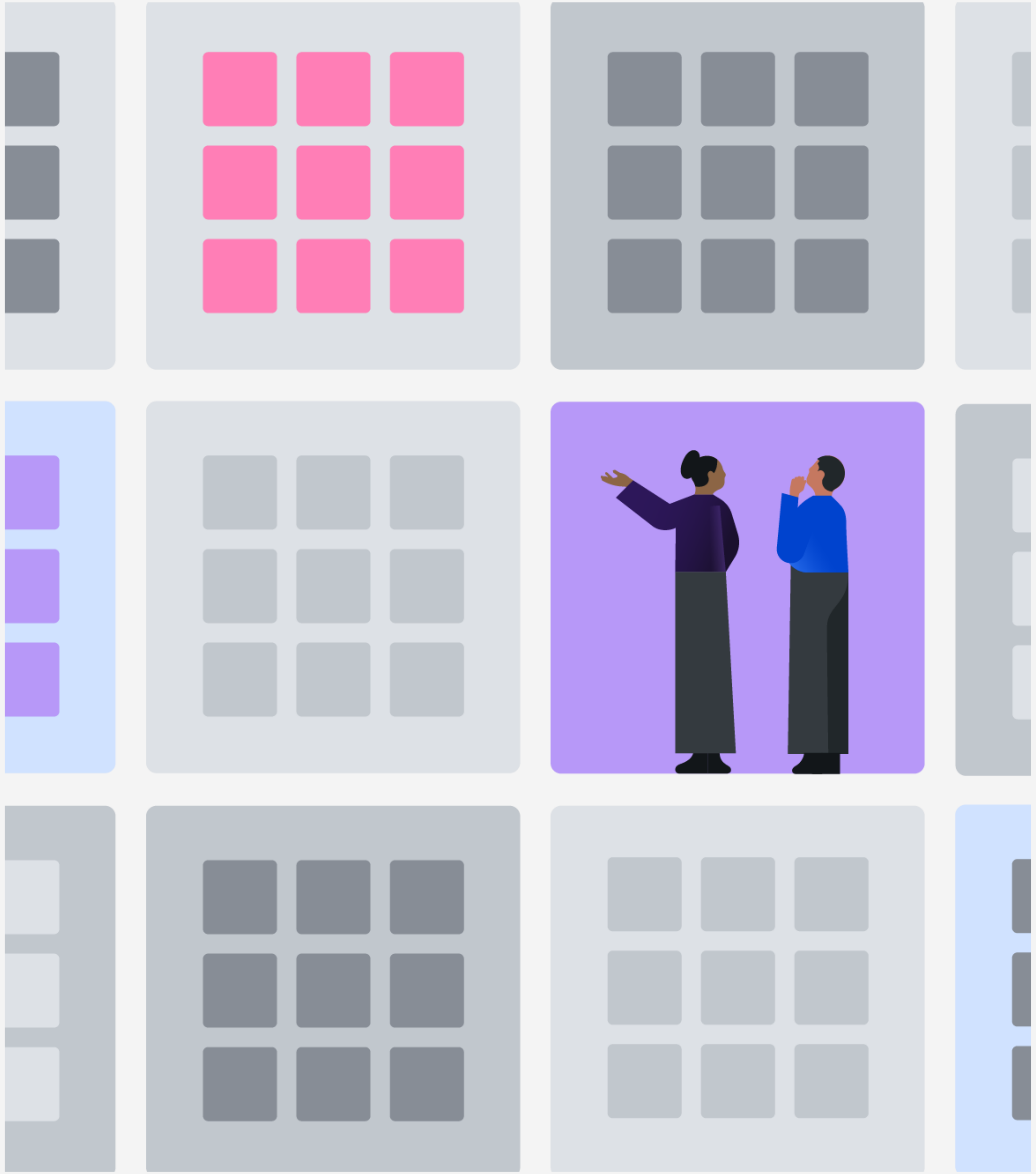
Understanding quantum information and computation

By John Watrous



Lesson 9

Density matrices

IBM

Motivation

- Density matrices represent a *broader class* of quantum states than quantum state vectors.
- Density matrices can describe states of *isolated parts* of systems, such as the state of one system that happens to be entangled with another system that we wish to ignore.
- Probabilistic states can be represented by density matrices, allowing quantum and classical information to be described together within a *single mathematical framework.*

Definition of density matrices

Suppose that X is a system and Σ is its classical state set.

A *density matrix* describing a state of X is matrix with complex number entries whose rows and columns have been placed in correspondence with Σ.

Typical names for density matrices: ρ, σ, ξ, ...

Requirements

1. Density matrices have unit trace: $\operatorname{Tr}(\rho) = 1$
2. Density matrices are positive semidefinite: $\rho \geq 0$

Definition of density matrices

Requirements

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2. Density matrices are positive semidefinite: $\rho \geq 0$

The first condition refers to the *trace,* which is defined for all square matrices as the sum of the diagonal entries.

$$\mathrm{Tr}\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-1} \end{pmatrix} = \alpha_{0,0} + \alpha_{1,1} + \cdots + \alpha_{n-1,n-1}$$

The trace is a linear function:

$$\mathrm{Tr}(\alpha A + \beta B) = \alpha\,\mathrm{Tr}(A) + \beta\,\mathrm{Tr}(B)$$

Definition of density matrices

Requirements

1. Density matrices have unit trace: $\mathrm{Tr}(\rho) = 1$
2. Density matrices are positive semidefinite: $\rho \geq 0$

The second requirement refers to ρ being *positive semidefinite.* This is a property that can be expressed in several different ways, including these:

- There exists a matrix M such that $\rho = M^\dagger M$.
- The matrix ρ is Hermitian, meaning that $\rho = \rho^\dagger$, and all of its eigenvalues are nonnegative real numbers.
- For every complex vector $|\psi\rangle$ we have $\langle\psi|\rho|\psi\rangle \geq 0$.

Note that the notation $\rho \geq 0$ means that ρ is positive semidefinite — not that each entry of ρ is nonnegative.

Examples

Requirements

1. Density matrices have unit trace: $\mathrm{Tr}(\rho) = 1$
2. Density matrices are positive semidefinite: $\rho \geq 0$

Example of a positive semidefinite matrix

We can generate an example of a positive semidefinite matrix by first choosing a matrix M arbitrarily and computing $M^\dagger M$.

$$M = \begin{pmatrix} -4-9i & 8 & 2+9i \\ -7 & -4-9i & -9+7i \\ 1-5i & 8-6i & -4i \end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix} 172 & 34+169i & -6-71i \\ 34-169i & 261 & 13-69i \\ -6+71i & 13+69i & 231 \end{pmatrix}$$

Examples

Requirements

1. Density matrices have unit trace: $\mathrm{Tr}(\rho) = 1$
2. Density matrices are positive semidefinite: $\rho \geq 0$

Examples of density matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{4} & \frac{i}{8} \\ -\frac{i}{8} & \frac{1}{4} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\frac{1}{664} \begin{pmatrix} 172 & 34 + 169i & -6 - 71i \\ 34 - 169i & 261 & 13 - 69i \\ -6 + 71i & 13 + 69i & 231 \end{pmatrix}$$

Interpretation

Requirements

1. Density matrices have unit trace: $\mathrm{Tr}(\rho) = 1$
2. Density matrices are positive semidefinite: $\rho \geq 0$

Intuitive meaning of density matrix entries

For a given density matrix both the rows and the columns correspond to classical states.

- *Diagonal* entries are the probabilities for each classical state to appear from a standard basis measurement.
- *Off-diagonal* entries describe how the two corresponding classical states are *in quantum superposition.*

Connection to state vectors

A quantum state vector $|\psi\rangle$ is a column vector having Euclidean norm 1.

