PROBLEM SET FOR FINAL EXAM.

Exercise. 1.4.8. Solve the following problems:

(a) Let V be an affine variety. Prove that $\mathbb{I}(V)$ is radical

(b) Prove that $\langle x^2, y^2 \rangle$ in not radical.

Solution: (a) Let f be a polynomial s.t. $f^m \in \mathbb{I}(V)$ for some m.

Note that $k[X]$ is an integer domain. Then $\forall \vec{a} \in V$,

$$f^m \in \mathbb{I}(V) \Rightarrow f(\vec{a})^m = 0 \Rightarrow f(\vec{a}) = 0$$

This shows that $f \in \mathbb{I}(V)$ and thus $\mathbb{I}(V)$ is radical.

(b) Consider $f(x,y) = x$. Then

$$f^2(x,y) = x^2 \in \langle x^2, y^2 \rangle \text{ but } f(x,y) \text{ does not.} \qquad \square$$

Exercise 1.5.17. Find a basis for the ideal,

$$\mathbb{I}(\mathbb{V}(x^5 - 2x^4 + 2x^2 - x,\ x^5 - x^4 - 2x^3 + 2x^2 + x - 1)).$$

Solution: Consider the unique factorization of these polynomials.

$$x^5 - 2x^4 + 2x^2 - x = x(x-1)^3(x+1)$$

$$x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = (x-1)^3(x+1)$$

Then, we have

$$\begin{aligned} \mathbb{I}(\mathbb{V}(\cdots)) &= \mathbb{I}(\{-1, 1\}) \\ &= \{ f \in k[x] : f \text{ vanishes at } 1 \text{ and } -1 \} \\ &= \langle x^2 - 1 \rangle. \qquad \square \end{aligned}$$

Exercise 2.8.11. Suppose that the numbers a, b, c s.t.

$$a+b+c=3$$
$$a^2+b^2+c^2=5$$
$$a^3+b^3+c^3=7$$

Prove that $a^4+b^4+c^4=9$.

Solution: Consider the elemental symmetric polynomials

$$\begin{cases} e_1 = a+b+c \\ e_2 = ab+bc+ac \\ e_3 = abc. \end{cases}$$

By solving the following system of equations:

$$\begin{cases} a+b+c = e_1 = 3 \\ a^2+b^2+c^2 = e_1^2 - 2e_2 = 5 \\ a^3+b^3+c^3 = e_1^3 - 3e_1e_2 + 3e_3 = 7, \end{cases}$$

we obtain $e_1 = 3$, $e_2 = 2$, $e_3 = -2/3$.

Therefore, we conclude

$$a^4+b^4+c^4 = e_1^4 - 4e_1^2e_2 + 4e_1e_3 + 2e_2^2 = 9 \qquad \square$$

Exercise 3.2.5. Suppose that $I \subseteq \mathbb{C}[x,y]$ is an ideal s.t. $I_1 \neq 0$.

Prove that $\mathbb{V}(I_1) = \pi_1(\mathbb{V}(I))$.

Solution: By the closure theorem, $\mathbb{V}(I_1)$ is the smallest variety containing $\pi_1(\mathbb{V}(I))$.

Note that $\mathbb{V}(I_1)$ is a finite set. Then $\pi_1(\mathbb{V}(I))$ is also a finite set and thus a variety which is equal to $\mathbb{V}(I_1)$. $\square$

Theorem (Polynomial implicitization) Let k be an infinite field. Suppose that F is the map

$$\begin{aligned} F: \quad k^m &\longrightarrow k^n \\ (t_1,\cdots,t_m) &\longmapsto (f_1(t_1,\cdots,t_m),\cdots,f_n(t_1,\cdots,t_m)) \end{aligned}$$

defined by the polynomial parametrization and I is the ideal

$$\langle x_1-f_1,\cdots,x_n-f_n\rangle \subseteq k[x_1,\cdots,x_n,t_1,\cdots,t_m].$$

Then $\mathbb{V}(I_m)$ is the smallest variety in k^m containing $F(k^m)$.

Solution: Firstly, we show that $\mathbb{V}(I_m)$ containing $F(k^m)$. Indeed, note that there is a commutative diagram

$$\begin{tikzcd} & k^{m+n} \arrow[dr, "\pi_m"] & \\ k^m \arrow[ur, "i"] \arrow[rr, "F"'] & & k^n \end{tikzcd}$$

Here, i is the map given by

$$i(t_1,\cdots,t_m) = (t_1,\cdots,t_m,f_1(t_1,\cdots,t_m),\cdots,f_n(t_1,\cdots,t_m)).$$

Then, we have

By lemma 3.2.1.

$$F(k^m) = \pi_m \circ i(k^m) = \pi_m(\mathbb{V}(I)) \subseteq \mathbb{V}(I_m).$$

<u>Claim</u>. If $h\in k[x_1,\cdots,x_n]$ vanishes on $F(k^m)$, then $h\in I_m$.

