LINMA2171

Homework I

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October 27, 2024

Projected Gradient Method

(a) Computation of $\nabla f_l(.)$

$$
\begin{aligned}
\nabla f_l(X)_{k,l} &= \frac{\partial}{\partial_{k,l}}\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j}-Y_{i,j})^2 + \frac{\partial}{\partial_{k,l}}\lambda\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}(X_{i+1,j}-X_{i,j})^2+(X_{i,j+1}-X_{i,j})^2 \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial}{\partial_{k,l}}(X_{i,j}-Y_{i,j})^2 + \lambda\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}\frac{\partial}{\partial_{k,l}}(X_{i+1,j}-X_{i,j})^2+\frac{\partial}{\partial_{k,l}}(X_{i,j+1}-X_{i,j})^2 \\
&= (X_{k,l}-Y_{k,l}) + \lambda\Big[2(X_{k,l}-X_{k-1,l})-2(X_{k+1,l}-X_{k,l})+2(X_{k,l}-X_{k,l-1})-2(X_{k,l+1}-X_{k,l})\Big] \\
&= (X_{k,l}-Y_{k,l}) + \lambda\Big[8X_{k,l}-2X_{k-1,l}-2X_{k+1,l}-2X_{k,l-1}-2X_{k,l+1}\Big]
\end{aligned}
$$

This is a general expression for the kl component of the gradient, but we give a more precise expression hereafter to consider the ”corner cases”

$$
\nabla f_l(X)_{k,l} = \begin{cases}
(X_{k,l}-Y_{k,l}) & \text{if } k=1 \text{ and/or } l=1 \\
 & \text{or if } k=m \text{ and } l=n \\
(X_{k,l}-Y_{k,l})+\lambda[4X_{k,l}-2X_{k+1,l}-2X_{k,l+1}] & \text{if i=2 and j=2} \\
(X_{k,l}-Y_{k,l})+\lambda[6X_{k,l}-2X_{k+1,l}-2X_{k,l-1}-2X_{k,l+1}] & \text{k=2 and } 3\le l\le n-1 \\
(X_{k,l}-Y_{k,l})+\lambda[6X_{k,l}-2X_{k,l+1}-2X_{k-1,l}-2X_{k+1,l}] & \text{l=2 and } 3\le k\le m-1 \\
(X_{k,l}-Y_{k,l})+\lambda[2X_{k,l}-2X_{k-1,l}] & \text{k=m and } 2\le l\le n-1 \\
(X_{k,l}-Y_{k,l})+\lambda[2X_{k,l}-2X_{k,l-1}] & l=n \text{ and } 2\le k\le m-1 \\
(X_{k,l}-Y_{k,l})+\lambda[8X_{k,l}-2X_{k-1,l}-2X_{k+1,l}-2X_{k,l-1}-2X_{k,l+1}] & \text{else}
\end{cases}
$$

(b) $\nabla f_l(.)$ L-Lipschitz derivation

We recall that a function $F:\mathbb{R}^n\to\mathbb{R}^n$ is L-Lipschitz if there exists a constant L such that $||F(x)-F(y)||\le L||x-y|| \quad \forall x,y\in\mathbb{R}^n$.
To find L for $\nabla f_l(.)$, we will consider the general case of the derivation above, since it will lead to upper bound for the corner cases and we will assume a 0 value for the terms with out of bounds index. We will also work on the following form to avoid square roots: $||F(x)-F(y)||^2\le L^2||x-y||^2$.
This leads to:

$$||\nabla f_l(X) - \nabla f_l(Y)||^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(|X_{i,j} - Y_{i,j}| + \lambda\Big[8|X_{i,j} - Y_{i,j}| + 2|X_{i,j+1} - Y_{i,j+1}| \tag{1}$$

$$+ 2|X_{i,j-1} - Y_{i,j-1}| + 2|X_{i+1,j} - Y_{i+1,j}| + 2|X_{i-1,j} - Y_{i-1,j}|\Big]\Big)^2 \tag{2}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(|\Delta_{ij}| + \lambda\Big[8|\Delta_{ij}| + 2|\Delta_{i-1,j}| + 2|\Delta_{i+1,j}| + 2|\Delta_{i,j-1}| + 2|\Delta_{i,j+1}|\Big]\Big)^2 \tag{3}$$

$$\leq 5\sum_{i=1}^{m}\sum_{j=1}^{n}(|\Delta_{i,j}| + 16\lambda|\Delta_{i,j}|)^2 \tag{4}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(\sqrt{5}(1+16\lambda))^2(\Delta_{i,j})^2 \tag{5}$$

Where we use the fact that $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2 \leq (\alpha_1 + \alpha_2 + \alpha_3)^2(x_1^2 + x_2^2 + x_3^3)$ as all alphas are ≥ 0 to go from (3) to (4). Then we can identify the Lipschitz constant, $L = \sqrt{5}(1 + 16\lambda)$.

