CSCI 4100 Fall 2018
Assignment 5 Answers

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Exercise 2.8

(a) According to the equation $\bar{g}(x) = \frac{1}{k}\sum_{k=1}^{K} g_k(x)$, then if H is closed under linear combination, then $\bar{g} \in H$

(b) Suppose there is a binary classification model. The first data set only have 1, and the second data set only have 0. Then $g_1(x) = 1$, $g_2(x) = 0$, and $H = 1, 2$.
However $\bar{g} = \frac{1}{2}(g_1(x) + g_2(x)$ and $\bar{g} \notin H$.

(c) No, because if it was true, the data will be seperated into either all 1 or all 0. This is a very bad result.

Problem 2.14

(a) Because $H = \cup_{k=1}^{K} H_k$, $m_H(N) \leq \sum_{k=1}^{K} m_{H_k}(N)$ because if H can shattere a dataset, H_k can do it too.

The equation $m_{H_k}(d_{vc}+1) < 2^{d_{vc}+1}$ shows that

$$m_{H_k}(d_{vc}+1) \leq \sum_{k=1}^{K} m_{H_k}(d_{vc}+1) < K2^{d_{vc}+1} < 2^{K(d_{vc}+1)}$$

So $d_{vc}(H) < K(d_{vc}+1)$.

(b) Becasue $d_{vc}(H) \leq K(d_{vc}+1)$, $m_H(l) \leq k(l^{d_{vc}}+1)$, and $m_{H_k}(l) \leq l^{d_{vc}}+1$.
Which means,

$$m_{H_k}(l) \leq l^{d_{vc}}+1 \leq 2l^{d_{vc}}$$

Then Let $H = H_1 \cup H_2 \cup H_3 ... \cup H_K$,

$$m_H(l) \leq \sum_{k=1}^{K}(d_{vc}+1) = K(d_{vc}+1) \leq 2Kl^{d_{vc}}$$

From the question, we assume $2^l \geq 2Kl^{d_{vc}}$, so

$$m_H(l) \leq 2Kl^{d_{vc}} \leq 2^l$$

Therefore, H can never shatter l points, and $d_{vc} < l$

(c) First assume $K \geq 2$, and let $l = 7(d_{vc}+K)log_2(d_{vc}K)$.
So,

$$2^l = 2^{7(d_{vc}+K)log_2(d_{vc}K)}$$

$$2Kl^{d_{vc}} = 2K(7(d_{vc}+K)log_2(d_{vc}K))^{d_{vc}}$$

Let $x = 7(d_{vc}+K)log_2(d_{vc}K)$. Here we need to show that

$$2^x > 2K(x)^{d_{vc}}$$

which is,

$$x > 1 + log_2K + d_{vc}log_2(x)$$

Let $x = 7(d_{vc}+K)log_2(d_{vc}K)$ There could be 2 situation:

(1) First $d_{vc} = 1$, $x = 7(1+K)log_2K$. We only need to show that $2^x > 2Klx$.

$$2^x = (1+1)^x \geq x + \frac{x(x-1)}{2} > \frac{x^2}{2} = \frac{x(7(1+K)log_2K)}{2}$$

Because $K \geq 2$,$log_2(K) \geq 1$

$$2^x \geq \frac{x(7(1+K))}{2} > \frac{x(4K)}{2} = 2Kx^{d_{vc}}$$

(2) Then when $d_{vc} \geq 2$, $x = 7(d_{vc}+K)log_2(d_{vc}K)$, so we need to show $x > 1+log_2K+d_{vc}log_2x$.

