Competitive Programming and Mathematics Society

PROBLEM SET 3

1 Are there $n \times n$ matrices A, B such that $AB - BA = \mathcal{I}_n$?

2 Let A and B be real 3×3 matrices such that $\det A = \det B = \det(A + B) = \det(A - B) = 0$. Show that $\det(xA + yB) = 0$ for any $x, y \in \mathbb{R}$.

3 Let A be an $n \times n$ matrix such that $\sum_{j=1}^{n} |A_{i,j}| < 1$ for each i. Prove that $\mathcal{I}_n - A$ is invertible.

4 Solve the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ \cdots \quad & \quad . \\ x_{99} + x_{100} + x_1 &= 0 \\ x_100 + x_1 + x_2 &= 0 \end{aligned}$$

5 Let P be an n-th degree polynomial with complex coefficients such that $P(0), P(1), \ldots P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients.

6 Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.