CSCI 4100 Fall 2018
Assignment 4 Answers

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Exercise 2.4

(a) First build a $(d+1) \times (d+1)$ matrix:

$$M = \begin{bmatrix} 1^0 & 1^1 & 1^2 & ... & 1^d \\ 2^0 & 2^1 & 2^2 & ... & 2^d \\ 3^0 & 3^1 & 3^2 & ... & 3^d \\ & & ... & & \\ (d+1)^0 & (d+1)^1 & (d+1)^2 & ... & (d+1)^d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1^2 & ... & 1^d \\ 1 & 2 & 2^2 & ... & 2^d \\ 1 & 3 & 3^2 & ... & 3^d \\ & & ... & & \\ 1 & (d+1) & (d+1)^2 & ... & (d+1)^d \end{bmatrix}$$

Such matrix M is a kind of Vandermonde determinant.
Because the value of Vandermonde determinant cannot be 0, the system

$$M \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ ... \\ x_{(d+1)} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_4 \\ ... \\ y_{d+1} \end{bmatrix}$$ will always have solutions.

So $(d+1)$ points can always be shattered, and it is true that $d_{vc} \geq d+1$.

(b) Suppose any point in a set of $(d+2)$ pints X can be reprense as a vector of length $(d+1)$: $[x_1, x_2, x_3, ..., x_{d+1}]$.
Then any $(d+2)$ vector of length $(d+1)$ have to be linear dependent, which means:

$$x_j = \sum_{i \neq j} a_i \times x_i$$

and not all $a_i = 0$
Then construct a dichotomy that cannot be generated:

$$y = \begin{cases} sign(a_i), & \text{if } i \neq j \\ -1, & \text{if } i = j \end{cases}$$

For all $i \neq j$, assume the labes are correct:

$$sign(a_i) = sign(w^T x_i) \Rightarrow a_i w^T x_i > 0$$

For jth data, $w^T x_j = \sum_i \neq a_i w^T x_i > 0$, and $y_j = 1$ and $y_j \neq -1$.
Thus no set of $d+2$ points in X can be shattered by the perceptron, and $d_v c \leq d+1$.

Problem 2.3

(a) From exercise 2.1 (a), we know for positive ray, $m_H(N) = N + 1$. Then for negative rays, $m_H(N) = N - 1$. Combine these two situations, we get $m_H(N) = N + 1 + N - 1 = 2 = 2N$.

So, $d_{vc} = 2$ because $m_H(3) = 6 < 2^3$

(b) From exercise 2.1 (b), we know for positive interval, $m_H(N) = \frac{N^2}{2} + \frac{N}{2} + 1$. If negative intervals are counted too, for the case $N > 2$: we can simply get $m_H(H)$ by double the growth function for positive interval and minus the growth function for positive or negative rays because this part is count twice.
So,

$$m_H(N) = 2(\frac{N^2}{2} + \frac{N}{2} + 1) - 2N = N^2 - N + 2 \text{ for } N > 2$$

And if $N = 1$ *or* 2, the groth function is the same as positive intervals. Thus,

$$m_H(N) = \begin{cases} \frac{N^2}{2} + \frac{N}{2} + 1, & \text{if } N \leq 2 \\ N^2 - N + 2, & \text{if } N > 2 \end{cases}$$

So $d_{vc} = 3$ because $m_H(4) = 14 < 2^4$

(c) Since they H contains the functions which are +1 in the sphere, there are two situations: first is +1 in sphere and -1 outside; and the other is there is only -1 out of the sphere but no +1.

It is similar to the positive intervals, so

$$m_H(N) = \frac{N(N+1)}{2} + 1$$

So $d_{vc} = 2$ because $m_H(3) = 7 < 2^3$

Problem 2.8
Because $m_H(N) \leq 2^N$, then $m_H(N)$ is either infinite and equal to 2^N or is finite and bounded by a polynomial. Therefore, 2^N is a possible growth function $m_H(N)$ because it is the upper bound of $m_H(N) \leq 2^N$;
$1 + N$,$1 + N + \frac{N(N-1)}{2}$ and $1 + N + \frac{N(N-1)(N-2)}{6}$ are possible growth functions $m_H(N)$ becasue they are all polynominals, and snaller than 2^N.

Problem 2.10
First assume $m_H(N) = x$.

Suppose there are $2N$ points, and seperate these $2N$ points into 2 sets of N points. Clearly, each set of N points can produce no more than x dichotomies, so

$$m_H(2N) = m_H(N + N) \leq x \times x$$

which means,

$$m_H(2N) \leq m_H(N)^2$$

Also because $E_{out}(g) \leq E_{in}(g) + \sqrt{\frac{8}{N}\ln\frac{4m_H(2N)}{\delta}}$, and $m_H(2N) \leq m_H(N)^2$, it is clearly

$$E_{out}(g) \leq E_{in}(g) + \sqrt{\frac{8}{N}\ln\frac{4m_H(N)^2}{\delta}}$$

Problem 2.12

$$E_{out}(g) \leq E_{in}(g) + \sqrt{\frac{8}{N}\ln\frac{4((2N)^{d_{vc}} + 1)}{\delta}}$$

Since $d_{vc} = 10$, confidence $= 95\%(\delta = 0.05)$, generalization error at most 0.05

$$\sqrt{\frac{8}{N}\ln\frac{4((2N)^{10} + 1)}{0.05}} \leq 0.05$$

Sovle the inequality by iterating from $N = 1$: $N \leq 452957$
So the sample size is 452957.