

- xinvert: A Python package for inversion problems in
- geophysical fluid dynamics
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Statement of need

Many problems in meteorology and oceanography can be cast into a balanced model in the form of a partial differential equation (PDE). The most well-known one is the relation between the streamfunction ψ and the vertical vorticity ζ (also known as Poisson equation):

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} = \zeta$$

Once ζ is given as a known, one need to get the unknown ψ with proper boundary conditions, which is essentially an inversion problem (Figure 1). Many geophysical fluid dynamical (GFD) problems can be described by such a balanced model, which is generally of second (or fourth) order in spatial derivatives and do not depend explicitly on time. Therefore, it is also known as a steady-state model. Early scientists tried to find analytical solutions by simplified the parameters of these models (e.g., assuming constant coefficients). Nowadays, with the new developments in numerical algorithms and parallel-computing programming, one may need a modern solver, written in a popular programming language like Python, to invert all these models in a numerical fashion. More specifically, the following needs should be satisfied:

- A unified numerical solver: It can solve all the classical balanced GFD models, even in a domain with irregular boundaries like the ocean. New models can also be adapted straightforwardly to fit the solver;
- Thinking and coding in equations: Users focus naturally on the key inputs and outputs
 of the GFD models, just like thinking of the knowns and unknowns of the PDEs;
- Flexible parameter specification: Coefficients of the models can be either constant, 1D vector, or ND array. This allows an easy reproduce of early simplified results and also an extension to more general/realistic results;
- Fast and efficient: The algorithm should be fast and efficient. Also, the codes can be compiled first and then executed as fast as C or FORTRAN, instead of executed in the slow pure Python interpreter. In addition, it can leverage the multi-core and out-of-core computational capabilities of modern computers;
- xinvert is then designed to satisfy all the above needs, based on the ecosystem of Python.



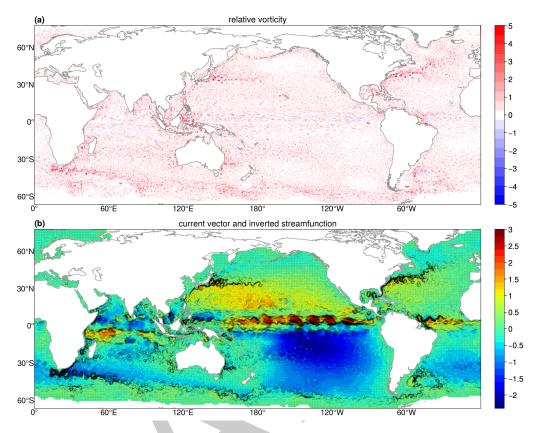


Figure 1: (a) Vertical relative vorticity ζ and (b) the inverted streamfunction ψ (shading) with current vector superimposed. Note the irregular boundaries over the global ocean.

Mathematics

This package, xinvert, is designed to invert or solve the following PDE in an abstract form:

$$L\left(\psi\right) = F\tag{1}$$

- where L is a second-order partial differential operator, ψ the unknown to be inverted for, and
- F a prescribed forcing function. There could be also some coefficients or parameters in the
- definition of L, which should be specified before inverting ψ .
- For the 2D case, the general form of Eq. (1) is:

$$L\left(\psi\right) \equiv A_{1} \frac{\partial^{2} \psi}{\partial y^{2}} + A_{2} \frac{\partial^{2} \psi}{\partial y \partial x} + A_{3} \frac{\partial^{2} \psi}{\partial x^{2}} + A_{4} \frac{\partial \psi}{\partial y} + A_{5} \frac{\partial \psi}{\partial x} + A_{6} \psi = F \tag{2}$$

- where coefficients A_1-A_6 are all known variables. When the condition $4A_1A_3-A_2^2>0$ is met everywhere in the domain, the above equation is an elliptic-type equation. In this
- case, one can invert ψ using the successive over relaxation (SOR) iteration method. When
- $4A_1A_3-A_2^2=0$ or $4A_1A_3-A_2^2<0$, it is a parabolic or hyperbolic equation. In either case, SOR would *fail* to converge to the solution.
- Sometimes the general form of Eq. (2) can be transformed into the standard form (i.e.,
- standarization):



$$L\left(\psi\right)\equiv\frac{\partial}{\partial y}\left(A_{1}\frac{\partial\psi}{\partial y}+A_{2}\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial x}\left(A_{3}\frac{\partial\psi}{\partial y}+A_{4}\frac{\partial\psi}{\partial x}\right)+A_{5}\psi=F\tag{3}$$

- In this case, $A_1A_4-A_2A_3>0$ should be met to insure its ellipticity. The elliptic condition
- has its own physical meaning in the problems of interest. That is, the system is in a steady (or
- balanced) state that is stable to any small perturbation.
- Many problems in meteorology and oceanography can be cast into the forms of either Eq. (2)
- or Eq. (3). However, some of them are formulated in 3D case (like the QG-omega equation):

$$L(\psi) \equiv \frac{\partial}{\partial z} \left(A_1 \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial y} \left(A_2 \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(A_3 \frac{\partial \psi}{\partial x} \right) = F \tag{4}$$

or in fourth-order case (the Munk model):

$$L(\psi) \equiv A_1 \frac{\partial^4 \psi}{\partial y^4} + A_2 \frac{\partial^4 \psi}{\partial y^2 \partial x^2} + A_3 \frac{\partial^4 \psi}{\partial x^4}$$

$$+ A_4 \frac{\partial^2 \psi}{\partial y^2} + A_5 \frac{\partial^2 \psi}{\partial y \partial x} + A_6 \frac{\partial^2 \psi}{\partial x^2} + A_7 \frac{\partial \psi}{\partial y} + A_8 \frac{\partial \psi}{\partial x} + A_9 \psi = F$$
(5)

So we implements four basic solvers to take into account the above four Eqs. (2)-(5) cases.

Most of the problems fit one of these four types of solver. However, it is **NOT** clear which form, the genral form Eq. (2) or the standard form Eq. (3), is preferred for the inversion if a problem can be cast into either one.