(c) $f_l(.)$ convexity derivation

$f_l(.)$ is convex. To show this, we use some basic properties already known:

1. The norm functions are convex
2. The composed of a convex function with a nondecreasing convex function is convex
3. The sum of convex functions is convex
4. $g : \mathbb{R} \to \mathbb{R} : t \to t^2$ is convex
5. if $f(x)$ is convex, then $c.f(x)$ is assuming c positive

Then it follows that from 1. and 2. $||X - Y^{(l)}||_F^2$ is convex. Now we show that every term of the regularizer, noted h, is convex using the definition of convexity:
Consider any X and $Y \in [0, 255]^{m\times n}$ and $h(Z) = (Z_{i+1,j} - Z_{i,j})^2$.Then we want to show that

$$h(\alpha X + (1-\alpha)Y) \leq \alpha h(X) + (1-\alpha)h(Y)$$
$$(\alpha X_{i+1,j} + (1-\alpha)Y_{i+1,j} - (\alpha X_{i,j} + (1-\alpha)Y_{i,j}))^2 \leq \alpha(X_{i+1,j} - X_{i,j})^2 + (1-\alpha)(Y_{i+1,j} - Y_{i,j})^2$$
$$(\alpha t_1 + (1-\alpha)t_2)^2 \leq \alpha t_1^2 + (1-\alpha)t_2^2$$

Where we posed $t_1 = (X_{i+1,j} - X_{i,j})$ and $t_2 = (Y_{i+1,j} - Y_{i,j})$. And since we find the classic definition of the convexity of t^2, we have that $h(Z)$ is convex. So now we can observe that $f_l(.)$ is the sum of weighted convex functions and by 3. and 5., It is also convex.

(d) Implementation of ProjGM

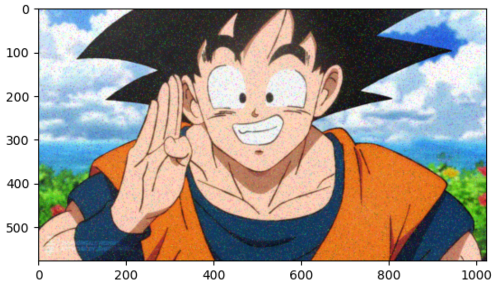
See file `code devoir 1.py`. We used numba to accelerate the execution of the code, so if you run the program it will take more time at the beginning but then it will be much faster.

(e) Result of PGM

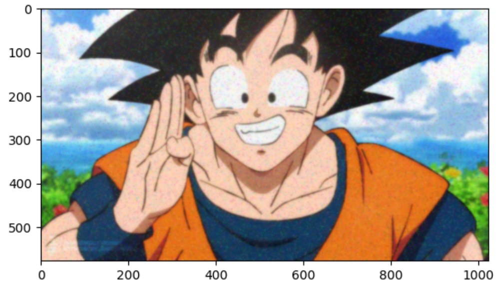
We applied our code with $\lambda \in \{0.1, 1, 5, 10\}$ and we got the following images (see next page).

(f) Relation between λ and the quality

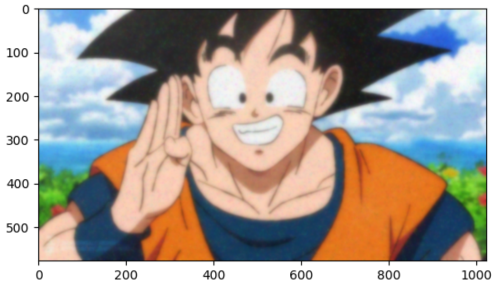
Based on these results, we can observe that when λ is small, then the regularizer term will have a small impact on the optimization problem. So the obtained image after solving the problem will be close to the original one, and exactly the same with $\lambda = 0$. When increasing λ, the regularizer term will take more importance and the solution will reduce the difference between the pixels. It will result in remooving the noise, but a blurred image.



