (0, T), \tag{6}$$

where $\mathcal{A}_i$ is a diffusion operator in Ω_i, $\mathcal{B}_i$ is a (boundary or distributed) control operator in Ω_i, I is the $n \times n$ identity matrix, C is a $n \times n$ coupling matrix and B a $n \times m$ control matrix.

Our main aim being to analyze semi-discrete versions of (6), we shall also consider linear ordinary differential equations of the similar form

$$\partial_t y + \mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I}\, y + \mathrm{I} \otimes \mathrm{A}_2 \otimes \mathsf{I}\, y + \mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C} y = \mathrm{B}_1 \otimes \mathrm{B}_2 \otimes \mathsf{B} v, \quad \text{in } (0,T), \tag{7}$$

where the unknown y and the control v belong to a finite dimensional tensor space, A_i and B_i are linear operators on those space. All those objects depend, by nature, on discretization parameters, as in the example of Section 1.1. Therefore, it will be crucial to take care of all the constants in the estimates so as to obtain, at the end, controllability results for (7), that will not depend on those parameters.

Remark 1.1. We have used in (6) and (7) a convention that will be used all along the paper:

- Operators acting in infinite dimensional function spaces are written with calligraphic letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$, ...
- Operators (matrices) acting in finite dimensional spaces coming from discretization issues (whose dimension may be large and depends on the discretization parameters) are written with upright letters: A, B, I, ...
- Matrices acting in finite dimensional spaces coming from the number of components or controls in the system (their dimension is fixed and independent of discretization parameters) are written with sans serif letters: B, C, I, ...

The outline of the paper is the following. Section 2 is dedicated to recall the main results in the controllability theory for ODEs while paying a particular attention to the discrete functional setting that will be adapted to the analysis of finite difference approximations of parabolic PDEs. In section 3, we first review the material we need concerning tensorized operators that are central in the present work, then we state the precise assumptions we need and our main abstract result (Theorem 3.1). In short, we will assume for our tensorized system (6) (in the continuous setting) or (7) (in the discrete setting):

- that the associated sub-problem concerning only the first coordinate of the tensor product which is
$$\partial_t y + \mathcal{A}_1 \otimes \mathsf{I}\, y + \mathcal{I} \otimes \mathsf{C} y = \mathcal{B}_1 \otimes \mathsf{B} v, \quad \text{in } (0,T), \tag{8}$$
or
$$\partial_t y + \mathrm{A}_1 \otimes \mathsf{I}\, y + \mathrm{I} \otimes \mathsf{C} y = \mathrm{B}_1 \otimes \mathsf{B} v, \quad \text{in } (0,T), \tag{9}$$
is null-controllable (or a relaxed version of this) at any time T with a precise control of the cost of the control.

  If we come back to our motivating example, this subsystem reads
$$\begin{cases} \partial_t \alpha - \partial_{x_1}^2 \alpha \quad = 0, & \text{in } (0,T) \times \Omega_1, \\ \partial_t \beta - \partial_{x_1}^2 \beta + \alpha = 0, & \text{in } (0,T) \times \Omega_1, \\ \qquad \alpha = 1_{\{0\}} v, & \text{on } (0,T) \times \partial\Omega_1, \\ \qquad \beta = 0, & \text{on } (0,T) \times \partial\Omega_1. \end{cases} \tag{10}$$
  Note that this system does not depend on the control set ω_2.
- that the diffusion operator $\mathcal{A}_2$ (resp. A_2) and the control operator $\mathcal{B}_2$ (resp. B_2) in the other direction satisfy a suitable spectral estimate similar to the

Lebeau-Robbiano spectral inequality, except that we allow the inequality to hold only for a portion of the spectrum.

For system (2), this amounts to ask that the Lebeau-Robbiano spectral inequality holds, relative to the control set ω_2, for the eigenfunctions of the operator $-\Delta_2$ in Ω_2 with homogeneous boundary condition.

The complete proof of the theorem is given in Section 3.3. It consists, following the strategy developed in [5], to implement a construction of the control similar to the one originally proposed by Lebeau and Robbiano in [15]. We split the time interval into a suitable number of subintervals whose length is carefully chosen and, on each of those subintervals, we construct a partial control (obtained by combining the two assumptions above) that is able to damp out exponentially the part of the solution corresponding to the frequencies less than some threshold. Note that, contrary to the usual construction, we are not necessary able to drive this part of the solution exactly to zero at this stage. This threshold is then increased while the construction progresses towards the final time T; this eventually gives the expected control.

The main novelties in the present work are that : we allow relaxed controllability and spectral inequalities in our assumptions and moreover we precisely take care of the dependence of all the quantities of interest (norms, constants, ...) with respect to parameters on which the problem may depend. Those two refinements of the proof in [5] are mandatory since we want to apply this abstract result to systems obtained by semi-discretization processes.

To conclude the paper, in section 4, we precisely explain how to use the abstract formalism developed here to achieve the uniform controllability results of semi-discrete coupled parabolic systems as announced in this introduction. As another example, we also show how to deduce the result of [5] (slightly generalized to variable coefficients operators) from the present abstract result and give some insights on other possible applications.

2. **Controllability for linear ODEs.**

Before studying discrete versions of System (6), which is the main aim of the paper, we start by introducing the main notations and results that we shall use in the sequel concerning the controllability of linear ODEs. Most of this material is already well-known, however we propose a specific point of view adapted to our needs.

2.1. **Framework.**

Let $(E, \langle \bullet, \bullet \rangle_0)$ and $(U, [\bullet, \bullet]_0)$ be two (finite dimensional) Euclidean spaces (each of them being identified with its own dual space). The corresponding norms are denoted by $\|\bullet\|_0$ and $[\![\bullet]\!]_0$. The presence of a subscript 0 in the notation is related to the fact that, in Section 2.2, those norms will be embedded in a scale of Sobolev-like norms.

We consider for the moment a general linear autonomous controlled system of the form

$$\begin{cases} y' + \mathrm{L}y = \mathrm{B}v, \text{ on } (0,T), \\ \quad y(0) = y_0, \end{cases} \tag{11}$$

where $\mathrm{L} : E \to E$ and $\mathrm{B} : U \to E$ are two linear operators, $y : [0,T] \to E$ is the state and $v : [0,T] \to U$ is the control we are looking for.

In the sequel of this paper, different such systems will be considered, coming in particular from the discretization of multi-D parabolic control problems such as (7) or their reduced version (9). In particular, the spaces E, U and the operators L and B will depend on some discretization parameter h. We will be interested in properties of those systems that are uniform with respect to h, that is the reason why we will pay, in this section, a particular attention to the various constants appearing in the estimates. In section 4 we will propose a suitable framework ensuring that all those constants will be uniform with respect to h.

2.2. **Well-posedness.** It is clear that (11) is well-posed for any choice of y_0 and v and that

$$\sup_{t\in[0,T]} \|y(t)\|_0 \leq e^{T\|\mathrm{L}\|}\big(\|y_0\|_0 + \sqrt{T}\|\mathrm{B}\| \, [\![v]\!]_{L^2(0,T;U)} \big), \tag{12}$$

where $\|\mathrm{L}\|$ (resp. $\|\mathrm{B}\|$) is the operator norm of $\mathrm{L} : E \to E$ (resp. $\mathrm{B} : U \to E$). However, in the framework we are interested in which comes from the semi-discretization in space of an evolution PDEs, those operator norms will not be bounded in general with respect to the discretization parameter. This is the consequence at the discrete level of the fact that differential operators are naturally unbounded operators in Sobolev spaces. For example if L the discrete Laplace operator on a uniform mesh of size h and B the boundary control operator as defined in section 4.1, then $\|\mathrm{B}\|$ and $\|\mathrm{L}\|$ both behave like C/h^2. Thus, inequality (12) will not give usable estimates.

Therefore, we need to introduce adapted estimates and some kind of discrete Sobolev norms to take into account the particular geometry of the (discrete) control operators under study. To this end, we introduce $\mathrm{D} : E \to E$ a self-adjoint definite positive operator on E (one can think of the discrete Laplace operator for instance) and we define a scale of inner products in E defined, for any $s \in \mathbb{R}$, by

$$\langle u, v\rangle_{s,\mathrm{D}} = \langle \mathrm{D}^s u, v\rangle_0 \, ,$$

and $\|\bullet\|_{s,\mathrm{D}}$ is the associated norm. Observe that $\|\bullet\|_0 = \|\bullet\|_{0,\mathrm{D}}$ and that the space $(E, \langle \bullet, \bullet\rangle_{-s,\mathrm{D}})$ is naturally isometric to the dual of $(E, \langle \bullet, \bullet\rangle_{s,\mathrm{D}})$ since

$$\|u\|_{-s,\mathrm{D}} = \sup_{\psi\in E} \frac{\langle u, \psi\rangle_0}{\|\psi\|_{s,\mathrm{D}}}. \tag{13}$$

We shall now define, for given $s \in \mathbb{R}$, the two constants $M_{s,\mathrm{adm}}, M_{s,\mathrm{cont}} > 0$ that satisfy

$$\sup_{t\in[0,T]} \left\| e^{-t\mathrm{L}^*}\psi \right\|_{s,\mathrm{D}} \leq M_{s,\mathrm{cont}} \, \|\psi\|_{s,\mathrm{D}} \, , \quad \forall \psi \in E, \tag{14}$$

$$\left(\int_0^T \left[\!\left[\mathrm{B}^* e^{-\tau \mathrm{L}^*}\psi \right]\!\right]_0^2 d\tau \right)^{\frac{1}{2}} \leq M_{s,\mathrm{adm}} \, \|\psi\|_{s,\mathrm{D}} \, , \quad \forall \psi \in E, \tag{15}$$

where the adjoint operators L^* and B^* are relative to the ambient inner product on E and U. Observe that (14) and (15) automatically holds since we consider finite dimensional spaces, and the only interesting point is the uniformity (or not) of the constants with respect to the spaces and the operators involved. Depending on the targeted application (distributed control, Dirichlet boundary control or Neumann boundary control for instance) we will need to choose a convenient value of s and of the operator D to ensure that those constants are actually uniform with respect to the discretization parameter.

Proposition 2.1. *For any $v \in L^2(0, T; U)$ and any $y_0 \in E$, there exists a unique solution y to (11) and it satisfies*

$$\sup_{t\in[0,T]} \|y(t)\|_{-s,\mathrm{D}} \leq M_{s,\mathrm{cont}} \, \|y_0\|_{-s,\mathrm{D}} + M_{s,\mathrm{adm}} \, [\![v]\!]_{L^2(0,T;U)} \, .$$

Proof. We write the Duhamel formula

$$y(t) = e^{-t\mathrm{L}} y_0 + \int_0^t e^{-(t-s)\mathrm{L}} \mathrm{B} v(s)\, ds,$$

then we take the inner product with any $\psi \in E$

$$\langle y(t), \psi \rangle_0 = \left\langle e^{-t\mathrm{L}} y_0, \psi \right\rangle_0 + \int_0^t \left\langle e^{-(t-\tau)\mathrm{L}} \mathrm{B} v(\tau), \psi \right\rangle_0 d\tau.$$

It follows that

$$\langle y(t), \psi \rangle_0 = \left\langle y_0, e^{-t\mathrm{L}^*} \psi \right\rangle_0 + \int_0^t \left[v(\tau), \mathrm{B}^* e^{-(t-\tau)\mathrm{L}^*} \psi \right]_0 d\tau,$$

and then

$$|\langle y(t), \psi \rangle_0| \leq \| y_0 \|_{-s,\mathrm{D}} \left\| e^{-t\mathrm{L}^*} \psi \right\|_{s,\mathrm{D}} + \int_0^t [\![v(\tau)]\!]_0 \left[\!\!\left[\mathrm{B}^* e^{-(t-\tau)\mathrm{L}^*} \psi \right]\!\!\right]_0 d\tau.$$

By (14) and (15) we deduce that

$$|\langle y(t), \psi \rangle_0| \leq \left(M_{s,\mathrm{cont}} \| y_0 \|_{-s,\mathrm{D}} + M_{s,\mathrm{adm}} [\![v]\!]_{L^2(0,T;U)} \right) \| \psi \|_{s,\mathrm{D}}.$$

Since this is valid for any $\psi \in E$, we deduce the expected estimate by the duality property (13). □

Remark 2.1. During this work we will often use the following very standard duality formula that was given in the proof above

$$\langle y(T), \psi \rangle_0 - \left\langle y_0, e^{-T\mathrm{L}^*} \psi \right\rangle_0 = \int_0^T \left[v(t), \mathrm{B}^* e^{-(T-t)\mathrm{L}^*} \psi \right]_0 dt, \quad \forall \psi \in E. \tag{16}$$

2.3. **Relaxed observability inequalities.** It is well-known (see [11, 18] for instance) that System (11) is null controllable at time T for any initial condition y_0 if and only if there exists $C > 0$, such that the following observability inequality for the adjoint problem is satisfied

$$\left\| e^{-T\mathrm{L}^*} q_T \right\|_0^2 \leq C^2 \int_0^T \left[\!\!\left[\mathrm{B}^* e^{-(T-t)\mathrm{L}^*} q_T \right]\!\!\right]_0^2 dt, \quad \forall q_T \in E. \tag{17}$$

The value of the constant C in this inequality is crucial since it appears in the measure of the control cost.

It happens that, when (11) comes from a discretization of a parabolic equation with the finite difference method then the null controllability of the semi-discrete system may not hold (in particular in a multi dimensional setting, see the example given by Kavian and reported in [19]). To tackle this problem, it was proposed (in [12, 8, 9, 7] for instance) to relax the controllability requirements by considering instead the $\varphi(h)$-*null-controllability* of (11). It consists in constructing uniformly bounded controls such that the solution $y(T)$ does not identically vanish but is small enough with respect to the discretization parameter h. This approach is based on the penalized HUM construction, where the penalization parameter is a given function $h \to \varphi(h)$ of the discretization parameter, given its name to this notion. Note that, the spaces E, U and the operators L and B all depend on h, in particular the dimensions of E and U may increase when h tends to zero. This is one of the main difficulty that we need to take care of in the analysis.

The $\varphi(h)$-null-controllability property is equivalent to a relaxed version of inequality (17) and the following Lemma 2.2, whose proof is given in appendix A, aims at establishing such an equivalence. This lemma is somehow related to [16, Lemma 3.4] or [4, Proposition 1] and is stated in a quite general framework : the constant s and the operator D can be chosen arbitrarily. Moreover using appropriate spaces F_0 and F_T (which are defined below), one can show that Lemma 2.2 encompasses some already known situations (see Remark 2.3). Roughly speaking, this Lemma is about controlling the components in the final state space F_T of the solution of system (24) which starts from an initial condition y_0 in the initial state space F_0. Even though we only state it in a finite dimensional setting, it is clear that infinite dimensional versions also hold, as in the references quoted above.

Let F_0 and F_T be two subspaces of E and P_{F_0} (resp. P_{F_T}) the orthogonal projection onto F_0 (resp. F_T) with respect to the inner product $\langle \bullet, \bullet \rangle_{-s,\mathrm{D}}$.

We will denote the adjoint operators of the projectors P_{F_0} and P_{F_T} for the inner product $\langle \bullet, \bullet \rangle_{0,\mathrm{D}}$ by $\mathrm{P}^*_{F_0}$ and $\mathrm{P}^*_{F_T}$. Observe that $\mathrm{P}^*_{F_0}$ and $\mathrm{P}^*_{F_T}$ are also the orthogonal projectors in $(E, \langle \bullet, \bullet \rangle_{s,\mathrm{D}})$ onto $\mathrm{D}^{-s}F_0$ and $\mathrm{D}^{-s}F_T$ respectively. In particular, we have

$$\left\| \mathrm{P}^*_{F_0} y \right\|_{s,\mathrm{D}} \leq \|y\|_{s,\mathrm{D}}, \quad \forall y \in E,$$

$$\left\| \mathrm{P}^*_{F_T} y \right\|_{s,\mathrm{D}} \leq \|y\|_{s,\mathrm{D}}, \quad \forall y \in E.$$

Remark 2.2. Observe that, if F_0 (resp. F_T) is stable by D then P_{F_0} (resp. P_{F_T}) does not depend on s. In particular, those projections are orthogonal for the inner product $\langle \bullet, \bullet \rangle_0$.

Dealing with such subspaces will be crucial in the sequel when we will apply the Lebeau-Robbiano strategy since it requires to be able to control some precise components of the solution at each step (depending on eigenspaces of A_2), see Section 3.3.

Lemma 2.2 (Relaxed observability inequalities and controllability)**.** *We use the above notations and assume that s, D, F_0 and F_T are given. Let $M_{\mathrm{obs}} > 0$ and $M_{\mathrm{rel}} \geq 0$ be two given numbers.*

The following two propositions are equivalent.

1. *For any $y_0 \in F_0$ there exists a $v \in L^2(0,T;U)$ satisfying*
$$\frac{1}{M^2_{\mathrm{obs}}} \llbracket v \rrbracket^2_{L^2(0,T;U)} + \frac{1}{M^2_{\mathrm{rel}}} \left\| \mathrm{P}_{F_T}(y(T)) \right\|^2_{-s,\mathrm{D}} \leq \|y_0\|^2_{-s,\mathrm{D}}, \tag{18}$$
where y is the corresponding solution of (11).
2. *For any $q_T \in \mathrm{D}^{-s}F_T$, the following relaxed observability inequality holds:*
$$\left\| \mathrm{P}^*_{F_0}(e^{-T\mathrm{L}^*} q_T) \right\|^2_{s,\mathrm{D}} \leq M^2_{\mathrm{obs}} \int_0^T \left\llbracket \mathrm{B}^* e^{-(T-t)\mathrm{L}^*} q_T \right\rrbracket^2_0 dt + M^2_{\mathrm{rel}} \|q_T\|^2_{s,\mathrm{D}}. \tag{19}$$

Of course, for $M_{\mathrm{rel}} = 0$, the inequality (18) should be understood as

$$\begin{cases} \dfrac{1}{M^2_{\mathrm{obs}}} \llbracket v \rrbracket^2_{L^2(0,T;U)} \leq \|y_0\|^2_{-s,\mathrm{D}}, \\ \left\| \mathrm{P}_{F_T}(y(T)) \right\|_{-s,\mathrm{D}} = 0. \end{cases}$$

Remark 2.3. Throughout this paper F_0 will always be equal to the whole space E. However it is worth noticing that by specifying spaces F_0 and F_T, one can recover usual inequalities related to different notions of controllability.

- When $F_0 = F_T = E$, inequality (19) is the usual relaxed inequality. If (19) holds with $M_{\text{rel}} = 0$ then system (11) is null controllable.

