Series and Analytic Number Theory

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1 Pre-requisites

- MATH1141,MATH1241 (Calculus section).
- MATH1081/Number theory workshops of CPMSoc :) .

1.1 Notation

- $\mathbb{N} = \{1, 2, \cdots\}$
- $\mathbb{N}^* = \{0, 1, 2, \cdots\}$

2 Series

> With the exception of the geometric series, there does not exist in all of mathematics a single infinite series whose sum has been determined rigorously.
>
> — Niels Henrik Abel

2.1 Definition of a Series

Consider a sequence of terms $\{a_n\}_{n=1}^{k}$, then a finite series is defined as

$$\sum_{n=1}^{k} a_n.$$

We can extend the definition above for an infinite sequence $\{a_n\}_{n=1}^{\infty}$ to an infinite series as follows:

$$\lim_{k\to\infty} \sum_{n=1}^{k} a_n = \sum_{n=1}^{\infty} a_n.$$

A series is said to converge to $L \in \mathbb{R} \iff \forall \epsilon > 0, \exists K > 0$ such that

$$\left| \sum_{n=1}^{k} a_n - L \right| \leq \epsilon, \forall k > K.$$

Which in the limiting form basically states that

$$\lim_{k \to \infty} \sum_{n=1}^{k} a_n = L, \ L \in \mathbb{R}.$$

A series is either convergent or divergent.

We now present a very important theorem, before which we note the following definition,

Definition (Uniform Convergence): A sequence of function $f_n(x) : X \to Y$ is said to be uniform convergent to $f(x)$ iff for every $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, x \in X$

$$\|f_n(x) - f(x)\| < \epsilon.$$

Theorem 2.1 (Interchange of Limit and Integral) Let $f_n(x) \in C([a, b])$ and converge uniformly on $[a, b]$. Then

$$\lim_{n \to \infty} \int_a^b f_n(x) \mathrm{d}x = \int_a^b \lim_{n \to \infty} f_n(x) \mathrm{d}x$$

Proof. Let f be the limit of f_n. Note that f is continuous.
Write $I_n = \int_a^b f_n(x)\mathrm{d}x$ and $I = \int_a^b f(x)\mathrm{d}x$. Given $\varepsilon > 0$, we select k such that

$$\|f - f_n\| < \frac{\varepsilon}{(b-a)}, \ x \in [a, b], n > k.$$

Then

$$|I - I_n| \leq \int_a^b |f - f_n| \, \mathrm{d}x < (b-a) \frac{\varepsilon}{(b-a)} = \varepsilon, \ n > k.$$

Hence $I_n \to I$.

Corollary: if instead of f_n had we considered $f_1 + \cdots + f_n$, then we would have obtained: if $f_n \in C([a, b])$ and uniform convergent on $[a, b]$ then,

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x.$$

Note: The above gives insight into why in generality we can't exchange summation and integrals or limits and integrals. Though as a heuristic in competitions one doesn't check these things completely (time constraints).
There are two ways to deal with this either we could familiarize ourselves with common functions that are uniform convergent and continuous, or simply interchange the operators and see weather one gets a valid answer. □

2.2 Telescoping Series

Telescoping is one of the celebrated ways of evaluating a series at hand, and is often a technique that is used in solving problems pertaining to Series. Lets take a look at one of the most basic problem that uses the principle.

Problem: Prove

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Proof. The above sum is equivalent to the following,

$$\begin{aligned}\lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1\end{aligned}$$

□

Quite evidently the principle used in the above argument was as follows; If (WLOG)

$$S = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} (b_n - b_{n+1}),$$

then we have

$$\sum_{n=1}^{k} (b_n - b_{n+1}) = (b_1 - \cancel{b_2}) + (\cancel{b_2} - \cancel{b_3}) + \cdots + (\cancel{b_k} - b_{k+1}) = b_1 - b_{k+1}.$$

Note: A road block that one runs against immediately is the fact that recognizing b_n is not mechanical(or algorithmic), much like how recognition of $g(x)$ such that $g'(x) = f(x)$ for $\int f(x)\mathrm{d}x$. This is where it becomes an end unto itself, rather than being a means to an end.