Here's the density matrix representation of the same state:

$$|\psi\rangle\langle\psi|$$

States that are represented by density matrices of this form are called *pure states.*

Examples

$$|+i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$|+i\rangle\langle +i| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

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Examples

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |+\rangle\langle +| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad |-\rangle\langle -| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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Examples

$$|v\rangle = \frac{1+2i}{3}\,|0\rangle - \frac{2}{3}\,|1\rangle$$

$$|v\rangle\langle v| = \begin{pmatrix} \frac{5}{9} & \frac{-2-4i}{9} \\ \frac{-2+4i}{9} & \frac{4}{9} \end{pmatrix}$$

Connection to state vectors

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Here's the density matrix representation of the same state:

$$|\psi\rangle\langle\psi|$$

States that are represented by density matrices of this form are called *pure states.*

In general

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \quad \Rightarrow \quad |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\overline{\alpha_1} & \cdots & \alpha_0\overline{\alpha_{n-1}} \\ \alpha_1\overline{\alpha_0} & |\alpha_1|^2 & \cdots & \alpha_1\overline{\alpha_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1}\overline{\alpha_0} & \alpha_{n-1}\overline{\alpha_1} & \cdots & |\alpha_{n-1}|^2 \end{pmatrix}$$

Connection to state vectors

A quantum state vector $|\psi\rangle$ is a column vector having Euclidean norm 1.

Here's the density matrix representation of the same state:

$$|\psi\rangle\langle\psi|$$

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Remark

There is no *global phase* degeneracy for density matrices: two quantum states are identical if and only if their density matrix representations are equal.

Suppose $|\psi\rangle$ and $|\phi\rangle$ are quantum state vectors that differ by a global phase:

$$|\phi\rangle = e^{i\theta}|\psi\rangle$$

The corresponding density matrices are identical:

$$|\phi\rangle\langle\phi| = \big(e^{i\theta}|\psi\rangle\big)\big(e^{i\theta}|\psi\rangle\big)^{\dagger} = e^{i(\theta-\theta)}|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$$

Probabilistic selections

Key property of density matrices

Convex combinations of density matrices represent probabilistic selections of quantum states.

Let ρ and σ be density matrices representing quantum states of a system, and suppose we prepare the system in state ρ with probability $p \in [0, 1]$ and σ with probability $1 - p$.

The resulting state is represented by this density matrix:

$$p\rho + (1 - p)\sigma$$

More generally, let $\rho_0, \ldots, \rho_{m-1}$ be density matrices, let $(p_0, \ldots, p_{m-1})$ be a probability vector, and suppose we prepare a system in state ρ_k with probability p_k.

The resulting state is represented by this density matrix:

$$\sum_{k=0}^{m-1} p_k \rho_k$$

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The resulting state is represented by this density matrix:

$$\sum_{k=0}^{m-1} p_k \rho_k$$

If $|\psi_0\rangle, \ldots, |\psi_{m-1}\rangle$ are quantum state vectors and we prepare a system in state $|\psi_k\rangle$ with probability p_k, then the resulting state is represented by this density matrix:

$$\sum_{k=0}^{m-1} p_k |\psi_k\rangle\langle\psi_k|$$

Probabilistic selections

Key property of density matrices

Convex combinations of density matrices represent probabilistic selections of quantum states.

Example

A qubit is prepared in the state $|0\rangle$ with probability $1/2$ and in the state $|+\rangle$ with probability $1/2$.

Here's the density matrix representation of the resulting state:

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Completely mixed state

Suppose we set the state of a qubit to be $|0\rangle$ or $|1\rangle$ randomly, each with probability $1/2$. Here's the density matrix representation of its state.

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{\mathbb{1}}{2}$$

This is known as the *completely mixed state* — it represents complete uncertainty about the state of a qubit.

Now suppose that we change the procedure: in place of the states $|0\rangle$ and $|1\rangle$ we'll use the states $|+\rangle$ and $|-\rangle$.

$$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{\mathbb{1}}{2}$$

It's the same state as before, the completely mixed state. This is a feature, not a bug! The two procedures can't be distinguished by measuring the qubit.

Probabilistic states

Classical states can be represented by density matrices.

Example

Density matrices for the classical states 0 and 1 of a qubit:

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In general

If a system has classical state set Σ, then each classical state $a \in \Sigma$ is represented by the density matrix $|a\rangle\langle a|$.

Probabilistic states

In general

If a system has classical state set Σ, then each classical state $a \in \Sigma$ is represented by the density matrix $|a\rangle\langle a|$.

Probabilistic states are represented by *convex combinations* of such density matrices.

Example

Suppose the classical state set of X is $\{0, \ldots, n-1\}$.

We can identify the probabilistic state of X represented by the probability vector $(p_0, \ldots, p_{n-1})$ with this density matrix:

$$\rho = \sum_{k=0}^{n-1} p_k |k\rangle\langle k| = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ 0 & p_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Spectral theorem

Spectral theorem for positive semidefinite matrices

Suppose P is an $n \times n$ positive semidefinite matrix.

There exists an orthonormal basis $\{|\psi_0\rangle, \ldots, |\psi_{n-1}\rangle\}$ of vectors along with *nonnegative real* numbers $\lambda_0, \ldots, \lambda_{n-1}$ such that

$$P = \sum_{k=0}^{n-1} \lambda_k |\psi_k\rangle\langle\psi_k|$$

Implication for density matrices

Any $n \times n$ density matrix ρ can be expressed as

$$\rho = \sum_{k=0}^{n-1} p_k |\psi_k\rangle\langle\psi_k|$$

for an orthonormal basis $\{|\psi_0\rangle, \ldots, |\psi_{n-1}\rangle\}$ and a probability vector $(p_0, \ldots, p_{n-1})$.

Spectral theorem

Example

Recall the example where a qubit is prepared in the state $|0\rangle$ with probability $1/2$ and $|+\rangle$ with probability $1/2$.

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

This density matrix can alternatively be expressed as follows:

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \cos^2\left(\frac{\pi}{8}\right)|\psi_{\pi/8}\rangle\langle\psi_{\pi/8}| + \sin^2\left(\frac{\pi}{8}\right)|\psi_{5\pi/8}\rangle\langle\psi_{5\pi/8}|$$

$$|\psi_\alpha\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$$

Qubit quantum state vectors

Up to a *global phase,* every qubit quantum state vector is equivalent to

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

for $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

Almost uniqueness

- If $\theta = 0$ or $\theta = \pi$, then ϕ is *irrelevant.*
- If $\theta \in (0, \pi)$, then ϕ is *unique.*

Pure states of a qubit

$$|\psi\rangle = \cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

Here's this state's density matrix representation:

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\left(\tfrac{\theta}{2}\right) & e^{-i\phi}\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right) & \sin^2\left(\tfrac{\theta}{2}\right) \end{pmatrix}$$

Like all 2×2 matrices, there's a unique way to express this density matrix as a *linear combination of Pauli matrices.*

