Competitive Programming and Mathematics Society

PROBLEM SET 2

1 How many three-digit numbers contain no zeros or nines as digits?

2 How many three-digit numbers have a digit sum of 10?

3 Let $p_n(k)$ denote the number of permutations of n objects with k fixed points.

Show that

$$\sum_{k=0}^{n} k p_n(k) = n!$$

4 A permutation $(x_1, x_2, \ldots x_{2n})$ of the set $\{1, 2, \ldots 2n\}$, where n is a positive integer, is said to be *good* if

$$\exists i < n : |x_i - x_{i+1}| = n,$$

and is otherwise said to be *bad*. Show that, for any particular n, there are more *good* permutations than *bad* permutations.

5 At Mario's Magnificent Pizza, there are n customers and n tables, and any customer can sit at any table.

At Luigi's Luxurious Pizza, there are n customers and $(2n-1)$ tables such that customer k can sit at tables $1, 2, 3, \ldots, (2k-1)$, but not at any other table.

Let $M(n, r)$ denote the number of different ways in which r customers at Mario's can be seated at r tables, forming r customer-table pairs.

Similarly, let $L(n, r)$ denote the number of different ways in which r customers at Luigi's can be seated at r tables, forming r customer-table pairs.

Prove that $M(n, r) = L(n, r)$, for $r = 1, 2, \ldots, n$.

6 Prove the following from first principles, without invoking known combinatorial identities:

(a) Pascal's Rule: Not counting permutations, the number of ways to choose k objects from a set of n is equal to the number of ways to choose $(k-1)$ objects from a set of $(n-1)$, plus the number of ways to choose k objects from a set of $(n-1)$.

(b) Not counting permutations, the number of ways to choose an odd number of objects from a set is the same as the number of ways to choose an even number of objects from that set (where zero objects can be chosen).

(c) Binomial Theorem: For any two real numbers x and y,

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^k.$$

(d) n Choose k Formula: Not counting permutations, there are $\frac{n!}{k!(n-k)!}$ ways to choose k objects from a set of n.

(e) Hockey Stick Identity: Not counting permutations, the number of ways to choose $(k+1)$ objects from a set of $(n+1)$ is the sum of the number of ways to choose k objects from a set of 1, a set of 2, and so on up to n.

(f) The number of ways to choose k objects from a set of n is $\frac{n}{k}$ times the number of ways to choose $(k-1)$ objects from a set of $(n-1)$, not counting permutations (so long as both quantities are well-defined).