GOPH 420 – Inversion and Parameter Estimation for Geophysicists (W2025)
Midterm Exam #1

Instructor: Brandon Karchewski
Duration: ~2.5 hours
Due Date: 23:59 on March 14, 2025
Format: Online asynchronous, posted March 10, 2025

<u>Note:</u> This test includes ***3 pages*** and ***3 questions*** with a total of ***45 marks***. Please check that all pages/questions are present. You are responsible for ensuring that your copy of the paper is complete. **PLEASE ENSURE THAT YOUR NAME AND UCID IS ON THE COVER OF EACH BOOKLET.**

SPECIAL INSTRUCTIONS:

1. You ***are allowed*** open access to ***any aid*** including, but not limited to, ***course textbook***, ***course notes***, ***Internet access***, ***any calculator*** (including software packages such as ***Python*** or ***spreadsheet software***).
2. This is a ***collaborative exam***. You ***are permitted*** to communicate with classmates and your instructor or TA during the exam, including verbal, written or electronic communication.
3. Clearly ***state any assumptions*** that you make and ***show/document all steps*** in the calculations to obtain full marks. For any code files and/or spreadsheet tools that you use to compute your solutions, please include sufficient comments/labels to explain each step of the calculation.
4. Any ***written work*** can be submitted in ***typed format*** (e.g. .docx, .pdf) and/or ***hand-written*** and uploaded as image files (e.g. .jpg, .png).
5. Any ***code and/or spreadsheet files*** that you create/use to compute your solutions should either be ***uploaded*** and submitted to the Dropbox folder on the course D2L page or ***shared as a repository on GitHub*** with the link clearly documented in your submission file.

<u>Question 1 – Taylor Series and Error Estimation [15 marks total]:</u>

The Stefan-Boltzmann law allows estimation of the heat energy H radiating from a surface [in W]. For a sphere, the law is

$$H = 4\pi R^2 e\sigma T^4 \tag{1}$$

where R is the radius of the sphere [in m], e is the emissivity [dimensionless], $\sigma = 5.67\times10^{-8}\ \mathrm{W\cdot m^{-2}\cdot K^{-4}}$ is the Stefan-Boltzmann constant and T is the absolute temperature [in K]. To a first approximation, Earth is a sphere with an average radius $R \approx (6.371 \pm 0.021)\times10^{6}$ m, average emissivity $e \approx (0.612 \pm 0.015)$ and average temperature $T \approx (285 \pm 5)$ K.

a) **[5 marks]** Using a first-order Taylor series, write an expression for total error in H resulting from error in the radius R, error in the emissivity e and error in the temperature T. Clearly indicate which term represents error owing to each variable.
b) **[7 marks]** Using the values given and your result from (a), compute:
   i. the relative error in the radius R,
   ii. the relative error in the emissivity e,
   iii. the relative error in the temperature T,
   iv. the expected value of the heat energy H,
   v. the total error in H (clearly indicating how much is contributed by errors in R, e, and T) and,
   vi. the relative error in H.
c) **[3 marks]** Based on your results from (b) and your understanding of Taylor series and error propagation, explain which variable requires contributes most to the error in the heat energy H. Do you think that the significance of its contribution is mostly due to the sensitivity of H to its value, or mostly due to the relative error in that variable?

Question 2 – Numerical Integration [15 marks total]:

You are part of the design team for the landing system for a Mars probe. The probe will reach the ground by first entering the atmosphere of Mars, decelerate to approximately 100 m/s using atmospheric drag, then deploy a parachute for the final landing. Your design requires that the parachute decelerate the landing vehicle from 100 m/s to a terminal velocity less than 8 m/s within 20 s of parachute deployment. An object in free fall in the atmosphere experiences forces owing to gravity and owing to drag as shown in the diagram to the left. From Newton's second law, we have that

$$m\frac{dv}{dt} = mg - cv \tag{2}$$

where m is the mass of the object [in kg], v is the velocity [in m/s], t is time [in s], g is the acceleration owing to gravity [in m/s^2], and c is a drag coefficient [in kg/s]. Near the landing site, the gravity is $g \approx 3.718$ m/s^2. The mass of your probe and landing system is $m \approx 93.0$ kg and you estimate that a drag coefficient of $c \approx 47.0$ kg/s will satisfy the design requirements. Supposing that the probe parachute and lander system reaches terminal velocity within 20 s, you would like to estimate how far the system travels during that time. You can do this by integrating the velocity function $v(t)$ from $t = 0$ s $v_0 = 100$ m/s to $t = 20$ s when $v_f \approx gm/c$. The analytical solution to equation (2) if the object starts from $v(0) = v_0$ is

$$v(t) = v_f + \left(v_0 - v_f\right)e^{-\left(\frac{c}{m}\right)t} \tag{3}$$

a) **[4 marks]** Integrate equation (3) from $t = 0$ s to $t = 20$ s using trapezoidal rule with $\Delta t = 2$ s and $\Delta t = 1$ s. Clearly state the integration result for both cases.
b) **[4 marks]** Integrate equation (3) from $t = 0$ s to $t = 20$ s using Simpson's 1/3 rule with $\Delta t = 2$ s and $\Delta t = 1$ s. Clearly state the integration result for both cases.
c) **[4 marks]** Integrate equation (3) from $t = 0$ s to $t = 20$ s using a 5-point Gauss-Legendre quadrature.
d) **[3 marks]** Compare your results from parts a)-c). What order of polynomial is being used to approximate the function in each case? Which result do you expect to be most accurate? Why? Consider both h and p refinement in interpreting your results.

Question 3 – Root Finding [15 marks total]:

You are retained as a consulting geoscientist to estimate the long term (i.e. steady state) pollution levels in a lake in the presence of a proposed industrial facility. The mass balance of a pollutant is given by

$$V\frac{dc}{dt} = W - Qc - kV\sqrt{c} = f(c) \tag{4}$$

where $V = 1.38\times10^6$ m^3 is the volume of the lake, $W = 2.15\times10^6$ g/yr is the input rate of a pollutant from an industrial facility, $Q = 1.29\times10^5$ m^3/yr is the average outflow rate from the lake and $k = 0.825$ m$^{0.5}$/g$^{0.5}$/yr is a reaction rate. [**Note:** At steady state $dc/dt = 0$, so $f(c) = 0$.]

a) **[4 marks]** At steady state, one can solve for c [in g/m^3] using fixed point iteration using one of the following expressions. For initial guesses in the range $1.0 < c < 5.0$, show whether $g_1(c)$ and/or $g_2(c)$ will always converge. Why or why not?

$$c = \left(\frac{W - Qc}{kV}\right)^2 = g_1(c) \tag{5}$$

$$c = \frac{W - kV\sqrt{c}}{Q} = g_2(c) \tag{6}$$

b) **[4 marks]** Using an initial guess $c_0 = 4.0$, use the fixed point iteration equation from a) that will converge the fastest (or at all) to determine the value of c such that at least 4 significant figures of precision have converged.

c) **[4 marks]** Using an initial guess $c_0 = 4.0$ and the first equation $f(c)$ above, use the Newton-Raphson method to determine the value of c such that at least 4 significant figures of precision have converged.

d) **[3 marks]** Compare your results from b) and c). Which method converged in fewer iterations? Why might this be the case?

THE END 😊