Exercise 5.1.4. Let $V = \mathbb{V}(z^2-(x^2+y^2+1)(4-x^2-y^2)) \subseteq \mathbb{R}^3$ and

$$\pi : V \to \mathbb{R}^2, \quad (x,y,z) \mapsto (x,y)$$

(a)-(c). Describe V.

Solution: Indeed, for each $(a,b) \in \mathbb{R}^2$, consider the equation

$$f_{a,b}(z) = z^2-(a^2+b^2-1)(4-a^2-b^2) = 0.$$

$$\#\{\text{roots of } f_{a,b}(z)=0\} = \#\pi^{-1}(a,b)$$
$$= \begin{cases} 2, & 1<x^2+y^2<4 \\ 1, & x^2+y^2 = 1 \text{ or } 4 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$V = \{(a,b,c) : c \text{ is a root of } f_{a,b}(z)=0, \text{ for all } (a,b)\in\mathbb{R}^2\} \qquad \square$$

Exercise 5.2.15 Consider the ring $\mathbb{R}[x,y]$.

(a) Describle $\mathbb{R}[x,y]/\langle x^2,y^2\rangle$.

Solution: $\mathbb{R}[x,y]/\langle x^2,y^2\rangle = \{a[1]+b[x]+c[y]+d[xy] : a,b,c,d\in\mathbb{R}\}$.

(b) Is $\mathbb{R}[x,y]/\langle x^3,y\rangle \cong \mathbb{R}[x,y]/\langle x^2,y^2\rangle$

Solution: No. Indeed, we have

$$3 = \dim_{\mathbb{R}} \mathbb{R}[x,y]/\langle x^3,y\rangle < \dim_{\mathbb{R}} \mathbb{R}[x,y]/\langle x^2,y^2\rangle = 4.$$

Exercise 5.3.5. Let $I = \langle y+x^2-1,\ xy-2y^2+2y\rangle \subseteq \mathbb{R}[x,y]$.

(a) Construct an $\mathbb{R}$-isomorphism $\mathbb{R}[x,y]/I \cong \mathbb{R}^4$

Solution: Note that $G := \{x^2+y-1,\ xy-2y^2+2y,\ y^3-(7/4)y^2+(3/4)y\}$ is a Gröbner basis for I with lex order $x>y$. Then

$$\mathbb{R}[x,y]/I \cong \operatorname{span}\{x^\alpha y^\beta \mid x^\alpha y^\beta \notin \langle LT(G)\rangle\}$$

$\cong \operatorname{span}\{1, x, y, y^2\} \cong \mathbb{R}^4$

(b) Using the lex ordered Gröbner basis for I in (2), compute a multiplication table of $\{[1], [x], [y], [y^2]\}$ in $\mathbb{R}[x, y]/I$.

Solution:

	$[1]$	$[x]$	$[y]$	$[y^2]$
$[1]$	$[1]$	$[x]$	$[y]$	$[y^2]$
$[x]$	$[x]$	$[0]$	$[0]$	$[0]$
$[y]$	$[y]$	$[0]$	$[0]$	$[0]$
$[y^2]$	$[y^2]$	$[0]$	$[0]$	$[0]$

(c) Is $\mathbb{R}[x, y]/I$ a field?

Solution. $\mathbb{R}[x, y]/I$ is not a field. Note that $[x], [y], [y^2]$ are zero divisors in $\mathbb{R}[x, y]/I$.

(d) Compute $\mathbb{V}(I)$.

Solution. $I_1 = \langle G_1 \rangle = \langle y^3 - (7/4)y^2 + (3/4)y \rangle$
$= \langle y(y-1)(y-3/4) \rangle$

Since $\# \mathbb{V}(I) \leq 4$ (ref. Proposition 5.3.7), we have
$V = \{(1, 0), (-1, 0), (0, 1), (-1/2, 3/4)\}$.

(e) What bound does Proposition 5.3.7(ii) gives for $\#\mathbb{V}(I)$? Does Proposition 5.3.7(i) gives a better bounded?

