CSCI 4100 Fall 2018
Assignment 6 Answers

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Exercise 3.4

(a) From the question, $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_N]^T$, $y = [y_1, y_2, ..., y_N]$, $X = [x_1, x_2, ..., x_3]$, so, $y = X\omega^* + \epsilon$.

Because $w_{lin} = ((X^TX)^{-1}X^T)y$ and $H = X(X^TX)^{-1}X^T$,
so $\hat{y} = Xw_{lin} = X(X^TX)^{-1}X^Ty = Hy = H(X\omega^* + \epsilon) = X(X^TX)^{-1}X^TX + H\epsilon = 1X\omega^* + H\epsilon$
$= X\omega^* + H\epsilon$

(b) Since $\hat{y} = X + H\epsilon$,
$\hat{y} = X\omega^* + H\epsilon - y = X\omega^* + H\epsilon - (X\omega^* + \epsilon)$
$= \epsilon(H - I)$

(c)

$$
\begin{aligned}
E_{in}(w) &= \frac{1}{N}||Xw - y||^2 \\
&= \frac{1}{N}||Xw^* + H\epsilon - y||^2 \\
&= \frac{1}{N}||\hat{y} - y||^2 \\
&= \frac{1}{N}||(H - I)\epsilon||^2 \\
&= \frac{1}{N}((H - I)\epsilon)^T((H - I)\epsilon) \\
&= \frac{1}{N}\epsilon^T(H - I)^2\epsilon \\
&= \frac{1}{N}\epsilon^T(I - H)^2\epsilon
\end{aligned}
$$

From exercise(c),$(I - H)^K = I$ for any K,
So,

$$E_{in}(w) = \frac{1}{N}\epsilon^T(I - H)\epsilon$$

(d) because $trace(H) = d + 1$,

$$
\begin{aligned}
E_D[E_{in}(w)] &= \frac{1}{N} E_D(\epsilon^T (I - H)^2 \epsilon) \\
&= \frac{1}{N} E_D trace(\epsilon^T (I - H)^2 \epsilon) \\
&= \frac{1}{N}(N\sigma^2 - (\sum_{i=1}^{N} H_{ii})\sigma^2) \qquad = \frac{1}{N}(N\sigma^2 - trace(H)\sigma^2) \\
&= \frac{1}{N}(N\sigma^2 - (d+1)\sigma^2) \\
&= \sigma^2(1 - \frac{d+1}{N})
\end{aligned}
$$

(e) First from part (a), $\hat{y} = X\omega^* + \epsilon$,
So,

$$
E_{test}(w_{lin}) = \frac{1}{N}||\hat{y} - y'||^2
$$

$$
\begin{aligned}
&= \frac{1}{N}(\epsilon^T H - \epsilon'^T (H\epsilon - \epsilon')) \\
&= \frac{1}{N}(\epsilon^T H H \epsilon - 2\epsilon'^T H \epsilon + \epsilon'^T \epsilon') \\
&= \frac{1}{N}(\epsilon^T H \epsilon - 2\epsilon'^T H \epsilon + \epsilon'^T \epsilon')
\end{aligned}
$$

Then $E_{D,\epsilon'}[E_{in}(w_{lin})] = E_{D,\epsilon'}[\frac{1}{N}(\epsilon^T H\epsilon - 2\epsilon'^T H\epsilon + \epsilon'^T\epsilon')]$
From part (d), $E_D(\epsilon^T\epsilon) = N\sigma^2$ and $E_D(\epsilon^T H\epsilon) = (d+1)\sigma^2$,

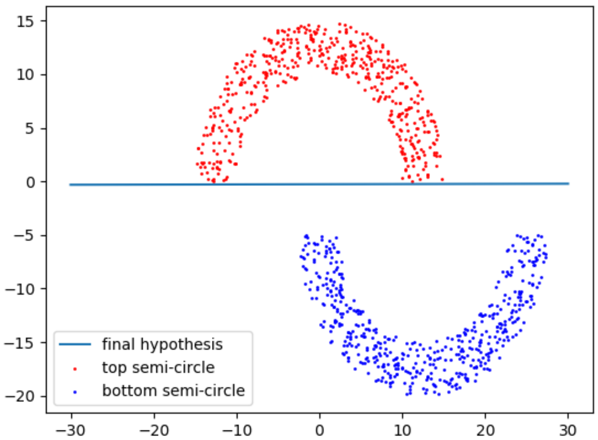
So, $E_{D\epsilon'}[E_{in}(w_{lin})] = \sigma^2(1 + \frac{d+1}{N}) - \frac{2}{N} E_{D,\epsilon'}(\epsilon'^T H\epsilon)]$

Because $E_{D,\epsilon'}(\epsilon'^T H\epsilon) = \sum_{i=1}^{N}(E(\epsilon'_i)H_{ii}E(\epsilon_i)) = 0$,

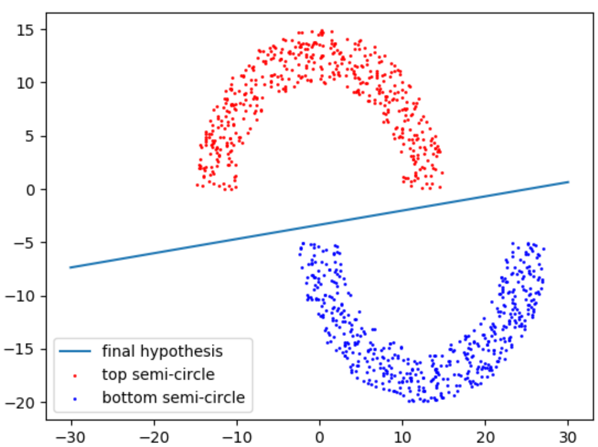
$$
E_{D\epsilon'}[E_{in}(w_{lin})] = \sigma^2(1 + \frac{d+1}{N})
$$