55 Summary

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xinvert is an open-source and uesr-friendly Python package that enables GFD scientists or interested amateurs to solve all possible GFD problems in a numerical fashion. With the ecosystem of open-source Python packages, in particular xarray (Hoyer & Hamman, 2017), dask (Rocklin, 2015), and numba (Lam et al., 2015), it is able to satisfy the above requirements:

- All the classical balanced GFD models can be inverted by this unified numerical solver;
- User APIs (Table 1) are very close to the equations: unknowns are on the left-hand side
 of the equal sign =, whereas the known forcing functions are on its right-hand side (other
 known coefficients are also on the left-hand side but are passed in through mParams);
- Passing a single xarray. DataArray is usually enough for the inversion. Coordinates information is already encapsulated and thus reducing the length of the parameter list. In addition, parameters in mParams can be either a constant, or varying with a specific dimension (like Coriolis parameter f), or fully varying with space and time, due to the use of xarray's (Hoyer & Hamman, 2017) broadcasting capability;
- This package leverages numba (Lam et al., 2015) and dask (Rocklin, 2015) to support Just-In-Time (JIT) compilation, multi-core, and out-of-core computations, and therefore greatly increases the speed and efficiency of the inversion.

Here we summarize the inversion problems in meteorology and oceanography into Table 1. The table can be extended further if one finds more problems that fit the abstract form of Eq. (1).



Table 1: Classical inversion problems in GFD. The model names, equations, typical references and function calls are listed

