Quantum Information Theory

Mid Semester Exam

Wednesday, 24 September 2025

Notations and nomenclature are for problems below are from class lectures and discussion. **Attempt questions accordingly. Provide justification for your arguments and steps.** [40 points = 15 marks]

1. Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then for any quantum state ρ_A, prove the following identity:

$$\begin{aligned}\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(\rho_A) &= \mathrm{tr}_A[(\rho_A^T \otimes \mathbb{1}_B)\Gamma^{\mathcal{N}}_{AB}] &(1)\\ &= \langle\Gamma|_{A'A}\, \rho_{A'} \otimes \Gamma^{\mathcal{N}}_{AB}\, |\Gamma\rangle_{A'A}\,, &(2)\end{aligned}$$

   where A' is isomorphic to A, id_R denotes the identity channel, and $\Gamma^{\mathcal{N}}_{AB} = \mathrm{id}_{R\to A}\otimes\mathcal{N}_{A\to B}(\Gamma_{RA})$ is the Choi operator of the channel $\mathcal{N}$. [10+5 points]

2. What is the adjoint of the partial trace channel tr_A? [10 points]

3. Write a purification of the following classical-quantum state:

$$\rho_{XA} := \sum_x p_X(x)\, |x\rangle\langle x|_X \otimes \rho_A^x, \qquad (3)$$

   where p_X denotes the probability distribution and $\{|x\rangle\}_x$ forms an orthonormal basis. [5 points]

4. An erasure channel is given as

$$\mathcal{E}_{A\to B}(\rho_A) = p\rho_B + (1-p)\, |e\rangle\langle e|_B\,. \qquad (4)$$

   Find an isometric extension channel of the erasure channel. It is fine if you simply provide isometric operator $V_{A\to BE}$ such that $\mathrm{tr}_E[V(\cdot)V^\dagger] = \mathcal{E}(\cdot)$. [10 points]