Finitely generated algebras

Assumption: k is an algebraically closed field;

Every algebra is commutative and contains identity.

* A k-algebra A is said to be finitely generated if

$\exists$ (finitely many) $x_1, \cdots, x_s \in A$

s.t. every $a \in A$ can be expressed as a polynomial in $x_1, \cdots, x_s$

Integrality over a ring

Definition: Let $B \subseteq A$ be a k-subalgebra. A is said to be integral over B if

$\forall a \in A$, $f(a) = 0$ for some monic polynomial $f \in B[x]$

Lemma [Lem 2.21] If A is integral over B, then

A is a field $\Rightarrow$ B is a field

Proof: Taking $a \in B$,

$\exists$ monic polynomial $f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ s.t. $f(a^{-1}) = 0$.

Multiplying both sides $f(a^{-1}) = 0$ by a^{n-1}, we obtain

$$-\frac{1}{a} = b_{n-1} + b_{n-2} \cdot a + \cdots + b_0 \cdot a^{n-1} \in B. \quad \square$$

Zariski's Lemma [Lem 2.23] Let K/k be a field extension.

If K is a finitely generated k-algebra, then $[K:k]$ is finite.

Proof: Assume that $\{y_1, \cdots, y_N\}$ is a generating set of K s.t.

(i) $y_1, \cdots, y_M$ are algebraically independent.

(ii) $y_{M+1}, \cdots, y_N$ are algebraical over $k(y_1, \cdots, y_M)$

Need to show $M=0$.

Claim: $\exists$ nonzero polynomial $f \in k[y_1, \cdots, y_M]$

s.t. K is integral over $k[y_1, \cdots, y_M, f(y_1, \cdots, y_M)^{-1}]$

Proof of claim: For each y_i $(i = M+1, \cdots, N)$,

$\exists$ a nonzero polynomial

$$\varphi_i(x) = a_{d_i}^{(i)} x^{d_i} + \cdots + a_1^{(i)} x + a_0 \in k[y_1, \cdots, y_M][x]$$

s.t. $\varphi_i(y_i) = 0 \Rightarrow y_i$ is integral over $k[y_1, \cdots, y_M, a_{d_i}^{(i)\,-1}]$

Now, it suffices to take $f = \prod_{i=M+1}^{N} a_{d_i}^{(i)}$. $\square$

By the previous lemma,

$k[y_1, \cdots, y_M, f(y_1, \cdots, y_M)^{-1}]$ is a field

So, for any polynomial $g(y_1, \cdots, y_M)$,

$$\frac{1}{g(y_1, \cdots, y_M)} = \frac{h(y_1, \cdots, y_M)}{f^m(y_1, \cdots, y_M)}$$ for some $m \in \mathbb{Z}$ and polynomial h.

(In other words, $g \mid f^m$ for some $m \in \mathbb{Z}$).

Infinitely many choices for irreducible $g \Rightarrow M = 0$ $\square$

Corollaries [Cor 2.24 – 2.25] Suppose that A is a finitely generated k-algebra and $\mathfrak{m} \trianglelefteq A$ maximal.

(1) $k \to A/\mathfrak{m}$ is a finite extension.

(Because, $A/\mathfrak{m}$ is still a finitely generated k-algebra)

(2) If $B \subseteq A$ is a k-subalgebra, then $B \cap \mathfrak{m} \trianglelefteq B$ is maximal.

Proof: Consider the chain of inclusions:

$$k \hookrightarrow B/B\cap\mathfrak{m} \hookrightarrow A/\mathfrak{m}$$

$$x + B\cap\mathfrak{m} \mapsto x + \mathfrak{m}$$

(The second map is well-defined:

$$\forall x \in B,\ x + \underline{m} = 0 \Rightarrow x \in \underline{m} \Rightarrow x \in B \cap \underline{m})$$

Note that the extension $k \to A/\underline{m}$ is finite (and thus algebraic).

$\Rightarrow A/\underline{m}$ is integral over $B/B\cap\underline{m}$

$\Rightarrow B/B\cap\underline{m}$ is a field (by Lem 2.21). $\square$

<u>Theorem (Nilpotents)</u> Let A be a finitely generated k-algebra.

If $a \in A$ belongs to all maximal ideals of A, then a is nilpotent

<u>Proof:</u> Consider $1 - ax \in A[x]$.

For any maximal $\underline{m} \trianglelefteq A[x]$, by Lem 2.21, $A \cap \underline{m} \trianglelefteq A$ is maximal.

So, $a \in \underline{m} \Rightarrow 1 - ax \notin \underline{m}$ (otherwise $1 \in \underline{m}$)

$\Rightarrow 1 - ax$ is invertible in $A[x]$

(otherwise, $1-ax$ belongs to some maximal ideal of $A[x]$).

Suppose that $(1-ax)(c_n x^n + \cdots + c_1 x + c_0) = 1$.

By comparing the coefficients, we obtain

$$\begin{aligned} &c_0 = 1 \\ &c_i - a c_{i-1} = 0 \quad (i = 1, \cdots, n) \quad \Rightarrow \quad a^{n+1} = 0. \qquad \square \\ &a c_n = 0 \end{aligned}$$

<u>Hilbert's 14th problem.</u>

Let $A = k[x_1, \cdots, x_n]$ and G be a group. Consider the algebra of invariants

$$A^G := \{ f \in A : f(g \cdot \vec{x}) = f(\vec{x}) \text{ for each } g \in G \}$$

<u>Problem</u> Is A^G always finitely generated?

Nagata's counterexample Let $A = k[x_1, y_1, \cdots, x_N, y_N]$ $(N \geq 3)$.

Define the group G as follows:

$$G := \left\{ \operatorname{diag}\left\{ \begin{pmatrix} c_1 & c_1 t_1 \\ 0 & c_1 \end{pmatrix}, \cdots, \begin{pmatrix} c_N & c_N t_N \\ 0 & c_N \end{pmatrix} \right\} : (c_1, \cdots c_N) \in C,\ (t_1, \cdots, t_N) \in T \right\}$$

Here, fixing a full-rank $3 \times N$ matrix (a_{ij}),

$$T := \{ (t_1, \cdots, t_N) : a_{i1} t_1 + \cdots + a_{iN} t_N = 0,\ i = 1, 2, 3 \}$$

$$C := \{ (c_1, \cdots, c_N) : \prod_{i=1}^{N} c_i = 1 \}.$$

The proof is very lengthy!