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2}$$

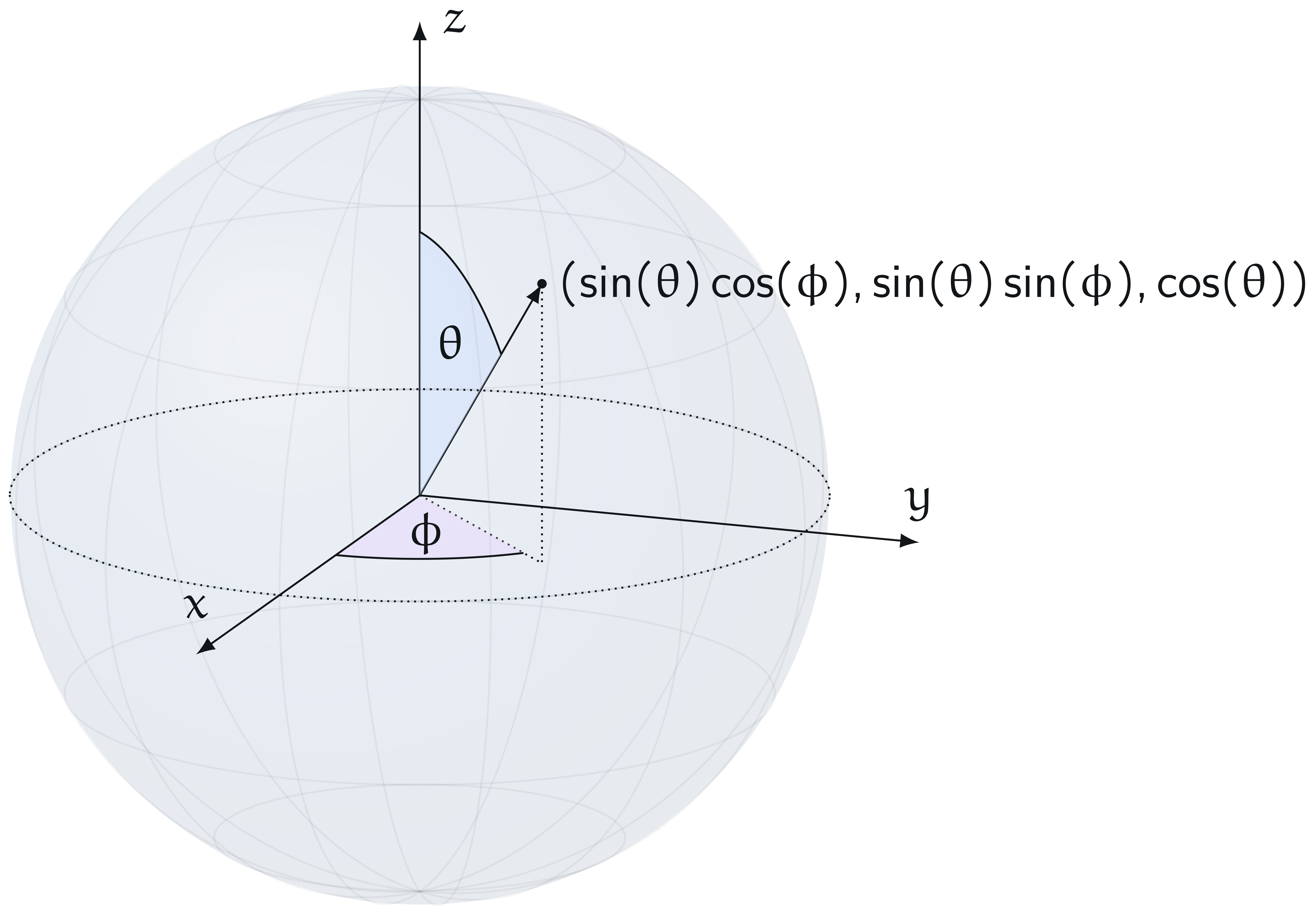
$$\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{2}$$

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



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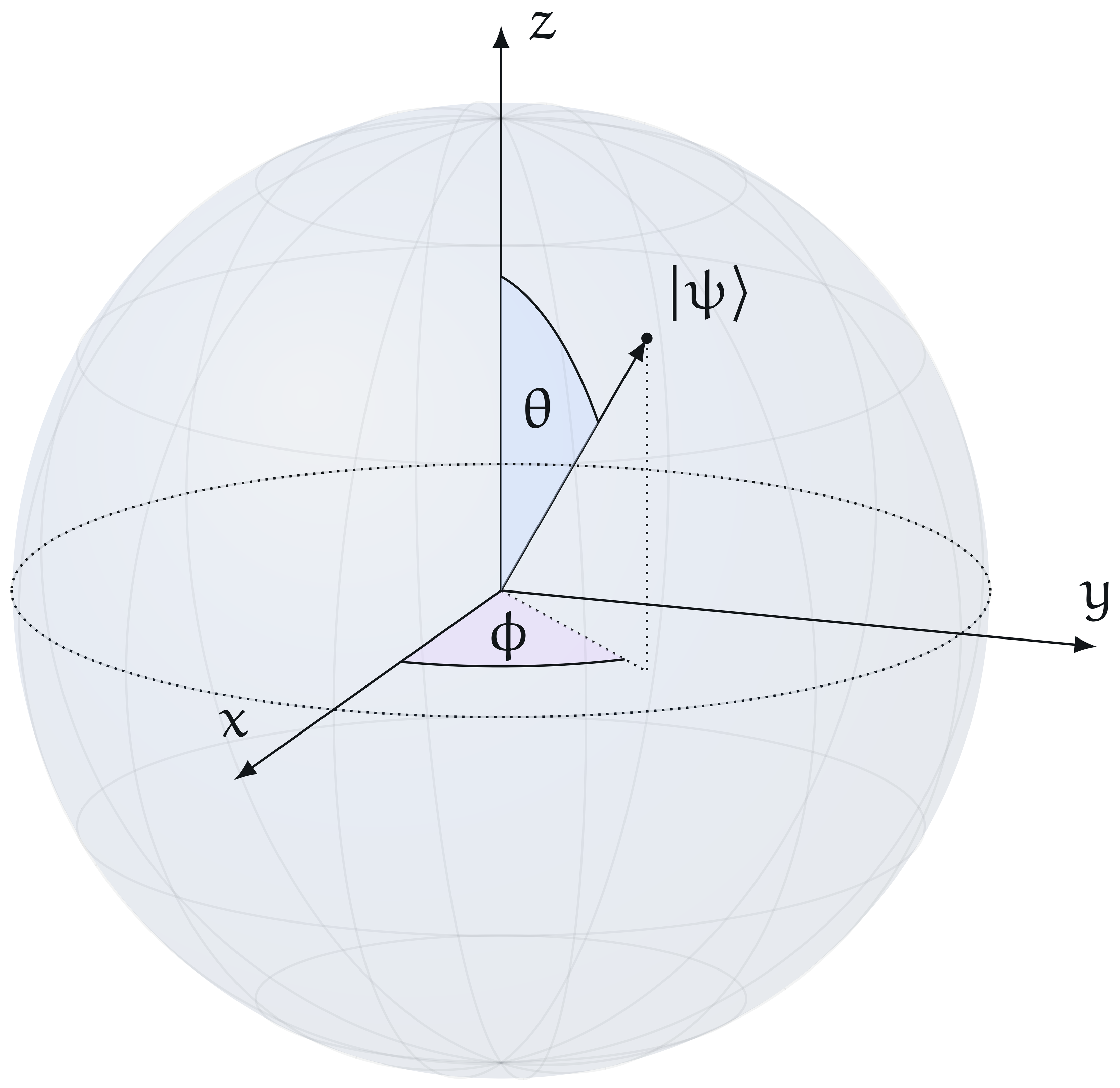
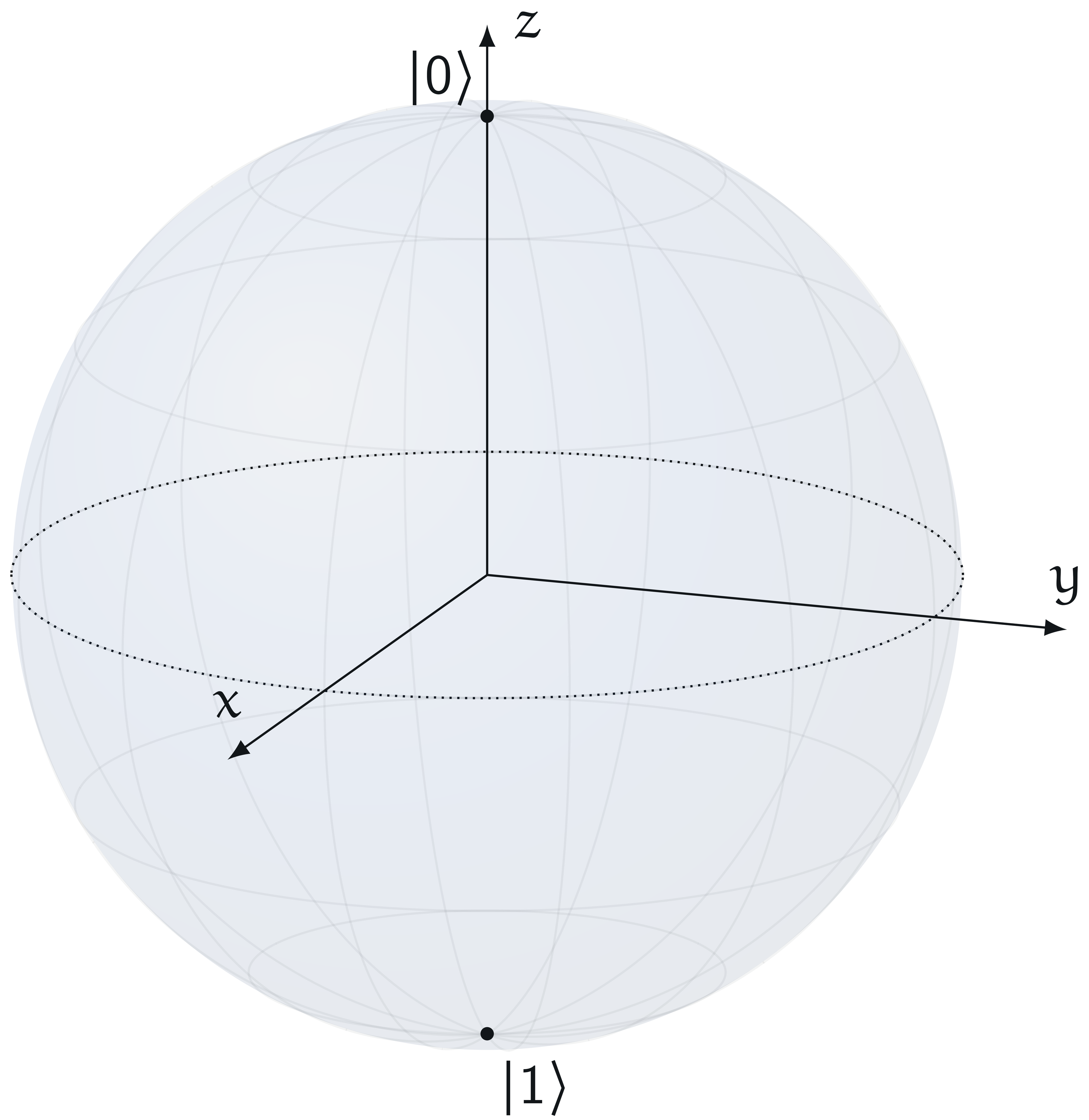