$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \zeta_k \qquad \qquad \text{dims=['Y','X'],} \\ \text{mParams=None)}$	Problem names and equations	Function call
$ \begin{array}{lll} \text{MOC streamfunction} \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta_i \end{array} & \text{psi} &= \text{invert_Poisson(vori,} \\ \text{dims=['7','Y'],} \\ \text{mParams=None)} \\ \text{Walker streamfunction} \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \end{array} & \text{psi} &= \text{invert_Poisson(vorj,} \\ \text{dims=['7','X'],} \\ \text{mParams=None)} \\ \text{Balanced mass field (Yuan et al., 2008)} \\ \nabla^2\Phi &= \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F \end{array} & \text{psi} &= \text{invert_Poisson(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams=None)} \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(f\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial x} \left(f\frac{\partial\psi}{\partial x}\right) = \nabla^2\Phi \end{array} & \text{psi} &= \text{invert_Poisson(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams=Anone} \\ \text{Beliassen model (Eliassen 1952)} \\ \frac{\partial}{\partial p} \left(A\frac{\partial\psi}{\partial p} + B\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y} \left(B\frac{\partial\psi}{\partial p} + C\frac{\partial\psi}{\partial y}\right) = F \end{array} & \text{psi} &= \text{invert_geostrophic(LapPhi,} \\ \text{dims=['Y','X'],} \\ \text{mParams=Angm,} &= \text{Thm}\} \\ \text{Degine for vortex (Hoskins et al., 1985)} \\ \frac{\partial}{\partial p} \left(\frac{f^2}{\partial \psi}\right) + \frac{\partial}{\partial y} \left(\frac{f^2}{\partial \psi}\right) + \frac{\partial}{\partial y^2} = q \end{array} & \text{angM} &= \text{invert_RefState(PV,} \\ \text{dims=['Y','Y'],} \\ \text{mParams=Ango,} &= \text{Gamma}\} \\ \text{psi} &= \text{invert_Pv2D(PV,} \\ \text{dims=['Y','X'],} \\ \text{mParams=ff,} &= \text{N2}) \\ \text{Gill-Matsuno model (Gill, 1980; Matsuno,} \\ \text{1966} \\ \text{AD} \nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\hat{\tau}}{\rho_0D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla \cdot (S\nabla\omega) = F \end{aligned} & \text{psi} &= \text{invert_StommelMunk} \\ \text{Glowell (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(c_3\frac{\partial\psi}{\partial p}\right) + \nabla \cdot (S\nabla\omega) = F \end{aligned} & \text{psi} &= \text{invert_Domega(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','X'],} \\ \text{mParams={Ango,} Gamma} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GommelMunk} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ mPa$	Horizontal streamfunction	<pre>psi = invert_Poisson(vork,</pre>
$ \begin{array}{lll} \text{MOC streamfunction} \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta_i \end{array} & \text{psi} &= \text{invert_Poisson(vori,} \\ \text{dims=['7','Y'],} \\ \text{mParams=None)} \\ \text{Walker streamfunction} \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \end{array} & \text{psi} &= \text{invert_Poisson(vorj,} \\ \text{dims=['7','X'],} \\ \text{mParams=None)} \\ \text{Balanced mass field (Yuan et al., 2008)} \\ \nabla^2\Phi &= \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F \end{array} & \text{psi} &= \text{invert_Poisson(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams=None)} \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(f\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial x} \left(f\frac{\partial\psi}{\partial x}\right) = \nabla^2\Phi \end{array} & \text{psi} &= \text{invert_Poisson(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams=Anone} \\ \text{Beliassen model (Eliassen 1952)} \\ \frac{\partial}{\partial p} \left(A\frac{\partial\psi}{\partial p} + B\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y} \left(B\frac{\partial\psi}{\partial p} + C\frac{\partial\psi}{\partial y}\right) = F \end{array} & \text{psi} &= \text{invert_geostrophic(LapPhi,} \\ \text{dims=['Y','X'],} \\ \text{mParams=Angm,} &= \text{Thm}\} \\ \text{Degine for vortex (Hoskins et al., 1985)} \\ \frac{\partial}{\partial p} \left(\frac{f^2}{\partial \psi}\right) + \frac{\partial}{\partial y} \left(\frac{f^2}{\partial \psi}\right) + \frac{\partial}{\partial y^2} = q \end{array} & \text{angM} &= \text{invert_RefState(PV,} \\ \text{dims=['Y','Y'],} \\ \text{mParams=Ango,} &= \text{Gamma}\} \\ \text{psi} &= \text{invert_Pv2D(PV,} \\ \text{dims=['Y','X'],} \\ \text{mParams=ff,} &= \text{N2}) \\ \text{Gill-Matsuno model (Gill, 1980; Matsuno,} \\ \text{1966} \\ \text{AD} \nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\hat{\tau}}{\rho_0D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla \cdot (S\nabla\omega) = F \end{aligned} & \text{psi} &= \text{invert_StommelMunk} \\ \text{Glowell (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(c_3\frac{\partial\psi}{\partial p}\right) + \nabla \cdot (S\nabla\omega) = F \end{aligned} & \text{psi} &= \text{invert_Domega(F,} \\ \text{dims=['Y','X'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','X'],} \\ \text{mParams={Ango,} Gamma} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GillMatsuno(Q,} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ \text{psi} &= \text{invert_GommelMunk} \\ \text{dims=['Y','Y'],} \\ \text{mParams={Ango,} Gamma} \\ mPa$	$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = \zeta_L$	
$\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta_i \qquad \qquad \text{dims} = \begin{bmatrix} \cdot Z', \cdot Y' \end{bmatrix}, \\ \text{mParams} = \text{None} \\ \nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \qquad \qquad \text{psi} = \text{invert} _\text{Poisson}(\text{vorj}, \text{dims} = \begin{bmatrix} \cdot Z', \cdot Y' \end{bmatrix}, \\ \text{mParams} = \text{None} \\ \nabla^2\Phi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = F \qquad \qquad \qquad \text{psi} = \text{invert} _\text{Poisson}(\text{F}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot X' \end{bmatrix}, \\ \text{mParams} = \text{None} \\ \nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F \qquad \qquad \qquad \text{Phi} = \text{invert} _\text{Poisson}(\text{F}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot X' \end{bmatrix}, \\ \text{mParams} = \text{None} \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(f \frac{\partial\psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(f \frac{\partial\psi}{\partial x} \right) = \nabla^2\Phi \qquad \qquad \qquad \text{psi} = \text{invert} _\text{geostrophic}(\text{LapPhi}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot X' \end{bmatrix}, \\ \text{mParams} = \{f\} \\ \end{pmatrix}$ Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p} \left(A \frac{\partial\psi}{\partial p} + B \frac{\partial\psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial\psi}{\partial p} + C \frac{\partial\psi}{\partial y} \right) = F \qquad \qquad \text{psi} = \text{invert} _\text{Resson}(\text{F}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot X' \end{bmatrix}, \\ \text{mParams} = \{f, \} \\ \end{pmatrix}$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial p} \left(\frac{f^2\partial\psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \qquad \qquad \qquad \text{angM} = \text{invert} _\text{ResState}(\text{PV}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot Y' \end{bmatrix}, \\ \text{mParams} = \{ang\Theta, \text{Gamma} \} \\ \end{pmatrix}$ PV inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2\partial\psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \qquad \qquad \qquad \text{angM} = \text{invert} _\text{PV2D}(\text{PV}, \text{dims} = \begin{bmatrix} \cdot Y', \cdot Y' \end{bmatrix}, \\ \text{mParams} = \{f, \text{N2}\} \\ \end{pmatrix}$ SGIII-Matsuno model (Gill, 1980; Matsuno, 1966) $A \frac{\partial^2\psi}{\partial y^2} + B \frac{\partial^2\psi}{\partial y^2} + C \frac{\partial\psi}{\partial y} + D \frac{\partial\psi}{\partial x} + E\phi = Q \\ \text{Stommel-Munk model (Munk, 1950;} \\ \text{Stommel-Munk model (Munk, 1950;} \\ \text{Stommel-Munk model (Munk, 1950;} \\ \text{QG-Omega equation (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial\omega}{\partial p} \right) + \nabla \cdot (S\nabla\omega) = F \qquad \qquad \text{psi} = \text{invert} _\text{GillMatsuno} \\ \text{unimal mass} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = \text{invert} _\text{Goldmats} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = \text{invert} _\text{Goldmats} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = \text{invert} _\text{Goldmats} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = \text{invert} _\text{Goldmats} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = \text{invert} _\text{Goldmats} = \{f, \text{psi} = \text{lone} \}$ $\text{vs.} = invert$	$\partial y^2 \partial x^2 \mathcal{S}^k$	mParams=None)
$\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta_i \qquad \qquad \text{dims=}[\cdot ?', \cdot Y'], \\ \text{mParams=None})$ $\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \qquad \qquad \text{psi} = \text{invert_Poisson(vorj,} \\ \text{dims=}[\cdot ?', \cdot X'], \\ \text{mParams=None})$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} = F \qquad \qquad \text{psi} = \text{invert_Poisson(F,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=None})$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} = F \qquad \qquad \text{psi} = \text{invert_Poisson(F,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \text{None})$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} = F \qquad \qquad \text{psi} = \text{invert_poisson(F,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{f\}\}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y} = \nabla^2\phi \qquad \qquad \text{psi} = \text{invert_geostrophic(LapPhi,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{f\}\}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} = \nabla^2\phi \qquad \qquad \text{psi} = \text{invert_geostrophic(LapPhi,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{f\}\}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} = 0 \qquad \qquad \text{psi} = \text{invert_geostrophic(LapPhi,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{ang\emptyset, \text{Gamma}\} \}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} = 0 \qquad \qquad \text{psi} = \text{invert_geostrophic(LapPhi,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{ang\emptyset, \text{Gamma}\} \}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} = 0 \qquad \qquad \text{psi} = \text{invert_geostrophic(LapPhi,} \\ \text{dims=}[\cdot Y', \cdot X'], \\ \text{mParams=} = \{ang\emptyset, \text{Gamma}\} \}$ $\nabla^2\phi = \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} $	MOC streamfunction	<pre>psi = invert_Poisson(vori,</pre>
Walker streamfunction $\nabla^2\psi=\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial x^2}=\zeta_j$ $psi=\text{invert}_P\text{oisson}(\text{vor}j,\text{dims}=['2','\times'1],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','\times'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','X'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','X'],\text{mParams}=\text{None})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','X'],\text{mParams}=\text{Angm},\text{Thm}})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','X'],\text{mParams}=\text{Angm},\text{Inm}})$ $Phi=\text{invert}_P\text{oisson}(F,\text{dims}=['Y','X'],\text{mParams}=\text{Angm}})$	$\nabla^2 \psi = \partial^2 \psi + \partial^2 \psi - \zeta$	
$\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \qquad \qquad \text{dims} = [\cdot Z\cdot , \cdot X\cdot], \\ \text{mParams} = \text{None}) \\ \text{Balanced mass field (Yuan et al., 2008)} \\ \nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F \qquad \qquad \qquad \text{Phi = invert_Poisson(F, dims} = [\cdot Y\cdot , \cdot X\cdot], \\ \text{mParams} = \text{None}) \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(J \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(J \frac{\partial \psi}{\partial x} \right) = \nabla^2\Phi \\ \text{Eliassen model (Eliassen, 1952)} \\ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F \\ \text{PV inversion for vortex (Hoskins et al., 1985)} \\ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 \\ \text{PV inversion for QQ flow} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \psi}{r^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \\ \text{Gill-Matsuno model (Gill, 1980; Matsuno, 1966)} \\ \text{Gill-Matsuno model (Munk, 1950; Stommel, 1948)} \\ A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \hat{\tau}}{\rho_0 D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S\nabla\omega) = F \\ \text{Signature} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (J \nabla \psi - J \sqrt{2} \hat{\psi}) = F \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (J \nabla \psi - J \sqrt{2} \hat{\psi}) = F \\ \text{dims} = [\cdot Z', \cdot Y', \cdot Y'], \\ \text{mParams} = \{\Lambda, R, D, \text{ beta, rho}\}) \\ \text{mParams} = \{A, R, D, \text{ beta, rho}\}) \\ m$	$\nabla \Psi - \frac{\partial}{\partial z^2} + \frac{\partial}{\partial y^2} - \zeta_i$	<pre>mParams=None)</pre>
$\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \zeta_j \qquad \qquad \text{dims} = [\cdot Z\cdot , \cdot X\cdot], \\ \text{mParams} = \text{None}) \\ \text{Balanced mass field (Yuan et al., 2008)} \\ \nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F \qquad \qquad \qquad \text{Phi = invert_Poisson(F, dims} = [\cdot Y\cdot , \cdot X\cdot], \\ \text{mParams} = \text{None}) \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(J \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(J \frac{\partial \psi}{\partial x} \right) = \nabla^2\Phi \\ \text{Eliassen model (Eliassen, 1952)} \\ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F \\ \text{PV inversion for vortex (Hoskins et al., 1985)} \\ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 \\ \text{PV inversion for QQ flow} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \psi}{r^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \\ \text{Gill-Matsuno model (Gill, 1980; Matsuno, 1966)} \\ \text{Gill-Matsuno model (Munk, 1950; Stommel, 1948)} \\ A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \hat{\tau}}{\rho_0 D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S\nabla\omega) = F \\ \text{Signature} \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (J \nabla \psi - J \sqrt{2} \hat{\psi}) = F \\ \frac{\partial}{\partial p} \left(J \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (J \nabla \psi - J \sqrt{2} \hat{\psi}) = F \\ \text{dims} = [\cdot Z', \cdot Y', \cdot Y'], \\ \text{mParams} = \{\Lambda, R, D, \text{ beta, rho}\}) \\ \text{mParams} = \{A, R, D, \text{ beta, rho}\}) \\ m$		
