

Finitely generated algebras

Assumption: k is an algebraically closed field;

Every algebra is commutative and contains identity.

* A k -algebra A is said to be **finitely generated** if

\exists (finitely many) $x_1, \dots, x_s \in A$

s.t. every $a \in A$ can be expressed as a polynomial in x_1, \dots, x_s

Integrality over a ring

Definition: Let $B \subseteq A$ be a k -subalgebra. A is said to be integral over B if

$\forall a \in A, f(a) = 0$ for some monic polynomial $f \in B[X]$

Lemma [Lem 2.21] If A is integral over B , then

A is a field $\Rightarrow B$ is a field

Proof: Taking $a \in B$,

\exists monic polynomial $f(x) = x^n + b_{n-1}x^{n-1} + \dots + b_1x + b_0$ s.t. $f(a^{-1}) = 0$.

Multiplying both sides $f(a^{-1}) = 0$ by a^n , we obtain

$$- \frac{1}{a} = b_{n-1} + b_{n-2} \cdot a + \dots + b_0 \cdot a^{n-1} \in B. \quad \square$$

Zariski's Lemma [Lem 2.23] Let K/k be a field extension.

If K is a finitely generated k -algebra, then $[K:k]$ is finite.

Proof: Assume that $\{y_1, \dots, y_n\}$ is a generating set of K s.t.

(i) y_1, \dots, y_n are algebraically independent.

(ii) y_{n+1}, \dots, y_n are algebraical over $k(y_1, \dots, y_n)$

Need to show $M=0$.

Claim: \exists nonzero polynomial $f \in k[y_1, \dots, y_M]$

s.t. K is integral over $k[y_1, \dots, y_M, f(y_1, \dots, y_M)^{-1}]$

Proof of claim: For each y_i ($i=M+1, \dots, N$),

\exists a nonzero polynomial

$$y_i(x) = a_{d_i}^{(i)} x^{d_i} + \dots + a_1^{(i)} x + a_0 \in k[y_1, \dots, y_M][x]$$

s.t. $y_i(y_i) = 0 \Rightarrow y_i$ is integral over $k[y_1, \dots, y_M, a_{d_i}^{(i)^{-1}}]$

Now, it suffices to take $f = \prod_{i=M+1}^N a_{d_i}^{(i)}$. \square

By the previous lemma,

$k[y_1, \dots, y_M, f(y_1, \dots, y_M)^{-1}]$ is a field

So, for any polynomial $g(y_1, \dots, y_M)$,

$$\frac{1}{g(y_1, \dots, y_M)} = \frac{h(y_1, \dots, y_M)}{f^m(y_1, \dots, y_M)} \text{ for some } m \in \mathbb{Z} \text{ and polynomial } h.$$

(In other words, $g \mid f^m$ for some $m \in \mathbb{Z}$).

Infinitely many choices for irreducible $g \Rightarrow M=0$ \square

Corollaries [Cor 2.24 - 2.25] Suppose that A is a finitely generated k -algebra and $\mathfrak{m} \triangleleft A$ maximal.

(1) $k \rightarrow A/\mathfrak{m}$ is a finite extension.

(Because, A/\mathfrak{m} is still a finitely generated k -algebra)

(2) If $B \subseteq A$ is a k -subalgebra, then $B \cap \mathfrak{m} \triangleleft B$ is maximal.

Proof: Consider the chain of inclusions:

$$k \hookrightarrow B/B \cap \mathfrak{m} \hookrightarrow A/\mathfrak{m}$$

$$x + B \cap \mathfrak{m} \mapsto x + \mathfrak{m}$$

(The second map is well-defined:

$$\forall x \in B, x + \mathfrak{m} = 0 \Rightarrow x \in \mathfrak{m} \Rightarrow x \in B \cap \mathfrak{m})$$

Note that the extension $k \rightarrow A/\mathfrak{m}$ is finite (and thus algebraic).

$\Rightarrow A/\mathfrak{m}$ is integral over $B/B \cap \mathfrak{m}$

$\Rightarrow B/B \cap \mathfrak{m}$ is a field (by Lem 2.21). \square

Theorem (Nilpotents) Let A be a finitely generated k -algebra.

If $a \in A$ belongs to all maximal ideals of A , then a is nilpotent

Proof: Consider $1 - ax \in A[x]$.

For any maximal $\mathfrak{m} \trianglelefteq A[x]$, by Lem 2.21, $A \cap \mathfrak{m} \trianglelefteq A$ is maximal.

So, $a \in \mathfrak{m} \Rightarrow 1 - ax \notin \mathfrak{m}$ (otherwise $1 \in \mathfrak{m}$)

$\Rightarrow 1 - ax$ is invertible in $A[x]$

(otherwise, $1 - ax$ belongs to some maximal ideal of $A[x]$).

Suppose that $(1 - ax)(c_n x^n + \dots + c_1 x + c_0) = 1$.

By comparing the coefficients, we obtain

$$c_0 = 1$$

$$c_i - ac_{i-1} = 0 \quad (i = 1, \dots, n) \Rightarrow a^{n+1} = 0. \quad \square$$

$$ac_n = 0$$

Hilbert's 14th problem.

Let $A = k[x_1, \dots, x_n]$ and G be a group. Consider the algebra of invariants

$$A^G := \{ f \in A : f(g \cdot \vec{x}) = f(\vec{x}) \text{ for each } g \in G \}$$

Problem Is A^G always finitely generated?

Nagata's counterexample Let $A = k[x_1, y_1, \dots, x_N, y_N]$ ($N \geq 3$).

Define the group G as follows:

$$G := \left\{ \text{diag} \left\{ \begin{pmatrix} c_1 & c_1 t_1 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} c_N & c_N t_N \\ 0 & c_N \end{pmatrix} \right\} : (c_1, \dots, c_N) \in C, (t_1, \dots, t_N) \in T \right\}$$

Here, fixing a full-rank $3 \times N$ matrix (a_{ij}) ,

$$T := \left\{ (t_1, \dots, t_N) : a_{i1}t_1 + \dots + a_{iN}t_N = 0, i = 1, 2, 3 \right\}$$

$$C := \left\{ (c_1, \dots, c_N) : \prod_{i=1}^N c_i = 1 \right\}.$$

The proof is very lengthy!