2.3 Problems

1. Evaluate the following:
$$\sum_{n=1}^{\infty} \left(\frac{n}{n^4+4}\right)$$

2. Evaluate
$$\sum_{n=4}^{\infty}\left(\frac{2S_n}{n!}-\frac{1}{(n-2)!}\right),$$
where $S_n = \sum_{k=1}^{n-1} k(n-k),\ n \geq 4$.

3. Evaluate
$$\sum_{n=1}^{\infty} 2^n \left(1-2^{\frac{1}{2^n}}\right)^2.$$

4. Evaluate
$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}.$$

5. (Putnam A3 2014) Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute
$$\prod_{k=0}^{\infty}\left(1-\frac{1}{a_k}\right).$$

3 Arithmetic Series

3.1 Arithmetic Functions

An arithmetic functions f are defined to be $f : \mathbb{N} \to \mathbb{C}$. These can be either used to count the primes or count units of a natural number n ($\varphi(n)$), or express some arithmetical properties of n.

Additive functions: An arithmetic function f is additive if $f(m+n) = f(m) + f(n), \forall \gcd(m,n) = 1$.

Multiplicative functions: An arithmetic function f is multiplicative if $f(mn) = f(m)f(n), \forall \gcd(m,n) = 1$.

An example of a multiplicative function that we have encountered is $\varphi(n)$. We introduce some other familiar arithmetic functions. Note that $n = \prod_{i=1}^{m} p_i^{a_i}$.

Theorem 3.1 Let $\sigma : \mathbb{N} \to \mathbb{C}$ such that
$$\sigma(n) = \sum_{d|n} d = \prod_{i=1}^{m}\left(\frac{p_i^{a_i}-1}{p-1}\right),$$
gives the sum of the divisors for n.

Theorem 3.2 The number of divisors of n is given by $\tau : \mathbb{N} \to \mathbb{C}$,

$$\tau(n) = \sum_{d|n} 1 = \prod_{i=1}^{m} (a_i + 1).$$

Corollary: For any positive integer n,

$$\prod_{d|n} d = n^{\tau(n)/2}.$$

Definition (Mobius function): The mobius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & p^2|n \text{ for some } \ p > 1 \\ (-1)^k & n = p_1 \cdots p_k \text{ where } p_k \in \mathbb{P} \end{cases}.$$

Theorem 3.3 The mobius function μ is multiplicative.

Proof. Let $m, n \in \mathbb{N}$ s.t. $\gcd(m, n) = 1$. If $n = 1$ then μ is clearly multiplicative. If $p^2|m$ then $p^2|mn$ implying that $\mu(m) = \mu(mn) = 0 = \mu(m)\mu(n)$. Consider the last case $m = p_1 \cdots p_k,\ n = q_1 \cdots q_h$, then $\mu(mn) = (-1)^{k+h} = \mu(m)\mu(n)$. □

3.2 Arithmetic Series

In number theory we are interested in evaluating series such as

$$F(s) = \sum_{s \in S} f(s), \text{ and } F(d) = \sum_{d|n} f(d),$$

where f is an arithmetic function and S is at most countably infinite set. We consider one of the basic series related to the totient function, namely the evaluation of the following.

Theorem 3.4 (Gauss) If φ is the Euler's totient function then,

$$\sum_{d|n} \varphi(d) = n,\ n \geq 1.$$

Proof. Consider

$$(\varphi(1)+\varphi(p_1)+\cdots+\varphi(p_1^{a_1}))\cdots(\varphi(1)+\cdots+\varphi(p_m^{a_m})) = d_1+\cdots+d_{\tau(n)} = \sum_{d|n}\varphi(d),$$

due $\varphi(mn)=\varphi(m)\varphi(n),\ \gcd(m,n)=1$. Subsequently, we see that

$$\varphi(1)+\varphi(p_1)+\cdots+\varphi(p_1^{a_1}) = p_1^{a_1},$$

through telescoping and hence the result follows. □

Lets formalize the ideas behind the summation of a multiplicative function.