Solution: Part (i) $\Rightarrow \# \mathbb{V}(I) \leq 4$

Part (ii) $\Rightarrow \# \mathbb{V}(I) \leq 6$. $\square$

Exercise 5.3.13 Let $I \subseteq \mathbb{C}[x_1, \cdots, x_n]$ be a binomial ideal.

(a) Prove that I has a reduced lex Gröbner basis consists of binomials.

Solution: Note that the S-polynomial of two binomial is a binomial. According to the procedure of Bechberger's algorithm, there is a lex reduced Gröbner basis.

(b) Assume that $\mathbb{V}(I) \subseteq \mathbb{C}^n$ is finite. Prove that every coordinate of a solution is either zero or a root of unity.

Solution: By theorem 5.3.6., the reduced Gröbner basis G for I is of the form

$$G = \{ x_1^{m_1} - f_1(x_1, \cdots x_n),\ x_2^{m_2} - f_2(x_2, \cdots x_n),\ \cdots,\ x_n^{m_n} - f_n(x_n), \cdots \}$$

Note that either $x_n = 0$ or $|x_n| = 1$.

This part will be proved by induction.

(c) Explain how part (b) relates to equation (4) (see p. 254)

Solution. Since $I = \langle y - x^7, x^{12} - x \rangle$ is a binomial ideal, each coordinate of a point in $\mathbb{V}$ is either zero or a root of unity. $\square$

<u>Exercise 6.4.12.</u> Let $V = \mathbb{V}(y^2 - 3x^2z + 2) \subseteq \mathbb{R}^3$ and L_A the linear map on $\mathbb{R}^3$ given by the matrix.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Verify that $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism

Solution. Indeed, L_A^{-1} is given by the matrix

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

(b) Find the equation of the image of V under L_A.

Solution. Suppose that $(\bar{x}, \bar{y}, \bar{z}) = L_A(x, y, z)$. By (a),

$$x = \frac{1}{3}(\bar{x} + \bar{y} - \bar{z})$$
$$y = \frac{1}{3}(-\bar{x} + 2\bar{y} - \bar{z})$$
$$z = \frac{1}{3}(\bar{x} - 2\bar{y} + 2\bar{z})$$

Then $\bar{x}, \bar{y}, \bar{z}$ satisfies

$$2 + \frac{1}{9}(-x + 2y - z)^2 - \frac{1}{9}(x + y - z)^2(x - 2y + 2z). \qquad \square$$

<u>Exercise 5.4.14.</u> Let $V = \mathbb{V}(y^5 - x^2) \subseteq \mathbb{R}^2$.

(a) Explain how each element of $\mathbb{R}[V]$ can be uniquely represented by a polynomial of the form $a(y) + b(y)x$.

<u>Solution</u> Using the lex order $x > y$, we obtain a k-isomorphism

$$\begin{aligned} \mathbb{R}[V] &\cong \mathbb{R}[x, y] / \langle -x^2 + y^5 \rangle \qquad \text{(see p. 265)} \\ &\cong \operatorname{span}\{x^i y^j : 0 \leq i \leq 1,\ 0 \leq j\} \\ &= \{a(y) + b(y)x : a, b \in \mathbb{R}[y]\}. \end{aligned}$$

(b) Express $(a + bx)(a' + b'x)$ in $\mathbb{R}[V]$ in the form given in (a).

Solution. Note that a, a', b, b' are polynomials in $\mathbb{R}[y]$. Then,

$$\begin{aligned} (a + bx)(a' + b'x) &= aa' + (ab' + ba')x + bb'x^2 \\ &= aa' + bb'y^5 + (ab' + ba')x \quad \operatorname{mod} \langle -x^2 + y^5 \rangle \end{aligned}$$

(c) Assume that there were some ring isomorphism $\Phi : \mathbb{R}[t] \to \mathbb{R}[V]$. Deduce a contradiction.