  As explained above, we cannot always expect M_{rel} to be equal to zero when system (11) is discretized by finite differences method with a space domain of dimension greater than one. However, if this inequality holds with $M_{\text{rel}}^2 = \varphi(h)$ and with M_{obs} independent of h, we recover the $\varphi(h)$-null-controllability notion briefly described above.
- When $dim(F_0) = 1$ and $F_T = E$, then proving (19) amounts to drive only one given initial condition to zero (when $M_{\text{rel}} = 0$) or close to zero (when M_{rel}^2 is small, like $\varphi(h)$ for instance). This question is tackled for instance in [7].
- The *partial null-controllability* consists in driving to zero only some components of the solution of a system of parabolic PDEs. It amounts to prove inequality (19) for $M_{\text{rel}} = 0$ and to choose an appropriate subspace F_T. This kind of controllability is studied for instance in [4] where related $\varphi(h)$-partial-null-controllability is also investigated.

3. **Controllability of tensorized systems.**

3.1. **Notations.** We introduce in this section the main notations used all along this paper. They mostly rely on usual notations and properties of tensor products (see for instance [13] for the algebraic properties of such structures, and [17] for related Euclidean/Hilbertian properties)

Let $(E_i, \langle \bullet, \bullet \rangle_{0,i})$, $i = 1, 2$ be two finite dimensional Euclidean spaces of dimensions N_1 and N_2. The associated norms are denoted by $\|\bullet\|_{0,i}$. Let D_i be two positive definite self-adjoint operators in those spaces and for any $s \in \mathbb{R}$, we introduce the following scalar products

$$\langle u_i, v_i \rangle_{s,\mathrm{D}_i} = \langle \mathrm{D}_i^s u_i, v_i \rangle_{0,i}\,, \quad \forall u_i, v_i \in E_i,$$

and the associated norms $\|\bullet\|_{s,\mathrm{D}_i}$.

In the case of a finite difference approximate system, the two spaces E_i have to be understood as the spaces of discrete in space (scalar) functions defined on a grid of Ω_i, $i = 1, 2$. We will then consider the tensor product space $E_1 \otimes E_2$ as a natural discretization space for functions defined on the tensor product grid of Ω. This space is equipped with the natural Euclidean structure defined by

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_0 = \langle u_1, v_1 \rangle_{0,1} \langle u_2, v_2 \rangle_{0,2}\,. \tag{20}$$

Remark 3.1. If E_1 and E_2 are infinite dimensional Hilbert spaces, the algebraic tensort product $E_1 \otimes E_2$ equipped with the above inner product is only a prehilbertian space. The natural functional space to consider in that case is the completion of $E_1 \otimes E_2$, which is then an Hilbert space denoted by $E_1 \widehat{\otimes} E_2$, see [17].

On $E_1 \otimes E_2$ we consider the operator $\mathrm{D} = \mathrm{D}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{D}_2$, where I stands for the identity operator in E_1 and E_2. This is a positive definite self-adjoint operator, from which we can define the following natural inner products and associated norms

$$\langle u, v \rangle_{s,\mathrm{D}} = \langle \mathrm{D}^s u, v \rangle_0\,, \quad \forall u, v \in E_1 \otimes E_2$$

Note, in particular that we have

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{1,\mathrm{D}} = \langle u_1, v_1 \rangle_{1,\mathrm{D}_1} \langle u_2, v_2 \rangle_{0,2} + \langle u_1, v_1 \rangle_{0,1} \langle u_2, v_2 \rangle_{1,\mathrm{D}_2}\,, \quad \forall u_1, v_1 \in E_1, \forall u_2, v_2 \in E_2. \tag{21}$$

Since we will be interested in vector-valued discrete functions that aim at being approximations of the solution (at any time t) of system (6), we will naturally work with the space

$$E = E_1 \otimes E_2 \otimes \mathbb{R}^n,$$

which corresponds to the fact that all the n components of the system are approximated at each grid point.

We recall that if L_i is a linear operator in E_i and L a linear operator in $\mathbb{R}^n$, the tensor product $\mathrm{L}_1 \otimes \mathrm{L}_2 \otimes \mathsf{L}$ is a linear operator on E defined by

$$(\mathrm{L}_1 \otimes \mathrm{L}_2 \otimes \mathsf{L})(u_1 \otimes u_2 \otimes z) = \mathrm{L}_1(u_1) \otimes \mathrm{L}_2(u_2) \otimes \mathsf{L}(z), \quad \forall u_i \in E_i, \forall z \in \mathbb{R}^n.$$

Let $(\bullet, \bullet)$ be the Euclidean inner product in $\mathbb{R}^n$ and $|\bullet|$ the associated norm. The previous definitions are naturally extended to the space E as follows

$$\langle u \otimes y, v \otimes z \rangle_{s,\mathrm{D}} = \langle u, v \rangle_{s,\mathrm{D}} (y, z), \quad \forall u, v \in E_1 \otimes E_2, \quad \forall y, z \in \mathbb{R}^n,$$

and we do similar extensions for spaces $E_1 \otimes \mathbb{R}^n$ and $E_2 \otimes \mathbb{R}^n$.

Recall that E can be identified to $E_1 \otimes \mathbb{R}^n \otimes E_2$ and $E_2 \otimes \mathbb{R}^n \otimes E_1$. Therefore, if $u := u_i \otimes y \in E_i \otimes \mathbb{R}^n$ and $v_j \in E_j$ where $i \neq j \in \{1, 2\}$, one is allowed to consider $u \otimes v_j$ as an element of E, although one should instead deal with the element $u_i \otimes v_j \otimes y$.

We now introduce two other finite dimensional Euclidean spaces $(U_i, [\bullet, \bullet]_{0,i})$, $i = 1, 2$, with the associated norms $[\![\bullet]\!]_{0,i}$ that correspond to the control space for each subproblem.

We define $U := U_1 \otimes U_2 \otimes \mathbb{R}^m$ and its inner product $[\bullet, \bullet]_0$, whose definition on $U_1 \otimes U_2$ is analogous to (20) :

$$[u_1 \otimes u_2, v_1 \otimes v_2]_0 = [u_1, v_1]_{0,1} [u_2, v_2]_{0,2}, \tag{22}$$

and is extended to U as before.

3.2. **Main theorem.** Let A_i be a symmetric definite positive operator in E_i. One can think of A_i as an approximation of the continuous operator $\mathcal{A}_i$ but the statement of our result is generic and does not explicitly make use of such an assumption. The eigenvalues of A_i will be denoted by $(\lambda_{i,k})_{k=1}^{N_i}$ and the corresponding eigenfunctions are $(\phi_{i,k})_{k=1}^{N_i}$. Those form an orthonormal family for $\langle \bullet, \bullet \rangle_{0,i}$.

In our estimates, the following discrete Poincaré inequality for A_1, will be needed

$$\|u\|_0 = \|u\|_{0,\mathrm{A}_1} \leq M_{\mathrm{P},1} \|u\|_{1,\mathrm{A}_1}, \forall u \in E_1, \tag{23}$$

for some $M_{\mathrm{P},1}$. Without loss of generality, we will assume that $M_{\mathrm{P},1} \geq 1$. Note that the best possible value for $M_{\mathrm{P},1}$ is $\lambda_{1,1}^{-1/2}$ but it is not sure that this value is greater than 1. Note also that the following generalized Poincaré estimate holds

$$\|u\|_0 \leq M_{\mathrm{P},1}^s \|u\|_{s,\mathrm{A}_1}, \forall u \in E_1, \forall s \geq 0.$$

For each $i = 1, 2$, we consider a control operator $\mathrm{B}_i : U_i \to E_i$, a coupling matrix C and a control matrix B.

Our goal is to analyze the controllability properties (uniform with respect to any parameter on which the system may depend) of the following tensorized ODE

$$\begin{cases} \partial_t y + Ly = \mathrm{B}_1 \otimes \mathrm{B}_2 \otimes \mathsf{B} v, \\ y(0) = y_0 \in E, \end{cases} \tag{24}$$

where the control v belongs to $L^2(0, T; U)$ and

$$L := \mathrm{A} + \mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C} \text{ and } \mathrm{A} := \mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I} + \mathrm{I} \otimes \mathrm{A}_2 \otimes \mathsf{I}.$$

Here, the symbol I stands for the identity operator on E_1 or E_2 and the symbol I is the identity operator of $\mathbb{R}^n$.

To this end, we will take benefit from the tensor product structure of the system and make two main assumptions:

1. the first one concerns the vector-valued sub-control-system in E_1 defined by

$$\begin{cases} \partial_t y + \mathrm{A}_1 \otimes \mathsf{I}\, y + \mathrm{I} \otimes \mathsf{C} y = \mathrm{B}_1 \otimes \mathsf{B} v, \\ y(0) = y_0 \in E_1 \otimes \mathbb{R}^n, \end{cases} \tag{25}$$

with $y \in \mathcal{C}^0([0,T], E_1 \otimes \mathbb{R}^n)$ and $v \in L^2(0,T; U_1 \otimes \mathbb{R}^m)$.

Assumption 1. *For some $M_{\mathrm{obs},1} > 0$, $\mu_{E_1} \in (0, +\infty]$, we have that for any $T > 0$, for any $y_0 \in E_1 \otimes \mathbb{R}^n$, there exists a control $v \in L^2(0,T; U_1 \otimes \mathbb{R}^m)$ for (25) such that*

$$\begin{cases} [\![v]\!]_{L^2(0,T;U_1 \otimes \mathbb{R}^m)} \leq e^{M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \|y_0\|_{-s,\mathrm{A}_1}, \\ \qquad \|y(T)\|_{-s,\mathrm{A}_1} \leq e^{M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} e^{-\mu_{E_1} T} \|y_0\|_{-s,\mathrm{A}_1}. \end{cases}$$

2. the second one concerns a spectral property related with the pair of operators $(\mathrm{A}_2, \mathrm{B}_2)$ similar to the well-known Lebeau-Robbiano inequality, excepted that it only holds for a certain portion of the spectrum of A_2.

Assumption 2. *For some $M_{\mathrm{LR},2} > 0$, $\mu_{E_2} \in (0, +\infty]$, we have the following partial Lebeau-Robbiano spectral inequality*

$$\|\psi\|_0 \leq e^{M_{\mathrm{LR},2}(1+\sqrt{\mu})} [\![\mathrm{B}_2^* \psi]\!]_{0,2}, \quad \forall \psi \in \operatorname{span}(\phi_{2,j},\ \lambda_{2,j} \leq \mu), \tag{26}$$

for any $0 < \mu < \mu_{E_2}$.

The main theorem of this paper is the following.

Theorem 3.1. *Assume that Assumptions 1 and 2 hold. Let $\mu^* = \min(\mu_{E_1}, \mu_{E_2})$.*

There exist a $M_{\mathrm{obs}} > 0$ depending only on $M_{\mathrm{obs},1}$, $M_{\mathrm{LR},2}$, $M_{s,\mathrm{cont}}$, $M_{s,\mathrm{adm}}$ and $\|\mathsf{C}\|$ such that for any $y_0 \in E$ and any $T > 0$ there exists a control $v \in L^2(0,T;U)$ satisfying

$$\begin{cases} [\![v]\!]_{L^2(0,T;U)} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} \|y_0\|_{-s,\mathrm{A}}, \\ \|y(T)\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} e^{-\mu^* T/4} \|y_0\|_{-s,\mathrm{A}}. \end{cases}$$

Note that in this theorem the constant $M_{\mathrm{obs},1}$ is built upon the Sobolev norms associated with $\mathrm{D}_1 = \mathrm{A}_1$ and the constants $M_{s,\mathrm{cont}}$, $M_{s,\mathrm{adm}}$ with the Sobolev norms associated with $\mathrm{D} = \mathrm{A}$. Those norms are thus problem dependent.

In the case where A_i are discrete versions of diffusion operators, it can be tempting to use instead usual discrete Sobolev norms, that are defined by using for D_1 and D the discrete Laplace operators. However, the equivalence between those norms, for large values of s, is uniform with respect to the discretization parameter if and only if the diffusion coefficients γ_i are smooth enough. In the case of non smooth diffusion coefficients, using the norms given in the theorem is mandatory.

3.3. **Proof of the main result.** In this section, we will prove Theorem 3.1.

3.3.1. *Preliminary estimates.* We start with some preliminary results.

Lemma 3.2 (Norms comparison)**.** *For any $s \geq 0$, and any*

$$q = \sum_{j=1}^{N_2} q_j \otimes \phi_{2,j}, \quad q_j \in E_1 \otimes \mathbb{R}^n,$$

we have

$$\sum_{j=1}^{N_2} \|q_j\|_{s,\mathrm{A}_1}^2 \leq \|q\|_{s,\mathrm{A}}^2 \leq 2^s M_{\mathrm{P},1}^{2s} \sum_{j=1}^{N_2} (1+\lambda_{2,j}^s) \|q_j\|_{s,\mathrm{A}_1}^2 .$$

Proof. We write each q_j under the form

$$q_j = \sum_{k=1}^{N_1} \phi_{1,k} \otimes q_{k,j},$$

with $q_{k,j} \in \mathbb{R}^n$ in such a way that

$$\mathrm{A}^s q = \sum_{\substack{1\leq k\leq N_1\\ 1\leq j\leq N_2}} (\lambda_{1,k} + \lambda_{2,j})^s \phi_{1,k} \otimes \phi_{2,j} \otimes q_{k,j}.$$

It follows that

$$\|q\|_{s,\mathrm{A}}^2 = \sum_{\substack{1\leq k\leq N_1\\ 1\leq j\leq N_2}} (\lambda_{1,k} + \lambda_{2,j})^s |q_{k,j}|^2,$$

and we immediately obtain

$$\|q\|_{s,\mathrm{A}}^2 \geq \sum_{j=1}^{N_2} \left(\sum_{k=1}^{N_1} \lambda_{1,k}^s |q_{k,j}|^2 \right) = \sum_{j=1}^{N_2} \|q_j\|_{s,\mathrm{A}_1}^2 .$$

Moreover, we have

$$\begin{aligned}
\|q\|_{s,\mathrm{A}}^2 \leq& 2^s \sum_{\substack{1\leq k\leq N_1\\ 1\leq j\leq N_2}} (\lambda_{1,k}^s + \lambda_{2,j}^s) |q_{k,j}|^2, \\
=& 2^s \sum_{j=1}^{N_2} \|q_j\|_{s,\mathrm{A}_1}^2 + 2^s \sum_{j=1}^{N_2} \lambda_{2,j}^s \|q_j\|_{0,\mathrm{A}_1}^2 \\
\leq& 2^s M_{\mathrm{P},1}^{2s} \sum_{j=1}^{N_2} \left(1+\lambda_{2,j}^s\right) \|q_j\|_{s,\mathrm{A}_1}^2 ,
\end{aligned}$$

and the claim follows. ◻

We define, for any $\mu \geq 0$, the following subspaces of E_2

$$E_{\mu,2} := \mathrm{span}\{\phi_{2,j} : \lambda_{2,j} \leq \mu\}, \text{ and } E_{\mu,2}^\perp := \mathrm{span}\{\phi_{2,j} : \lambda_{2,j} > \mu\}$$

and the corresponding subspaces of E

$$E_\mu := E_1 \otimes E_{\mu,2} \otimes \mathbb{R}^n, \text{ and } E_\mu^\perp := E_1 \otimes E_{\mu,2}^\perp \otimes \mathbb{R}^n.$$

Let P_μ (resp. $\mathrm{P}_\mu^\perp$) be the orthogonal projection in E onto E_μ (resp. $E_\mu^\perp$) for the inner product $\langle \bullet, \bullet \rangle_{-s,\mathrm{A}}$. By construction we have $\mathrm{P}_\mu + \mathrm{P}_\mu^\perp = \mathrm{I}$.

Let us now state some properties of the uncontrolled system

$$\begin{cases} \partial_t y + Ly = 0, \\ y(0) = y_0 \in E. \end{cases} \tag{27}$$

Proposition 3.3 (Dissipation estimates)**.**

1. *For any $\mu \geq 0$, if $y_0 \in E_\mu^\perp$, then the unique solution to (27) satisfies $y(t) \in E_\mu^\perp$ for any t and moreover, for any $s \in \mathbb{R}$, we have*
$$\|y(t)\|_{-s,\mathrm{A}} \leq e^{\|\mathsf{C}\| t} e^{-t\mu} \|y_0\|_{-s,\mathrm{A}}, \quad \forall t \geq 0.$$
2. *For any $\mu \geq 0$, and any $y_0 \in E$, the unique solution to (27) satisfies, for any $s \in \mathbb{R}$,*
$$\|\mathrm{P}_\mu y(t)\|_{-s,\mathrm{A}} \leq e^{\|\mathsf{C}\| t} \|\mathrm{P}_\mu y_0\|_{-s,\mathrm{A}},$$
$$\left\|\mathrm{P}_\mu^\perp y(t)\right\|_{-s,\mathrm{A}} \leq e^{\|\mathsf{C}\| t - \mu t} \left\|\mathrm{P}_\mu^\perp y_0\right\|_{-s,\mathrm{A}} \leq e^{\|\mathsf{C}\| t - \mu t} \|y_0\|_{-s,\mathrm{A}}.$$

Proof. 1. The first point is just a consequence of the fact that E_μ and $E_\mu^\perp$ are stable by $\mathrm{L} = \mathrm{A} + \mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C}$. A straightforward computation, using that A commutes with $\mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C}$ shows that
$$\frac{1}{2}\partial_t \left\langle \mathrm{A}^{-s} y, y\right\rangle_0 + \left\langle \mathrm{A}\mathrm{A}^{-\frac{s}{2}} y, \mathrm{A}^{-\frac{s}{2}} y\right\rangle_0 = -\left\langle \mathrm{A}^{-s}(\mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C}) y, y\right\rangle_0$$
Then we observe that, $\mathrm{A}^{-\frac{s}{2}} y(t) \in E_\mu^\perp$ for any t, and that by definition of $E_\mu^\perp$ we have
$$\langle \mathrm{A} z, z\rangle_0 \geq \mu \langle z, z\rangle_0, \quad \forall z \in E_\mu^\perp.$$
It follows that
$$\frac{1}{2}\partial_t \left\langle \mathrm{A}^{-s} y, y\right\rangle_0 \leq (\|\mathsf{C}\| - \mu) \left\langle \mathrm{A}^{-s} y, y\right\rangle_0,$$
and the claim follows by the differential form of Gronwall's Lemma.
2. We simply observe that $\mathrm{P}_\mu y$ and $\mathrm{P}_\mu^\perp y$ solve the same equation as y with initial conditions $\mathrm{P}_\mu y_0 \in E_0^\perp$, $\mathrm{P}_\mu^\perp y_0 \in E_\mu^\perp$. It is then enough to use the inequality of the first point to conclude.