Balanced mass field (Yuan et al., 2008) $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F$ Phi = invert_Poisson(F, dims=['Y','X'], mParams=None) Geostrophic streamfunction $\frac{\partial}{\partial y}\left(f\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(f\frac{\partial\psi}{\partial x}\right) = \nabla^2\Phi$ Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p}\left(A\frac{\partial\psi}{\partial p} + B\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial\psi}{\partial p} + C\frac{\partial\psi}{\partial y}\right) = F$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta}\left(\frac{2\Lambda_0}{r^2}\frac{\partial\Lambda}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\frac{\Gamma g}{Qr}\frac{\partial\Lambda}{\partial r}\right) = 0$ PV inversion for QG flow $\frac{\partial}{\partial p}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial p}\right) + \frac{\partial^2\psi}{\partial y^2} = q$ $\frac{\partial}{\partial p}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial p}\right) + \partial$	Walker streamfunction	
Balanced mass field (Yuan et al., 2008) $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F$ Phi = invert_Poisson(F, dims=['Y','X'], mParams=None) Geostrophic streamfunction $\frac{\partial}{\partial y}\left(f\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(f\frac{\partial\psi}{\partial x}\right) = \nabla^2\Phi$ Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p}\left(A\frac{\partial\psi}{\partial p} + B\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial\psi}{\partial p} + C\frac{\partial\psi}{\partial y}\right) = F$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta}\left(\frac{2\Lambda_0}{r^2}\frac{\partial\Lambda}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\frac{\Gamma g}{Qr}\frac{\partial\Lambda}{\partial r}\right) = 0$ PV inversion for QG flow $\frac{\partial}{\partial p}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial p}\right) + \frac{\partial^2\psi}{\partial y^2} = q$ $\frac{\partial}{\partial p}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial p}\right) + \partial$	$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = \zeta_i$	
$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F$ $\begin{array}{ll} \text{dims=}['Y', 'X'], \\ \text{mParams=None}) \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(f \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(f \frac{\partial \psi}{\partial x} \right) = \nabla^2\Phi \\ \text{Eliassen model} \left(\text{Eliassen, 1952} \right) \\ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F \\ \text{PV inversion for vortex} \left(\text{Hoskins et al., 1985} \right) \\ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 \\ \text{PV inversion for QG flow} \\ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \\ \text{Gill-Matsuno model} \left(\text{Gill, 1980; Matsuno, 1966} \right) \\ \text{Stommel, 1948} \\ A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} + C\frac{\partial\psi}{\partial x} + E\phi = Q \\ \text{Stommel-Munk model} \left(\text{Munk, 1950; Stommel model} \left(\text{Hoskins et al., 1978} \right) \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S\nabla\omega) = F \\ \text{QG-Omega equation} \left(\text{Hoskins et al., 1978} \right) \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert_3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} $	$\partial z^2 = \partial x^2 = i f$	mParams=None)
$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} = F$ $\begin{array}{ll} \text{dims=}['Y', 'X'], \\ \text{mParams=None}) \\ \text{Geostrophic streamfunction} \\ \frac{\partial}{\partial y} \left(f \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(f \frac{\partial \psi}{\partial x} \right) = \nabla^2\Phi \\ \text{Eliassen model} \left(\text{Eliassen, 1952} \right) \\ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F \\ \text{PV inversion for vortex} \left(\text{Hoskins et al., 1985} \right) \\ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 \\ \text{PV inversion for QG flow} \\ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2\psi}{\partial y^2} = q \\ \text{Gill-Matsuno model} \left(\text{Gill, 1980; Matsuno, 1966} \right) \\ \text{Stommel, 1948} \\ A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} + C\frac{\partial\psi}{\partial x} + E\phi = Q \\ \text{Stommel-Munk model} \left(\text{Munk, 1950; Stommel model} \left(\text{Hoskins et al., 1978} \right) \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S\nabla\omega) = F \\ \text{QG-Omega equation} \left(\text{Hoskins et al., 1978} \right) \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} \right) \\ \text{psi = invert_3DFlow(F, dims=['Z','Y', X'], mParams=} \{f, N2, \text{ epsilon}\} $	Balanced mass field (Yuan et al., 2008)	<pre>Phi = invert_Poisson(F,</pre>
Geostrophic streamfunction $\frac{\partial}{\partial y}\left(f\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(f\frac{\partial \psi}{\partial x}\right) = \nabla^2\Phi$ $\text{psi} = \text{invert_geostrophic}(\text{LapPhi}, \text{dims=['Y', 'X']}, \text{mParams={f}})$ $\frac{\partial}{\partial p}\left(A\frac{\partial \psi}{\partial p} + B\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial \psi}{\partial p} + C\frac{\partial \psi}{\partial y}\right) = F$ $\text{psi} = \text{invert_geostrophic}(\text{LapPhi}, \text{dims=['Y', 'X']}, \text{mParams={f}})$ $\text{mParams={f}} = \text{f}$ $\frac{\partial}{\partial p}\left(A\frac{\partial \psi}{\partial p} + B\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial \psi}{\partial p} + C\frac{\partial \psi}{\partial y}\right) = F$ $\text{psi} = \text{invert_geostrophic}(\text{LapPhi}, \text{dims=['Y', 'X']}, \text{mParams={f}})$ $\text{psi} = \text{invert_geostrophic}(\text{LapPhi}, \text{dims=['Y', 'X']}, \text{mParams={f, logorith}})$ $\text{dims=['Y', 'X']}, \text{mParams={f, logorith}}$ $\text{dims=['Y', 'X']}, \text{mParams={ango, Gamma}})$ $\text{psi} = \text{invert_pv2D}(\text{PV}, \text{dims=['Y', 'X']}, \text{mParams={f, logorith}})$ $\text{dims=['Y', 'X']}, \text{mParams={f, logorith}}$ $\text{dims=['Y', 'Y', 'X']}, \text{mParams={f, logorith}}$ $\text{dims=['Y', 'Y', 'X']}, \text{mParams={f, logorith}}$ $\text{dims=['Y', 'Y', 'X']}, \text{mParams={f, logorith}}$		
