Competitive Programming and Mathematics Society

CLUBS
SUPPORTED BY Arc
INDEPENDENTLY RUN

Arithmetic Series and Generating Functions

Sarthak Sahoo and Gordon Ye

Welcome

- All workshops shall be 2 hours long.
- The notes of the contents in the workshops shall be provided on the CPMSoc website.
- Each workshop will have an accompanying problem set, which can be found in the notes.
- There will be workshops on odd numbered weeks from week 5 on wards (That's this week).
- Hope you enjoy yourselves and feel free to ask questions during the workshops ☺.

Table of contents

CPMSOC

Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

1 **Welcome**

2 **Notation**

3 **Series**
- Definition of a Series
- Telescoping Series

4 **Arithmetic Series**
- Arithmetic Functions
- Arithmetic Series

5 **Elucidation On Formal Power Series**

6 **Example Problems**

Notation

1. $\mathbb{N} = \left\{1, 2, 3 \cdots\right\}$
2. $\mathbb{P}_n = \left\{p \in \mathbb{P} : p | n,\ n \in \mathbb{N}\right\}$
3. $C[a, b] = \left\{\text{set of continuous functions on the interval [a,b]}\right\}$

Definition of a Series

Consider a sequence of terms $\{a_n\}_{n=1}^{k}$, then a finite series is defined as $\sum_{n=1}^{k} a_n$. We can extend the definition above for an infinite sequence $\{a_n\}_{n=1}^{\infty}$ to an infinite series as follows: $\lim_{k\to\infty} \sum_{n=1}^{k} a_n = \sum_{n=1}^{\infty} a_n$.

Definition

A series is said to converge to $L \in \mathbb{R} \iff \forall \epsilon > 0, \exists K > 0$ such that $\left|\sum_{n=1}^{k} a_n - L\right| \leq \epsilon, \forall k > K$. Which in the limiting form basically states that $\lim_{k\to\infty} \sum_{n=1}^{k} a_n = L$.

Note: A series is either convergent or divergent.

Definition of a Series

We now present a very important Theorem (Weak Form), this is one of the tools that frequently used to evaluate series.

Theorem (Interchange of Limit and Integral)

Let $f_n(x) \in C([a,b])$ and converge uniformly on $[a,b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\mathrm{d}x$$

Proof.

See the notes for the proof. One can skip the proof of this if the individual has yet to experience real analysis.

The definition of uniform convergent can be found there. ∎

Note: The above gives insight into why in generality we can't exchange summation and integrals or limits and integrals. Though as a heuristic in competitions one doesn't check these things completely (time constraints).
There are two ways to deal with this either we could familiarize ourselves with common functions that are uniform convergent and continuous, or simply interchange the operators and see weather one gets a valid answer.

Example

Evaluate the following, by switching the sum and the integral:

$$\int_0^1 \frac{\log(1-x)}{x} \mathrm{d}x$$

An example where integral and sum can't be interchanged.

CPMSOC

Example

Try evaluating the following, by switching the summation and the integral:

$$\int_0^\infty xe^{-x}\mathrm{d}x$$

Telescoping Series

Telescoping is one of the celebrated ways of evaluating a series at hand, and is often a technique that is used in solving problems pertaining to Series.
The principle use in telescoping is the following:
If (WLOG)

$$S = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} \left(b_n - b_{n+1}\right),$$

then we have

$$\sum_{n=1}^{k} \left(b_n - b_{n+1}\right) = \left(b_1 - b_2\right) + \left(b_2 - b_3\right) + \cdots + \left(b_k - b_{k+1}\right) = b_1 - b_{k+1}.$$

Note: A road block that one runs against immediately is the fact that recognizing b_n is not mechanical(or algorithmic), much like how recognition of $g(x)$ such that $g'(x) = f(x)$ for $\int f(x)\mathrm{d}x$. This is where it becomes an end unto itself, rather than being a means to an end.

Telescoping Series

CPMSOC

Example

Find a formula for

$$\sum_{i=1}^{n}(2i-1) = 1 + 3 + \cdots + (2n-1).$$

What can we cancel out?

Arithmetic Functions

An arithmetic functions f are defined to be $f : \mathbb{N} \to \mathbb{C}$. These can be either used to count the primes or count units of a natural number n ($\varphi(n)$), or express arithmetical properties of n.

Additive functions: An arithmetic function f is additive if $f(m+n) = f(m) + f(n), \forall \gcd(m,n) = 1$.

Multiplicative functions: An arithmetic function f is multiplicative if $f(mn) = f(m)f(n), \forall \gcd(m,n) = 1$.

An example of a multiplicative function that we have encountered is $\varphi(n)$. We introduce some other familiar arithmetic functions. Note that $n = \prod_{i=1}^{m} p_i^{a_i}$ canonically.

Important Arithmetic Functions

CPMSOC

Definition

Let $\sigma : \mathbb{N} \to \mathbb{C}$ such that

$$\sigma(n) = \sum_{d|n} d = \prod_{i=1}^{m} \left(\frac{p_i^{a_i+1} - 1}{p_i - 1} \right)$$

gives the sum of the divisors for n. **Note**: σ is multiplicative

Definition

The number of divisors of n is given by $\tau : \mathbb{N} \to \mathbb{C}$,

$$\tau(n) = \sum_{d|n} 1 = \prod_{i=1}^{m} (a_i + 1).$$

Note: τ is multiplicative

Important Arithmetic Functions

Definition (Mobius function)

The mobius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & p^2 | n \text{ for some } \; p > 1 \\ (-1)^k & n = p_1 \cdots p_k \text{ where } p_k \in \mathbb{P} \end{cases} .$$

Theorem

The mobius function μ is multiplicative.

