MATRIX COMPUTATIONS: HOMEWORK 2

Andre Theo, Dejean Maxime, Doat Damien, Nouidei Safiya

October 2024

1 Exercise A: Krylov subspaces

(A1)

As assumed in the assignement, $K_{r+1}(A,b) \subseteq K_r(A,b)$. If we can prove that we also have $K_{r+1}(A,b) \supseteq K_r(A,b)$, it means that we proved that $K_{r+1}(A,b) = K_r(A,b)$. Since, by definition of the krylov subspaces as $span\{b, Ab, ..., A^r b\}$, all elements of $K_r(A,b)$ are in $K_{r+1}(A,b)$, we know that, indeed, $K_{r+1}(A,b) \supseteq K_r(A,b)$. And thus, we have proved the equality between the two spaces. We then need to show that, for $s \geq r$, $K_s(A,b) = K_r(A,b)$. It can be done with a proof by induction. Indeed, since the eqaulity holds for all r, it means that $K_{r+2}(A,b) = K_{r+1}(A,b) = K_r(A,b)$ etc. This means that for all i such that $s - r \leq i$, $K_{r+i}(A,b) = K_{r+(i-1)}(A,b) = ... = K_r(A,b)$. This shows that $K_s(A,b) = K_r(A,b)$.

(A2)

Let's prove that $dim(K_r(A,b)) = r$ by induction. Since $dim(K_n(A,b)) = s$ and $r \leq s$, it means that, at each iteration i so that $i \leq r$, the dimension of $dim(K_i(A,b)) = i$. Indeed, no combination will already be in the span since r is smaller than s. We thus have:

- $i = 1$: $K_i(A,b) = \{b\}$ and $dim(K_i(A,b)) = 1$
- $i = 2$: $K_i(A,b) = \{b, Ab\}$ and $dim(K_i(A,b)) = 2$
- ...
- $i = r$: $K_i(A,b) = K_r(A,b)$ and $dim(K_i(A,b)) = r$

2 Exercise B: Arnoldi's iteration

(B1)

We want to show that R_s has non-zero elements on its diagonal. Fisrt, since $K_s(A, b) = [b, Ab, \ldots, A^{s-1}b]$, it consists of s linearly independent columns. Thus, $K_s(A, b)$ has full column rank, so $rank(K_s(A, b)) = s$. In any QR decomposition $M = QR$ where $M \in R^{m \times s}$ and M has full column rank, R is invertible. Therefore, R is nonsingular, meaning all its diagonal elements are non-zero. Finally, in our case, since $K_s(A, b)$ has full column rank, R_s is invertible and hence has non-zero elements on its diagonal. Thus, we conclude that R_s has non-zero elements on its diagonal otherwise $dim(K_s(A, b)) \neq s$.

(B2)

The goal is to show that for each $1 \leq r \leq s - 1$,

$$AQ_s[1 : r]R_s[1 : r, 1 : r] = Q_s[1 : r']R_s[1 : r', 2 : r'],$$

where $r' = r + 1$. Let's do it step by step :

1. **Krylov Subspace Properties:**
   The Krylov subspace $K_r(A, b)$ is defined as:

   $$K_r(A, b) = span\{b, Ab, A^2b, \ldots, A^{r-1}b\}.$$

   By construction, applying A to $K_r(A, b)$ generates $K_{r+1}(A, b)$:

   $$K_{r+1}(A, b) = span\{b, Ab, A^2b, \ldots, A^r b\}.$$

   Therefore, $K_{r+1}(A, b)$ extends $K_r(A, b)$ by including the new vector $A^r b$, making it a shifted version of $K_r(A, b)$ with an additional dimension.

2. **QR Decomposition of $K_r(A, b)$:**
   Given the QR decomposition of $K_s(A, b)$, we have:

   $$K_s(A, b) = Q_s R_s,$$

   where $Q_s \in R^{n \times s}$ is an orthonormal basis for $K_s(A, b)$ and $R_s \in R^{s \times s}$ is upper-triangular. The submatrices $Q_s[1 : r]$ and $R_s[1 : r, 1 : r]$ provide a basis and the associated coefficients for the subspace $K_r(A, b)$.

3. **Applying A to $Q_s[1 : r]R_s[1 : r, 1 : r]$:**
   When we apply A to $Q_s[1 : r]R_s[1 : r, 1 : r]$, we obtain a matrix that spans the extended subspace $K_{r+1}(A, b)$. This new subspace can be represented by:

   $$AQ_s[1 : r]R_s[1 : r, 1 : r] = Q_s[1 : r']R_s[1 : r', 2 : r'],$$

   where $r' = r + 1$.

(B3)

(B4)

(B5)

(B6)

(B7)

3 Exercise C: GMRES for linear system solution approximation

(C1)

(C2)

(C3)

(C4)

(C5)

4 Exercise D: Arnoldi's method for eigenvalue approximation

(D1)

5 Exercise E: Implementation