$\begin{split} \frac{\partial}{\partial y} \left(f \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(f \frac{\partial \psi}{\partial x} \right) &= \nabla^2 \Phi \\ & \text{dims=}['Y', 'X'], \\ & \text{mParams} = \{f\} \end{split} \end{split}$ Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) &= F \\ & \text{psi} &= \text{invert}_\text{Eliassen}(F, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{\text{Angm}, \text{ Thm}\} \} \end{split}$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) &= 0 \\ & \text{angM} &= \text{invert}_\text{RefState}(PV, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} \end{split}$ PV inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} &= q \\ & \text{psi} &= \text{invert}_\text{PV2D}(PV, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{f, \text{ N2}\} \} \end{split}$ Significantly and the property of the prope	$\nabla^2 \Phi = \frac{\partial}{\partial y^2} + \frac{\partial}{\partial x^2} = F$	· ·
$\begin{split} \frac{\partial}{\partial y} \left(f \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left(f \frac{\partial \psi}{\partial x} \right) &= \nabla^2 \Phi \\ & \text{dims=}['Y', 'X'], \\ & \text{mParams} = \{f\} \end{split} \end{split}$ Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) &= F \\ & \text{psi} &= \text{invert}_\text{Eliassen}(F, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{\text{Angm}, \text{ Thm}\} \} \end{split}$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) &= 0 \\ & \text{angM} &= \text{invert}_\text{RefState}(PV, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} \end{split}$ PV inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} &= q \\ & \text{psi} &= \text{invert}_\text{PV2D}(PV, \\ & \text{dims=}['Z', 'Y'], \\ & \text{mParams} = \{f, \text{ N2}\} \} \end{split}$ Significantly and the property of the prope		
Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0$ pv inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q$ angM = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_Pv2D(PV, dims=['Z','Y'], mParams={f, N2}) phi = invert_GilMatsuno(Q, dims=['Y','X'], mParams={f, epsilon, Phi}) psi = invert_StommelMunk(curl, dims=['Y','X'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','X'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho})		
Eliassen model (Eliassen, 1952) $\frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F$ PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0$ pv inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q$ angM = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) psi = invert_Pv2D(PV, dims=['Z','Y'], mParams={f, N2}) phi = invert_GilMatsuno(Q, dims=['Y','X'], mParams={f, epsilon, Phi}) psi = invert_StommelMunk(curl, dims=['Y','X'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','X'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho}) psi = invert_StommelMunk(curl, dims=['Y','Y'], mParams={A, R, D, beta, rho})	$\frac{\partial}{\partial x}\left(f\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial x}\left(f\frac{\partial \psi}{\partial x}\right) = \nabla^2\Phi$	
$ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F $ $ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{Angm, Thm}\} \} $ $ \text{angM} = \text{invert} \text{_RefState}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} $ $ \text{angM} = \text{invert} \text{_RefState}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} $ $ \text{PV inversion for QG flow} $ $ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{psi} = \text{invert} \text{_PV2D}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{f, N2\} \} $ $ \text{mParams} = \{f, N2\} \} $ $ \text{mParams} = \{f, N2\} \} $ $ \text{phi} = \text{invert} \text{_GillMatsuno}(\text{Q}, \\ \text{dims} = ['Y', 'X'], \\ \text{mParams} = \{f, \text{epsilon}, \text{Phi}\} \} $ $ \text{psi} = \text{invert} \text{_StommelMunk}(\text{curl}, \\ \text{dims} = ['Y', 'X'], \\ \text{mParams} = \{A, R, D, \text{beta, rho}\} \} $ $ \text{psi} = \text{invert} \text{_Omega}(\text{F}, \\ \text{dims} = ['Y', 'Y'], \\ \text{mParams} = \{f, S\} \} $ $ m$	$\partial y \setminus \partial y \cap \partial x \setminus \partial x \cap \partial x$	mparams={T})
$ \frac{\partial}{\partial p} \left(A \frac{\partial \psi}{\partial p} + B \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(B \frac{\partial \psi}{\partial p} + C \frac{\partial \psi}{\partial y} \right) = F $ $ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{Angm, Thm}\} \} $ $ \text{angM} = \text{invert} \text{_RefState}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} $ $ \text{angM} = \text{invert} \text{_RefState}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{\text{ang0}, \text{ Gamma}\} \} $ $ \text{PV inversion for QG flow} $ $ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{psi} = \text{invert} \text{_PV2D}(\text{PV}, \\ \text{dims} = ['Z', 'Y'], \\ \text{mParams} = \{f, N2\} \} $ $ \text{mParams} = \{f, N2\} \} $ $ \text{mParams} = \{f, N2\} \} $ $ \text{phi} = \text{invert} \text{_GillMatsuno}(\text{Q}, \\ \text{dims} = ['Y', 'X'], \\ \text{mParams} = \{f, \text{epsilon}, \text{Phi}\} \} $ $ \text{psi} = \text{invert} \text{_StommelMunk}(\text{curl}, \\ \text{dims} = ['Y', 'X'], \\ \text{mParams} = \{A, R, D, \text{beta, rho}\} \} $ $ \text{psi} = \text{invert} \text{_Omega}(\text{F}, \\ \text{dims} = ['Y', 'Y'], \\ \text{mParams} = \{f, S\} \} $ $ m$	Eliassen model (Eliassen, 1952)	<pre>psi = invert_Eliassen(F,</pre>
PV inversion for vortex (Hoskins et al., 1985) $\frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta}\right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r}\right) = 0$ angM = invert_RefState(PV, dims=['Z','Y'], mParams={ang0, Gamma}) $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p}\right) + \frac{\partial^2 \psi}{\partial y^2} = q$ psi = invert_PV2D(PV, dims=['Z','Y'], mParams={f, N2}) $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p}\right) + \frac{\partial^2 \psi}{\partial y^2} = q$ phi = invert_GillMatsuno(Q, dims=['Y','X'], mParams={f, epsilon, Phi}) $\frac{\partial}{\partial p} \left(\frac{\partial^2 \psi}{\partial p}\right) + \frac{\partial^2 \psi}{\partial x^2} + C \frac{\partial \psi}{\partial y} + D \frac{\partial \psi}{\partial x} + E \phi = Q$ Stommel-Munk model (Munk, 1950; Stommel, 1948) $A\nabla^4 \psi - \frac{R}{D}\nabla^2 \psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \vec{\tau}}{\rho_0 D}$ QG-Omega equation (Hoskins et al., 1978) $\frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p}\right) + \nabla \cdot (S\nabla \omega) = F$ psi = invert_StommelMunk(curl, dims=['Y', 'X'], mParams={f, S}) $\frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p}\right) + \nabla \cdot (c_1 \nabla \psi - c_2 \hat{\nabla} \psi) = F$ psi = invert_3DFlow(F, dims=['Z', 'Y', 'X'], mParams={f, N2, epsilon})	$\partial \left(A \partial \psi + B \partial \psi \right) + \partial \left(B \partial \psi + C \partial \psi \right) = E$	dims=['Z','Y'],
$ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 $ $ \qquad \qquad \text{dims=['Z','Y'], mParams=\{ang0, Gamma\})} $ $ \text{PV inversion for QG flow} $ $ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{psi = invert_PV2D(PV, dims=['Z','Y'], mParams=\{f, N2\})} $ $ \text{dims=['Y','Y'], mParams=\{f, N2\})} $ $ \text{phi = invert_GilMatsuno(Q, dims=['Y','X'], mParams=\{f, epsilon, Phi\})} $ $ \text{PStommel-Munk model (Munk, 1950; Stommel-Munk model (Munk, 1950; Stommel, 1948)} $ $ \text{A}\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D} $ $ \text{QG-Omega equation (Hoskins et al., 1978)} $ $ \text{W = invert_Omega(F, dims=['Z','Y','X'], mParams=\{f, S\})} $ $ \text{W = invert_Dmega(F, dims=['Z','Y','X'], mParams=\{f, S\})} $ $ \text{Psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams=\{f, N2, epsilon\})} $	$\frac{\partial}{\partial p} \left(A \frac{\partial}{\partial p} + D \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left(D \frac{\partial}{\partial p} + O \frac{\partial}{\partial y} \right) = I$	mParams={Angm, Thm})