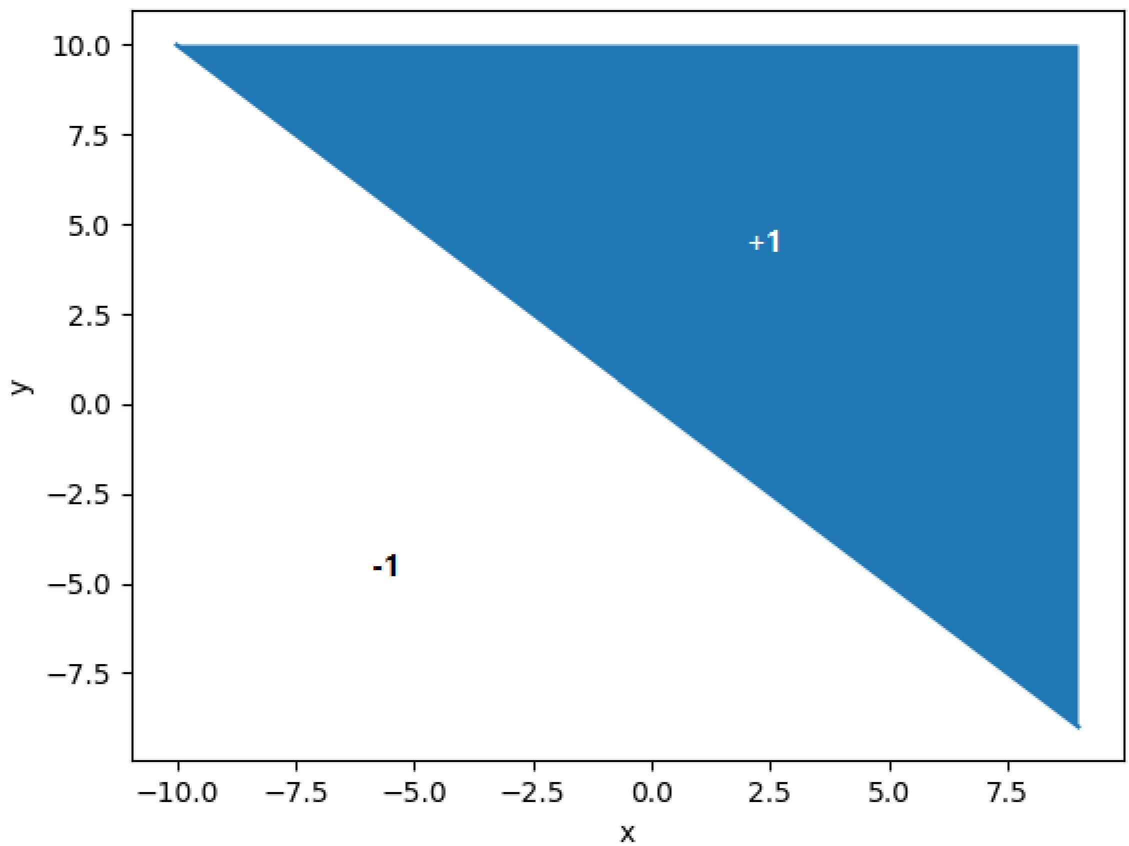
$$\begin{aligned}
1 + log_2K + d_{vc}log_2x &= log_2(2K) + d_{vc}log_2(7(d_vc + K)log_2(d_{vc}K)) \\
&\leq log_2(d_{vc}K) + d_{vc}log_2(7(d_{vc} + K)) + d_{vc}log_2(d_{vc}K) \\
&< log_2(d_{vc}K) + d_{vc}log_2((d_{vc} + K)^6) + d_{vc}log_2(d_{vc}K) \\
&= log_2(d_{vc}K) + 7d_{vc}log_2(d_{vc} + K) \\
&< 7(d_{vc} + K)log_2(d_{vc}K) \\
&= s
\end{aligned}$$

So, $d_{vc}(H) \leq 7(d_{vc} + K)log_2(d_{vc}K)$
Also, from part (a), we know that $d_{vc}(H) \leq K(d_{vc} + 1)$. Therefore,

$$d_{vc}(H) \leq min(K(d_{vc} + 1), 7(d_{vc} + K)log_2(d_{vc}K))$$

Problem 2.15

(a) Here is the plot of the example.



(b) Due to the hint, we can construct the first point randomly, and the generate the second point with larger x-component and smaller y-component. The third point we can generate will have a even larger x-component and even smaller y-component than the previous point. In this way we can generate infinite number of points, which means that H can always shatter infinite number of points. SO, $d_{vc} = \infty$ and $m_H(N) = 2^N$.

Problem 2.24

(a) For any two randomly chosen $D = (x_1, x_1^2), (x_2, x_2^2)$, $g(x)$ should be the line connect this two points.

Therefore,

$$g(x) = kx + b$$
$$x_2^1 = kx_1 + b$$
$$x_2^2 = kx_2 + b$$

So,

$$g(x) = (x_1 + x_2)x - x_1x_2$$

According to the formula $\bar{g}(x) = \frac{1}{K}\sum_{k=1}^{K} g_k(x)$,

$$\bar{g}(x) = \frac{1}{K}\sum_{k=1}^{K}[(x_{k1} + x_{k2})x - x_{k1}x_{k2}] = 0$$

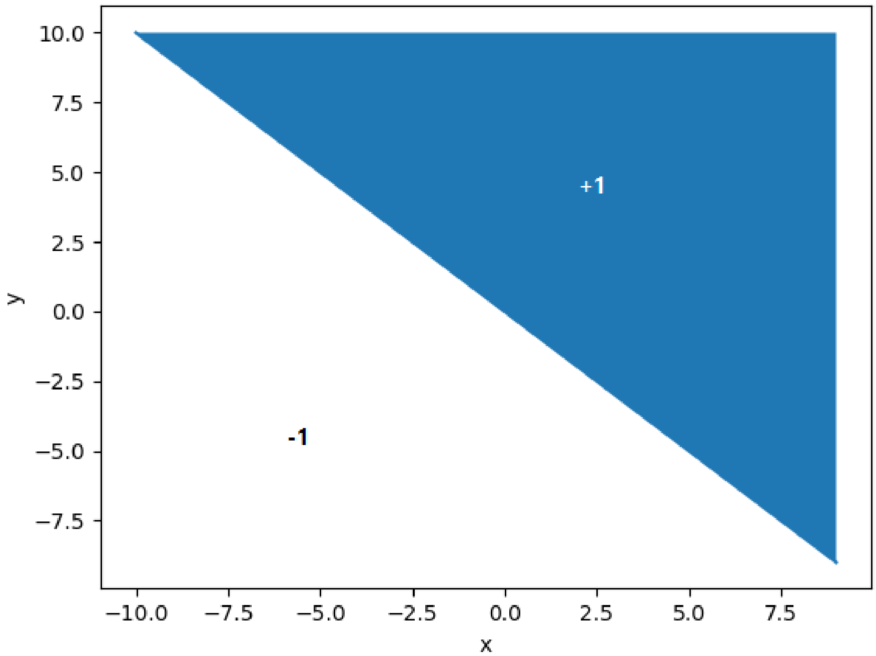
(b) (1) To compute $\bar{g}(x)$, randomly select 2 points on [-1,1] and calculate $g(x)$ for this pair of points. Do the above step 10000 times and then calculate $\bar{g}(x)$ by using formula $\bar{g}(x) = \frac{1}{K}\sum_{k=1}^{K} g_k(x)$.

(2) To compute $E_{out}(g^D)$, there is a formula $E_{out}(g^D) = E_x[(g^D(x) - f(x))^2]$. So we can get E_{out} by use this formula.

(3) for $bias$, use the formula $bias = (\bar{g}(x) - f(x))^2$ and $bias = E_x[bias(x)]$

(4) for var, use the formula $var(x) = E_x[var(x)]$ and $var = E_x[var(x)]$

(c) From the experiment: Eout is 0.5333333333333333
bias is 0.19569357380857
var is 0.33338149781948495
So, $E[E_{out}] = bias + var$



(d) According to the formula mentioned in part(b),

$$\begin{aligned} E_{out} &= E_x[((x_1+x_2)x - x_1x_2 - x^2)^2] \\ &= \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[((x_1+x_2)x - x_1x_2 - x^2)^2]dx_1dx_2dx \\ &= 0.533 \end{aligned}$$

$$\begin{aligned} bias &= E_x[(\bar{g}(x) - f(x))^2] \\ &= \frac{1}{2}\int_{-1}^{1}(0 - x^2)^2dx \\ &= 0.2 \end{aligned}$$

From part(a), we get $\bar{g}(x) = 0$, so

$$\begin{aligned} var(x) &= E_x[E_D[(g^D(x) - \bar{g}(x))^2]] \\ &= \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[(x_1+x_2)x - x_1x_2]^2dx_1dx_2dx \\ &= \frac{2}{3}x^2 + \frac{1}{9} \end{aligned}$$

$$\begin{aligned} var &= E_x[\frac{2}{3}x^2 + \frac{1}{9}] \\ &= \frac{1}{2}\int_{-1}^{1}\frac{2}{3}x^2 + \frac{1}{9} \\ &= \frac{1}{3} \end{aligned}$$

Therefrore,

$$E_{out} = 0.533 = 0.2 + \frac{1}{3} = bias + var$$