This is the *Bloch sphere* representation of the qubit state $|\psi\rangle$.

Bloch sphere examples

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



Examples

The standard basis $\{|0\rangle, |1\rangle\}$

$$|0\rangle = \cos(0)|0\rangle + e^{i\phi}\sin(0)|1\rangle$$

$$|1\rangle = \cos\left(\frac{\pi}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\pi}{2}\right)|1\rangle$$

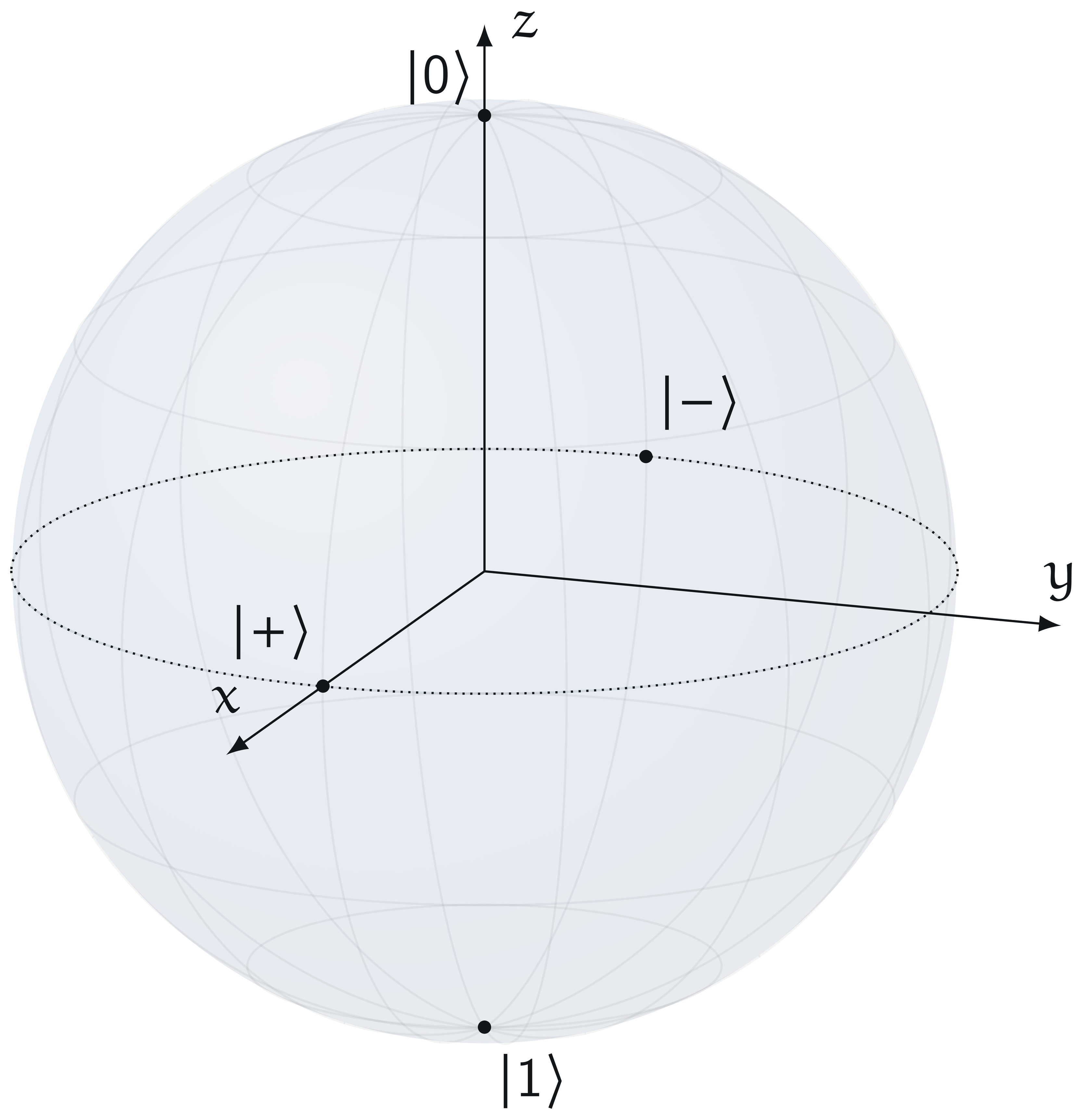
$$|0\rangle\langle 0| = \frac{\mathbb{1} + \sigma_z}{2}$$

$$|1\rangle\langle 1| = \frac{\mathbb{1} - \sigma_z}{2}$$

Bloch sphere examples

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



Examples

The basis $\{|+\rangle, |-\rangle\}$

$$|+\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i0}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$|-\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\pi}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