□

3.3.2. *Partial controllability results.* The following proposition is a partial version of Theorem 3.1. More precisely, we establish the existence of a control v that reduces the norm of the projection on E_μ of the final state $y(T)$ as much as possible. However, the bound on the control is not uniform with respect to μ yet.

Proposition 3.4. *There exists a positive number M_{part} which depends only on s, $M_{\mathrm{P},1}$, $M_{\text{obs},1}$, $M_{\mathrm{LR},2}$, $M_{s,\text{cont}}$, $M_{s,\text{adm}}$ such that for any $\mu \in (0, \mu_{E_2})$ and any $y_0 \in E$, there exists a control $v \in L^2(0,T;U)$ that satisfies*

$$\begin{cases} [\![v]\!]_{L^2(0,T;U)} \leq e^{M_{\text{part}}(1+\frac{1}{T}+\sqrt{\mu})} \|y_0\|_{-s,\mathrm{A}} \\ \|\mathrm{P}_\mu(y(T))\|_{-s,\mathrm{A}} \leq e^{M_{\text{part}}(1+\frac{1}{T})} e^{-\mu_{E_1} T} \|y_0\|_{-s,\mathrm{A}}, \end{cases} \tag{28}$$

and

$$\|y(T)\|_{-s,\mathrm{A}} \leq e^{M_{\text{part}}(1+\frac{1}{T}+\sqrt{\mu})} \|y_0\|_{-s,\mathrm{A}}. \tag{29}$$

Proof. Let $q_T \in E_\mu$ and q be the solution of the backward equation

$$\begin{cases} -\partial_t q + \mathrm{L}^* q = 0 \\ q(T) = q_T. \end{cases}$$

We recall that A_1 (resp. A_2) is self-adjoint in E_1 (resp. E_2) so that

$$\mathrm{L}^* = \mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I} + \mathrm{I} \otimes \mathrm{A}_2 \otimes \mathsf{I} + \mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C}^*.$$

Since E_μ is stable by L^*, we can decompose q in the following way

$$q(t) = \sum_{\lambda_{2,j} \leq \mu} q_j(t) \otimes \phi_{2,j}, \text{ with } q_j(t) \in E_1 \otimes \mathbb{R}^n$$

where q_j satisfies

$$\begin{cases} -\partial_t q_j + (\mathrm{A}_1 \otimes \mathsf{I}) q_j + \lambda_{2,j} q_j + (\mathrm{I} \otimes \mathsf{C}^*) q_j = 0 \\ q_j(T) = q_{T,j} \in E_1 \otimes \mathbb{R}^n. \end{cases}$$

The new variable $z_j(t) := q_j(t) e^{\lambda_{2,j}(T-t)}$ satisfies the following system

$$\begin{cases} -\partial_t z_j + (\mathrm{A}_1 \otimes \mathsf{I}) z_j + (\mathrm{I} \otimes \mathsf{C}^*) z_j(t) = 0 \\ z_j(T) = q_{T,j}. \end{cases}$$

Thanks to Assumption 1, we can apply Lemma 2.2 in the space $E_1 \otimes \mathbb{R}^n$, with $F_0 = F_T = E_1 \otimes \mathbb{R}^n$, $\mathrm{L} = \mathrm{A}_1 \otimes \mathsf{I} + \mathrm{I} \otimes \mathsf{C}$, $\mathrm{B} = \mathrm{B}_1 \otimes \mathsf{B}$, $\mathrm{D} = \mathrm{A}_1 \otimes \mathsf{I}$, $M_{\mathrm{obs}} = e^{M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)}$ and $M_{\mathrm{rel}} = e^{M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} e^{-\mu_{E_1} T}$. Inequality (19) with $q_T = q_{T,j}$ gives

$$\|z_j(0)\|_{s,\mathrm{A}_1}^2 \leq e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \int_0^T [\![\mathrm{B}_1^* \otimes \mathsf{B}^* z_j(t)]\!]_{0,1}^2 \, dt + e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} e^{-\mu_{E_1} 2T} \|q_{T,j}\|_{s,\mathrm{A}_1}^2,$$

and therefore, coming back to the variable q_j, we get

$$\|q_j(0)\|_{s,\mathrm{A}_1}^2 e^{2\lambda_{2,j} T} \leq e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \int_0^T [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 \, e^{2\lambda_{2,j}(T-t)} dt + e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} e^{-\mu_{E_1} 2T} \|q_{T,j}\|_{s,\mathrm{A}_1}^2,$$

and thus,

$$\|q_j(0)\|_{s,\mathrm{A}_1}^2 \leq e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \int_0^T [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 \, dt + e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} e^{-\mu_{E_1} 2T} e^{-2\lambda_{2,j} T} \|q_{T,j}\|_{s,\mathrm{A}_1}^2.$$

Using the second inequality in Lemma 3.2, we get

$$\|q(0)\|_{s,\mathrm{A}}^2 \leq 2^s M_{\mathrm{P},1}^{2s} e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \sum_{\lambda_{2,j} \leq \mu} \Bigg((1+\lambda_{2,j}^s) \int_0^T [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 \, dt + (1+\lambda_{2,j}^s) e^{-2\lambda_{2,j} T} e^{-\mu_{E_1} 2T} \|q_{T,j}\|_{s,\mathrm{A}_1}^2 \Bigg).$$

Applying now the first inequality in Lemma 3.2 and the following inequality $\lambda^s e^{-2\lambda T} \leq \frac{(se^{-1})^s}{(2T)^s}$, we get

$$\|q(0)\|_{s,\mathrm{A}}^2 \leq 2^s M_{\mathrm{P},1}^{2s}(1+\mu^s)e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \sum_{\lambda_{2,j}\leq\mu} \int_0^T [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 \, dt$$
$$+ 2^s M_{\mathrm{P},1}^{2s} e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \left(1 + \frac{(se^{-1})^s}{(2T)^s}\right) e^{-\mu_{E_1} 2T} \|q_T\|_{s,\mathrm{A}}^2 .$$

We shall now apply Assumption 2. To this end we choose any orthonormal basis $(\Psi_k)_{k=1,\ldots,K}$ of $U_1 \otimes \mathbb{R}^n$ and we decompose each $\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)$ in this basis, the coefficients being denoted by $a_{k,j}(t)$. It follows

$$\sum_{\lambda_{2,j}\leq\mu} [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 = \sum_{\lambda_{2,j}\leq\mu} \left[\!\!\left[\sum_{k=1}^K a_{k,j}(t)\Psi_k\right]\!\!\right]_{0,1}^2 = \sum_{k=1}^K \sum_{\lambda_{2,j}\leq\mu} (a_{k,j}(t))^2.$$

For any $k \in \{1, ..., K\}$, given that $0 < \mu < \mu_{E_2}$, we can apply the discrete Lebeau-Robbiano inequality given by Assumption 2 to the vector $\psi = \sum_{\lambda_{2,j}\leq\mu} a_{k,j}(t)\phi_{2,j}$, to obtain

$$\sum_{\lambda_{2,j}\leq\mu} [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2 \leq \sum_{k=1}^K e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})} \left[\!\!\left[\sum_{\lambda_{2,j}\leq\mu} a_{k,j}(t)\mathrm{B}_2^*\phi_{2,j}\right]\!\!\right]_{0,2}^2$$
$$\leq e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})} \sum_{k=1}^K \sum_{\lambda_{2,j}\leq\mu} \sum_{\lambda_{2,j'}\leq\mu} [a_{k,j}(t)\mathrm{B}_2^*\phi_{2,j}, a_{k,j'}(t)\mathrm{B}_2^*\phi_{2,j'}]_{0,2} .$$

Apply now (22) and the fact that for any k, $[\![\Psi_k]\!]_{0,1} = 1$:

$$\sum_{\lambda_{2,j}\leq\mu} [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j(t)]\!]_{0,1}^2$$
$$\leq e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})} \sum_{k=1}^K \sum_{\lambda_{2,j}\leq\mu} \sum_{\lambda_{2,j'}\leq\mu} [a_{k,j}(t)\Psi_k \otimes \mathrm{B}_2^*\phi_{2,j}, a_{k,j'}(t)\Psi_k \otimes \mathrm{B}_2^*\phi_{2,j'}]_0$$
$$\leq e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})} \sum_{\substack{\lambda_{2,j}\leq\mu \\ \lambda_{2,j'}\leq\mu}} [\mathrm{B}_1^* \otimes \mathsf{B}^* \otimes \mathrm{B}_2^*(q_j(t)\otimes\phi_{2,j}), \mathrm{B}_1^* \otimes \mathsf{B}^* \otimes \mathrm{B}_2^*(q_{j'}(t)\otimes\phi_{2,j'})]_0$$
$$\leq e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})} [\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q(t)]\!]_0^2 .$$

Hence,

$$\|q(0)\|_{s,\mathrm{A}}^2 \leq 2^s M_{\mathrm{P},1}^{2s}(1+\mu^s)e^{2M_{\mathrm{LR},2}(1+\sqrt{\mu})+2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \int_0^T [\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q(t)]\!]_0^2 \, dt$$
$$+ 2^s M_{\mathrm{P},1}^{2s} e^{2M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \left(1 + \frac{(se^{-1})^s}{(2T)^s}\right) e^{-\mu_{E_1} 2T} \|q_T\|_{s,\mathrm{A}}^2 . \tag{30}$$

Note that, by construction, E_μ is stable by A so that (30) is valid for any $q_T \in F_T = \mathrm{D}^{-s}E_\mu$. Thus, we can apply Lemma 2.2 in the space E with $\mathrm{D} = \mathrm{A}$, $F_0 = E$

and $F_T = E_\mu$. It follows that there exists a control $v \in L^2(0,T;U)$ such that

$$\begin{cases} [\![v]\!]_{L^2(0,T;U)} \leq 2^{\frac{s}{2}} M_{\mathrm{P},1}^s \sqrt{1+\mu^s} e^{M_{\mathrm{LR},2}(1+\sqrt{\mu})+M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \|y_0\|_{-s,\mathrm{A}}, \\ \|\mathrm{P}_\mu(y(T))\|_{-s,\mathrm{A}} \leq 2^{\frac{s}{2}} M_{\mathrm{P},1}^s e^{M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)} \sqrt{1+\dfrac{(se^{-1})^s}{(2T)^s}} e^{-\mu_{E_1} T} \|y_0\|_{-s,\mathrm{A}}. \end{cases}$$

Finally, inequality (29) comes from Proposition 2.1 and the estimate on the control cost just proved which give

$$\begin{aligned} &\|y(T)\|_{-s,\mathrm{A}} \\ &\quad \leq \left(M_{s,\mathrm{cont}} + M_{s,\mathrm{adm}} 2^{\frac{s}{2}} M_{\mathrm{P},1}^s \sqrt{1+\mu^s} e^{M_{\mathrm{LR},2}(1+\sqrt{\mu})+M_{\mathrm{obs},1}\left(1+\frac{1}{T}\right)}\right) \|y_0\|_{-s,\mathrm{A}}. \end{aligned}$$

Those estimates can easily be put into the expected form for a suitable choice of M_{part}. □

The following corollary contains the main idea of Lebeau and Robbiano's strategy: during the first half of a given time interval, we control the lowest frequencies then, during the second half of the time interval, we turn the control to zero to take advantage of the natural dissipation of the problem.

Corollary 3.5. *There exists a positive number M_{LR} which depends only on $M_{\mathrm{obs},1}$, $M_{\mathrm{LR},2}$, $M_{s,\mathrm{cont}}$, $M_{s,\mathrm{adm}}$ and $\|\mathsf{C}\|$ such that for any $\tau \in (0,T)$, $\mu \in (0,\mu_{E_2})$ and any $y_0 \in E$, there exists a control $v \in L^2(0,\tau;U)$ that satisfies*

$$\begin{cases} [\![v]\!]_{L^2(0,\tau;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{1}{\tau}+\sqrt{\mu})} \|y_0\|_{-s,\mathrm{A}} \\ \|y(\tau)\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{\tau}+\tau\right)} \left(e^{-\mu_{E_1}\tau/2} + e^{-\mu\tau/2+M_{\mathrm{LR}}\sqrt{\mu}}\right) \|y_0\|_{-s,\mathrm{A}}. \end{cases}$$

Proof. We apply Proposition 3.4 on the time interval $(0,\tau/2)$. We get a control v and a solution y which satisfy (28) (with T replaced by $\tau/2$). Then, on the interval $(\tau/2,\tau)$, we set the control to zero. Thus, we have constructed the following control

$$\bar{v}(t) := \begin{cases} v(t) & \text{for } t \in (0,\tau/2) \\ 0 & \text{for } t \in (\tau/2,\tau), \end{cases}$$

and the associated solution of the system is still denoted by y. Clearly, the L^2 norm of $\bar{v}$ on $(0,\tau)$ is equal to that of v on $(0,\tau/2)$.

1. Since the control is 0 on $(\tau/2,\tau)$, the dissipation properties given in Proposition 3.3 yield

$$\|\mathrm{P}_\mu(y(\tau))\|_{-s,\mathrm{A}} \leq e^{\|\mathsf{C}\|\frac{\tau}{2}} \|\mathrm{P}_\mu(y(\tau/2))\|_{-s,\mathrm{A}}.$$

Then, we use (28) to get

$$\|\mathrm{P}_\mu(y(\tau))\|_{-s,\mathrm{A}} \leq e^{2M_{\mathrm{part}}\left(1+\frac{1}{\tau}\right)} e^{\|\mathsf{C}\|\frac{\tau}{2}} e^{-\mu_{E_1}\frac{\tau}{2}} \|y_0\|_{-s,\mathrm{A}}. \tag{31}$$

2. By the dissipation properties given in Proposition 3.3 we get

$$\left\|\mathrm{P}_\mu^\perp(y(\tau))\right\|_{-s,\mathrm{A}} \leq e^{(\|\mathsf{C}\|-\mu)\frac{\tau}{2}} \|y(\tau/2)\|_{-s,A}.$$

Then inequality (29) of Proposition 3.4 (with still T replaced by $\tau/2$) leads to

$$\left\|\mathrm{P}_\mu^\perp(y(\tau))\right\|_{-s,\mathrm{A}} \leq e^{2M_{\mathrm{part}}\left(1+\frac{1}{\tau}+\sqrt{\mu}\right)} e^{(\|\mathsf{C}\|-\mu)\frac{\tau}{2}} \|y_0\|_{-s,\mathrm{A}}. \tag{32}$$

We combine (31) and (32) to get the result for a suitable value of M_{LR} depending on M_{part}. □

Observe that Corollary 3.5 is slightly different from the similar result in the classical Lebeau and Robbiano's strategy. Indeed, the modes corresponding to frequencies less than μ of the final state $y(\tau)$ are not cancelled; they still exist but are controlled by the small term $e^{-\mu_{E_1}\frac{\tau}{2}}$. When applying the complete strategy on the interval $(0,T)$, one has to make sure that the term $e^{-\mu_{E_1}\tau/2}$ is smaller than $e^{-\mu\tau/2}$. This constraint is fulfilled in Theorem 3.1 by dealing with frequencies μ smaller than μ_{E_1}.

3.3.3. *Conclusion of the proof of the main theorem.* We can now apply Lebeau-Robbiano's strategy, with a well chosen finite number of steps, and prove Theorem 3.1.

Proof. Without loss of generality we suppose that $M_{\mathrm{LR}} \geq \ln(2)$. Let $\alpha > 0$ such that

$$\frac{\alpha}{8M_{\mathrm{LR}}} - 3\frac{\sqrt{\alpha}}{2} - 5 > 0. \tag{33}$$

Note that α only depends on M_{LR}. Let $\mu_j = \frac{\alpha}{T^2}(2^j)^2$, $\tau_j = \frac{T}{2}\frac{1}{2^j}$ and $T_j = \sum_{k=1}^{j}\tau_k$.

Suppose that $\mu^* \leq \mu_1$. In this case we apply Corollary 3.5 with $\mu = \mu^*$ on the whole time interval $(0,T)$. There exists a control $v \in L^2(0,T,U)$ that satisfies

$$\begin{cases} [\![v]\!]_{L^2(0,T;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{1}{T}+\sqrt{\mu^*})}\left\|y_0\right\|_{-s,\mathrm{A}}, \\ \left\|y(T)\right\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{T}+T\right)}\left(e^{-\mu_{E_1}T/2} + e^{-\mu^*T/2+M_{\mathrm{LR}}\sqrt{\mu^*}}\right)\left\|y_0\right\|_{-s,\mathrm{A}}. \end{cases}$$

Since $e^{-\mu_{E_1}T/2} \leq e^{-\mu^*T/2}$ we have

$$\left\|y(T)\right\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{T}+T\right)}e^{-\mu^*T/2}\left(1 + e^{M_{\mathrm{LR}}\sqrt{\mu^*}}\right)\left\|y_0\right\|_{-s,\mathrm{A}},$$

and moreover $e^{M_{\mathrm{LR}}\sqrt{\mu^*}} \leq e^{M_{\mathrm{LR}}\sqrt{\mu_1}} \leq e^{\frac{2}{T}M_{\mathrm{LR}}\sqrt{\alpha}}$. Therefore,

$$\left\|y(T)\right\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+2\sqrt{\alpha}\right)\left(1+\frac{1}{T}+T\right)}e^{-\mu^*T/2}\left\|y_0\right\|_{-s,\mathrm{A}}.$$

Thus, if $\mu^* \leq \mu_1$, the proof of Theorem 3.1 is complete since α depends only on M_{LR}, which itself depends only on $M_{\mathrm{obs},1}$, $M_{\mathrm{LR},2}$, $M_{s,\mathrm{cont}}$, $M_{s,\mathrm{adm}}$ and $\|\mathsf{C}\|$.

Suppose now that $\mu_1 < \mu^*$ and define $j^* \in \mathbb{N}^*$ such that $\mu_{j^*} < \mu^* \leq \mu_{j^*+1}$.