Solution: Since Φ is onto, we may suppose

$$\Phi(f(t)) = x, \qquad \Phi(g(t)) = y.$$

Consider the unique factorization of f and g:

$$f = f_1^{c_1} \cdots f_s^{c_s}, \quad g = g_1^{d_1} \cdots g_t^{d_t}.$$

Since Φ is injective,

$$\begin{aligned} y^5 - x^2 \equiv 0 \text{ in } \mathbb{R}[V] &\Rightarrow x^2 = \Phi(f^2) = \Phi(g^5) = y^5 \\ &\Rightarrow f^2 = f_1^{2c_1} \cdots f_s^{2c_s} = g_1^{5d_1} \cdots g_t^{5d_t} = g^5. \end{aligned}$$

This is contrary to the uniqueness. $\square$

Exercise 5.5.7 Let $S = \mathbb{V}(x^2+y^2+z^2-1)$ and $W = \mathbb{V}(z)$ the (x,y)-plane

(a) Derive the parametric equations for the line $L_q \subseteq \mathbb{R}^3$ passing through $(0,0,1)$ in S to $(x_0, y_0, z_0) \subseteq S \setminus (0,0,1)$.

Solution.
$$\begin{cases} x = tx_0 \\ y = ty_0 \\ z = 1 + t(z_0 - 1) \end{cases}$$

(b) Show that $\varphi(q) = L_q \cap W$ defines a rational mapping $\varphi : S \to \mathbb{R}^2$

Solution: Since $\varphi(q) \subseteq W$, we must have
$$0 = 1 + t(z_0 - 1) \Rightarrow t = 1/(1 - z_0).$$
Then the rational mapping $\varphi : S \to \mathbb{R}^2$ is given by setting
$$\varphi(x_0, y_0, z_0) := \left(x_0/(1-z_0),\ y_0/(1-z_0) \right).$$

(c) Show that the parametrization given by
$$x = \frac{2u}{u^2+v^2+1}, \quad y = \frac{2v}{u^2+v^2+1}, \quad z = \frac{u^2+v^2-1}{u^2+v^2+1}$$
is the inverse map of φ.

Solution: Indeed, the above parametrization gives the map
$$\phi : \mathbb{R}^2 \to S$$
$$\phi(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$
A direct verification shows that $\phi \circ \varphi = id_S$ and $\varphi \circ \phi = id_{\mathbb{R}^2}$.

(d) Deduce that S and W are birationally equivalent and find subvarieties $S' \subseteq S$ and $W' \subseteq W$ s.t. φ and ϕ put $S \setminus S'$ and $W \setminus W'$ into one-one coorrespondence.

Solution: The birational equivalence follows from (c). Moreover,
$$S' = \{(0,0,1)\}$$
$$W' = \emptyset. \qquad \square$$

<u>Exercise 5.6.1.</u> Show that for any f.g. k-algebra R, there is an algebra isomorphism for some n and ideal $I \subseteq k[x_1, \cdots, x_n]$

$$k[x_1, \cdots, x_n]/I \to R$$

Solution. Let $\{r_1, \cdots, r_n\}$ be a generating set for R.

Define the algebra homomorphism:

$$\Phi : k[x_1, \cdots, x_n] \to R, \quad f(x_1, \cdots, x_n) \mapsto f(r_1, \cdots, r_n).$$

(Clearly, Φ is onto)

Therefore, there is an isomorphism $k[x_1, \cdots, x_n]/\ker\Phi \cong R$ $\square$

<u>Exercise 5.6.5.</u> Show that $k[x_1, \cdots x_n]/I$ is finite over k iff it is a finite dimensional k-vector space.

Solution: We need only to prove "$\Rightarrow$".

Suppose that $F = \{[f_1(x)], \cdots, [f_p(x)]\} \subseteq k[x_1, \cdots, x_n]/I$

s.t. every $[f] \in k[x_1, \cdots, x_n]/I$ can be written as

$$[f] = k_1[f_1] + \cdots + k_p[f_p], \quad k_i \in k$$

Then, the basis of $k[x_1, \cdots, x_n]/I$ is some subset of F. $\square$

Exercise 5.6.18 Let $A = k[x] \times k[x]$ with $k = \bar{k}$.