$ \frac{\partial}{\partial \theta} \left(\frac{2\Lambda_0}{r^2} \frac{\partial \Lambda}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{\Gamma g}{Qr} \frac{\partial \Lambda}{\partial r} \right) = 0 $ $ \qquad \qquad \text{dims=['Z','Y'], mParams=\{ang0, Gamma\})} $ $ \text{PV inversion for QG flow} $ $ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{psi = invert_PV2D(PV, dims=['Z','Y'], mParams=\{f, N2\})} $ $ \text{dims=['Y','Y'], mParams=\{f, N2\})} $ $ \text{phi = invert_GilMatsuno(Q, dims=['Y','X'], mParams=\{f, epsilon, Phi\})} $ $ \text{PStommel-Munk model (Munk, 1950; Stommel-Munk model (Munk, 1950; Stommel, 1948)} $ $ \text{A}\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D} $ $ \text{QG-Omega equation (Hoskins et al., 1978)} $ $ \text{W = invert_Omega(F, dims=['Z','Y','X'], mParams=\{f, S\})} $ $ \text{W = invert_Dmega(F, dims=['Z','Y','X'], mParams=\{f, S\})} $ $ \text{Psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams=\{f, N2, epsilon\})} $	PV inversion for vortex (Hoskins et al., 1985)	angM = invert RefState(PV.
PV inversion for QG flow $\frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q$ $\text{psi} = \text{invert}_{\text{PV2D}}(\text{PV}, \text{dims}=['\text{Z'},'\text{Y'}], \text{mParams}=\{f, N2\})}$ $\text{gill-Matsuno model (Gill, 1980; Matsuno, 1966)}$ $A \frac{\partial^2 \phi}{\partial y^2} + B \frac{\partial^2 \phi}{\partial x^2} + C \frac{\partial \phi}{\partial y} + D \frac{\partial \phi}{\partial x} + E \phi = Q$ $\text{phi} = \text{invert}_{\text{GillMatsuno}}(Q, \text{dims}=['\text{Y'},'\text{X'}], \text{mParams}=\{f, \text{epsilon, Phi}\})}$ $\text{psi} = \text{invert}_{\text{Stommel-Munk}}(\text{curl}, \text{dims}=['\text{Y'},'\text{X'}], \text{mParams}=\{A, R, D, \text{beta, rho}\})}$ $A \nabla^4 \psi - \frac{R}{D} \nabla^2 \psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \vec{\tau}}{\rho_0 D}$ $QG-\text{Omega equation (Hoskins et al., 1978)}$ $W = \text{invert}_{\text{Omega}}(F, \text{dims}=['\text{Z'},'\text{Y'},'\text{X'}], \text{mParams}=\{f, S\})}$ $W = \text{invert}_{\text{Omega}}(F, \text{dims}=['\text{Z'},'\text{Y'},'\text{X'}], \text{mParams}=\{f, S\})}$ $\text{psi} = \text{invert}_{\text{Omega}}(F, \text{dims}=['\text{Z'},'\text{Y'},'\text{X'}], \text{mParams}=\{f, S\})}$ $\text{psi} = \text{invert}_{\text{Omega}}(F, \text{dims}=['\text{Z'},'\text{Y'},'\text{X'}], \text{mParams}=\{f, S\})}$		
$ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{dims=['Z','Y'], } \\ \text{mParams=}\{f, N2\}) $ $ \text{Gill-Matsuno model (Gill, 1980; Matsuno, } \\ 1966) $ $ \text{phi = invert_GillMatsuno(Q, } \\ \text{dims=['Y','X'], } \\ \text{mParams=}\{f, \text{ epsilon, Phi}\}) $ $ \text{Stommel-Munk model (Munk, 1950; } \\ \text{Stommel, 1948}) $ $ A\nabla^4 \psi - \frac{R}{D}\nabla^2 \psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \vec{\tau}}{\rho_0 D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} $ $ \text{mParams=}\{A, R, D, \text{ beta, rho}\}) $ $ \text{w = invert_Omega(F, } \\ \text{dims=['Z','Y','X'], } \\ \text{mParams=}\{f, S\}) $ $ \text{mParams=}\{f, S\}) $ $ \text{psi = invert_3DFlow(F, } \\ \text{dims=['Z','Y','X'], } \\ \text{mParams=}\{f, N2, \text{ epsilon}\}) $	$\frac{\partial \theta}{\partial \theta} \left(\frac{1}{r^2} \frac{\partial \theta}{\partial \theta} \right) + \frac{\partial r}{\partial r} \left(\frac{\partial r}{\partial r} \frac{\partial r}{\partial r} \right) = 0$	
$ \frac{\partial}{\partial p} \left(\frac{f^2}{N^2} \frac{\partial \psi}{\partial p} \right) + \frac{\partial^2 \psi}{\partial y^2} = q $ $ \text{dims=['Z','Y'], } \\ \text{mParams=}\{f, N2\}) $ $ \text{Gill-Matsuno model (Gill, 1980; Matsuno, } \\ 1966) $ $ \text{phi = invert_GillMatsuno(Q, } \\ \text{dims=['Y','X'], } \\ \text{mParams=}\{f, \text{ epsilon, Phi}\}) $ $ \text{Stommel-Munk model (Munk, 1950; } \\ \text{Stommel, 1948}) $ $ A\nabla^4 \psi - \frac{R}{D}\nabla^2 \psi - \beta \frac{\partial \psi}{\partial x} = -\frac{\hat{\nabla} \cdot \vec{\tau}}{\rho_0 D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} $ $ \text{mParams=}\{A, R, D, \text{ beta, rho}\}) $ $ \text{w = invert_Omega(F, } \\ \text{dims=['Z','Y','X'], } \\ \text{mParams=}\{f, S\}) $ $ \text{mParams=}\{f, S\}) $ $ \text{psi = invert_3DFlow(F, } \\ \text{dims=['Z','Y','X'], } \\ \text{mParams=}\{f, N2, \text{ epsilon}\}) $	PV inversion for QG flow	nsi = invert PV2D(PV
Gill-Matsuno model (Gill, 1980; Matsuno, 1966) $A\frac{\partial^2\phi}{\partial y^2} + B\frac{\partial^2\phi}{\partial x^2} + C\frac{\partial\phi}{\partial y} + D\frac{\partial\phi}{\partial x} + E\phi = Q$ Stommel-Munk model (Munk, 1950; Stommel, 1948) $A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D}$ $QG-Omega \ equation \ (Hoskins \ et \ al., 1978)$ $\frac{\partial}{\partial p}\left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla\cdot(S\nabla\omega) = F$ $\frac{\partial}{\partial p}\left(c_3\frac{\partial\psi}{\partial p}\right) + \nabla\cdot\left(c_1\nabla\psi - c_2\hat{\nabla}\psi\right) = F$ $phi = invert_GillMatsuno(Q, dims=['Y','X'], mParams={f, epsilon, Phi})$ $psi = invert_StommelMunk(curl, dims=['Y','X'], mParams={A, R, D, beta, rho})$ $w = invert_Omega(F, dims=['Z','Y','X'], mParams={f, S})$ $psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams={f, N2, epsilon})$	$\partial \left(f^2 \partial \psi \right) \partial^2 \psi$	
$\begin{array}{ll} \operatorname{dims}=['Y','X'], \\ \operatorname{dims}=['Y','X'], \\ \operatorname{dims}=['Y','X'], \\ \operatorname{MParams}=\{f,\operatorname{epsilon},\operatorname{Phi}\}) \\ \operatorname{Stommel-Munk model}(\operatorname{Munk},\operatorname{1950};\\ \operatorname{Stommel},\operatorname{1948}) \\ \operatorname{A}\nabla^4\psi-\frac{R}{D}\nabla^2\psi-\beta\frac{\partial\psi}{\partial x}=-\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D} \\ \operatorname{QG-Omega}\operatorname{equation}(\operatorname{Hoskins}\operatorname{et}\operatorname{al.},\operatorname{1978}) \\ \operatorname{QG-Omega}\operatorname{equation}(\operatorname{Hoskins}\operatorname{et}\operatorname{al.},\operatorname{1978}) \\ \operatorname{dims}=['Z','Y','X'], \\ \operatorname{mParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\}, \\ \operatorname{MParams}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\},$	$\frac{\partial}{\partial p} \left(\frac{\partial}{\partial r} \frac{\partial}{\partial p} \right) + \frac{\partial}{\partial y^2} = q$	