- During the time interval $(0,T_1) = (0,\tau_1)$, we apply a control v_1 as given by Corollary 3.5 with $\mu = \mu_1$ (which applies here since $\mu_1 < \mu^* \leq \mu_{E_2}$) and we get

$$\begin{cases} [\![v_1]\!]_{L^2(0,T_1;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{1}{\tau_1}+\sqrt{\mu_1})}\left\|y_0\right\|_{-s,\mathrm{A}}, \\ \left\|y(T_1)\right\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{\tau_1}+\tau_1\right)}\left(e^{-\mu_{E_1}\tau_1/2} + e^{-\mu_1\tau_1/2+M_{\mathrm{LR}}\sqrt{\mu_1}}\right)\left\|y_0\right\|_{-s,\mathrm{A}}. \end{cases}$$

- For any index $j \leq j^*$, we continue this procedure by applying Corollary 3.5 on time interval $(T_{j-1},T_j) = (T_{j-1},T_{j-1}+\tau_j)$ with $\mu = \mu_j$. We get a control v_j that satisfies

$$\begin{cases} [\![v_j]\!]_{L^2(T_{j-1},T_j;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{1}{\tau_j}+\sqrt{\mu_j})}\left\|y(T_{j-1})\right\|_{-s,\mathrm{A}}, \\ \qquad \left\|y(T_j)\right\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{\tau_j}+\tau_j\right)}\times \\ \qquad\qquad \left(e^{-\mu_{E_1}\tau_j/2} + e^{-\mu_j\tau_j/2+M_{\mathrm{LR}}\sqrt{\mu_j}}\right)\left\|y(T_{j-1})\right\|_{-s,\mathrm{A}}. \end{cases} \tag{34}$$

Let us focus on the estimation of the term $y(T_j)$. First, we use that $e^{-\mu_{E_1}\tau_j/2} \leq e^{-\mu_j\tau_j/2}$ and the relations

$$\begin{cases} \mu_j\tau_j = \dfrac{\alpha}{2T}2^j = \dfrac{\alpha}{4}\dfrac{1}{\tau_j}, \\ \sqrt{\mu_j} = \dfrac{\sqrt{\alpha}}{2\tau_j}. \end{cases} \tag{35}$$

It follows

$$e^{-\mu_j\tau_j/2+M_{\mathrm{LR}}\sqrt{\mu_j}} \leq e^{\frac{1}{\tau_j}\left(-\frac{\alpha}{8}+M_{\mathrm{LR}}\frac{\sqrt{\alpha}}{2}\right)}.$$

Hence

$$\|y(T_j)\|_{-s,\mathrm{A}} \leq 2e^{M_{\mathrm{LR}}}e^{M_{\mathrm{LR}}(\frac{1}{\tau_j}+\tau_j)+\frac{1}{\tau_j}(-\frac{\alpha}{8}+M_{\mathrm{LR}}\frac{\sqrt{\alpha}}{2})}\|y(T_{j-1})\|_{-s,\mathrm{A}}, \tag{36}$$

since $\ln(2) \leq M_{\mathrm{LR}}$, inequality $1 \leq \frac{1}{\tau_j}+\tau_j$ leads to

$$2e^{M_{\mathrm{LR}}}e^{M_{\mathrm{LR}}(\frac{1}{\tau_j}+\tau_j)} \leq e^{2M_{\mathrm{LR}}}e^{M_{\mathrm{LR}}(\frac{1}{\tau_j}+\tau_j)} \leq e^{3M_{\mathrm{LR}}(\frac{1}{\tau_j}+\tau_j)},$$

which, associated with (36), finally yields

$$\|y(T_j)\|_{-s,\mathrm{A}} \leq e^{3M_{\mathrm{LR}}(\frac{1}{\tau_j}+\tau_j)+\frac{1}{\tau_j}(-\frac{\alpha}{8}+M_{\mathrm{LR}}\frac{\sqrt{\alpha}}{2})}\|y(T_{j-1})\|_{-s,\mathrm{A}}.$$

Now we set $\beta := \frac{\alpha}{8M_{\mathrm{LR}}} - \frac{\sqrt{\alpha}}{2} - 3$, in such a way that the previous estimate reads

$$\|y(T_j)\|_{-s,\mathrm{A}} \leq e^{3M_{\mathrm{LR}}\tau_j}e^{-M_{\mathrm{LR}}\frac{\beta}{\tau_j}}\|y(T_{j-1})\|_{-s,\mathrm{A}}.$$

According to (33), we have $\beta > 0$ since $\frac{\alpha}{8M_{\mathrm{LR}}} - (\frac{\sqrt{\alpha}}{2}+3) > \frac{\alpha}{8M_{\mathrm{LR}}} - (\frac{3\sqrt{\alpha}}{2}+5) > 0$.

Thus, we can estimate $y(T_j)$ by using the previous upper-bound recursively,

$$\begin{aligned} \|y(T_j)\|_{-s,\mathrm{A}} &\leq \left(e^{3M_{\mathrm{LR}}\sum_{k=1}^{j}\tau_k}\right)\left(e^{-\beta M_{\mathrm{LR}}\sum_{k=1}^{j}\frac{1}{\tau_k}}\right)\|y_0\|_{-s,\mathrm{A}} \\ &\leq e^{3M_{\mathrm{LR}}T_j}e^{-\frac{2}{T}\beta M_{\mathrm{LR}}\sum_{k=1}^{j}2^k}\|y_0\|_{-s,\mathrm{A}} \\ &\leq e^{3M_{\mathrm{LR}}T_j}e^{-\frac{2\beta M_{\mathrm{LR}}2^j}{T}}\|y_0\|_{-s,\mathrm{A}}. \end{aligned}$$

Therefore,

$$\|y(T_j)\|_{-s,\mathrm{A}} \leq e^{3M_{\mathrm{LR}}T_j}e^{-\frac{\beta M_{\mathrm{LR}}}{\tau_j}}\|y_0\|_{-s,\mathrm{A}}, \tag{37}$$

and with (35),

$$\|y(T_j)\|_{-s,\mathrm{A}} \leq e^{3M_{\mathrm{LR}}T_j}e^{-2\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu_j}}\|y_0\|_{-s,\mathrm{A}}.$$

When $j = j^*$, we end up with (recall that $\mu^* < \mu_{j^*+1} = 4\mu_{j^*}$)

$$\|y(T_{j^*})\|_{-s,\mathrm{A}} \leq e^{3M_{\mathrm{LR}}T_{j^*}}e^{-\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu^*}}\|y_0\|_{-s,\mathrm{A}}. \tag{38}$$

Let us now estimate the control v. Taking back (34) for $j \geq 2$, combined with (37) and (35),

$$[\![v_j]\!]_{L^2(T_{j-1},T_j;U)} \leq e^{M_{\mathrm{LR}}\left[1+\frac{1}{\tau_j}\left(1+\frac{\sqrt{\alpha}}{2}-\frac{\beta}{2}\right)+3T_{j-1}\right]}\|y_0\|_{-s,\mathrm{A}}.$$

Note that, (33) gives that $\bar{\beta} := \beta/2-1-\frac{\sqrt{\alpha}}{2} = \frac{1}{2}(\beta-2-\sqrt{\alpha}) = \frac{1}{2}(\frac{\alpha}{8M_{\mathrm{LR}}}-3\frac{\sqrt{\alpha}}{2}-5) > 0$.

This implies that

$$\begin{aligned}
[\![v]\!]^2_{L^2(T_1,T_{j^*};U)} &= \sum_{j=2}^{j^*} [\![v_j]\!]^2_{L^2(T_{j-1},T_j;U)} \\
&\leq e^{2M_{\mathrm{LR}}(1+3T)} \sum_{j\geq 0} e^{-\frac{M_{\mathrm{LR}}\bar{\beta}}{T}2^{j+2}} \|y_0\|^2_{-s,\mathrm{A}} \\
&\leq \left(1+\frac{T}{M_{\mathrm{LR}}\bar{\beta}}\right) e^{2M_{\mathrm{LR}}(1+3T)} \|y_0\|^2_{-s,\mathrm{A}},
\end{aligned}$$

and the last inequality came from $\sum_{j\geq 0} e^{-\frac{M_{\mathrm{LR}}\bar{\beta}}{T}2^{j+2}} \leq \sum_{j\geq 0} e^{-j\frac{M_{\mathrm{LR}}\bar{\beta}}{T}} \leq \frac{1}{1-e^{-\frac{M_{\mathrm{LR}}\bar{\beta}}{T}}}$ and $\frac{1}{1-e^{-x}} \leq 1+\frac{1}{x}$, for $x>0$.

Using that

$$[\![v_1]\!]^2_{L^2(0,T_1;U)} \leq e^{2M_{\mathrm{LR}}(1+\frac{1}{\tau_1}+\tau_1+\sqrt{\mu_1})} \|y_0\|^2_{-s,\mathrm{A}},$$

we have built a control v on $(0,T_{j^*})$ that satisfies

$$[\![v]\!]_{L^2(0,T_{j^*};U)} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} \|y_0\|_{-s,\mathrm{A}}, \tag{39}$$

with M_{obs} depending on M_{LR} but not on T.

Notice that $T_{j^*} = \sum_{j=1}^{j^*}\tau_j = \frac{T}{2}\sum_{j=1}^{j^*}\frac{1}{2^j} \leq \frac{T}{2}$. The last step of the proof consists in applying Corollary 3.5 on the interval (T_{j^*},T) to recover the estimates of Theorem 3.1.

Corollary 3.5 with $\mu=\mu^*$ gives now a control $v\in L^2(T_{j^*},T)$ such that

$$\begin{cases}
[\![v]\!]_{L^2(T_{j^*},T;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{1}{T-T_{j^*}}+T-T_{j^*}+\sqrt{\mu^*})} \|y(T_{j^*})\|_{-s,\mathrm{A}} \\
\|y(T)\|_{-s,\mathrm{A}} \leq e^{M_{\mathrm{LR}}\left(1+\frac{1}{T-T_{j^*}}+T-T_{j^*}\right)} \times \\
\qquad \left(e^{-\mu_{E_1}(T-T_{j^*})/2} + e^{-\mu^*(T-T_{j^*})/2+M_{\mathrm{LR}}\sqrt{\mu^*}}\right) \|y(T_{j^*})\|_{-s,\mathrm{A}}.
\end{cases} \tag{40}$$

First, with (38), we have

$$[\![v]\!]_{L^2(T_{j^*},T;U)} \leq e^{M_{\mathrm{LR}}(1+\frac{2}{T}+T-T_{j^*}+\sqrt{\mu^*})} e^{3M_{\mathrm{LR}}T_{j^*}} e^{-\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu^*}} \|y_0\|_{-s,\mathrm{A}},$$

and by (33), we can check that

$$\beta > \sqrt{\alpha}. \tag{41}$$

Hence

$$[\![v]\!]_{L^2(T_{j^*},T;U)} \leq e^{2M_{\mathrm{LR}}(1+\frac{1}{T}+T)} \|y_0\|_{-s,\mathrm{A}}.$$

Finally this estimation and (39) give the first inequality of Theorem 3.1.

According to (40) and (38),

$$\begin{aligned}
\|y(T)\|_{-s,\mathrm{A}} \leq & e^{M_{\mathrm{LR}}\left(1+\frac{1}{T-T_{j^*}}+T-T_{j^*}\right)} \left(e^{-\mu_{E_1}(T-T_{j^*})/2} + e^{-\mu^*(T-T_{j^*})/2+M_{\mathrm{LR}}\sqrt{\mu^*}}\right) \times \\
& e^{3M_{\mathrm{LR}}T_{j^*}} e^{-\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu^*}} \|y_0\|_{-s,\mathrm{A}} \\
\leq & e^{2M_{\mathrm{LR}}\left(1+\frac{1}{T}+T\right)} \left(e^{-\mu_{E_1}T/4} + e^{-\mu^*T/4+M_{\mathrm{LR}}\sqrt{\mu^*}}\right) e^{-\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu^*}} \|y_0\|_{-s,\mathrm{A}} \\
\leq & 2e^{2M_{\mathrm{LR}}\left(1+\frac{1}{T}+T\right)} e^{-\mu^*T/4} e^{M_{\mathrm{LR}}\sqrt{\mu^*}} e^{-\frac{\beta M_{\mathrm{LR}}}{\sqrt{\alpha}}\sqrt{\mu^*}} \|y_0\|_{-s,\mathrm{A}}.
\end{aligned}$$

Using (41), this reduces to

$$\|y(T)\|_{-s,\mathrm{A}} \leq 2e^{2M_{\mathrm{LR}}\left(1+\frac{1}{T}+T\right)}e^{-\mu^* T/4}\,\|y_0\|_{-s,\mathrm{A}}\,,$$

and this concludes the proof. $\square$

4. Applications.

4.1. Dirichlet boundary null control of a semi-discrete cascade system on a rectangle. In this section, we will apply the general framework introduced above to prove the $\phi(h)$ null controllability properties for semi-discrete versions of the boundary control problem of the coupled system (6) in the case of a cascade form (that is for particular B and C given below). The penalization term $\phi(h)$ will be exponentially small in h, just like in similar results obtained in the literature and quoted in the introduction section 1.1.

The results are valid for the finite difference discretization in space of the system in any dimension. However, for the simplicity of the presentation we will only state and prove our theorem in dimension $d = 2$. Observe that other usual techniques (based on Carleman estimates, or on moments methods as in [2, 10, 9, 8]) do not directly apply in this setting since we are considering boundary controls for multi-dimensional coupled systems with less controls than components in the system.

4.1.1. *Presentation of the semi-discrete system.* Let us rewrite the assumptions in the precise setting we consider here. Let $a_1, a_2 \in \mathbb{R}_+^*$, $\Omega_1 = (0, a_1)$, $\Omega_2 = (0, a_2)$ and $\Omega := \Omega_1 \times \Omega_2$. Let ω_2 be a non empty open subset of Ω_2. Without loss of generality, we assume that

$$\overline{\omega_2} \subset \Omega_2. \tag{42}$$

Let $\gamma_1 \in \mathcal{C}^1(\overline{\Omega}_1)$ and $\gamma_2 \in \mathcal{C}^1(\overline{\Omega}_2)$ be two functions which satisfy:

$$\gamma_{1,\min} := \inf_{\overline{\Omega}_1} \gamma_1 > 0 \text{ and } \gamma_{2,\min} := \inf_{\overline{\Omega}_2} \gamma_2 > 0.$$

Let $\mathcal{A}_1 = -\partial_{x_1}(\gamma_1(x_1)\partial_{x_1}\bullet)$ (resp. $\mathcal{A}_2 = -\partial_{x_2}(\gamma_2(x_2)\partial_{x_2}\bullet)$) be the self-adjoint unbounded operator in $L^2(\Omega_1)$ (resp. $L^2(\Omega_2)$) whose domain is $D(\mathcal{A}_1) = H_0^1(\Omega_1) \cap H^2(\Omega_1)$ (resp. $D(\mathcal{A}_2) = H_0^1(\Omega_2) \cap H^2(\Omega_2)$). Let $n \geq 1$, we can define an unbounded operator in $L^2(\Omega_1)\widehat{\otimes}L^2(\Omega_2) \otimes \mathbb{R}^n$ (see Remark 3.1) on the domain $D(\mathcal{A}_1) \otimes D(\mathcal{A}_2) \otimes \mathbb{R}^n$ as follows

$$\mathcal{A} := \mathcal{A}_1 \otimes I \otimes \mathsf{I} + I \otimes \mathcal{A}_2 \otimes \mathsf{I}.$$

This operator is closable and we use the same notation $(\mathcal{A}, D(\mathcal{A}))$ for its closure. Since $L^2(\Omega_1)\widehat{\otimes}L^2(\Omega_2) \otimes \mathbb{R}^n$ is canonically isomorphic to $(L^2(\Omega))^n$, it is easily seen that $\mathcal{A}$ can in fact been understood as a self-adjoint operator in $(L^2(\Omega))^n$ with domain $(H_0^1(\Omega) \cap H^2(\Omega))^n$. The control matrix $\mathsf{B} \in M_{n,1}(\mathbb{R})$ and the coupling matrix $\mathsf{C} \in M_n(\mathbb{R})$ are chosen in the so-called cascade form, given as follows

$$\mathsf{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots 0 & 1 & 0 \end{pmatrix}.$$

Let us now define the finite difference approximation that we consider for this system. For simplicity we present the results on uniform grids even though the same result will hold for regular enough families of meshes (see for instance the discussion in [2, Remark 1.6] or [9, Section 5]). Let N_1 and N_2 be two integers greater or equal to 2. For $i \in \{1, 2\}$, we set $h_i := \frac{a_i}{N_i+1}$ and consider $(x_{1,j})_{j=0}^{N_1+1}$ and $(x_{2,j})_{j=0}^{N_2+1}$ a discretization of Ω_1 and Ω_2 respectively : $x_{1,j} := jh_1$ for $j \in \{0, \dots, N_1+1\}$, $x_{2,j} := jh_2$ for $j \in \{0, \dots, N_2+1\}$. We define also, for $j \in \{0, \dots, N_1\}$, $x_{1,j+1/2} = x_{1,j} + \frac{h_1}{2}$ and for $j \in \{0, \dots, N_2\}$, $x_{2,j+1/2} = x_{2,j} + \frac{h_2}{2}$.

The geometry of the grid and of the control domain is summarized in Figure 2: the interior grid points are shown in black circles ●, the homogeneous Dirichlet boundary points are shown in white circles ○ and the boundary points in the control region are shown in gray squares ■.