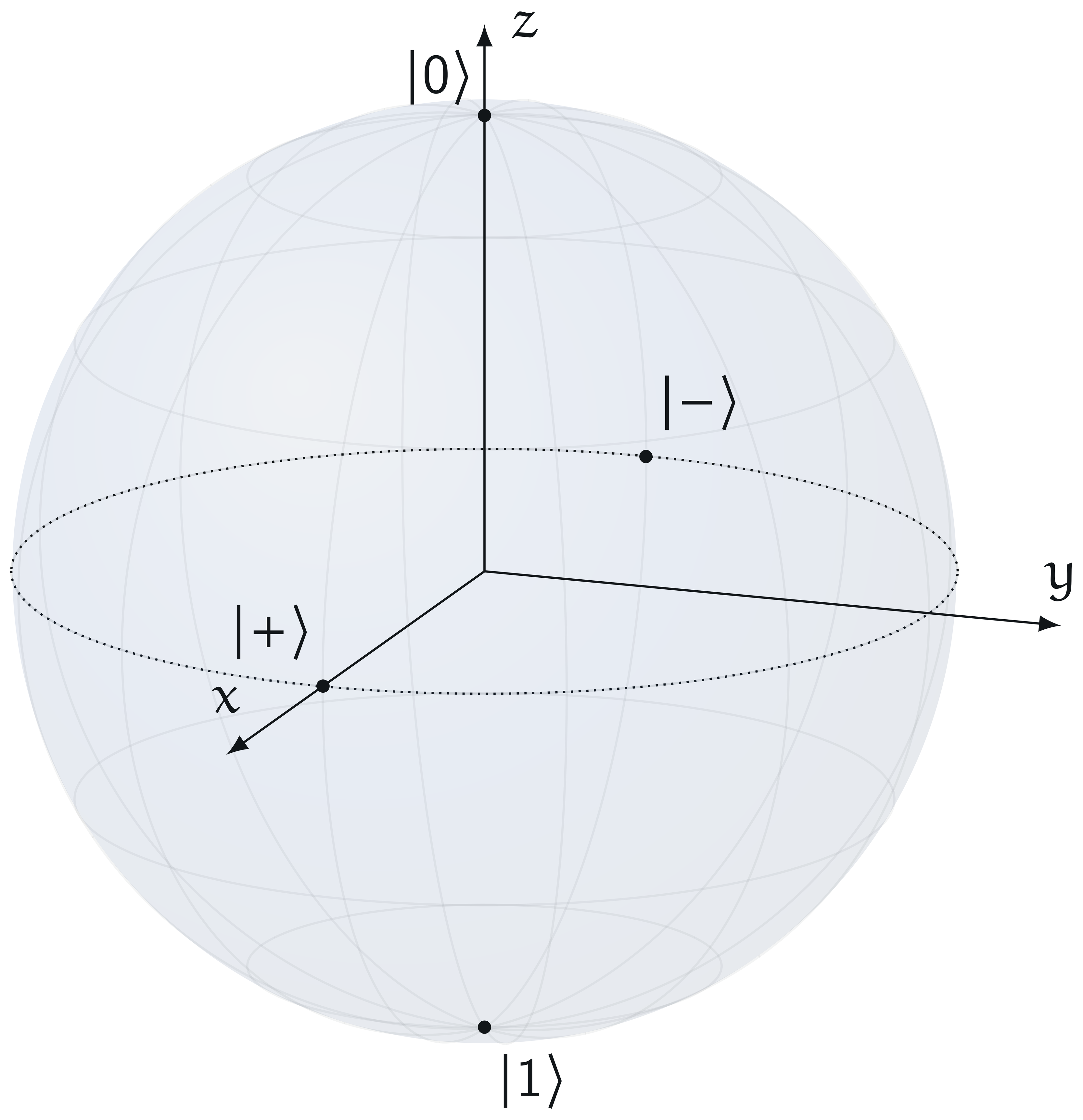
$$|+\rangle\langle+| = \frac{\mathbb{1} + \sigma_x}{2}$$

$$|-\rangle\langle-| = \frac{\mathbb{1} - \sigma_x}{2}$$

Bloch sphere examples

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



Examples

The basis $\{|{+i}\rangle, |{-i}\rangle\}$

$$|{+i}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

$$|{-i}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

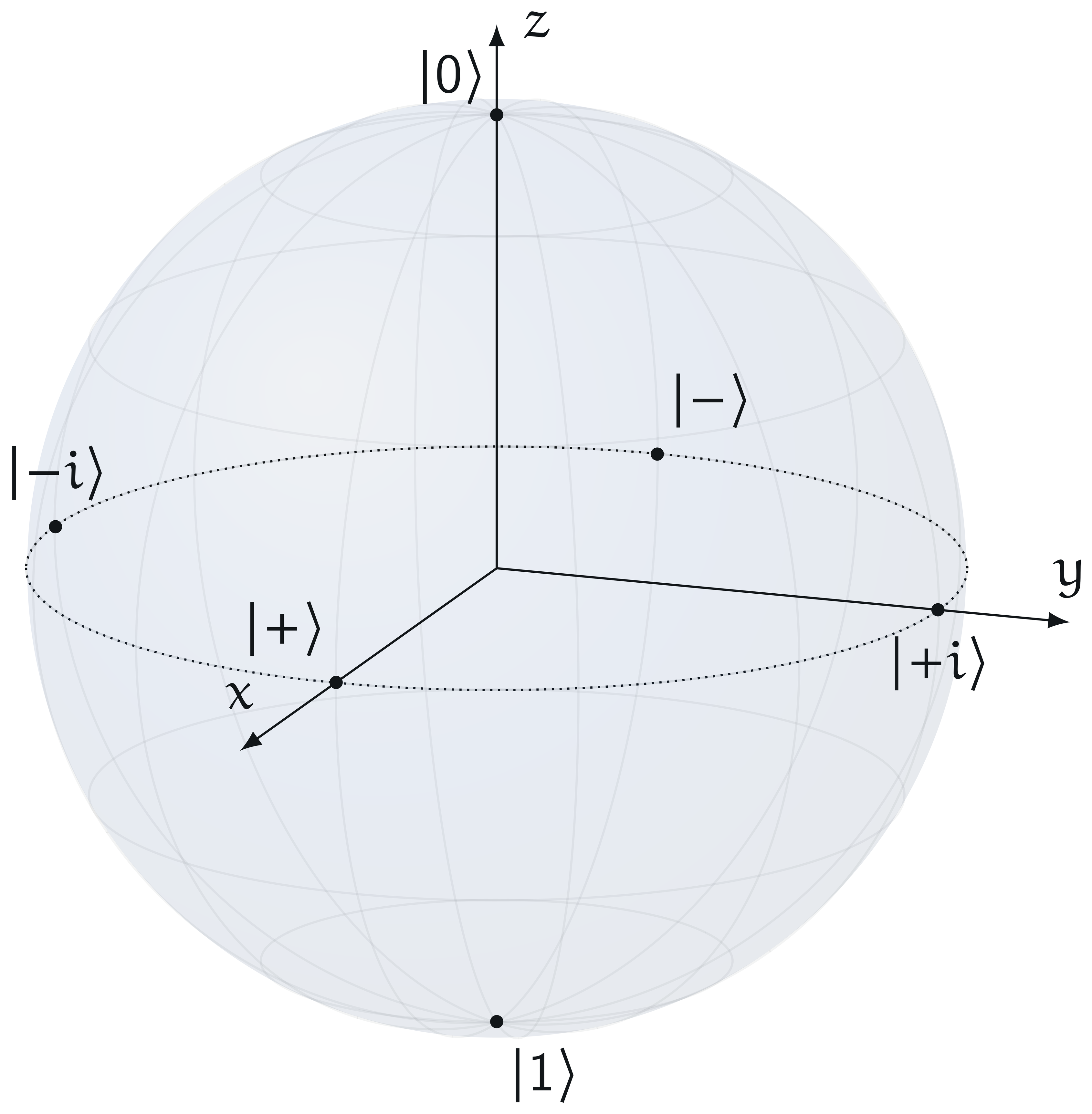
$$|{+i}\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$|{-i}\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\frac{3\pi}{2}}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

Bloch sphere examples

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



Examples

The basis $\{|+i\rangle, |-i\rangle\}$

$$|+i\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$|-i\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\frac{3\pi}{2}}\sin\left(\frac{\pi}{4}\right)|1\rangle$$