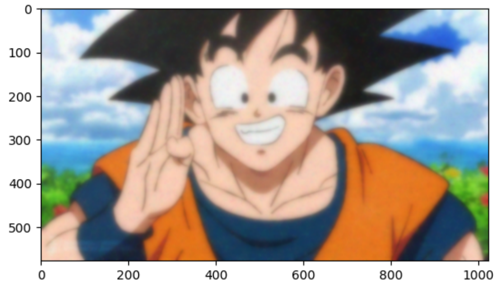
(a) $\lambda = 0.1$



(b) $\lambda = 1$



(c) $\lambda = 5$



(d) $\lambda = 10$

Proximal Gradient Method

(a) Solving SIR model

See file `code devoir 1.py` and data.txt.

(b) Implementation of ProxGM

See file `code devoir 1.py`.

(c) SINDy results

To apply SINDy, we first computed the gradient of $f(\alpha) = \frac{1}{2}||b^{(l)} - \Theta\alpha||_2^2$ and then L a Lipschitz constant of the gradient:

$$\nabla f(\alpha)_k = \sum_{i=1}^{m} \theta_{i,k}(-b_i + \sum_{j=1}^{p} \alpha_j \theta_{i,j})$$

$$
\begin{aligned}
||\nabla f(\alpha) - \nabla f(\beta)||^2 &= \sum_{k=1}^{p} \Big(\sum_{i=1}^{m} \theta_{i,k} \sum_{i=1}^{p} (\alpha_j - \beta_j)\theta_{i,j} \Big)^2 \\
&\leq \sum_{k=1}^{p} \Big(\sum_{i=1}^{m} |\theta_{i,k}| \sum_{i=1}^{p} |\alpha_j - \beta_j||\theta_{i,j}| \Big)^2 \\
&\leq p\bar{\theta}^2 \Big(\sum_{i=1}^{m} \sum_{j=1}^{p} |\alpha_j - \beta_j| \Big)^2 \\
&\leq p^2 \bar{\theta}^2 m^2 \sum_{j=1}^{p} (\alpha_j - \beta_j)^2
\end{aligned}
$$

Where $\bar{\theta}$ represents the maximum entry of the Θ matrix we can identify a Lipschitz constant, $L = p\bar{\theta}m$ Then we solve (5) from the homework statement with our implementation (b) and we get the following coefficients:

$$
\alpha^{(1)} = \begin{bmatrix} -8.54 \times 10^{-4} \\ 0.0 \\ 1.75 \times 10^{-5} \\ -1.94 \times 10^{-1} \\ 0.0 \end{bmatrix} \quad \alpha^{(2)} = \begin{bmatrix} 1.05 \times 10^{-3} \\ -4.14 \times 10^{-2} \\ 0.0 \\ 1.77 \times 10^{-1} \\ -9.87 \times 10^{-3} \end{bmatrix} \quad \alpha^{(3)} = \begin{bmatrix} 2.65 \times 10^{-4} \\ 4.96 \times 10^{-2} \\ -2.29 \times 10^{-5} \\ 0.0 \\ 0.0 \end{bmatrix}
$$

Removing the terms smaller than 10^{-3}, we get the following system of equations:

$$
\begin{aligned}
&\dot{x}_1(t) = -0.194 x_1(t) x_2(t) \\
&\dot{x}_2(t) = 0.001 x_1(t) - 0.041 x2(t) + 0.177 x_1(t) x_2(t) - 0.009 x_2(t) x_3(t) \\
&\dot{x}_3(t) = 0.049 x_2(t)
\end{aligned}
$$

We can see that SINDy is working well and giving good results.