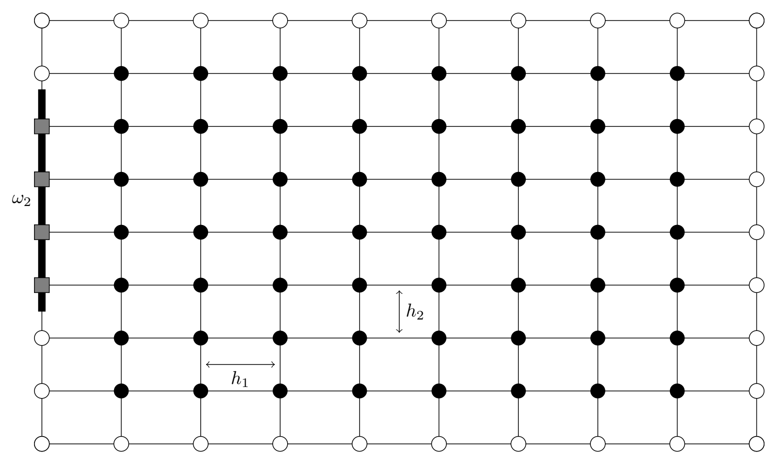


FIGURE 2. Grid geometry

Associated with those grids, we can introduce the discrete functional spaces $E_1 = \mathbb{R}^{N_1}$ (resp. $E_2 = \mathbb{R}^{N_2}$) equipped with the Euclidean inner product inherited from the L^2 inner product in Ω_1 (resp. in Ω_2) and defined by

$$\langle y, z \rangle_{0,i} := \sum_{j=1}^{N_i} h_i y_j z_j, \quad \forall y, z \in E_i, \forall i \in \{1, 2\}.$$

For any function $f : \Omega_i \to \mathbb{R}$, we will use the same letter f to denote the sampling of f on the grid $(f(x_{i,j}))_j \in E_i$ or the *multiplication by* f operator in E_i which means that, for any $y \in E_i$, the vector $fy \in E_i$ is defined by

$$(fy)_j = f(x_{i,j})y_j, \quad \forall j \in \{1, \dots, N_i\}. \tag{43}$$

Let A_i be the self-adjoint operator in E_i (that can be seen as a $N_i \times N_i$ matrix) corresponding to the discretization of the scalar 1D operator $\mathcal{A}_i$ by the finite difference method, which is defined for any $y \in E_i$ by

$$(\mathrm{A}_i y)_j := -\frac{1}{h_i}\left(\gamma_{i,j+1/2}\frac{y_{j+1}-y_j}{h_i} - \gamma_{i,j-1/2}\frac{y_j - y_{j-1}}{h_i}\right), \forall j \in [\![1, N_i]\!], \tag{44}$$

with the usual convention that $y_0 = y_{N_i+1} = 0$. Here, we have used the notation $\gamma_{i,j+1/2} := \gamma_i(x_{i,j+1/2})$, for $i = 1, 2$ and $j \in \{0, ..., N_i - 1\}$, for the sampling of the diffusion coefficient γ_i on the dual grid of the mesh of Ω_i. At some point we will also need to consider the discrete Laplace operators Δ_i defined by

$$(-\Delta_i y)_j := -\frac{y_{j+1} - 2y_j + y_{j-1}}{h_i^2}, \forall j \in [\![1, N_i]\!], \ \forall y \in E_i.$$

The discretization of the vector-valued 2D operator $\mathcal{A}$ we will be interested in is thus given by

$$\mathrm{A} = \mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I} + \mathrm{I} \otimes \mathrm{A}_2 \otimes \mathsf{I},$$

and, if we take into account the coupling terms, the complete discretization of the operator appearing in (6) is given by

$$\mathrm{L} := \mathrm{A} + \mathrm{I} \otimes \mathrm{I} \otimes \mathsf{C}.$$

We introduce now the discrete control spaces and operators associated with our boundary control problem. We set $U_2 = E_2$, and to avoid confusions, we will use the notation $[\bullet, \bullet]_{0,2}$ instead of $\langle \bullet, \bullet \rangle_{0,2}$ when dealing with objects in U_2.

Using the convention (43), we define the discrete control operator in Ω_2 by

$$\mathrm{B}_2 = 1_{\omega_2}.$$

This means that for any $y \in E_2$,

$$(\mathrm{B}_2 y)_j = \begin{cases} y_j, \text{ if } x_{2,j} \in \omega_2, \\ 0, \text{ if } x_{2,j} \notin \omega_2. \end{cases}$$

Observe that this operator is self-adjoint in E_2, that is $\mathrm{B}_2^* = \mathrm{B}_2$. Since the domain Ω_1 is a 1D interval, the corresponding boundary control v is in fact a scalar control that lives in the space $U_1 = \mathbb{R}$. However, the discretization A_1 of the operator $\mathcal{A}_1$ is built upon the assumption that we are dealing with homogeneous Dirichlet boundary condition. Therefore, in order to include non-homogeneous Dirichlet boundary condition on the left end-point of Ω_1 in the discretization, we need to add a source term in the discretization which is given by

$$\mathrm{B}_1 v := \begin{pmatrix} -\frac{\gamma_{1,1/2}}{h_1^2} v \\ 0 \\ | \\ 0 \end{pmatrix} \in E_1.$$

For the analysis, it will be sightly more convenient to work with this operator in the following form

$$\mathrm{B}_1 v = \gamma_{1,1/2} (\Delta_1 r) v, \quad \forall v \in \mathbb{R}, \tag{45}$$

where r is (the sampling of) the affine map defined by $x \in \Omega_1 \mapsto 1 - x/a_1$. This is the discrete counterpart of the boundary control operator as analyzed in [18, Chapter 2].

A simple computation shows that the adjoint of B_1 is given by

$$\mathrm{B}_1^* q = \gamma_{1,1/2} \partial_l q,$$

where $\partial_l q$ is the discrete *normal* derivative of $q \in E_1$ on the left boundary of the domain defined by

$$\partial_l q = \frac{0 - q_1}{h_1} = -\frac{q_1}{h_1}.$$

Note that this formula takes into account implicitly the homogeneous Dirichlet boundary condition for q.

We can now precisely write the semi-discrete control problem that we consider

$$\begin{cases} y'(t) + \mathrm{L} y(t) = \mathrm{B}_1 \otimes \mathrm{B}_2 \otimes \mathsf{B} v(t) \\ y(0) = y_0 \in E, \end{cases} \tag{46}$$

where $y(t) \in E = E_1 \otimes E_2 \otimes \mathbb{R}^n$ and $v(t) \in U = \mathbb{R} \otimes U_2 \otimes \mathbb{R}$. Note that, in this particular case, the control space U can be in fact identified with E_2.

The main theorem of this section is the following. The crucial point is that all the constants appearing in the estimates (47) do not depend on the discretization parameter h. In short, we prove that we can drive the semi-discrete system (46) to a target which is exponentially small with respect to h with controls that are uniformly bounded. Up to a subsequence, this results imply the weak convergence of the semi-discrete controls towards a control of the continuous problem which leads the solution to zero, as soon as the discrete initial data converges towards the suitable initial data.

Theorem 4.1. *There exist $C > 0$, $\widetilde{C} > 0$ and $h_0 > 0$, depending only on γ_1, γ_2 and ω_2 such that, for any $T > 0$, any mesh such that $h < h_0$, and any $y_0 \in E$ there exists a control $v \in L^2(0,T;U)$ satisfying*

$$\begin{cases} [\![v]\!]_{L^2(0,T;U)} \leq e^{\widetilde{C}(1+\frac{1}{T}+T)} \|y_0\|_{-1,\mathrm{A}}, \\ \|y(T)\|_{-1,\mathrm{A}} \leq e^{\widetilde{C}(1+\frac{1}{T}+T)} e^{-C/h^2} \|y_0\|_{-1,\mathrm{A}}, \end{cases} \tag{47}$$

where y is the corresponding solution to the semi-discrete problem (46) with control v and $h = \max(h_1, h_2)$ is the space discretization parameter.

Once such a theorem is proved, even with a non explicit/constructive proof, one can produce an optimization algorithm, based on the penalized HUM approach, that is able to compute a control v satisfying (47). We can even relax the requirements by replacing the exponential factor e^{-C/h^2} by any more convenient $\phi(h)$, such as $\phi(h) = h^p$ for some large enough integer p. Those questions are discussed in details for instance in [7] where some numerical illustrations are given.

4.1.2. *Additional notations and properties.* In order to simplify the presentation of the following proofs we need to introduce a few more notations. For any $i \in \{1, 2\}$, we define $\gamma_i^{\pm}$ to be the translated sampling of the diffusion coefficient γ_i defined by

$$(\gamma_i^{\pm})_j = \gamma_{i,j\pm 1/2}, \quad \forall j \in \{1, ..., N_i\}.$$

We also introduce the forward and backward difference operators $\nabla_i^{\pm}$ defined, for any $y \in E_i$, by

$$(\nabla_i^+ y)_j = \frac{y_{j+1} - y_j}{h_i}, \quad \forall j \in \{1, ..., N_i\},$$
$$(\nabla_i^- y)_j = \frac{y_j - y_{j-1}}{h_i}, \quad \forall j \in \{1, ..., N_i\}.$$

For any $i \in \{1, 2\}$, we define $(e_{i,j})_{1 \leq j \leq N_i}$ to be the canonical basis of E_i (each element corresponds to a point in the 1D grid of Ω_i). At some point in the forthcoming analysis, we shall need to work with *compactly supported discrete functions* in order to justify some discrete integrations by parts. To this end, we introduce the subspaces $E_i^{00} \subset E_i^0 \subset E_i$ defined by

$$E_i^0 = \mathrm{span}(e_{i,j}, \quad j = 2, \ldots, N_i - 1),$$
$$E_i^{00} = \mathrm{span}(e_{i,j}, \quad j = 3, \ldots, N_i - 2).$$

With those notations, we observe that the gradient operators defined above satisfy the duality property

$$\langle \nabla_i^+ y, z \rangle_{0,i} = - \langle y, \nabla_i^- z \rangle_{0,i}, \quad \forall y, z \in E_i^0, \tag{48}$$

which is obtained by a summation by parts.

Lemma 4.2. *There exists a $C > 0$ depending only on $\inf \gamma_i$, $\sup \gamma_i$, $\|\gamma_i'\|_{L^\infty}$, a_i such that: for any $i \in \{1, 2\}$, and any $y \in E_i$, we have*

$$\|y\|_{0,i} \leq C \left\|\nabla_i^{\pm} y\right\|_{0,i}, \tag{49}$$

$$\left\|\nabla_i^{+} y\right\|_{0,i}^2 + \left\|\nabla_i^{-} y\right\|_{0,i}^2 \leq C \left\langle \mathrm{A}_i y, y\right\rangle_{0,i}, \tag{50}$$

$$\|\Delta_i y\|_{0,i} \leq C \left\|\mathrm{A}_i y\right\|_{0,i}. \tag{51}$$

Remark 4.1. By combining the above properties, we can obtain the following estimate

$$\|y\|_{0,i} \leq C \left\|\mathrm{A}_i y\right\|_{0,i}, \quad \forall y \in E_i, \tag{52}$$

where C depends only on γ_i and a_i. This implies in particular that the Poincaré inequality (23) holds with a constant $M_{\mathrm{P},1}$ uniform with respect to the discretization parameters.

Proof. The first inequality is very classical. We recall the sketch of proof: for $i \in \{1, 2\}$, we first write

$$y_j = y_j - y_0 = \sum_{k=1}^{j} (y_k - y_{k-1}) = \sum_{k=1}^{j} h_i (\nabla_i^- y)_k,$$

and by the Cauchy-Schwarz inequality, we obtain

$$|y_j|^2 \leq a_i \left\|\nabla_i^- y\right\|_{0,i}^2, \quad \forall j \in \{0, \ldots, N_i\}.$$

The claim follows by multiplying by h_i and summing over j. The proof with the operator ∇_i^+, is done in the same way but starting from the equality

$$y_j = y_j - y_{N_i+1} = -\sum_{k=j}^{N_i} (y_{k+1} - y_k) = \sum_{k=j}^{N_i} h_i (\nabla_i^+ y)_k.$$

For the third estimate, a simple calculation shows that

$$\left\langle \mathrm{A}_i y, y\right\rangle_{0,i} = \sum_{j=0}^{N_i} h_i \gamma_{i,j+1/2} \left| \frac{y_{j+1} - y_j}{h_i} \right|^2, \ \forall y \in E_i,$$

and we deduce (50) with C depending only on $\min_{\Omega_i}(\gamma_i)$.

Finally, a straightforward algebraic computation gives

$$\mathrm{A}_i = \frac{\gamma_i^+ + \gamma_i^-}{2}(-\Delta_i) - \frac{\gamma_i^+ - \gamma_i^-}{h_i} \frac{\nabla_i^+ + \nabla_i^-}{2},$$

which implies

$$\|-\Delta_i y\|_{0,i} \leq C \left\|\mathrm{A}_i y\right\|_{0,i} + C\|\gamma_i'\|_{L^\infty} (\left\|\nabla_i^+ y\right\|_{0,i} + \left\|\nabla_i^- y\right\|_{0,i}),$$

where C only depends on $\min_{\Omega_i}(\gamma_i)$. By using (50), the Cauchy-Schwarz inequality and the Young inequality, we finally deduce that

$$\|-\Delta_i y\|_{0,i} \leq C \left\|\mathrm{A}_i y\right\|_{0,i} + C \left\|y\right\|_{0,i}, \quad \forall y \in E_i,$$

where C only depends on the function γ_i and we conclude by using (49) (where we take the square on both sides) and (50). $\square$

4.1.3. *Proof of the semi-discrete controllability result.* The proof of Theorem 4.1 consists essentially in applying the analysis above to a suitable setting. To achieve the results, it is just needed to check that all the assumptions of Theorem 3.1 are satisfied with constants that do not depend on the discretization parameter.

- In a first step, we will check that the discrete diffusion operator L and the discrete control operator $\mathrm{B} = \mathrm{B}_1 \otimes \mathrm{B}_2 \otimes \mathsf{B}$ are compatible in the sense that inequalities (14) and (15) are satisfied with $s = 1$, and $\mathrm{D} = \mathrm{A}$, uniformly with respect to the mesh size.
- In a second step, using previous results in the literature, we shall prove that the discrete Lebeau-Robbiano spectral inequality (Assumption 2) and the semi-discrete controllability of the 1D system (Assumption 1) hold.

Let us start by proving the following result.

Lemma 4.3. *There exist two positive constants M_{cont} and M_{adm} independent of h_1 and h_2 that satisfy:*

$$\sup_{t\in[0,T]} \left\| e^{-t\mathrm{L}^*}\psi \right\|_{1,\mathrm{A}} \leq M_{cont} \left\|\psi\right\|_{1,\mathrm{A}}, \quad \forall \psi \in E, \tag{53}$$

$$\left(\int_0^T \left[\!\left[\mathrm{B}^* e^{-\tau \mathrm{L}^*}\psi \right]\!\right]_0^2 d\tau \right)^{\frac{1}{2}} \leq M_{adm} \left\|\psi\right\|_{1,\mathrm{A}}, \quad \forall \psi \in E, \tag{54}$$

Proof. The proof of (53) is exactly the same as the one of the dissipation estimates of Proposition 3.3, for $\mu = 0$ and $s = -1$, except that we deal with the adjoint matrix C^*.

The more intricate part is now to prove (54). This will be a combination of discrete trace estimates and of discrete elliptic regularity properties for the operator A. In that proof the fact that we consider $s = 1$ in the definition of the norms is crucial, this is the discrete counter-part of the fact that, in the continuous setting, we need to take H_0^1 initial data to ensure that the normal derivative of the solution of the backward heat equation belongs to L^2.

1. First, we prove a **1D trace inequality**. More precisely, we show that there exists $C > 0$ independent of h_1 such that:

   $$[\![\mathrm{B}_1^* \otimes \mathsf{B}^* q]\!]_{0,1}^2 \leq C \left\|\mathrm{A}_1 \otimes \mathsf{I}\, q\right\|_{0,1}^2, \forall q \in E_1 \otimes \mathbb{R}^n. \tag{55}$$

   To this end, we first use (45) and the fact that Δ_1 is self-adjoint to get

   $$\mathrm{B}_1^* \otimes \mathsf{B}^* = \gamma_{1,1/2}(r \otimes \mathsf{B}^*)(\Delta_1 \otimes \mathsf{I}).$$

   It follows that, for any $q \in E_1 \otimes \mathbb{R}^n$,

   $$[\![\mathrm{B}_1^* \otimes \mathsf{B}^* q]\!]_{0,1} \leq \gamma_{1,1/2} \left\|r\right\|_{0,1} \left\|\mathsf{B}^*\right\| \left\|\Delta_1 \otimes \mathsf{I}\, q\right\|_{0,1} \leq C \left\|\Delta_1 \otimes \mathsf{I}\, q\right\|_{0,1},$$

   where C depends only on γ_1 and Ω_1.

   From (51) (that we apply on each component of the vector-valued unknown q), we deduce that

   $$\left\|\Delta_1 \otimes \mathsf{I}\, q\right\|_{0,1} \leq C \left\|\mathrm{A}_1 \otimes \mathsf{I}\, q\right\|_{0,1},$$

   which proves (55).
2. In the second step, we prove that the inequality (55) can be extended to **2D discrete unknowns**. More precisely, we show that there exists a constant $C > 0$ independent of h_1 and h_2 such that

   $$[\![\mathrm{B}^* q]\!]_0^2 = [\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q]\!]_0^2 \leq C \left\|\mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I}\, q\right\|_0^2, \forall q \in E. \tag{56}$$

We define $f_j = e_{2,j}/\sqrt{h_2}$, in such a way that $(f_{2,j})_{1\leq j\leq N_2}$ is an orthonormal basis of E_2. Note that the particular structure of B_2 implies that $\mathrm{B}_2 f_{2,j} = \alpha_j f_{2,j}$ with $\alpha_j \in \{0,1\}$ ($\alpha_j = 1$ if and only if the mesh point $x_{2,j}$ lies in ω_2).

We can decompose any $q \in E$ into the unique form

$$q = \sum_{j=1}^{N_2} q_j \otimes f_{2,j},$$

with $q_j \in E_1 \otimes \mathbb{R}^n$ for any j. We have, by orthogonality of $(f_{2,j})_j$,

$$[\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q]\!]_0^2 = \left[\!\!\left[\sum_{j=1}^{N_2} \alpha_j \Big[(\mathrm{B}_1^* \otimes \mathsf{B}^*) q_j\Big] \otimes f_{2,j}\right]\!\!\right]_0^2 = \sum_{j=1}^{N_2} [\![\mathrm{B}_1^* \otimes \mathsf{B}^* q_j]\!]_{0,1}^2 \, \alpha_j^2.$$

Using (55) for each j, and the fact that $\alpha_j^2 \leq 1$, we get

$$[\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q]\!]_0^2 \leq C \sum_{j=1}^{N_2} \|\mathrm{A}_1 \otimes \mathsf{I}\, q_j\|_{0,1}^2 = C \sum_{j=1}^{N_2} \left\| \Big[(\mathrm{A}_1 \otimes \mathsf{I}) \otimes \mathrm{I}\Big] q_j \otimes f_{2,j} \right\|_0^2,$$

and, still by orthogonality of $(f_{2,j})_j$, we obtain the claimed inequality (56), the constant C being the same as in (55).