Proof.

Let $m, n \in \mathbb{N}$ s.t. $\gcd(m, n) = 1$. If $n = 1$ then μ is clearly multiplicative. If $p^2 | m$ then $p^2 | mn$ implying that $\mu(m) = \mu(mn) = 0 = \mu(m)\mu(n)$. Consider the last case $m = p_1 \cdots p_k$, $n = q_1 \cdots q_h$, then $\mu(mn) = (-1)^{k+h} = \mu(m)\mu(n)$. ■

Example

Prove that the product of multiplicative functions is multiplicative.

Arithmetic Series

In number theory we are interested in evaluating series such as

$$F(s) = \sum_{s \in S} f(s), \text{ and } F(d) = \sum_{d|n} f(d),$$

where f is an arithmetic function and S is at most countably infinite set . We consider one of the basic series related to the totient function, namely the evaluation of the following.

Theorem (Gauss)

If φ is the Euler's totient function then,

$$\sum_{d|n} \varphi(d) = n, \forall n \geq 1.$$

Proof.

Consider

$$\left(\varphi(1) + \varphi(p_1) + \cdots + \varphi(p_1^{a_1})\right) \cdots \left(\varphi(1) + \cdots + \varphi(p_m^{a_m})\right) = \varphi(d_1) + \cdots + \varphi(d_{\tau(n)}) = \sum_{d|n} \varphi(d),$$

due $\varphi(mn) = \varphi(m)\varphi(n),\ \gcd(m,n) = 1$. Subsequently, we see that

$$\varphi(1) + \varphi(p_1) + \cdots + \varphi(p_1^{a_1}) = p_1^{a_1},$$

through telescoping and hence the result follows. ■

Definition (Summation function)

For an Arithmetic function, the summation function is defined as follows:

$$F(n) = \sum_{d|n} f(d).$$

Theorem

If f is multiplicative then so is F.

Proof.

Consider

$$F(mn) = \sum_{d|mn} f(d)$$

, where $\gcd(m, n) = 1$. Note that there must be integers k, h such that $d = kh$ and $k|m$ and $h|n$, with $\gcd(k, h) = 1$. Therefore $f(kh) = f(k)f(h)$. Hence

$$F(mn) = \sum_{d|mn} f(d) = \sum_{k|m,h|n} f(k)f(h) = \sum_{k|m} f(k) \sum_{h|n} f(h) = F(m)F(n).$$

■

Generating Functions

CPMSOC

Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

A formal power series is what a generating function is formally known as. A generating function is basically used to encode the terms of a sequence onto a power series. For instance the generating function for $\{F_n\}_{n\geq 0}$, where F_n is the Fibonacci sequence, is

$$\sum_{n=0}^{\infty} F_n x^n.$$

However we would also like a closed form for our generating functions to make things more compact and readily applicable to problems. The closed for the above is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2}.$$

Proof.

The Fibonacci sequence follows the recursion relation,

$$F_{n+2} = F_{n+1} + F_n, \;\; F_0 = F_1 = 1$$

Let $g(x) := \sum_{n=0}^{\infty} F_n x^n$, then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+2}x^n &= \sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} F_n x^n \\
\implies \frac{1}{x^2}\sum_{n=2}^{\infty} F_n x^n &= \frac{1}{x}\sum_{n=1}^{\infty} F_n x^n + g(x) \\
\implies g(x) &= \frac{1}{1-x-x^2}.
\end{aligned}$$

Note: We have already encountered generating function before (CPMSoc Workshop on Combinatorics)

Generating Functions

In general, if f is a function such that

$$f(x) = a_0u_0(x) + a_1u_1(x) + a_2u_2(x) + \cdots + a_nu_n(x) + \cdots,$$

where $a_i \in \mathbb{R}$, we say that f is the generating function of the sequence of real numbers

$$\{a_0, a_1, a_2, \cdots, a_n, \cdots\}$$

with respect to the sequence of functions

$$\{u_0, u_1, u_2, \cdots, u_n, \cdots\}.$$

For example $\sum_{n=0}^{\infty} F_nx^n = \frac{1}{1-x-x^2}$ is the generating function of $\{1, 1, 2, 3, 5, \cdots\}$ w.r.t $\{1, x, x^2, \cdots\}$.

We shall now present a list of common generating functions,

1. The Catalan numbers are given by $C_k = \frac{\binom{2k}{k}}{k+1}$,

$$\sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k}x^k = \frac{1-\sqrt{1-4x}}{2x}.$$

2. The generating function for a particular Binomial coefficient

$$\sum_{k\geq 0} \binom{2k}{k}x^k = \frac{1}{\sqrt{1-4x}}.$$

3. The Generalized Binomial Theorem

$$\sum_{k=0}^{\infty} \binom{n}{k}x^k = (1+x)^n, \quad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

where $n \in \mathbb{R}$.

Example Problems

CPMSOC

1. Evaluate the sum
$$\sum_{n\geq 1} \frac{7n+32}{n(n+2)}\left(\frac{3}{4}\right)^n.$$
2. By considering
$$\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right),$$
provide a proof for infinitude of primes.
3. Let S be the set of triples (i,j,k) of positive integers which satisfy $i+j+k=17$. Compute
$$\sum_{(i,j,k)\in S} ijk.$$
4. Prove that for any positive integer n,
$$\prod_{d|n} d = n^{\tau(n)/2}.$$