LINMA2171

Homework I

Damien Doat & Théo André

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Projected Gradient Method

(a) Computation of $\nabla f_l(.)$

$$\begin{aligned}
\nabla f_l(X)_{kl} &= \frac{\partial}{\partial_{kl}} \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (X_{ij} - Y_{ij})^2 + \frac{\partial}{\partial_{kl}} \lambda \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} (X_{i+1,j} - X_{i,j})^2 + (X_{i,j+1} - X_{i,j})^2 \\
&= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial}{\partial_{kl}} (X_{ij} - Y_{ij})^2 + \lambda \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} \frac{\partial}{\partial_{kl}} (X_{i+1,j} - X_{i,j})^2 + \frac{\partial}{\partial_{kl}} (X_{i,j+1} - X_{i,j})^2 \\
&= (X_{kl} - Y_{kl}) + \lambda [2(X_{k,l} - X_{k-1,l}) - 2(X_{k+1,l} - X_{k,l}) + 2(X_{k,l} - X_{k,l-1}) - 2(X_{k,l+1} - X_{k,l})] \\
&= (X_{k,l} - Y_{k,l}) + 8\lambda X_{kl} - 2\lambda X_{k-1,l} - 2\lambda X_{k+1,l} - 2\lambda X_{k,l-1} - 2\lambda X_{k,l+1}
\end{aligned}$$

This is a general expression for the kl component of the gradient, but we give a more precise expression hereafter to consider the "corner cases"

$$\nabla f_l(X)_{kl} = \begin{cases}
(X_{kl} - Y_{kl}) & \text{if } i = 1 \text{ and/or } j = 1 \text{ orif } k = m \text{ and} \\
(X_{kl} - Y_{kl}) + 4\lambda X_{kl} - 2\lambda X_{k+1,l} - 2\lambda X_{k,l+1} & \text{if i=2 and j=2} \\
(X_{kl} - Y_{kl}) + 6\lambda X_{kl} - 2\lambda X_{k+1,l} - 2\lambda X_{k,l-1} - 2\lambda X_{k,l+1} & \text{k=2 and l geq2} \\
(X_{kl} - Y_{kl}) + 6\lambda X_{kl} - 2\lambda X_{k,l+1} - 2\lambda X_{k-1,l} - 2\lambda X_{k+1,l} & \text{l=2 and k}\geq 2 \\
(X_{kl} - Y_{kl}) + 2\lambda X_{kl} - 2\lambda X_{k-1,l} & \text{k=m and } 2 \geq l \leq m \\
(X_{kl} - Y_{kl}) + 2\lambda X_{kl} - 2\lambda X_{k,l-1} & l = m \text{ and } 2 \geq k \leq n \\
(X_{kl} - Y_{kl}) + 8\lambda X_{kl} - 2\lambda X_{k-1,l} - 2\lambda X_{k+1,l} - 2\lambda X_{k,l-1} - 2\lambda X_{k,l+1} & \text{else}
\end{cases}$$

(b) $\nabla f_l(.)$ L-Lipschitz derivation

We recall that a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is L-Lipschitz if there exists a constant L such that $||F(x) - F(y)|| \leq L||x - y||\forall x, y \in \mathbb{R}^n$. To find L for $\nabla f_l(.)$, we will consider the general case of the derivation above, since it will lead to upper bound for the corner cases and we will assume a 0 value for out of bounds index. This leads to:

$$
\begin{aligned}
||F(x)-F(y)||^2 &\leq \sum_{i=1}^{m}\sum_{j=1} n(|X_{i,j}-Y_{i,j}| + 8\lambda|X_{i,j}-Y_{i,j}| + 2\lambda|X_{i,j+1}-Y_{i,j+1}| + 2\lambda|X_{i,j-1}-Y_{i,j-1}| + 2\lambda|X_{i+1,j}-Y_{i+1,} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}(|\Delta_{ij}| + 8|\lambda\Delta_{ij}| + 2\lambda|\Delta_{i-1,j}| + 2\lambda|\Delta_{i+1,j} + 2\lambda|\Delta_{i,j-1}| + 2\lambda|\Delta_{i,j+1})^2 \\
&\leq 5\sum_{i=1}^{m}\sum_{j=1}^{n}(|\Delta_{i,j} + 16\lambda|\Delta_{i,j})^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}(\sqrt{5}(1+16\lambda))^2(\Delta_{i,j})^2
\end{aligned}
$$

Where we can identify the L constant,

(c) $f_l(.)$ convexity derivation

(e) Result of PGM

(f) Relation between λ and the quality

Proximal Gradient Method