3. In the estimate (56) we only have the operator $\mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I}$ that appears in the right-hand side and not the complete discrete 2D elliptic operator A. The third step consists in proving a **discrete elliptic regularity property** that will allow us to get that there exists a constant $C > 0$ independent of h_1 and h_2 such that

$$[\![\mathrm{B}^* q]\!]_0^2 \leq C \, \|\mathrm{A}q\|_0^2 \, , \forall q \in E. \tag{57}$$

The main idea is based on the following well-known computation: for any smooth and compactly supported function f defined on $\mathbb{R}^2$, we can write

$$\begin{aligned}\iint |\partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f) + \partial_{x_2}(\gamma_2(x_2)\partial_{x_2} f)|^2 \\ = \iint |\partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f)|^2 + |\partial_{x_2}(\gamma_2(x_2)\partial_{x_2} f)|^2 \\ + 2\iint \partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f) \times \partial_{x_2}(\gamma_2(x_2)\partial_{x_2} f) \, .\end{aligned}$$

Moreover, by a double integration by parts, we find

$$\iint \partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f) \times \partial_{x_2}(\gamma_2(x_2)\partial_{x_2} f) = \iint \gamma_1(x_1)\gamma_2(x_2)\big|\partial_{x_1}\partial_{x_2} f\big|^2 \geq 0,$$

so that

$$\iint |\partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f)|^2 \leq \iint |\partial_{x_1}(\gamma_1(x_1)\partial_{x_1} f) + \partial_{x_2}(\gamma_2(x_2)\partial_{x_2} f)|^2 \, .$$

We want to apply the same idea to prove roughly speaking that

$$\|\mathrm{A}_1 \otimes \mathrm{I}\, q\|_0^2 \leq \|\mathrm{A}_1 \otimes \mathrm{I}\, q + \mathrm{I} \otimes \mathrm{A}_2\, q\|_0^2 \, ,$$

which, according to (56), would prove (57).

However, because of boundary terms, it is easier to prove this inequality for compactly supported discrete functions, that is for $q \in E_1^{00} \otimes E_2^{00}$. We will thus proceed by extension and truncation of the discrete functions under study.

Let $\xi_1 : (0, a_1) \to \mathbb{R}$ be a smooth function such that $\xi_1 = 1$ on $(0, \frac{a_1}{3})$, and $\xi_1 = 0$ on $(\frac{2a_1}{3}, a_1)$, and $\xi_2 : (0, a_2) \to \mathbb{R}$ be a smooth compactly supported function such that $\xi_2 = 1$ on ω_2. Such a function exists thanks to the assumption (42). We introduce now the truncation operator on $E_1 \otimes E_2$, defined by $\mathrm{T} = \xi_1 \otimes \xi_2$. The choice of ξ_1 and ξ_2 implies that

$$[\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q]\!]_0^2 = [\![(\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^*)(\mathrm{T} \otimes \mathsf{I})q)]\!]_0^2,$$

and thus by (56) we have

$$[\![\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q]\!]_0^2 \leq C \left\|(\mathrm{A}_1 \otimes \mathrm{I} \otimes \mathsf{I})(\mathrm{T} \otimes \mathsf{I})q\right\|_0^2.$$

The discrete function $(\mathrm{T} \otimes \mathsf{I})q$ vanishes near all the boundaries of the domain, except the one corresponding to $\{x_1 = 0\}$ because of the choice of ξ_1. We will now introduce a symmetrization procedure that will let us work with a compactly supported discrete function.

We start by introducing the extended space in the first variable defined by

$$\overline{E}_1 := \mathbb{R}^{\{-N_1,\dots,N_1\}},$$

which stands for discrete functions defined on a uniform discretization of $(-a_1, a_1)$ with a mesh size h_1. If we denote by $(\bar{e}_{1,j})_{-N_1 \leq j \leq N_1}$ the canonical basis of $\overline{E}_1$, we can define the odd symmetrization operator $\mathrm{S} : E_1 \to \overline{E}_1$ by

$$\mathrm{S} e_{1,j} = \bar{e}_{1,j} - \bar{e}_{1,-j}, \quad \forall j \in \{1, \dots, N_1\}.$$

With this notation, for $q \in E_1 \otimes E_2$, $(\mathrm{S} \otimes \mathrm{I})q$ is the odd symmetrization of q with respect to the x_1 variable in the extended 2D domain.

We consider $\bar{\gamma}_1$ to be the even extension of γ_1 to $(-a_1, a_1)$ and let $\overline{\mathrm{A}}_1$ be the discrete diffusion operator associated with $\bar{\gamma}_1$ and defined on $\overline{E}_1$ in the same way as in (44).

A simple computation shows that the symmetrization is compatible with the diffusion operator definitions, that is

$$\overline{\mathrm{A}}_1 \mathrm{S} q = \mathrm{S} \mathrm{A}_1 q, \quad \forall q \in E_1. \tag{58}$$

In particular, using the same notation for the norm in $\overline{E}_1$ as for the one in E_1, we have

$$\left\|\overline{\mathrm{A}}_1 \mathrm{S} q\right\|_{0,1}^2 = 2 \left\|\mathrm{A}_1 q\right\|_{0,1}^2, \quad \forall q \in E_1.$$

As a consequence, all the previous estimates lead to the following inequality

$$\left\|\mathrm{B}_1^* \otimes \mathrm{B}_2^* \otimes \mathsf{B}^* q\right\|_{0,2}^2 \leq C \left\|\overline{\mathrm{A}}_1 \otimes \mathrm{I} \otimes \mathsf{I}\, \bar{q}\right\|_0^2, \ \forall q \in E,$$

where, we have set $\bar{q} = (\mathrm{S} \otimes \mathrm{I} \otimes \mathsf{I})(\mathrm{T} \otimes \mathsf{I})q$. Observe now that, by construction of the symmetrization and truncation operators, the discrete function $\bar{q}$ belongs to $\overline{E}_1^{00} \otimes E_2^{00} \otimes \mathbb{R}^n$.

Note that for such *compactly supported* discrete functions we have the algebraic identities

$$(\overline{\mathrm{A}}_1 \otimes \mathrm{I})\, \bar{q} = \left((-\overline{\nabla}_1^- \bar{\gamma}_1^+ \overline{\nabla}_1^+) \otimes \mathrm{I}\right) \bar{q},$$

$$(\mathrm{I} \otimes \mathrm{A}_2)\, \bar{q} = \left(\mathrm{I} \otimes (-\nabla_2^- \gamma_2^+ \nabla_2^+)\right) \bar{q}.$$

We can now make the following computation

$$\begin{aligned}\left\|(\overline{\mathrm{A}}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)\, \bar{q}\right\|_0^2 &= \left\|\overline{\mathrm{A}}_1 \otimes \mathrm{I}\, \bar{q}\right\|_0^2 + \left\|\mathrm{I} \otimes \mathrm{A}_2\, \bar{q}\right\|_0^2 \\ &\quad + 2 \left\langle \overline{\mathrm{A}}_1 \otimes \mathrm{I}\, \bar{q}, \mathrm{I} \otimes \mathrm{A}_2\, \bar{q} \right\rangle_0,\end{aligned}$$

and the double product term can be evaluated as follows, the discrete integration by parts being justified by (48) and the fact that $\nabla_i^{\pm} E_i^{00} \subset E_i^0$,

$$\left\langle \overline{\mathrm{A}}_1 \otimes \mathrm{I}\,\bar{q}, \mathrm{I} \otimes \mathrm{A}_2\,\bar{q} \right\rangle_0 = \left\langle (\overline{\nabla}_1^- \bar{\gamma}_1^+ \overline{\nabla}_1^+) \otimes \mathrm{I}\,\bar{q}, \mathrm{I} \otimes (\nabla_2^- \gamma_2^+ \nabla_2^+)\,\bar{q} \right\rangle_0$$
$$= \left\langle (\bar{\gamma}_1^+ \otimes \gamma_2^+)(\overline{\nabla}_1^+ \otimes \nabla_2^+)\bar{q}, (\overline{\nabla}_1^+ \otimes \nabla_2^+)\bar{q} \right\rangle_0 \geq 0.$$

At the end, we obtain that

$$\left\| \overline{\mathrm{A}}_1 \otimes \mathrm{I}\,\bar{q} \right\|_0^2 \leq \left\| (\overline{\mathrm{A}}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)\,\bar{q} \right\|_0^2 .$$

To conclude the proof of the final claim, we just need to show that

$$\left\| (\overline{\mathrm{A}}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)\,\bar{q} \right\|_0^2 \leq C \left\| \mathrm{A}\bar{q} \right\|_0^2 ,$$

for a $C > 0$ independent of the mesh.

Recall that $\bar{q} = (\mathrm{S} \otimes \mathrm{I})\mathrm{T}q$ and by (58) we have

$$(\overline{\mathrm{A}}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)\,\bar{q} = (\mathrm{S} \otimes \mathrm{I})\mathrm{AT}q,$$

and in particular we have

$$\left\| (\overline{\mathrm{A}}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)\,\bar{q} \right\|_0^2 = 2 \left\| \mathrm{AT}q \right\|_0^2 .$$

It thus remains to evaluate the norm of $\mathrm{AT}q$ by the one of $\mathrm{A}q$. A simple algebraic computation leads to

$$\begin{aligned} \mathrm{AT}q =& (\mathrm{A}_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}_2)(\xi_1 \otimes \xi_2) q \\ =& (\xi_1 \otimes \xi_2)(\mathrm{A}q) + ((\mathrm{A}_1 \xi_1) \otimes \xi_2) q + (\xi_1 \otimes (\mathrm{A}_2 \xi_2)) q \\ &+ (\gamma_1^+ (\nabla_1^+ \xi_1) \otimes \mathrm{I})(\nabla_1^+ \otimes \mathrm{I}) q + \big(\gamma_1^- (\nabla_1^- \xi_1) \otimes \mathrm{I}\big)(\nabla_1^- \otimes \mathrm{I}) q \\ &+ \big(\mathrm{I} \otimes \gamma_2^+ (\nabla_2^+ \xi_2)\big)(\mathrm{I} \otimes \nabla_2^+) q + \big(\mathrm{I} \otimes \gamma_2^- (\nabla_2^- \xi_2)\big)(\mathrm{I} \otimes \nabla_2^-) q. \end{aligned}$$

Since ξ_1 and ξ_2 are smooth and γ_1 and γ_2 are bounded, we conclude, with the mean-value theorem that

$$\begin{aligned} \left\| \mathrm{AT}q \right\|_0 \leq C \left\| \mathrm{A}q \right\|_0 + C \left\| q \right\|_0 + C \left\| (\nabla_1^+ \otimes \mathrm{I}) q \right\|_0 \\ + C \left\| (\nabla_1^- \otimes \mathrm{I}) q \right\|_0 + C \left\| (\mathrm{I} \otimes \nabla_2^+) q \right\|_0 + C \left\| (\mathrm{I} \otimes \nabla_2^-) q \right\|_0 . \end{aligned}$$

The conclusion follows by (49), (50), (52) and the Cauchy-Schwarz and Young inequalities.

4. The last step of the proof consists in showing (54) by using (57) and a classical energy estimate. Let $\psi \in E$ and let $q(t) = e^{-t\mathrm{L}^*}\psi$. We set $z(t) = \mathrm{I} \otimes \mathrm{I} \otimes e^{t\mathsf{C}^*} q(t)$. One can check that z satisfies

$$\begin{cases} \partial_t z + \mathrm{A} z = 0, \\ \qquad z(0) = \psi. \end{cases} \tag{59}$$

We multiply (59) by $\mathrm{A}z$ and integrate on $(0, T)$,

$$\langle z(T), \mathrm{A}z(T) \rangle_0 - \langle \psi, \mathrm{A}\psi \rangle_0 + 2 \int_0^T \left\| \mathrm{A}z(t) \right\|_0^2 dt = 0,$$

thus, since the operator A is positive,

$$\int_0^T \left\| \mathrm{A}z(t) \right\|_0^2 dt \leq \left\| \psi \right\|_{1,\mathrm{A}}^2 .$$

Since A commutes with $\mathrm{I}\otimes\mathrm{I}\otimes e^{t\mathsf{C}^*}$, we find that

$$\int_0^T \left\|\mathrm{I}\otimes\mathrm{I}\otimes e^{t\mathsf{C}^*}\mathrm{A}q(t)\right\|_0^2 dt \leq \|\psi\|_{1,\mathrm{A}}^2 ,$$

Moreover, $\left\|\mathrm{I}\otimes\mathrm{I}\otimes e^{t\mathsf{C}^*}\mathrm{A}q(t)\right\|_0^2 \geq e^{-2T\|\mathsf{C}\|}\|\mathrm{A}q(t)\|_0^2$, for any $t\in(0,T)$ and so

$$\int_0^T \|\mathrm{A}q(t)\|_0^2\, dt \leq e^{2T\|\mathsf{C}\|}\|\psi\|_{1,\mathrm{A}}^2 .$$

Applying (57) to $q(t)$ for any $t\in(0,T)$ and integrating in time this inequality, we finally get

$$\int_0^T [\![\mathrm{B}^* q(t)]\!]_0^2\, dt \leq Ce^{CT}\|\psi\|_{1,\mathrm{A}}^2 .$$

and this concludes the proof of (54).

□

Remark 4.2. Note that we just proved that M_{adm} depends on T like e^{CT} and this is consistent with estimates of v and $y(T)$ given by Theorem 4.1.

Now we can prove Assumption 2

Proof of Assumption 2. In [8] the authors proved a discrete Lebeau-Robbiano inequality on quite general meshes.

We translate the statement of their Theorem 6.1 in our setting.

Theorem 4.4. *There exists $C>0$, $\varepsilon>0$ and $h_0>0$, depending on γ_2, ω_2 such that if $h_2\leq h_0$, we have for all $0<\mu\leq\varepsilon/h_2^2$,*

$$\sum_{k:\lambda_{2,k}\leq\mu}|\alpha_k|^2=\int_{\Omega_2}\left|\sum_{k:\lambda_{2,k}\leq\mu}\alpha_k\phi_{2,k}\right|^2\leq Ce^{C\sqrt{\mu}}\int_{\omega_2}\left|\sum_{k:\lambda_{2,k}\leq\mu}\alpha_k\phi_{2,k}\right|^2, \quad \forall(\alpha_k)_{1\leq k\leq N_1}\subset\mathbb{C}$$

This result exactly yields that Assumption 2 is fulfilled in this setting with

$$\mu_{E_2}=\frac{\varepsilon}{h_2^2}.$$

□

Finally, we prove that Assumption 1 holds.

Proof of Assumption 1. We base our proof on the strategy developed in [2], where the semi-discretized (on a uniform mesh) boundary null-control problem in space dimension 1 with operator A_1 is tackled by applying the moments method. However, the explicit dependence in T of the control cost by some bound in $e^{C/T}$ was not given in that work. This is crucial in the present analysis.

We can actually obtain this precised bound by using the expression of the control obtained with this method, in setting (S2) described in [2], and by using the refined bounds on biorthogonal families to exponential functions given by Theorem 1.5 of [5]. We will just describe here the new estimate that we need to adapt the results of [2] to our needs.

We first set $\varepsilon = 1/2$ for instance (any value between 0 and 1 would be acceptable), and we follow [2], to define

$$k^{h_1}_{max,\varepsilon} := \max\left\{k \in [\![1, N_1]\!]; \lambda_{1,k} < \frac{4}{h_1^2}\gamma_{1,\min}(1-\varepsilon)\right\}$$

and :

$$(\tilde{\Lambda}^{h_1}_{\varepsilon})_k := \begin{cases} \lambda_{1,k} \text{ for } k \in [\![1, k^{h_1}_{max,\varepsilon}]\!], \\ \lambda_{1,k^{h_1}_{max,\varepsilon}} + 4\gamma_{1,\min}k^2 \text{ for } k \geq k^{h_1}_{max,\varepsilon} + 1. \end{cases}$$

the sequence $\tilde{\Lambda}^{h_1}_{\varepsilon}$ satisfies the following items required to apply Theorem 1.5 of [5]

1. $(\tilde{\Lambda}^{h_1}_{\varepsilon})_k \neq (\tilde{\Lambda}^{h_1}_{\varepsilon})_n$, for all $k, n \in \mathbb{N}$ with $k \neq n$;

2,3 $(\tilde{\Lambda}^{h_1}_{\varepsilon})_k \in \mathbb{R}^*_+$, for every $k \geq 1$;

4. $(\tilde{\Lambda}^{h_1}_{\varepsilon})_k$ is non decreasing;
5. First, applying Theorem 3.2 of [2] there exists $C > 0$ such that
$$(\tilde{\Lambda}^{h_1}_{\varepsilon})_{k+1} - (\tilde{\Lambda}^{h_1}_{\varepsilon})_k \geq C\sqrt{(\tilde{\Lambda}^{h_1}_{\varepsilon})_k},\ \forall k \geq 1.$$
Second, note that according to Lemma 3.3 of [2], there exist c_1 and c_2 two positive constants such that :
$$c_1k^2 \leq \lambda_{1,k} \leq c_2k^2,\ \forall k \in [\![1, N_1]\!]. \tag{60}$$
Therefore,
$$(\tilde{\Lambda}^{h_1}_{\varepsilon})_{k+1} - (\tilde{\Lambda}^{h_1}_{\varepsilon})_k \geq Ck,\ \forall k \geq 1.$$
Take $k, n \in \mathbb{N}^*$, $k > n$, we have :
$$|(\tilde{\Lambda}^{h_1}_{\varepsilon})_k - (\tilde{\Lambda}^{h_1}_{\varepsilon})_n| \geq C\sum_{i=n}^{k-1} i \geq C(k^2 - n^2).$$
6. Finally, let $\mathcal{N}$ the *counting function* associated with the sequence $(\tilde{\Lambda}^{h_1}_{\varepsilon})_{k\geq 1}$, defined by
$$\mathcal{N}(r) = \sharp\{k : (\tilde{\Lambda}^{h_1}_{\varepsilon})_k \leq r\}.$$
There exist $\alpha > 0$, $p_{\min} > 0$ and $p_{\max} > 0$ such that for every $r > 0$,
$$-\alpha + p_{\min}\sqrt{r} \leq \mathcal{N}(r) \leq \alpha + p_{\max}\sqrt{r}.$$
Indeed, from item 4, it follows easily from the definition of $\mathcal{N}$ that $(\tilde{\Lambda}^{h_1}_{\varepsilon})_{\mathcal{N}(r)} \leq r$ and $(\tilde{\Lambda}^{h_1}_{\varepsilon})_{\mathcal{N}(r)+1} \geq r$, and according to (60),
$$c_1\mathcal{N}(r)^2 \leq r \leq c_2(\mathcal{N}(r)+1)^2,$$
thus,
$$\sqrt{\frac{r}{c_2}} - 1 \leq \mathcal{N}(r) \leq \sqrt{\frac{r}{c_1}}.$$

We can now apply Theorem 1.5 of [5] to conclude that there exists $T_0 > 0$, such that for every $0 < T < T_0$, there exists a family of functions in $L^2(0,T)$ denoted by $(q^{\tilde{\Lambda}^{h_1}_{\varepsilon}}_{j,k})_{k\geq 1, j\in[\![1,n-1]\!]}$, satisfying for all $k, l \geq 1$, and $i, j \in [\![0, n-1]\!]$,

$$\int_0^T q^{\tilde{\Lambda}^{h_1}_{\varepsilon}}_{j,k}(t)(t-T)^i \exp(-(\tilde{\Lambda}^{h_1}_{\varepsilon})_k(T-t))\,dt = \delta_{k,l}\delta_{i,j},$$

and for all $k \geq 1$ and $j \in [\![0, n-1]\!]$,

$$\|q^{\tilde{\Lambda}^{h_1}_{\varepsilon}}_{j,k}\|_{L^2(0,T)} \leq C\exp\left(C\sqrt{(\tilde{\Lambda}^{h_1}_{\varepsilon})_k} + \frac{C}{T}\right), \tag{61}$$

where C does not depend on h_2.