$\begin{array}{ll} \operatorname{dims}=['Y','X'], \\ \operatorname{dims}=['Y','X'], \\ \operatorname{dims}=['Y','X'], \\ \operatorname{MParams}=\{f,\operatorname{epsilon},\operatorname{Phi}\}) \\ \operatorname{Stommel-Munk model}(\operatorname{Munk},\operatorname{1950};\\ \operatorname{Stommel},\operatorname{1948}) \\ \operatorname{A}\nabla^4\psi-\frac{R}{D}\nabla^2\psi-\beta\frac{\partial\psi}{\partial x}=-\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D} \\ \operatorname{QG-Omega}\operatorname{equation}(\operatorname{Hoskins}\operatorname{et}\operatorname{al.},\operatorname{1978}) \\ \operatorname{QG-Omega}\operatorname{equation}(\operatorname{Hoskins}\operatorname{et}\operatorname{al.},\operatorname{1978}) \\ \operatorname{dims}=['Z','Y','X'], \\ \operatorname{mParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}) \\ \operatorname{MParams}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\}, \\ \operatorname{MParams}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\}, \\ \operatorname{MParamas}=\{f,S\},$	CILM 1000 M to	oli cara CillMata a (O
$A\frac{\partial^2 \dot{\phi}}{\partial y^2} + B\frac{\partial^2 \phi}{\partial x^2} + C\frac{\partial \phi}{\partial y} + D\frac{\partial \phi}{\partial x} + E\phi = Q$		
$\begin{array}{lll} \text{Stommel-Munk model (Munk, 1950;} \\ \text{Stommel, 1948)} & \text{psi = invert_StommelMunk(curl,} \\ \text{dims=['Y','X'],} \\ \text{MParams={A, R, D, beta, rho})} \\ \text{QG-Omega equation (Hoskins et al., 1978)} & \text{w = invert_Omega(F,} \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S \nabla \omega) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_5tommelMunk(curl,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(s \nabla \omega \right) = F \\ \text{3D ocean flow} & \text{psi = invert_5tommelMunk(curl,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(s \nabla \omega \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ psi = i$		
$\begin{array}{lll} \text{Stommel-Munk model (Munk, 1950;} \\ \text{Stommel, 1948)} & \text{psi = invert_StommelMunk(curl,} \\ \text{dims=['Y','X'],} \\ \text{MParams={A, R, D, beta, rho})} \\ \text{QG-Omega equation (Hoskins et al., 1978)} & \text{w = invert_Omega(F,} \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S \nabla \omega) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_5tommelMunk(curl,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(s \nabla \omega \right) = F \\ \text{3D ocean flow} & \text{psi = invert_5tommelMunk(curl,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(s \nabla \omega \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ psi = i$	$A\frac{\partial^{2} \psi}{\partial x^{2}} + B\frac{\partial^{2} \psi}{\partial x^{2}} + C\frac{\partial \psi}{\partial x} + D\frac{\partial \psi}{\partial x} + E\phi = Q$	iliraralis={1, epseton, rnt}
$\begin{array}{ll} \text{Stommel, 1948}) & \text{dims=['Y','X'],} \\ A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D} & \text{mParams=\{A, R, D, beta, rho\})} \\ \text{QG-Omega equation (Hoskins et al., 1978)} & \text{w = invert_Omega(F,} \\ \frac{\partial}{\partial p}\left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla\cdot(S\nabla\omega) = F & \text{dims=['Z','Y','X'],} \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p}\left(c_3\frac{\partial\psi}{\partial p}\right) + \nabla\cdot\left(c_1\nabla\psi - c_2\hat{\nabla}\psi\right) = F & \text{dims=['Z','Y','X'],} \\ \text{mParams=\{f, N2, epsilon\})} \end{array}$		psi = invert StommelMunk(curl.
$A\nabla^4\psi - \frac{R}{D}\nabla^2\psi - \beta\frac{\partial\psi}{\partial x} = -\frac{\hat{\nabla}\cdot\vec{\tau}}{\rho_0D}$ mParams={A, R, D, beta, rho}) $\frac{\partial}{\partial p}\left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla\cdot(S\nabla\omega) = F$ $\frac{\partial}{\partial p}\left(c_3\frac{\partial\psi}{\partial p}\right) + \nabla\cdot\left(c_1\nabla\psi - c_2\hat{\nabla}\psi\right) = F$ mParams={A, R, D, beta, rho}) $\frac{\partial}{\partial p}\left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla\cdot\left(S\nabla\omega\right) = F$ $\frac{\partial}{\partial p}\left(f^2\frac{\partial\omega}{\partial p}\right) + \nabla\cdot\left(f_1\nabla\psi - f_2\nabla\psi\right) = F$ mParams={f, S}) $\frac{\partial}{\partial p}\left(f_1\nabla\psi - f_2\nabla\psi\right) = F$ psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams={f, N2, epsilon})		
$\begin{array}{ll} \text{AV}^4\psi - \frac{1}{D} \text{V}^2\psi - \beta \frac{1}{\partial x} = -\frac{1}{\rho_0 D} \\ \text{QG-Omega equation (Hoskins et al., 1978)} & \text{w = invert_Omega(F,} \\ \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S \nabla \omega) = F & \text{dims} = [\ 'Z', \ 'Y', \ 'X'], \\ \text{mParams} = \{ f, \ S \}) \\ \text{3D ocean flow} & \text{psi = invert_3DFlow(F,} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F & \text{dims} = [\ 'Z', \ 'Y', \ 'X'], \\ \text{mParams} = \{ f, \ N2, \ \text{epsilon} \}) \end{array}$,	
$\begin{aligned} & \text{QG-Omega equation (Hoskins et al., 1978)} \\ & \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S \nabla \omega) = F \end{aligned} \\ & \text{dims=['Z','Y','X'], mParams=\{f, S\})} \\ & \text{3D ocean flow} \\ & \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \end{aligned} \\ & \text{psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams=\{f, N2, epsilon\})} \end{aligned}$	$A\nabla^4\psi - \frac{1}{D}\nabla^2\psi - \beta\frac{\tau}{\partial x} = -\frac{\tau}{\rho_0 D}$	
$\begin{split} \frac{\partial}{\partial p} \left(f^2 \frac{\partial \omega}{\partial p} \right) + \nabla \cdot (S \nabla \omega) &= F \\ & \text{dims=['Z','Y','X'],} \\ \text{mParams=}\{\text{f, S}\}) \end{split}$ $3D \text{ ocean flow} \\ \frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) &= F \\ & \text{dims=['Z','Y','X'],} \\ & \text{dims=['Z','Y','X'],} \\ & \text{mParams=}\{\text{f, N2, epsilon}\}) \end{split}$		<pre>w = invert_0mega(F,</pre>
3D ocean flow $\frac{\partial}{\partial p}\left(c_3\frac{\partial \psi}{\partial p}\right) + \nabla \cdot \left(c_1\nabla \psi - c_2\hat{\nabla}\psi\right) = F$ $\text{psi = invert_3DFlow(F, dims=['Z','Y','X'], mParams=\{f, N2, epsilon\})}$		
$\frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{mParams} = \{ \text{f, N2, epsilon} \})$	$\partial p \left(\frac{1}{2} \frac{\partial p}{\partial p} \right) + \mathbf{v} \cdot \left(\frac{\partial \mathbf{v} \omega}{\partial p} \right) = \mathbf{r}$	mParams={f, S})
$\frac{\partial}{\partial p} \left(c_3 \frac{\partial \psi}{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \hat{\nabla} \psi \right) = F \\ \text{mParams} = \{ \text{f, N2, epsilon} \})$	3D ocean flow	nsi = invert 3DFlow(F
op (op)		
	$\overline{\partial p} \left(c_3 \overline{\partial p} \right) + \nabla \cdot \left(c_1 \nabla \psi - c_2 \nabla \psi \right) = F$	



₄ Usage

- xinvert is designed in a functional-programming (FP) style. Users only need to import the
- function they are interested in and call it to get the inverted results (Table 1). Note that
- $_{77}$ the calculation of the forcing function F (e.g., calculating the vorticity using velocity vector)
- 78 on the right-hand side of the equation is NOT the core part of this package. But there is a
- 79 FiniteDiff utility module with which finite difference calculus can be readily performed.

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- 55 Chinese Academy of Sciences.

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