Indeed, applying the division algorithm w.r.t. the lex order

$$x_1>\cdots>x_n>t_1>\cdots>t_m$$

we obtain

$$h(x_1,\cdots,x_n) = q_1\cdot(x_1-f_1)+\cdots+q_n\cdot(x_n-f_n)+r(t_1,\cdots,t_m)$$

because $LT(x_i-f_i)=x_i$.

For any $\vec{a}\in k^m$, we set $t_i=a_i$ and $x_i=f_i(\vec{a})$ to obtain

$$h(f_1(\vec{a}),\cdots,f_n(\vec{a})) = 0+\cdots+0+r(\vec{a})$$

Since k is infinite, r is a zero polynomial. This shows that

$$h(x_1,\cdots,x_n) \subseteq I \cap k[x_1,\cdots,x_n] = I_m.$$

Remain to prove $\mathbb{V}(I_m)$ is the smallest.
Assume that $Z = \mathbb{V}(h_1,\cdots,h_s) \subseteq k^m$ is a variety containing $F(k^m)$
By our claim, we have.

$$\mathbb{V}(I_m) \subseteq \mathbb{V}(h_1,\cdots,h_s) = Z. \qquad \square$$

Exercise 4.2.4. Let $I \subseteq k[x_1,\cdots,x_n]$ be an ideal.

(a) Show that $\sqrt{I}$ is a radical ideal.

(b) Show that I is radical iff $I = \sqrt{I}$.

(c) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

Solution: (a) For any $f \in k[x_1,\cdots,x_n]$,
if $f^m \in \sqrt{I}$ for some $m>0$, then

$$f^{mp} \in I \text{ for some } p>0 \Rightarrow f \in \sqrt{I}$$

This proves that $\sqrt{I}$ is radical.

(b) It suffices to prove the necessity part.
Suppose that I is radical. It is clear that $I \subseteq \sqrt{I}$ and we need only to show the converse inclusion.
For any $f \in \sqrt{I}$, since I is radical,

$$f^m \in I \text{ for some } m>0 \Rightarrow f \in I.$$

(c) By part (b), $\sqrt{I}$ is radical shows that $\sqrt{\sqrt{I}} = \sqrt{I}$. $\square$

Exercise 4.5.2. Show that a prime ideal is radical.

Solution: Let $I \subseteq k[x_1, \cdots, x_n]$ be a prime ideal.

Suppose that $f \in k[x_1, \cdots, x_n]$ s.t. $f^m \in I$ for some $m > 0$.

Since I is prime,

$$f^m = f \cdot f^{m-1} \in I \Rightarrow \text{either } f \in I \text{ or } f^{m-1} \in I$$
$$\Rightarrow f^{m-1} \in I$$

Repeating the above procedure, we obtain $f \in I$.

This shows that I is radical. $\square$

Exercise 5.4.12. Let $V = \mathbb{V}(y^2 - 3x^2z + 2) \subseteq \mathbb{R}^3$ and L_A the linear mapping on $\mathbb{R}^3$ defined by the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Verify that $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism.

(b) Find the equation of $L_A(V)$.

Solution: Consider the inverse of A:

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

(a) Since $L_A \cdot L_{A^{-1}} = L_{A^{-1}} \cdot L_A = id_{\mathbb{R}^3}$, $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism.

(b) Suppose that $L_A(x, y, z)^t = A(x, y, z)^t$

$$= (2x + z,\ x + y,\ y + z)^t$$
$$:= (\bar{x}, \bar{y}, \bar{z})^t$$

Then $(x, y, z)^t = L_A^{-1}(\bar{x}, \bar{y}, \bar{z})^t$

$$= \left(\tfrac{1}{3}(\bar{x}+\bar{y}-\bar{z}),\ \tfrac{1}{3}(-\bar{x}+2\bar{y}+\bar{z}),\ \tfrac{1}{3}(\bar{x}-2\bar{y}+2\bar{z})\right).$$

Since x, y, z satisfy $y^2 - 3x^2z + 2$, the $L_A(V)$ is given by the equation

$$\tfrac{1}{9}(-\bar{x}+2\bar{y}+\bar{z})^2 - \tfrac{1}{9}(\bar{x}+\bar{y}-\bar{z})^2(\bar{x}-2\bar{y}+2\bar{z}) + 2. \qquad \square$$

Exercise 5.6.18. Let $A = k[x] \times k[x]$ be a k algebra with $k = \bar{k}$.