Remark 4.3. In [5], Theorem 1.5, hypothesis of 6 on the counting function is slightly different from item 6 given above. Indeed, in this reference the authors require the following condition : for some $p, \alpha > 0$,

$$|p\sqrt{r} - \mathcal{N}(r)| \leq \alpha, \forall r > 0. \tag{62}$$

Actually, looking carefully at the proof of Theorem 1.5, we realize that we can use item 6 above instead of hypothesis (62). Indeed, we can just replace p by $p_{\max}$ in the definition of the number d at page 2988 and in the proofs of Theorem 4.3 at page 2987, Lemma A.3 at page 2994 (note that the property $\lim_{t\to+\infty} \frac{\mathcal{N}(t)}{t} = 0$ is preserved), and Lemma A.4 at page 2996 and change p into $p_{\min}$ at page 2993 in the proof of Lemma A.1.

Now, in the proof of [2, Theorem 5.4], we can use the improved estimate (61) into the bound of the control (with $t_0 = \frac{T}{2}$), and find that

$$[\![v]\!]_{L^2(0,T;U)} \leq C_\varepsilon \left\|y_0\right\|_{-1,\mathrm{A}_1} \sum_{k=1}^{k^{h_1}_{max,\varepsilon}} e^{-\lambda_{1,k}\frac{T}{2}} e^{C\sqrt{\lambda_{1,k}}+\frac{C}{T}},$$

and using (60),

$$[\![v]\!]_{L^2(0,T;U)} \leq C_\varepsilon e^{\frac{C}{T}} \left\|y_0\right\|_{-1,\mathrm{A}_1} \sum_{k=1}^{+\infty} e^{-c_2k^2\frac{T}{2}+C\sqrt{c_1}k},$$

thus there exists $C_1 > 0$ depending only on ε (whose value has been arbitrarily set to $1/2$) and γ_1 such that

$$[\![v]\!]_{L^2(0,T;U)} \leq C_1 e^{\frac{C_1}{T}} \left\|y_0\right\|_{-1,\mathrm{A}_1}. \tag{63}$$

Indeed if $k \geq k_0 := \lceil \frac{C\sqrt{c_1}+1}{c_2T/2} \rceil$, we have $-c_2k^2\frac{T}{2} + C\sqrt{c_1}k \leq -k$, so

$$\begin{aligned}
\sum_{k=1}^{+\infty} e^{-c_2k^2\frac{T}{2}+C\sqrt{c_1}k} &\leq \sum_{k=1}^{k_0} e^{-c_2k^2\frac{T}{2}+C\sqrt{c_1}k} + \sum_{k=k_0+1}^{+\infty} e^{-k} \\
&\leq k_0 e^{C\sqrt{c_1}k_0} + \frac{1}{1-e},
\end{aligned}$$

which gives (63) since $k_0 \leq \frac{C_{\gamma_1,\varepsilon}}{T}$.

Still using the estimates in the proof of Theorem 5.4 of [2], we get

$$\left\|y(T)\right\|_{-1,\mathrm{A}_1} \leq e^{C_2(1+\frac{1}{T})} e^{-\mu_{E_1}T} \left\|y_0\right\|_{-1,\mathrm{A}_1},$$

with

$$\mu_{E_1} = \frac{C_1}{h_1^2}$$

and $C_1 > 0$ and $C_2 > 0$ depending only on γ_1. □

Now we can apply Theorem 3.1 with μ_{E_1} and μ_{E_2} both of the same order $\frac{C}{h^2}$. The proof of Theorem 4.1 is complete.

4.2. **Dirichlet boundary null control of a continuous n-dimensional system on a cylindrical domain of dimension d.** Our aim is to show how the finite dimensional framework developed in Section 3 actually applies to the study of null-controllability problems of tensorized parabolic systems.

As an illustration of this statement, we shall give a short proof of the main result of [5] by using Theorem 3.1 of the present article. This is of course not surprising since our approach is directly inspired from the one developed in [5]. However, it seems to us interesting to show how our general finite dimensional framework actually encompasses already known results through a spectral projection technique. This example should convince the reader that, by using the same strategy, one can easily adapt the proof to other kinds of tensorized controlled systems like, for instance, Neumann or Robin boundary controls, or even mixed (distributed and boundary) controls, as soon as we have in hand a suitable controllability result on the associated 1D system.

Finally, since we have taken care of all the constants in the proofs, this strategy can be used to derive controllability properties that are uniform with respect to some parameters present in the problem. As an illustration, in the case of a 1D Robin boundary control problem, we can prove estimates that are uniform in the Robin parameter (see [6]). By the present technique, those result will automatically be translated to the corresponding multi-D result, generalizing the one proved in [5].

Let us recall the statement of [5, Theorem 1.3] with the notation of the present paper.

Theorem 4.5. *Let $\Omega_1 = (0,1)$, and Ω_2 be a smooth bounded connected domain of $\mathbb{R}^{d-1}$ and set $\Omega = \Omega_1 \times \Omega_2$.*

We suppose given a coupling matrix $\mathsf{C} \in M_n(\mathbb{R})$ and a control matrix $\mathsf{B} \in M_{n,m}(\mathbb{R})$. Assume that the following system of n equations

$$\begin{cases} \partial_t y^1 + \mathcal{A}_1 \otimes \mathsf{I}\, y^1 + \mathcal{I} \otimes \mathsf{C} y^1 = 0, & \text{in } (0,T) \times \Omega_1, \\ \quad y^1 = 1_{\{0\}} \otimes (\mathsf{B} v^1) & \text{in } (0,T) \times \partial\Omega_1, \\ y^1(0) = y_0^1 & \text{in } \Omega_1, \end{cases}$$

is null controllable for any $y_0^1 \in H^{-1}(\Omega_1) \otimes \mathbb{R}^n$ and any time $T > 0$ with, in addition, the following bound:

$$[\![v^1]\!]_{L^2(0,T;\mathbb{R}^m)} \leq e^{M_{\text{obs},1}(1+\frac{1}{T})} \| y_0^1 \|_{H^{-1}(\Omega_1)\otimes\mathbb{R}^n}. \tag{64}$$

Then, for any nonempty open set $\omega_2 \subset \Omega_2$, the following system of n equations:

$$\begin{cases} \partial_t y + \mathcal{L} y = 0, & \text{in } (0,T) \times \Omega, \\ \quad y = (1_{\{0\}\times\omega_2} \otimes \mathsf{B}) v & \text{in } (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{65}$$

where $\mathcal{L} := \mathcal{A}_1 \otimes \mathcal{I} \otimes \mathsf{I} + \mathcal{I} \otimes \mathcal{A}_2 \otimes \mathsf{I} + \mathcal{I} \otimes \mathcal{I} \otimes \mathsf{C}$, is null controllable for any $y_0 \in H^{-1}(\Omega) \otimes \mathbb{R}^n$ and any time $T > 0$ with, in addition, the following bound:

$$[\![v]\!]_{L^2((0,T)\times\partial\Omega)^m} \leq e^{M_{\text{obs},1}(1+\frac{1}{T})} \| y_0 \|_{H^{-1}(\Omega)\otimes\mathbb{R}^n}. \tag{66}$$

Proof. Let $y_0 \in H^{-1}(\Omega)\otimes\mathbb{R}^n$, recall that, by definition, the solution y of (65) satisfies for any $\psi \in H_0^1(\Omega)\otimes\mathbb{R}^n$, the equality

$$\langle y(T),\psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \left\langle y_0, e^{-T\mathcal{L}^*}\psi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\left[\left(\frac{\partial}{\partial x_1}_{\big|x_1=0} \otimes 1_{\omega_2}\otimes \mathsf{B}^*\right) e^{-(T-\cdot)\mathcal{L}^*}\psi, v\right]_{L^2((0,T)\times\partial\Omega)^m}.$$

Therefore, v is a null-control for this system, if and only if, it satisfies

$$-\left\langle y_0, e^{-T\mathcal{L}^*}\psi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\left[\left(\frac{\partial}{\partial x_1}_{\big|x_1=0} \otimes 1_{\omega_2}\otimes \mathsf{B}^*\right) e^{-(T-\cdot)\mathcal{L}^*}\psi, v\right]_{L^2((0,T)\times\partial\Omega)^m},$$

for any $\psi \in H_0^1(\Omega)\otimes\mathbb{R}^n$. Actually, it is enough to check the previous equality for any ψ belonging to a total family of $H_0^1(\Omega)\otimes\mathbb{R}^n$.

Let us consider the total family of $H_0^1(\Omega)$ made of the eigenfunctions of $\mathcal{A}_1$ and $\mathcal{A}_2$. For $i=1,2$, we denote by $(\phi_{i,j},\lambda_{i,j})_{j\geq 1}$ the eigenfunctions and eigenvalues of the operator $\mathcal{A}_i$ with homogeneous Dirichlet boundary conditions. We choose them to form an orthonormal basis of $L^2(\Omega_i)$. We will thus consider the total family of $H_0^1(\Omega)$ defined by $(\phi_{1,j_1}\otimes\phi_{2,j_2})_{j_1,j_2}$ that will be tensorized with the canonical basis of $\mathbb{R}^n$, $(e_k)_{k\in\{1,\dots,n\}}$ to finally produce a total family of $H_0^1(\Omega)\otimes\mathbb{R}^n$.

We are thus led to find a control satisfying, $\forall j_1,j_2\geq 1$ and $\forall k\in\{1,\dots,n\}$,

$$\begin{aligned} &e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})}\left\langle y_0, \phi_{1,j_1}\otimes\phi_{2,j_2}\otimes(e^{-T\mathsf{C}^*}e_k)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &= \left[e^{-(T-\cdot)(\lambda_{1,j_1}+\lambda_{2,j_2})}(\phi_{1,j_1})'(0)\otimes(1_{\omega_2}\phi_{2,j_2})\otimes(\mathsf{B}^* e^{-(T-\cdot)\mathsf{C}^*}e_k), v\right]_{L^2((0,T)\times\partial\Omega)^m}. \end{aligned} \tag{67}$$

In order to apply Theorem 3.1, we need to consider a projection of (65) on a finite dimensional space. For $i=1,2$ and $J\geq 1$, we define the spaces

$$F_{i,J} = \operatorname{span}(\phi_{i,j},\, j\leq J),$$

and the operator $\mathrm{A}_i : F_{i,J}\to F_{i,J}$ as the restriction to $F_{i,J}$ of $\mathcal{A}_i$, that is the unique isomorphism of $F_{i,J}$ satisfying $\mathrm{A}_i\phi_{i,j}=\lambda_{i,j}\phi_{i,j}$, for any $j\leq J$.

Let $\mathcal{P}_{i,J}$ be the orthogonal projection from the space $H^{-1}(\Omega_i)$ equipped with the inner product $\langle\bullet,\bullet\rangle_{-1,\mathcal{A}_i}$ onto the space $F_{i,J}$.

We define also $\mathcal{P}_J := \mathcal{P}_{1,J}\otimes\mathcal{P}_{2,J}\otimes\mathsf{I}$ which is an orthogonal projection from $H^{-1}(\Omega)\otimes\mathbb{R}^n$ onto $F_{1,J}\otimes F_{2,J}\otimes\mathbb{R}^n$, with respect to $\langle\bullet,\bullet\rangle_{-1,\mathcal{A}}$.

Remark 4.4. Since the eigenfunctions $(\phi_{i,j})_{j\geq 1}$ are orthogonal with respect to the $L^2(\Omega_i)$, $H^{-1}(\Omega_i)$ and $H_0^1(\Omega_i)$ inner products, we deduce that the projection $\mathcal{P}_J$ (resp. $\mathcal{P}_{i,J}$) is orthogonal with respect to $\langle\bullet,\bullet\rangle_{s,\mathcal{A}}$ (resp. $\langle\bullet,\bullet\rangle_{s,\mathcal{A}_i}$), for any $s\in\{-1,0,1\}$.

Hypothesis (64) allows to apply Assumption 1 with $E_1=F_{1,J}$, $U_1=\mathbb{R}$, $\mathcal{B}_1=\Delta_1 r$, where we have used the affine function $r(x)=1-x$ (note that $\mathcal{B}_1^* = -1_{\{0\}}\partial_{x_1}$) and $\mu_{E_1}=+\infty$. Moreover, thanks to the Lebeau-Robbiano's spectral inequality in the domain Ω_2, (see [15, 14]), Assumption 2 is fulfilled with $E_2=F_{2,J}$, $U_2=F_{2,J}$,

$\mathcal{B}_2 : U_2 \to E_2$, $\mathcal{B}_2 = \mathcal{P}_{2,J} 1_{\omega_2}$, and $\mu_{E_2} = +\infty$. Indeed, the usual spectral inequality gives that there exists $C > 0$ such that for any $\mu > 0$, $J \in \mathbb{N}^*$

$$\|\psi\|^2_{L^2(\Omega_2)} \leq Ce^{C\sqrt{\mu}} \int_{\Omega_2} (1_{\omega_2}\psi)^2, \quad \forall \psi \in \operatorname{Span}(\phi_{2,j_2}, \quad \lambda_{2,j_2} \leq \mu) \cap E_2.$$

Moreover,

$$\int_{\Omega_2} (1_{\omega_2}\psi)^2 = \int_{\Omega_2} (1_{\omega_2}\psi)\psi,$$

and according to Remark 4.4 and given that $\psi \in E_2$,

$$\int_{\Omega_2} (1_{\omega_2}\psi)\psi = \int_{\Omega_2} (\mathcal{P}_{2,J} 1_{\omega_2}\psi)\psi = \int_{\Omega_2} (\mathcal{B}_2\psi)\psi = \int_{\Omega_2} (\mathcal{B}_2^*\psi)\psi,$$

since $\mathcal{B}_2$ is self-adjoint. It follows from the Cauchy-Schwarz inequality that

$$\|\psi\|_{L^2(\Omega_2)} \leq Ce^{C\sqrt{\mu}} \|\mathcal{B}_2^*\psi\|_{L^2(\Omega_2)}, \quad \forall \psi \in \operatorname{Span}(\phi_{2,j_2}, \quad \lambda_{2,j_2} \leq \mu) \cap E_2.$$

Thus, for any fixed J, Theorem 3.1 applies with μ^* as large as we want. Thus, letting μ^* go to infinity, we finally build a null control for the projected finite dimensional problem denoted by $v_J \in L^2(0,T;U_2) \otimes \mathbb{R}^m$ that drives solution y of (24) from $P_J y_0$ to zero and which satisfies the following bound, uniformly with respect to J:

$$[\![v_J]\!]_{L^2((0,T)\times\partial\Omega)^m} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} \left\| P_J y_0 \right\|_{-1,\mathrm{A}} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} \left\| y_0 \right\|_{-1,\mathcal{A}}. \tag{68}$$

where $\|\bullet\|^2_{-1,\mathrm{A}} = \langle \bullet, \bullet \rangle_{-1,\mathrm{A}} = \left\langle \bullet, \mathrm{A}^{-1}\bullet \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$.

From (68) we can infer that there exists a subsequence of $(v_J)_J$ denoted by $(v_{J'})_{J'}$ which weakly converges in $L^2((0,T)\times\partial\Omega)^m$ to a limit denoted by v, satisfying also

$$[\![v]\!]_{L^2((0,T)\times\partial\Omega)^m} \leq e^{M_{\mathrm{obs}}(1+\frac{1}{T}+T)} \left\| y_0 \right\|_{-1,\mathcal{A}}.$$

We claim that this limit drives the solution y of (65) from y_0 to zero. Indeed, let $k \in \{1, \dots, N\}$, $j_1, j_2 \geq 1$, and $J \geq \max(j_1, j_2)$.

First, we have by definition of a solution of (24)

$$\begin{aligned} &- \left\langle \mathcal{P}_J y_0, e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \left[e^{-(T-.)(\lambda_{1,j_1}+\lambda_{2,j_2})} (\mathrm{B}_1^*\phi_{1,j_1}) \otimes (\mathrm{B}_2^*\phi_{2,j_2}) \otimes (\mathsf{B}^* e^{-(T-.)\mathsf{C}^*} e_k), v_J \right]_{L^2((0,T)\times\partial\Omega)^m} \end{aligned} \tag{69}$$

Second, since $J \geq j_1$ and $J \geq j_2$, and by definition of $\mathcal{P}_J$, we have

$$\begin{aligned} &- \left\langle \mathcal{P}_J y_0, e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\quad = - \left\langle \mathcal{P}_J y_0, (\lambda_{1,j_1} + \lambda_{2,j_2}) e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{-1,\mathrm{A}} \\ &\quad = - \left\langle y_0, (\lambda_{1,j_1} + \lambda_{2,j_2}) e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{-1,\mathcal{A}}. \end{aligned}$$

Hence, we get

$$\begin{aligned} &- \left\langle \mathcal{P}_J y_0, e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\qquad = - \left\langle y_0, e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})} \phi_{1,j_1} \otimes \phi_{2,j_2} \otimes (e^{-T\mathsf{C}^*} e_k) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{70}$$

Third, since $F_{2,J}$ is stable by A_2, $\mathcal{P}_{2,J}$ is an orthogonal projection with respect to the $L^2(\Omega_2)$ norm, see Remark 2.2. So, since $v_J \in L^2(0,T;U_2)\otimes\mathbb{R}^m$ and $U_2 = F_{2,J}$, according to remark 4.4, we have for any t

$$[\mathcal{B}_2^*\phi_{2,j_2}, v_J(t)]_{L^2(\Omega_2)^m} = [\mathcal{P}_{2,J}1_{\omega_2}\phi_{2,j_2}, v_J(t)]_{L^2(\Omega_2)^m} = [1_{\omega_2}\phi_{2,j_2}, v_J(t)]_{L^2(\Omega_2)^m}.$$

Hence,

$$\begin{aligned}\Big[(\mathrm{B}_1^*\phi_{1,j_1})\otimes(\mathrm{B}_2^*\phi_{2,j_2})\otimes(\mathsf{B}^*e^{-(T-.)\mathsf{C}^*}e_k), v_J\Big]_{L^2((0,T)\times\partial\Omega)^m}\\ = \Big[-(\phi_{1,j_1})'(0)\otimes 1_{\omega_2}\phi_{2,j_2}\otimes(\mathsf{B}^*e^{-(T-.)\mathsf{C}^*}e_k), v_J\Big]_{L^2((0,T)\times\partial\Omega)^m}. \quad (71)\end{aligned}$$

Gathering (69), (70) and (71), we get

$$\begin{aligned}-\Big\langle y_0, e^{-T(\lambda_{1,j_1}+\lambda_{2,j_2})}\phi_{1,j_1}\otimes\phi_{2,j_2}\otimes(e^{-T\mathsf{C}^*}e_k)\Big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ = \Big[-\phi_{1,j_1}'(0)\otimes 1_{\omega_2}\phi_{2,j_2}\otimes(\mathsf{B}^*e^{-(T-.)\mathsf{C}^*}e_k), v_J\Big]_{L^2((0,T)\times\partial\Omega)^m}. \quad (72)\end{aligned}$$

Considering the subsequence $v_{J'}$ in (72) and letting J' go to infinity, we get (67) and this concludes the proof. $\square$

Appendix A. Proof of Lemma 2.2.