Def (Summation function): For an Arithmetic function, the summation function is defined as follows:

$$F(n)=\sum_{d|n} f(d).$$

Theorem 3.5 If f is multiplicative then so is F.

Proof. Consider

$$F(mn)=\sum_{d|mn} f(d)$$

, where $\gcd(m,n)=1$. Note that there must be integers k,h such that $d=kh$ and $k|m$ and $h|n$, with $\gcd(k,h)=1$. Therefore $f(kh)=f(k)f(h)$. Hence

$$F(mn)=\sum_{d|mn} f(d)=\sum_{k|m,h|n} f(k)f(h)=\sum_{k|m} f(k)\sum_{h|n} f(h)=F(m)F(n).$$

□

Theorem 3.6 (Mobius inversion) Let f be an arithmetic function and let F be its summation function. Then

$$f(n)=\sum_{d|n}\mu(d)F\left(\frac{n}{d}\right)$$

Proof. Left to the readers as an exercise. Hint: Consider the right hand side of the equality and try to use what you know to reduce it down to $f(n)$. □

The above theorem serves to recover f, when we have been given the summation function and wish to find out the arithmetic function. Therefore aptly named 'Inversion Formula'.

3.3 Problems

1. Prove that for any integer $n \geq 1$,

$$\sum_{d|n} (\tau(d))^3 = \left(\sum_{d|n} \tau(d) \right)^2 .$$

2. Let μ be the Möbius function. For $n \geq 1$, evaluate

$$\sum_{k=1}^{n} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor .$$

3. Let n be a positive integer. Prove that

$$\sum_{k \geq 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

4 Elucidation On Formal Power Series

> A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.
>
> — George Polya

A formal power series is what a generating function is formally known as. A generating function is basically used to encode the terms of a sequence onto a power series. For instance the generating function for $\{F_n\}_{n \geq 0}$, where F_n is the Fibonacci sequence, is

$$\sum_{n=0}^{\infty} F_n x^n. \tag{1}$$

However we would also like a closed form for our generating functions to make things more compact and readily applicable to problems. The closed for the above is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2}$$

Proof. The Fibonacci sequence follows the recursion relation,

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = F_1 = 1$$

Let $g(x) := \sum_{n=0}^{\infty} F_n x^n$, then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+2}x^n &= \sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} F_n x^n \\
\implies \frac{1}{x^2}\sum_{n=2}^{\infty} F_n x^n &= \frac{1}{x}\sum_{n=1}^{\infty} F_n x^n + g(x) \\
\implies g(x) &= \frac{1}{1-x-x^2}.
\end{aligned}$$

□

Note: We have already encountered generating function before (CPMSoc Workshop on Combinatorics)

In general, if f is a function such that

$$f(x) = a_0u_0(x) + a_1u_1(x) + a_2u_2(x) + \cdots + a_nu_n(x) + \cdots,$$

where $a_i \in \mathbb{R}$, we say that f if the generating function of the sequence of real numbers

$$\{a_0, a_1, a_2, \cdots, a_n, \cdots\}$$

with respect to the sequence of functions

$$\{u_0, u_1, u_2, \cdots, u_n, \cdots\}.$$

We shall now present a list of common generating functions,

1. The Catalan numbers are given by $C_k = \frac{\binom{2k}{k}}{k+1}$,

$$\sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k}x^k = \frac{1-\sqrt{1-4x}}{2x}.$$

2. The generating function for a particular Binomial coefficient

$$\sum_{k\geq 0} \binom{2k}{k}x^k = \frac{1}{\sqrt{1-4x}}.$$

3. The Generalized Binomial Theorem

$$\sum_{k=0}^{\infty} \binom{n}{k}x^k = (1+x)^n, \quad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

where $n \in \mathbb{R}$

4.1 Problems

1. Prove that
$$\sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{2x}$$

2. Prove that
$$\sum_{k\geq 0} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}.$$

3. Find the number of ways of distributing 15 apples to 5 students.