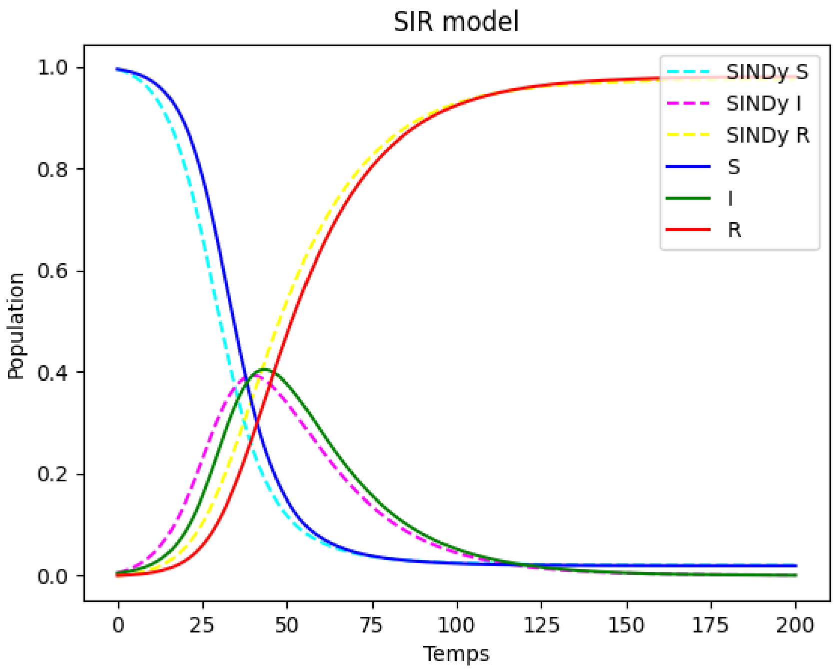


Figure 2: SINDy results comparison

(d) Impact of λ on SINDy

When running SYNDy with $\lambda = 10^{-4}$ we can retireve more precisely the coefficients of the SIR model:

$$\alpha^{(1)} = \begin{bmatrix} -5.11 \times 10^{-4} \\ 3.53 \times 10^{-3} \\ 0.0 \\ -2.04 \times 10^{-1} \\ -3.53 \times 10^{-3} \end{bmatrix} \quad \alpha^{(2)} = \begin{bmatrix} 4.82 \times 10^{-4} \\ -5.27 \times 10^{-2} \\ -2.77 \times 10^{-6} \\ 2.0 \times 10^{-1} \\ 4.13 \times 10^{-3} \end{bmatrix} \quad \alpha^{(3)} = \begin{bmatrix} 4.02 \times 10^{-5} \\ 4.88 \times 10^{-2} \\ -2.49 \times 10^{-6} \\ 4.3 \times 10^{-3} \\ 0.0 \end{bmatrix}$$

However when increasing λ to 10^{-2} or 10^{-1}, we get very bad results as shown in the following figure. This highlights the importance of chossing a nice value for λ

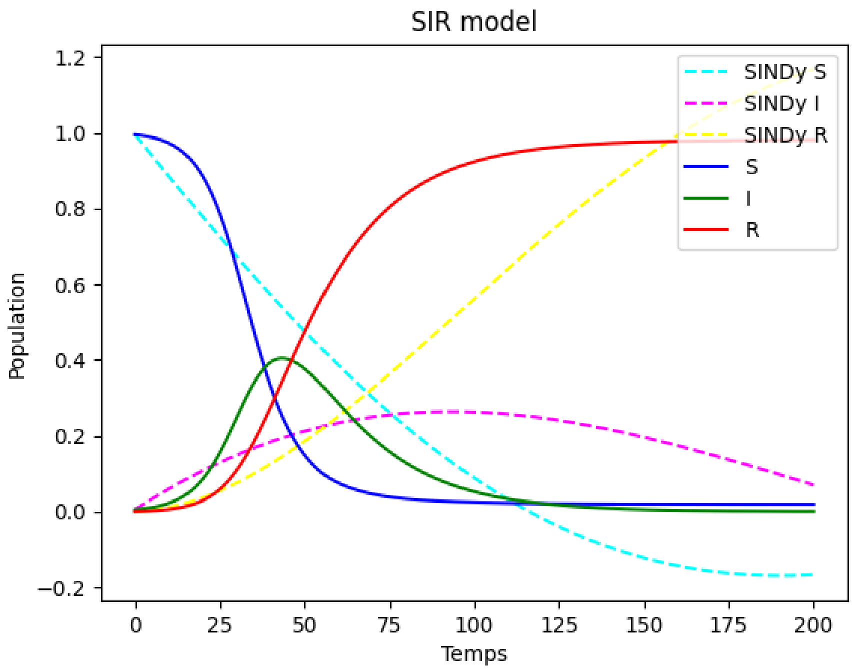


Figure 3: SINDy $\lambda = 0.1$