1. $\Rightarrow$ 2. For any $q_T \in \mathrm{D}^{-s}F_T$, we set $q(t) = e^{-(T-t)\mathrm{L}^*}q_T$ and we consider the initial data $y_0 := \mathrm{P}_{F_0}\mathrm{D}^s\mathrm{P}_{F_0}^*q(0)$; by assumption there exists a control v and an associated solution y satisfying (18).

By (16) we find

$$\begin{aligned}\left\langle \mathrm{P}_{F_0}\mathrm{D}^s\mathrm{P}_{F_0}^*q(0), q(0)\right\rangle_0 &= \langle y(T), q_T\rangle_0 - \int_0^T [v(t), \mathrm{B}^*q(t)]_0\,dt\\ \left\|\mathrm{P}_{F_0}^*(q(0))\right\|_{s,\mathrm{D}}^2 &= \langle y(T), \mathrm{D}^s q_T\rangle_{-s,\mathrm{D}} - \int_0^T [v(t), \mathrm{B}^*q(t)]_0\,dt.\end{aligned}$$

Since $\mathrm{D}^s q_T \in F_T$ and P_{F_T} is orthogonal with respect to $\langle\bullet,\bullet\rangle_{-s,\mathrm{D}}$, we deduce

$$\begin{aligned}\left\|\mathrm{P}_{F_0}^*(q(0))\right\|_{s,\mathrm{D}}^2 \le{}& \langle \mathrm{P}_{F_T}y(T), \mathrm{D}^s q_T\rangle_{-s,\mathrm{D}}\\ &+ [\![v]\!]_{L^2(0,T;U)}\left(\int_0^T [\![\mathrm{B}^*q(t)]\!]_0^2\,dt\right)^{1/2}.\end{aligned}$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}\left\|\mathrm{P}_{F_0}^*(q(0))\right\|_{s,\mathrm{D}}^2 \le{}& \left(\frac{1}{M_{\mathrm{rel}}^2}\left\|\mathrm{P}_{F_T}y(T)\right\|_{-s,\mathrm{D}}^2 + \frac{1}{M_{\mathrm{obs}}^2}[\![v]\!]_{L^2(0,T;U)}^2\right)^{\frac12}\\ &\times\left(M_{\mathrm{rel}}^2\left\|\mathrm{D}^s q_T\right\|_{-s,\mathrm{D}}^2 + M_{\mathrm{obs}}^2\int_0^T [\![\mathrm{B}^*q(t)]\!]_0^2\,dt\right)^{\frac12}.\end{aligned}$$

Using the estimate (18) given in the assumptions we get

$$\left\|\mathrm{P}_{F_0}^*(q(0))\right\|_{s,\mathrm{D}}^2 \le \|y_0\|_{-s,\mathrm{D}}\left(M_{\mathrm{rel}}^2\left\|\mathrm{D}^s q_T\right\|_{-s,\mathrm{D}}^2 + M_{\mathrm{obs}}^2\int_0^T [\![\mathrm{B}^*q(t)]\!]_0^2\,dt\right)^{\frac12}.$$

Recall that we have chosen $y_0 = \mathrm{P}_{F_0}\mathrm{D}^s\mathrm{P}^*_{F_0}q(0)$, and that P_{F_0} is the orthogonal projection onto F_0 with respect to the inner product $\langle\cdot,\cdot\rangle_{-s,\mathrm{D}}$. Therefore,

$$\|y_0\|_{-s,\mathrm{D}} \le \left\|\mathrm{D}^s\mathrm{P}^*_{F_0}q(0)\right\|_{-s,\mathrm{D}} = \left\|\mathrm{P}^*_{F_0}q(0)\right\|_{s,\mathrm{D}}$$

and thus

$$\left\|\mathrm{P}^*_{F_0}(q(0))\right\|_{s,\mathrm{D}} \le \left(M_{\mathrm{rel}}^2\,\|q_T\|^2_{s,\mathrm{D}} + M_{\mathrm{obs}}^2\int_0^T [\![\mathrm{B}^*q(t)]\!]_0^2\,dt\right)^{\frac{1}{2}},$$

which gives the claim.

2. $\Rightarrow$ 1. Let y_0 be an element of F_0. We apply the quite usual penalized HUM approach (as described for instance in [7]) to find such a control v. To this end, we introduce the functional defined on $\mathrm{D}^{-s}F_T$ by

$$J(q_T) := \frac{1}{2}\int_0^T [\![\mathrm{B}^*q(t)]\!]_0^2\,dt + \frac{M_{\mathrm{rel}}^2}{2M_{\mathrm{obs}}^2}\,\|q_T\|^2_{s,\mathrm{D}} + \langle y_0, q(0)\rangle_0\,.$$

It is clear that J is continuous, strictly convex and coercive; we denote by $q_T^{\mathrm{opt}} \in \mathrm{D}^{-s}F_T$ its minimizer and by $q^{\mathrm{opt}}(t) = e^{-(T-t)\mathrm{L}^*}q_T^{\mathrm{opt}}$ the solution to the backward equation with final condition q_T^{opt}. We set $v(t) := \mathrm{B}^*q^{\mathrm{opt}}(t)$ and introduce the associated solution y to (11). Let us show the relation

$$\mathrm{P}_{F_T}(y(T)) = -\frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\mathrm{D}^s q_T^{\mathrm{opt}}. \tag{73}$$

Take any $q_T \in \mathrm{D}^{-s}F_T$ and let $q(t) = e^{-(T-t)\mathrm{L}^*}q_T$ be the solution of the adjoint equation with final condition q_T. The associated Euler-Lagrange equation for J at the optimal point q_T^{opt} writes

$$\int_0^T \left[\mathrm{B}^*q^{\mathrm{opt}}(t), \mathrm{B}^*q(t)\right]_0 dt + \frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\left\langle q_T^{\mathrm{opt}}, q_T\right\rangle_{s,\mathrm{D}} + \langle y_0, q(0)\rangle_0 = 0. \tag{74}$$

Combining (74) with (16) gives that for any $q_T \in \mathrm{D}^{-s}F_T$,

$$\langle y(T), q_T\rangle_0 = -\frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\left\langle q_T^{\mathrm{opt}}, q_T\right\rangle_{s,\mathrm{D}},$$

and thus,

$$\langle y(T), \mathrm{D}^s q_T\rangle_{-s,\mathrm{D}} = -\frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\left\langle \mathrm{D}^s q_T^{\mathrm{opt}}, \mathrm{D}^s q_T\right\rangle_{-s,\mathrm{D}}$$

and using that $\mathrm{D}^s q_T^{\mathrm{opt}} \in F_T$, and that $\mathrm{D}^s q_T$ is an arbitrary element in F_T,we finally find the equality (73).

Now we apply (74) with $q_T = q_T^{\mathrm{opt}}$ to get

$$\begin{aligned}
[\![v]\!]^2_{L^2(0,T;U)} + \frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\left\|q_T^{\mathrm{opt}}\right\|^2_{s,\mathrm{D}} &= \left\langle y_0, q^{\mathrm{opt}}(0)\right\rangle_0 \\
&= \left\langle y_0, \mathrm{P}^*_{F_0}q^{\mathrm{opt}}(0)\right\rangle_0 \\
&\le \|y_0\|_{-s,\mathrm{D}}\left\|\mathrm{P}^*_{F_0}q^{\mathrm{opt}}(0)\right\|_{s,\mathrm{D}},
\end{aligned}$$

where the second equality comes from the fact that $y_0 \in F_0$. Using now the hypothesis (19), we obtain

$$[\![v]\!]^2_{L^2(0,T;U)} + \frac{M_{\mathrm{rel}}^2}{M_{\mathrm{obs}}^2}\left\|q_T^{\mathrm{opt}}\right\|^2_{s,\mathrm{D}} \le \|y_0\|_{-s,\mathrm{D}}\left(M_{\mathrm{obs}}^2\,[\![v]\!]^2_{L^2(0,T;U)} + M_{\mathrm{rel}}^2\left\|q_T^{\mathrm{opt}}\right\|^2_{s,\mathrm{D}}\right)^{1/2}$$

and thus

$$M_{\mathrm{obs}}^2 [\![v]\!]_{L^2(0,T;U)}^2 + M_{\mathrm{rel}}^2 \left\| q_T^{\mathrm{opt}} \right\|_{s,\mathrm{D}}^2 \leq M_{\mathrm{obs}}^4 \|y_0\|_{-s,\mathrm{D}}^2 . \tag{75}$$

Note that taking the norm $\|\cdot\|_{s,\mathrm{D}}$ in (73) gives

$$\left\| q_T^{\mathrm{opt}} \right\|_{s,\mathrm{D}}^2 = \frac{M_{\mathrm{obs}}^4}{M_{\mathrm{rel}}^4} \|\mathrm{P}_{F_T} y(T)\|_{-s,\mathrm{D}}^2$$

so that with (75) we end up with

$$M_{\mathrm{obs}}^2 [\![v]\!]_{L^2(0,T;U)}^2 + \frac{M_{\mathrm{obs}}^4}{M_{\mathrm{rel}}^2} \|\mathrm{P}_{F_T} y(T)\|_{-s,\mathrm{D}}^2 \leq M_{\mathrm{obs}}^4 \|y_0\|_{-s,\mathrm{D}}^2$$

and the claim is proved.

Appendix B. **Numerical Illustrations.** In this section we give a numerical illustration of the boundary controllability of a 2D parabolic system of the cascade form using the HUM approach (see [7] for an introduction to this method and also the last paragraph of section 4.1.1). We consider a discretization in time of system (4) of coupled heat-like equations, with a constant diffusion coefficients equal to 0.05 in both directions, on a unit square with a uniform discretization of 50 mesh points in both directions and 100 times steps. The time horizon is $T = 1$, the coupling coefficient is equal to 10 and the initial condition of the controlled equation is $\alpha_0 = 0$ and the initial condition of the second component is $\beta_0 = \sin(\pi x)\sin(\pi y)$. The boundary control is acting on a part of the boundary situated on three of the edges of the square namely

$$(0.2, 0.6) \times \{1\} \bigcup (0.4, 0.8) \times \{0\} \bigcup \{0\} \times (0.2, 0.8) .$$

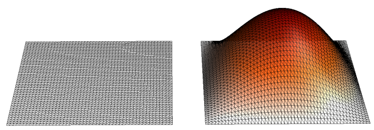
Observe that Theorem 4.1 ensures that the controllability properties of the discrete system holds even if the control only acts on one of those three parts.

In this fully discretized framework, we make the parameter ε of the (fully-discrete version of the) penalised functional $F_\varepsilon(v) := \frac{1}{2} [\![v]\!]_{L^2(0,T;U)}^2 + \frac{1}{2\varepsilon} \left(\|\alpha(T)\|^2 + \|\beta(T)\|^2 \right)$ depend on the mesh parameter h and we set $\varepsilon = 0.3 \times h^2$. We give two sets of simulations below for different times $t \in \{0, T/3, 2T/3, T\}$.

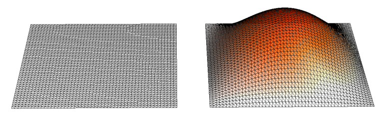
In figure 3, we set the control to zero and we plot solutions α and β. We see on figure 3d that the second component β of the system at time T is not equal to zero since the solution only decreases because of the dissipation of the heat equation. The first component α remains equal to zero since its initial condition is zero and it is not controlled.

In figure 4, however, we see how the control affects the first component of the system α so that it can drive both components of the system to zero.

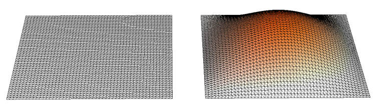
In figure 5 we plot the norms of the two components $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ as a function of time. When no control is applied to the system (dashed lines), we observe that the norm of the component β decreases until the final time T where it is close to 0.2, whereas α remains constant equal to zero. However, when applying the control (solid lines), α does not remain constant anymore, and both components α and β eventually get close to zero. We recall that we do not exactly reach zero since we used a penalized HUM approach with a penalty term depending on the mesh size.



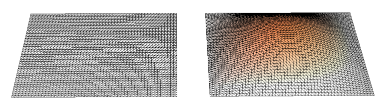
(A) t=0



(B) t=T/3

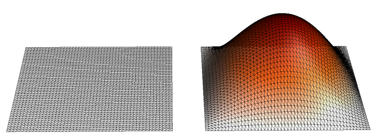


(C) t=2T/3

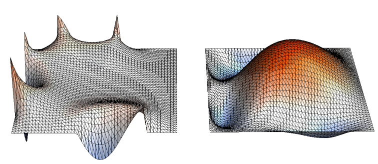


(D) t=T

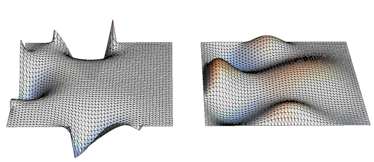
FIGURE 3. Component α (left) and β (right) of system (4) with no control



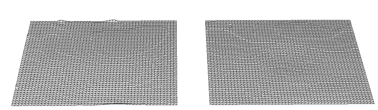
(A) t=0



(B) t=T/3

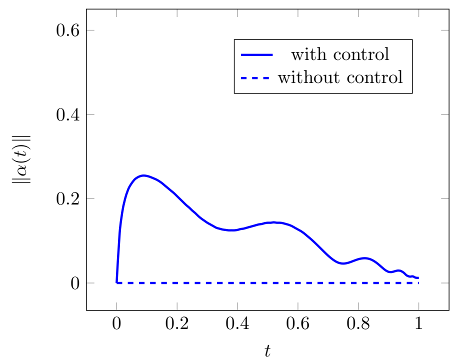


(C) t=2T/3

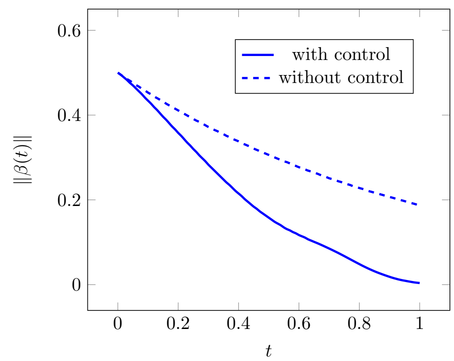


(D) t=T

FIGURE 4. Component α (left) and β (right) of system (4) with a boundary control



(A) Norm of α



(B) Norm of β

FIGURE 5. Norms of the components α, β of system (4) with and without control.

References

[1] F. Alabau-Boussouira and M. Léautaud, Indirect controllability of locally coupled wave-type systems and applications, *Journal de Mathématiques Pures et Appliquées*, **99** (2013), 544–576.

[2] D. Allonsius, F. Boyer and M. Morancey, Spectral analysis of discrete elliptic operators and applications in control theory, *Numerische Mathematik*, **140** (2018), 857–911.

[3] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. Teresa, Recent results on the controllability of linear coupled parabolic problems: A survey, *Mathematical Control and Related Fields*, **1** (2011), 267–306.

[4] F. Ammar Khodja, F. Chouly and M. Duprez, Partial null controllability of parabolic linear systems, *Math. Control Relat. Fields*, **6** (2016), 185–216.

[5] A. Benabdallah, F. Boyer, M. González-Burgos and G. Olive, Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the n-dimensional boundary null controllability in cylindrical domains, *SIAM Journal on Control and Optimization*, **52** (2014), 2970–3001.

[6] K. Bhandari and F. Boyer, Boundary null-controllability of one-dimensional coupled parabolic system with Robin conditions, *submitted*, `https://hal.archives-ouvertes.fr/hal-02091091` (2019).

[7] F. Boyer, On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems, in *CANUM 2012, 41e Congrès National d'Analyse Numérique*, vol. 41 of ESAIM Proc., EDP Sci., Les Ulis, 2013, 15–58.

[8] F. Boyer, F. Hubert and J. L. Rousseau, Discrete carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations, *Journal de Mathématiques Pures et Appliquées*, **93** (2010), 240–276.

[9] F. Boyer, F. Hubert and J. L. Rousseau, Discrete carleman estimates for elliptic operators in arbitrary dimension and applications, *SIAM Journal on Control and Optimization*, **48** (2010), 5357–5397.

[10] F. Boyer and J. L. Rousseau, Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations, *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, **31** (2014), 1035–1078.

[11] J.-M. Coron, *Control and nonlinearity*, vol. 136 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.

[12] S. Labbé and E. Trélat, Uniform controllability of semidiscrete approximations of parabolic control systems, *Systems Control Lett.*, **55** (2006), 597–609.

[13] S. Lang, *Algebra*, Springer New York, 2002.

[14] J. Le Rousseau and G. Lebeau, On carleman estimates for elliptic and parabolic operators. applications to unique continuation and control of parabolic equations, *ESAIM: COCV*, **18** (2011), 712–747.

[15] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, *Séminaire Équations aux dérivées partielles (Polytechnique)*, 1–11.

[16] L. Miller, A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 1465–1485.

[17] M. Reed and B. Simon, *Methods of modern mathematical physics. I*, 2nd edition, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis.

[18] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.

[19] E. Zuazua, Control and numerical approximation of the wave and heat equations, *International Congress of Mathematicians, Madrid, Spain*, **III** (2006), 1389–1417.

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