(a) Show that
* $k \cong \{(a, a) \mid a \in k\} \subseteq A$
* A is reduced (contains no nilpotent)

Solution: The algebra isomorphism is given by

$$\begin{aligned} k &\xrightarrow{\sim} \{(a,a) \mid a \in k\} \\ a &\longmapsto (a, a) \end{aligned}$$

Since $A = k[x] \times k[x]$ is a domain, A is reduced.

(b) Prove that A is generated by

$$s_1 = (1, 0), \quad s_2 = (0, 1) \quad \text{and} \quad s_3 = (x, x).$$

Solution: For any $(f(x), g(x)) \in A$,

$$\begin{aligned} (f(x), g(x)) &= f(x, x)(1, 0) + g(x, x)(0, 1) \\ &= f(s_3)s_1 + g(s_3)s_2. \end{aligned}$$

(c) Prove that $k[s_3] \subseteq A$ is a Nother normalization of A.

Solution: (b) shows that A is finite over $k[s_3]$.

(d) Define a k-algebra homomorphism.

$$\varphi : k[x_1, x_2, y] \to A \quad (x_1 \mapsto s_1, \ x_2 \mapsto s_2, \ y \mapsto s_3).$$

Prove that $\ker\varphi = \langle x_1 + x_2 - 1, \ x_2^2 - x_2 \rangle$

Solution: Clearly, $I := \langle x_1 + x_2 - 1, \ x_2^2 - x_2 \rangle \subseteq \ker\varphi$.

It suffices to prove the converse. Note that $\{x_1 + x_2 - 1, \ x_2^2 - x_2\}$ is a Gröbner basis for I.

* Assume that $h(x_1, x_2, y) \notin \ker\varphi$. Using the division algorithm with respect to the lex order $x_1 > x_2$, we obtain:

$$h(x_1, x_2, y) = u(x_1, x_2, y)(x_1 + x_2 - 1) + v(x_1, x_2, y)(x_2^2 - x_2) + r(x_1, x_2, y).$$

Here, $r(x_1, x_2, y) \neq 0$ and each monomial of $r(x_1, x_2, y)$ is not divisable by x_1 or x_2^2.

This shows that
$$r(x_1, x_2, y) = ax_2 + b,$$
Therefore, $\varphi(h(x_1, x_2, y)) = (b, a+b) \neq 0$.

(e) In the affine space k^3 with coordinate x_1, x_2, y, show that

* $V = \mathbb{V}(I)$ is a disjoint union of two lines
* $k[V] \cong k[x] \times k[x]$.

Solution: By soluting the system:
$$x_1 + x_2 - 1 = 0, \qquad x_2^2 - x_2 = 0$$
we obtain two lines in k^3:
$$L_1 = \{(1, 0, y) : y \in k\},$$
$$L_2 = \{(0, 1, y) : y \in k\}.$$

* I is radical: Indeed, every $f \in k[x_1, x_2, y]$ is of the form
$$f(x_1, x_2, y) = u(x_1, x_2, y)(x_1 + x_2 - 1) + v(x_1, x_2, y)(x_2^2 - x_2) + ax_2 + b$$
If $f^m(x_1, x_2, y) = A(x_1 + x_2 - 1) + B(x_2^2 - x_2) + (ax_2 + b)^m \in I$, then
$$(ax_2 + b)^m \in I$$
The procedure of the division algorithm shows that
$$x_2^2 - x_2 \mid (ax_2 + b)^m \Rightarrow b = 0 \Rightarrow a = 0$$

Therefore, $k[V] \cong k[x_1, x_2, y] / \mathbb{I}(V)$
$$= k[x_1, x_2, y]/I = k[x_1, x_2, y]/\ker\varphi \cong A. \qquad \square$$