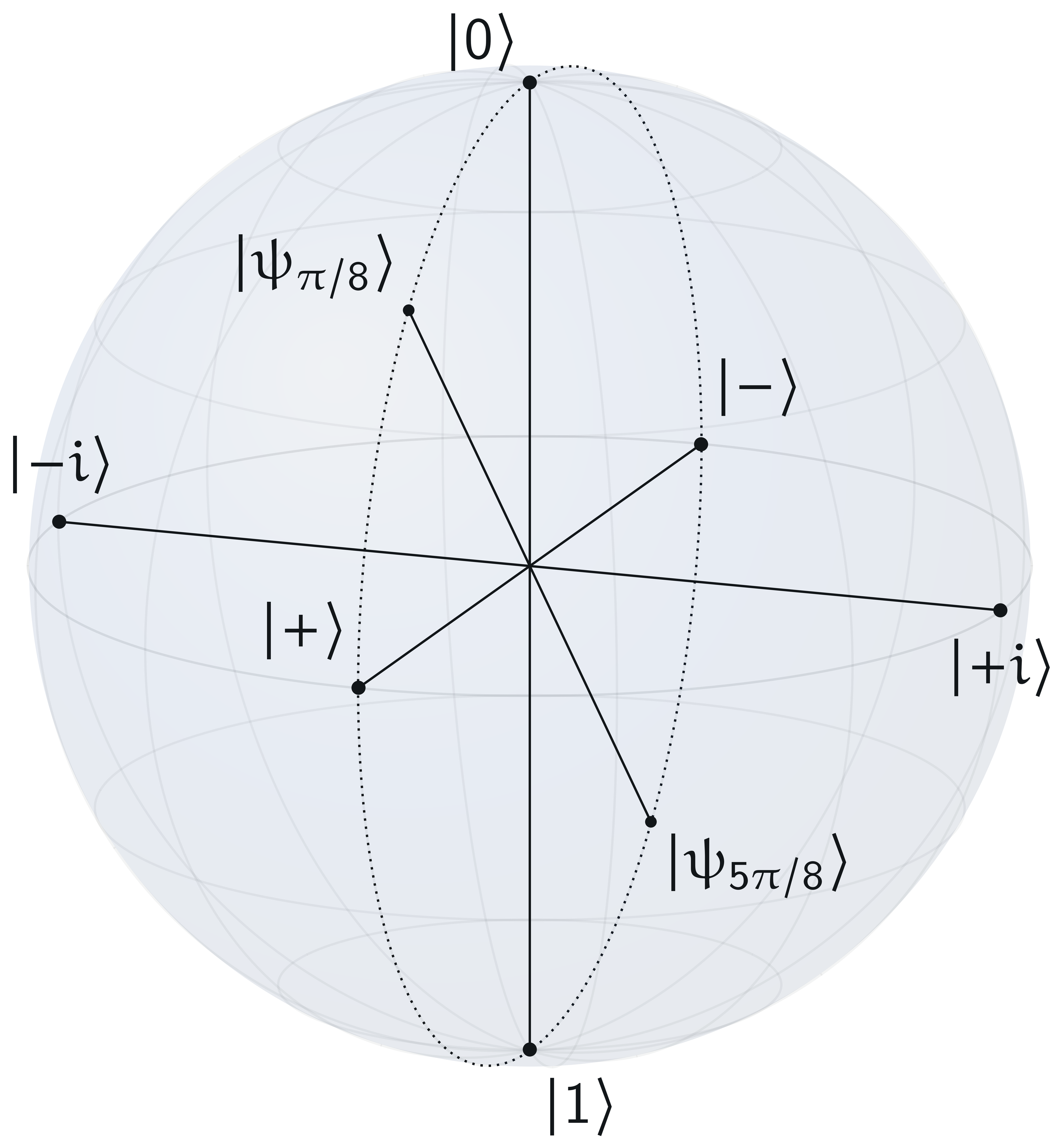
$$|+i\rangle\langle+i| = \frac{\mathbb{1} + \sigma_y}{2}$$

$$|-i\rangle\langle-i| = \frac{\mathbb{1} - \sigma_y}{2}$$

Bloch sphere examples

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [0,2\pi)$$

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z}{2}$$



Examples

Another class of state vectors:

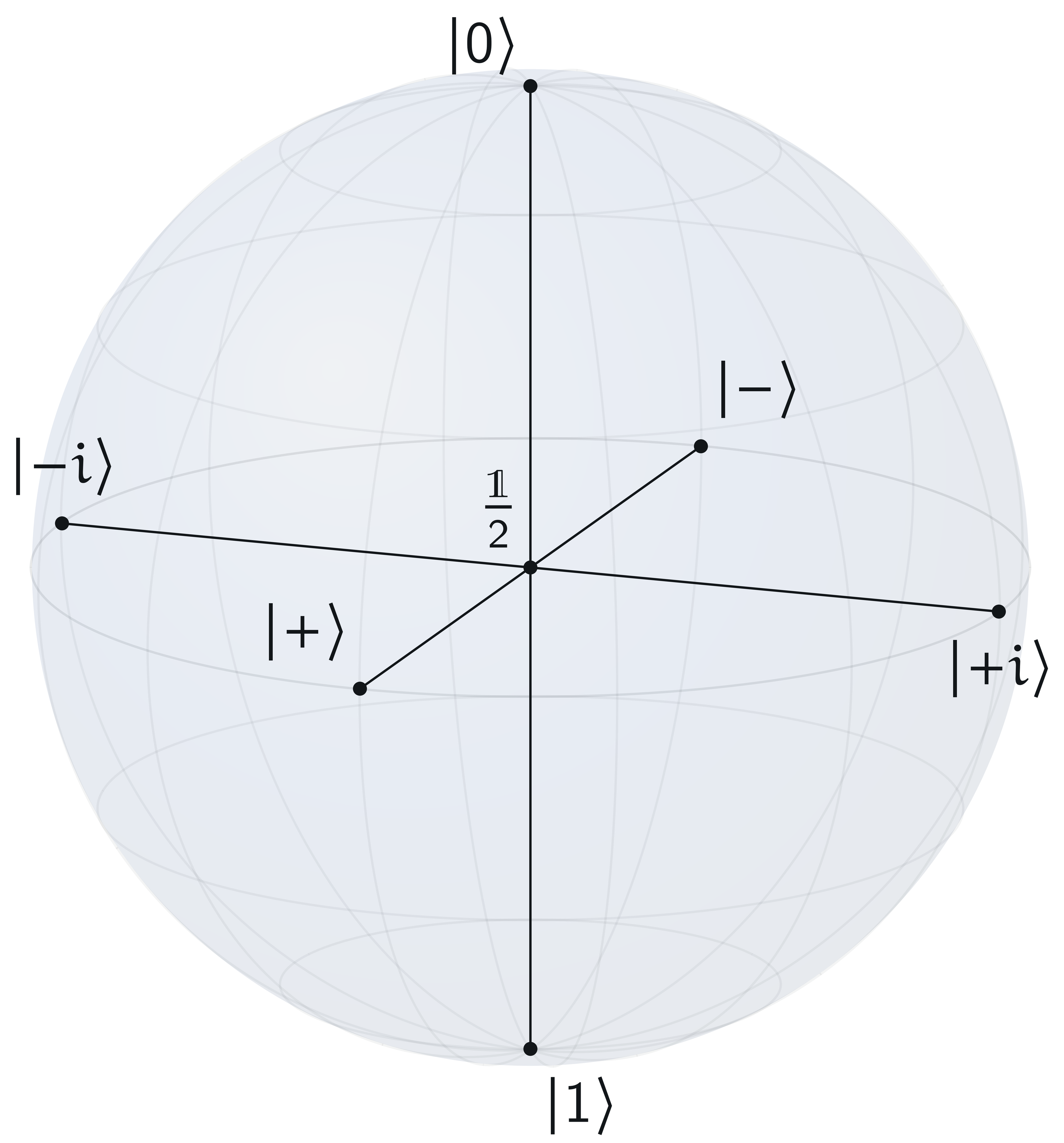
$$|\psi_\alpha\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle, \;\; \alpha \in [0,\pi)$$

$$|\psi_\alpha\rangle\langle\psi_\alpha| = \frac{\mathbb{1} + \sin(2\alpha)\sigma_x + \cos(2\alpha)\sigma_z}{2}$$

Bloch ball

The *Bloch ball* contains all of the points on the Bloch sphere as well as those inside it.

- Points in the interior of the Bloch ball correspond to qubit states that are not pure.
- Convex combinations of points in the Bloch ball correspond to convex combinations of qubit density matrices.



Examples

The completely mixed state lies at the center of the Bloch ball.

$$\begin{aligned}\frac{\mathbb{1}}{2} &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\ &= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i|\end{aligned}$$

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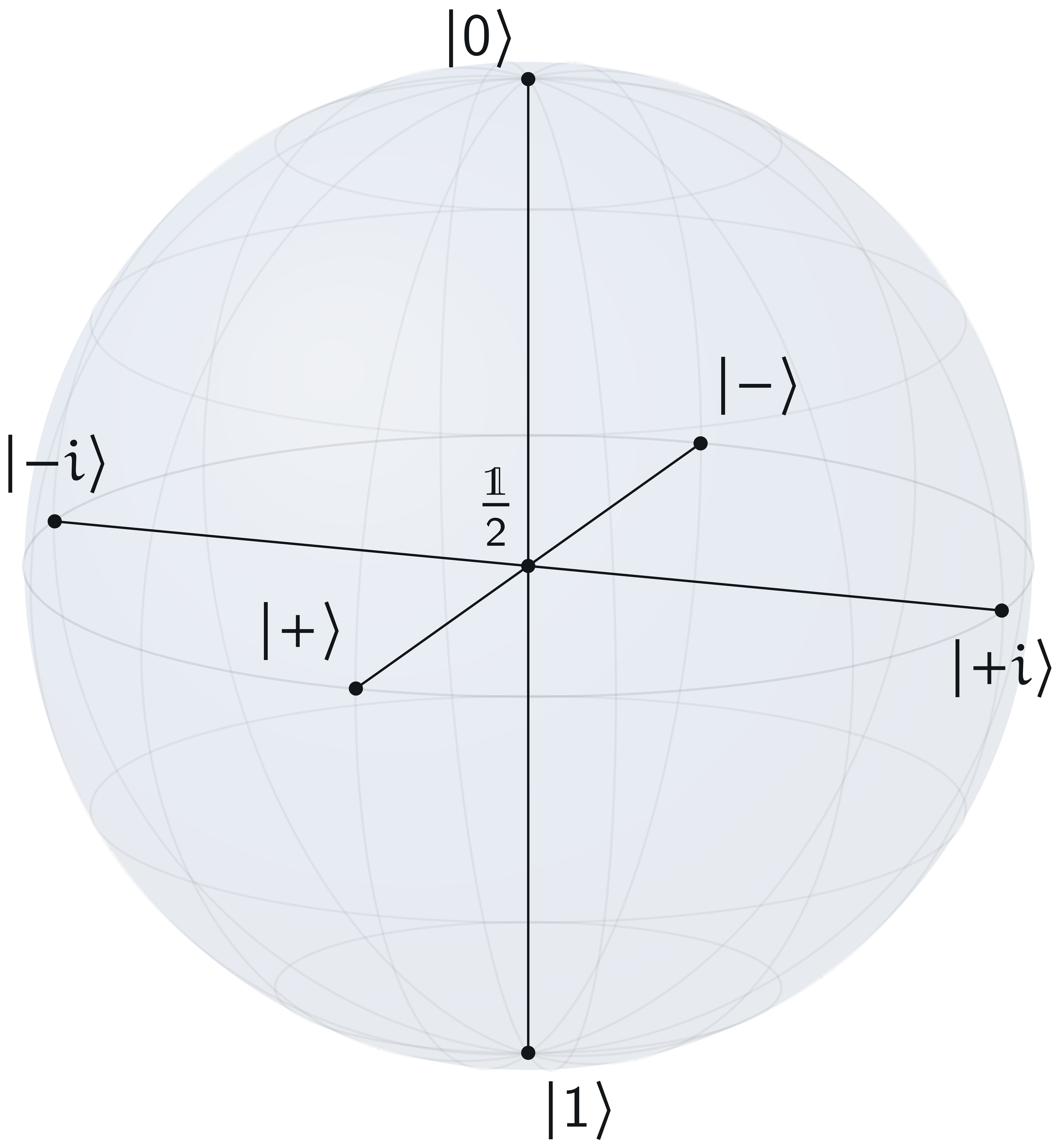

Examples

The completely mixed state lies at the center of the Bloch ball.

$$\frac{\mathbb{1}}{2} = \frac{\mathbb{1} + 0 \cdot \sigma_x + 0 \cdot \sigma_y + 0 \cdot \sigma_z}{2}$$

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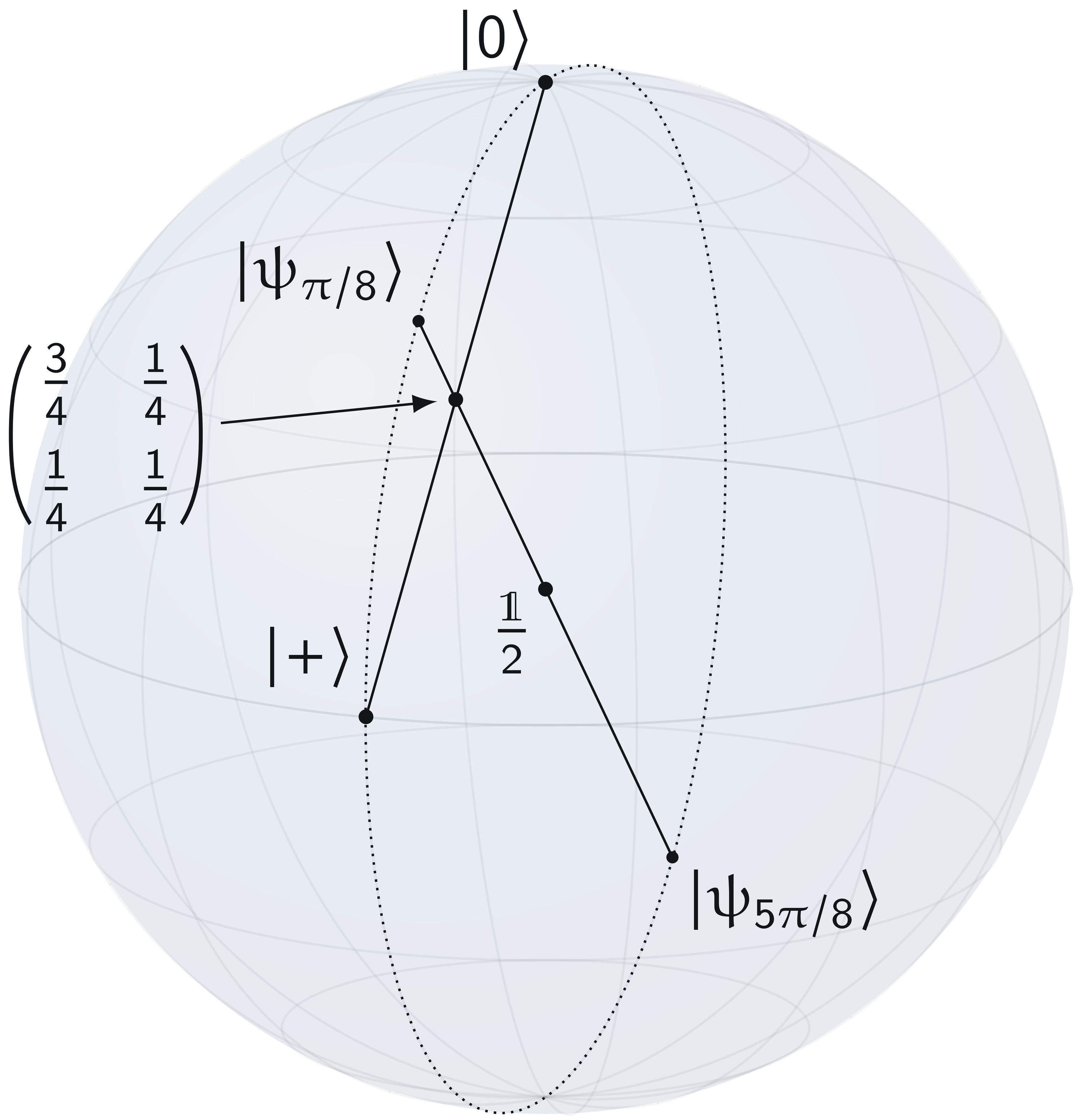
- Points in the interior of the Bloch ball correspond to qubit states that are not pure.
- Convex combinations of points in the Bloch ball correspond to convex combinations of qubit density matrices.