(a) Show that $*$ $k \cong \{(a, a) : a \in k\} \subseteq A$

$*$ A is reduced (i.e. it contains no nil. elem.)

Solution: The algebra isomorphism is given by

$$\varphi: k \longrightarrow \{(a, a) : a \in k\}$$
$$a \longmapsto (a, a)$$

Assume that (f, g) is a nonzero nilpotent element in A. Then

$$f^m = g^m = 0 \text{ for some } m > 0.$$

Since $k[x]$ in an integer domain, we have $f = g = 0$.

This is a contradiction.

(b) Prove that A is generated by

$$s_1 = (1, 0),\quad s_2 = (0, 1),\quad s_3 = (x, x).$$

Solution. Indeed, for any $(f(x), g(x)) \in A$,

$$(f(x), g(x)) = (f(x), f(x))(1, 0) + (g(x), g(x))(0, 1)$$
$$= f(s_3)s_1 + g(s_3)s_2.$$

(c) Prove that $k[s_3] \subseteq A$ is a Noether normalization of A.

Solution: By (b), A is finite over $k[s_3]$ and thus $k[s_3] \subseteq A$ is a Noether normalization of A.

(d) Define a k-algebra homomorphism

$$\varphi: k[x_1, x_2, y] \to A \text{ by}$$

$$x_1 \mapsto s_1, \quad x_2 \mapsto s_2, \quad y \mapsto s_3$$

Prove that $\ker\varphi = \langle x_1 + x_2 - 1,\ x_2^2 - x_2 \rangle$

Solution: It is clear that $I := \langle x_1 + x_2 - 1,\ x_2^2 - x_2 \rangle \subseteq \ker\varphi$. We need only to prove the converse inclusion.

Note that $G = \{x_1 + x_2 - 1,\ x_2^2 - x_2\}$ is a Gröbner basis for I w.r.t. the lex order $x_1 > x_2 > y$.

Assume that $h(x_1, x_2, y) \notin I$.

The division algorithm w.r.t. the lex order $x_1 > x_2 > y$ gives

$$h(x_1, x_2, y) = u(x_1, x_2, y)(x_1 + x_2 - 1) + v(x_1, x_2, y)(x_2^2 - x_2) + r(x_1, x_2, y)$$

where $r(x_1, x_2, y)$ is nonzero and is of the form: $f(y) + g(y)x_2$

Since $\varphi(f(y) + g(y)x_2) = f(s_3) + g(s_3)s_2$

$$= (f(x),\ g(x) + f(x)) \neq 0,$$

we conclude $h(x) \notin \ker\varphi$.

(e) Show that $V = \mathbb{V}(I) \subseteq k^3$ is a disjoint union of two lines. Then use this to explain why $k[V] \cong k[x] \times k[x] = A$

Solution: By solving the system of equations:

$$x_1 + x_2 - 1 = 0, \quad x_2^2 - x_2 = 0$$

we obtain $V = V_1 \sqcup V_2$, where

$$V_1 = \{(1, 0, y) : y \in k\}, \quad V_2 = \{(0, 1, y) : y \in k\}$$

To prove $k[V] \cong A$. We need only to show that I is radical and then we have

$$k[V] \cong k[x_1, x_2, y]/\mathbb{I}(V) = k[x_1, x_2, y]/I \cong A$$

Indeed, suppose that $f(x_1, x_2, y)$ is a polynomial satisfying $f^m \in I$ for some positive integer m.

Applying the division algorithm w.r.t. the lex order $x_1 > x_2 > y$,

$$f(x_1, x_2, y) = u(x_1, x_2, y)(x_1 + x_2 - 1) + v(x_1, x_2, y)(x_2^2 - x_2) + x_2 a(y) + b(y)$$

Then $f^m(x_1, x_2, y) \in (x_2 a(y) + b(y))^m + I$

Note that $(x_2 a(y) + b(y))^m \in I$. Accroding to the procedure of division algorithm, we have

$$x_2^2 - x_2 \mid (x_2 a(y) + b(y))^m \Rightarrow b(y) = 0 \Rightarrow a(y) = 0$$

This proves $f(x_1, x_2, y) \in I$. $\square$