Problem 3.1

(a)

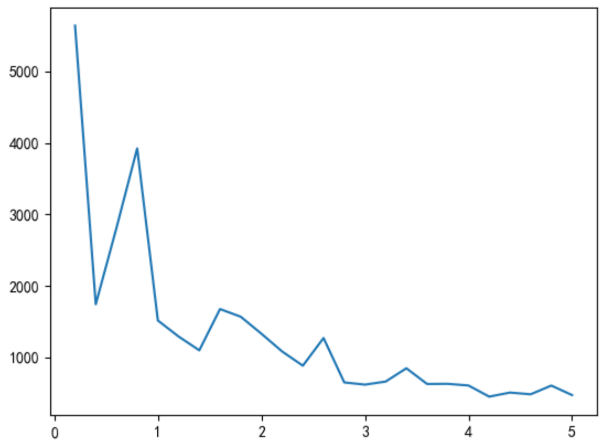


(b)



By linear regression, the final hypothesis is much more colse to the center of two semi-circles then by PLA, because linear regression consider y as real number instead of a sign, which makes result more precise.

Problem 3.2



As the graph showed, the number of iterations decreases as the *sep* increase.
From Problem 1.3, we know that

$$t \leq \frac{R^2||w^*||^2}{\rho^2}$$

Here because two semi-circles are fixed, so R is a constant.
Then As *sep* increases, the shortest distance between two points gets larger, which makes $\frac{||w^*||}{\rho}$ smaller. Hence t gets smaller, and the number of iteration is getting smaller.

Problem 3.8

(1) Show that among all hypotheses. the one that minimizes E_{out} is given by $h^*(x) = E[y|x]$:

$$\begin{aligned} E_{out}(h) &= E[(h(x) - y)^2] \\ &= E[((h(x) - h^*(x)) + (h^*(x) - y))^2] \\ &= E[(h(x) - h^*(x))^2] + E[(h^*(x) - y)^2] + 2E[(h(x) - h^*(x))(h^*(x) - y)] \end{aligned}$$

Hence, to minimize E_{out}, we need minimize $2E[(h(x) - h^*(x))(h^*(x) - y)]$.
Let $h^*(x) = E[y|x]$:

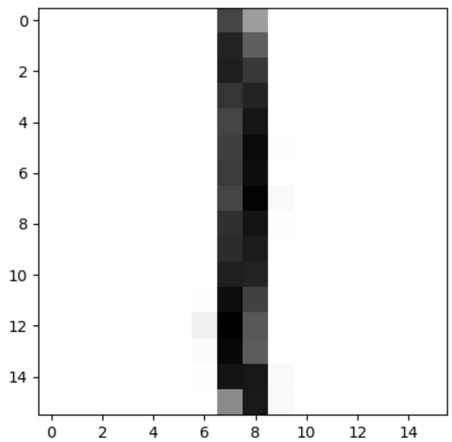
$$\begin{aligned} E[(h(x) - h^*(x))(h^*(x) - y)] = E[(h(x) - h^*(x))(h^*(x) - y)] \\ = E[(h(x) - h^*(x))(h^*(x) - h^*(x))] \\ = 0 \end{aligned}$$

Therefore, when $h^*(x) = E[y|x]$, E_{out} is minimized.

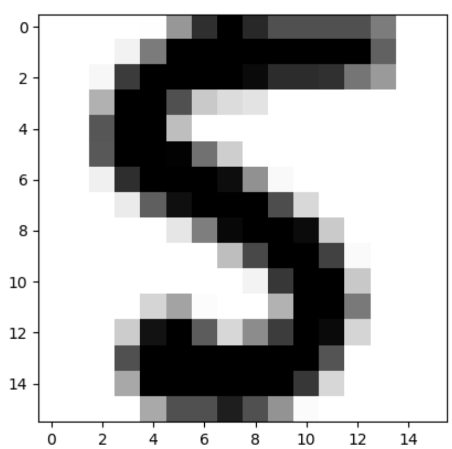
(2) Becasue $E(E(y|x)) = E(y)$,
$E[\epsilon(x)]$
$= E[E[\epsilon(x)|x]]$
$= E[E[y - h^*(x)|x]]$
$= E[E[E(y|x) - E(h^*(x)|x)]]$
$= E[h^*(x) - h^*(x)]$
$= 0$

Handwritten Digits Data - Obtaining Features

(a) This is plot for an 1 digit:



This is plot for a 5 digit:



(b) Feature 1: Let M be a matrix for a digit,

$$intensity = \sum_{i=0}^{15}\sum_{j=0}^{15} M_{i,j}$$

Feature 2: Let $M_{a,b}$ and $M_{c,d}$ be two position in a digit.
Let $(M_{a,b} == M_{c,d}) = 1$ if $M_{a,b} = M_{c,d}$;
Let $(M_{a,b} == M_{c,d}) = 0$ if $M_{a,b}! = M_{c,d}$.
So,

$$symmetry = \sum_{i=0}^{16}\sum_{j=0}^{7}((M_{i,j} == M_{i,15-j})$$

(c) This is a plot of features for each digits: