Squid Game (Part 3)

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February 12, 2022

1 Problem Statement

Find a simple expression in n equivalent to

$$S(n) = \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1}$$

2 Solution

$$\begin{aligned}
S(n) &= \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} \\
&= \sum_{k=0}^{n-2} \binom{n}{k+1} (k+1)^{k}(n-k-1)^{n-k-2} && \text{(Index adjusted)} \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{n(n-1)}{(k+1)(n-k-1)} (k+1)^{k}(n-k-1)^{n-k-2} \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} n(n-1)(k+1)^{k-1}(n-k-1)^{n-k-3} \\
&= (n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} [(k+1)+(n-k-1)](k+1)^{k-1}(n-k-1)^{n-k-3} \\
&= (n-1) \left[\sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)^{k}(n-k-1)^{n-k-3} + \sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)^{k-1}(n-k-1)^{n-k-2} \right] \\
&= 2(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)^{k}(n-k-1)^{n-k-3} && \text{(Index inverted in the second term)} \\
&= 2(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)(k+1)^{k-1}(n-k-1)^{n-k-3} \\
&= 2(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)P(k)P((n-2)-k) && \text{(Theorem 3.2)} \\
&= 2(n-1)n^{n-2} && \text{(Theorem 3.2)}
\end{aligned}$$

3 Appendix

3.1 Parking Functions

Theorem Statements
A series of n cars $(c_i, i = 1, 2, 3, ..., n)$ drive in a straight line into a one way lane (no U-turn) containing a series of n parking spaces $(s_i, i = 1, 2, 3, ..., n)$. Each car has a parking space they prefer (more than one car can have the same preferred space). They will drive until they reach their preferred space and park if the said space is empty. If the space is filled, they continue driving until they find the next closest empty space (only forward/without a U-turn) and park there. If a car reaches the end of the parking lot (space n) without being able to find a parking spot, the process ends.
Let $\alpha = (a_1, a_2, a_3, ..., a_n)$ be a function of parking space preference, i.e. car c_i prefers space a_i If all n cars are able to park, then α is considered a parking function of length n. If the process ends with some cars not being able to park, α is not a parking function.
Here, we are interested in the number of parking functions of each given length n

Lemma 3.1. *Every permutation of the entries of a parking function is also a parking function.*

Proof. Consider two consecutive entries $a_p = x$ and $a_{p+1} = y$. In the initial parking function α_1, the car c_p parks at space $m \leq x$ and the car c_{p+1} parks at space $n \leq y$. This implies that the spaces $i, 1 \leq i < m \vee m < i < n$ are filled prior to the car a_p arriving.
We proceed to swap their preferences ($a_p = y$ and $a_{p+1} = x$) to check whether this new function α_2 is still a parking function.

Case 1. $y \leq m$

The car c_p parks at the space m and the car c_{p+1} parks at the space n

Case 2. $m < y \leq n$

The car c_p parks at the space n and the car c_{p+1} parks at the space m
This does not affect parking positions of other cars, thus the function α_2 is a parking function.
We can swap the consecutive entries as many time as it takes to reach any possible permutations of the original function. □

Theorem 3.2 (Closed Form)**.** *The number of parking functions of length n is given by*

$$P(n) = (n+1)^{n-1}$$

Proof. Add an additional space $n+1$ and arrange the spaces in a circle instead of a straight line, allowing $n+1$ also as an option for a preferred space.
This allows the cars that reach the last space to start again from the first space. Hence, all cars is guaranteed to be able to park. Since there are $n+1$ spaces but only n cars, there will always be one empty space at the end of the process. α is a parking function of length n if and only if the said empty space is $n+1$. Here, it is clear that α does not contain $n+1$, otherwise the space $n+1$ must be parked in. If the empty spot is one other than $n+1$, there always exists a function with identical cyclic permutation. As a consequence of lemma 3.1, said function is a parking function. Each class of a cyclic permutation contains n+1 members (empty space from 1 to $n+1$) and in each class, there is only one viable parking function of length n.
Hence,

$$P(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$

Note: Modulo notations are omitted for simplicity $\square$

Theorem 3.3 (Recurrence Form). *The number of parking functions of length n is given by*

$$P(n) = \sum_{i=0}^{n-1} \binom{n-1}{i}(i+1)P(i)P(n-i-1)$$

Proof. Given n cars and n spaces, we split cases based on the space the car c_n parks at the end. Consider case i^{th}, where the car c_n parks at $i+1$. A subset S of i cars (not necessarily consecutively ordered) fill the space 1 to i. This implies two things:

- Each car in the set S must have a preference of space between 1 and i
- Cars in the set $S' - \{c_n\}$ must have a preference of space between $i+2$ and n, otherwise the space $i+1$ must have been filled by the time the car c_n arrives

There are

- $P(i)$ possible functions of preference for the cars in S
- $P(n-i-1)$ possible functions of preference for the cars in $S' - \{c_n\}$
- $\binom{n-1}{i}$ ways to split the cars into the two sets
- $i+1$ possible preference of the car c_n from 1 to $i+1$

Thus, the number of possible parking functions in this case is

$$\binom{n-1}{i}(i+1)P(i)P(n-i-1)$$

Summing over all cases, the overall total number of functions is given by

$$P(n) = \sum_{i=0}^{n-1} \binom{n-1}{i}(i+1)P(i)P(n-i-1)$$

$\square$

We can therefore equate closed form and the recurrence form for use in this problem

$$P(n-1) = n^{n-2} = \sum_{i=0}^{n-2} \binom{n-2}{k}(k+1)P(k)P((n-k-2)$$