Examples

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

$$= \cos^2\left(\frac{\pi}{8}\right)|\psi_{\pi/8}\rangle\langle\psi_{\pi/8}| + \sin^2\left(\frac{\pi}{8}\right)|\psi_{5\pi/8}\rangle\langle\psi_{5\pi/8}|$$

Multiple systems

Density matrices can represent states of multiple systems:

- Multiple system are viewed as single, compound systems.
- The rows and columns of density matrices for multiple systems correspond to *Cartesian products* of the classical state sets of the individual systems.

Example: Bell states

$$|\phi^+\rangle\langle\phi^+| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad |\phi^-\rangle\langle\phi^-| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$|\psi^+\rangle\langle\psi^+| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad |\psi^-\rangle\langle\psi^-| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Independence and correlation

Tensor products of density matrices represent *independence* between systems.

Product states

If X is prepared in the state ρ and Y is independently prepared in the state σ, then the state of (X,Y) is the tensor product $\rho \otimes \sigma$.

States of this form are called *product states.*

Density matrices that cannot be expressed as product states represent *correlations* between systems. Different types of correlations can be considered.

Example: a correlated classical state

A uniform random bit shared between Alice and Bob has this state.

$$\frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

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Example: ensembles of quantum states

An *ensemble* describes a random selection of a quantum state.

The following density matrix represents an ensemble, assuming $(p_0, \ldots, p_{m-1})$ is a probability vector and $\rho_0, \ldots, \rho_{m-1}$ are density matrices of the same size.

$$\sum_{k=0}^{m-1} p_k |k\rangle\langle k| \otimes \rho_k$$

Independence and correlation

Tensor products of density matrices represent *independence* between systems.

Product states

If X is prepared in the state ρ and Y is independently prepared in the state σ, then the state of (X,Y) is the tensor product $\rho \otimes \sigma$.

States of this form are called *product states.*

Density matrices that cannot be expressed as product states represent *correlations* between systems. Different types of correlations can be considered.

Example: separable states

Convex combinations of product states represent classical correlations among quantum states.

$$\sum_{k=0}^{m-1} p_k \, \rho_k \otimes \sigma_k$$

States that can be written like this are *separable;* all the rest are *entangled.*

Reduced states for an e-bit

Alice and Bob *share an e-bit:* Alice holds A, Bob holds B, and (A, B) is in this state:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Question

What is the state of Alice's qubit A in isolation?

Imagine Bob performs a standard basis measurement on B.

Outcome	Probability	Resulting state of A
0	$\frac{1}{2}$	$\lvert 0\rangle$
1	$\frac{1}{2}$	$\lvert 1\rangle$

This leaves the qubit A in the completely mixed state:

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{\mathbb{1}}{2}$$

Reduced states for an e-bit

Alice and Bob *share an e-bit:* Alice holds A, Bob holds B, and (A, B) is in this state:

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Question

What is the state of Alice's qubit A in isolation?

Answer

The completely mixed state $\frac{\mathbb{1}}{2}$.

Reduced states in general

Suppose we have a pair of systems (A, B) in a pure state $|\psi\rangle$.

Let Γ be the classical state set of B. There is a unique collection $\{|\phi_b\rangle : b \in \Gamma\}$ of vectors for which this expression is true:

$$|\psi\rangle = \sum_{b\in\Gamma} |\phi_b\rangle \otimes |b\rangle$$

In particular, each $|\phi_b\rangle$ is given by this formula:

$$|\phi_b\rangle = \big(\mathbb{1}_{\mathsf{A}} \otimes \langle b|\big)|\psi\rangle$$

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$$|\psi\rangle = \sum_{b\in\Gamma} |\phi_b\rangle \otimes |b\rangle$$

$$|\phi_b\rangle = (\mathbb{1}_{\mathsf{A}} \otimes \langle b|)|\psi\rangle$$

Suppose a standard basis measurement were performed on B:

- Each $b \in \Gamma$ appears with probability $\||\phi_b\rangle\|^2$.
- Conditioned on obtaining the outcome b, the state of A becomes $\frac{|\phi_b\rangle}{\||\phi_b\rangle\|}$.

The reduced state of A:

$$\sum_{b\in\Gamma} \||\phi_b\rangle\|^2 \frac{|\phi_b\rangle\langle\phi_b|}{\||\phi_b\rangle\|^2} = \sum_{b\in\Gamma} |\phi_b\rangle\langle\phi_b|$$

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Suppose we have a pair of systems (A, B) in a pure state $|\psi\rangle$.

Let Γ be the classical state set of B. There is a unique collection $\{|\phi_b\rangle : b \in \Gamma\}$ of vectors for which this expression is true:

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Suppose a standard basis measurement were performed on B:

- Each $b \in \Gamma$ appears with probability $\| |\phi_b\rangle \|^2$.
- Conditioned on obtaining the outcome b, the state of A becomes $\frac{|\phi_b\rangle}{\| |\phi_b\rangle \|}$.

The reduced state of A:

$$\sum_{b\in\Gamma} |\phi_b\rangle\langle\phi_b| = \sum_{b\in\Gamma} (\mathbb{1}_\mathsf{A} \otimes \langle b|)|\psi\rangle\langle\psi|(\mathbb{1}_\mathsf{A} \otimes |b\rangle)$$

Reduced states in general

Suppose we have a pair of systems (A, B) in a pure state $|\psi\rangle$.

The reduced state of A:

$$\sum_{b\in\Gamma} \bigl(\mathbb{1}_\mathsf{A} \otimes \langle b|\bigr)|\psi\rangle\langle\psi|\bigl(\mathbb{1}_\mathsf{A} \otimes |b\rangle\bigr)$$

Suppose (A, B) is in a state described by a density matrix ρ. The reduced state of A is described by this density matrix:

$$\rho_\mathsf{A} = \sum_{b\in\Gamma} \bigl(\mathbb{1}_\mathsf{A} \otimes \langle b|\bigr)\rho\bigl(\mathbb{1}_\mathsf{A} \otimes |b\rangle\bigr)$$

The reduced state of B is obtained in a similar way.

$$\rho_\mathsf{B} = \sum_{a\in\Sigma} \bigl(\langle a| \otimes \mathbb{1}_\mathsf{B}\bigr)\rho\bigl(|a\rangle \otimes \mathbb{1}_\mathsf{B}\bigr)$$

(Σ is the classical state set of A.)

Reduced states in general

Suppose (A, B) is in a state described by a density matrix ρ. The reduced state of A is described by this density matrix:

$$\rho_{\mathsf{A}} = \sum_{b \in \Gamma} \bigl(\mathbb{1}_{\mathsf{A}} \otimes \langle b \vert\bigr)\rho\bigl(\mathbb{1}_{\mathsf{A}} \otimes \vert b \rangle\bigr)$$

The reduced state of B is obtained in a similar way.

$$\rho_{\mathsf{B}} = \sum_{a \in \Sigma} \bigl(\langle a \vert \otimes \mathbb{1}_{\mathsf{B}}\bigr)\rho\bigl(\vert a \rangle \otimes \mathbb{1}_{\mathsf{B}}\bigr)$$

These notions can be generalized to three or more systems in a natural way.

Example

Suppose $(\mathsf{A}, \mathsf{B}, \mathsf{C})$ is in the state ρ. The reduced states of (A, C) and C:

$$\rho_{\mathsf{AC}} = \sum_{b \in \Gamma} \bigl(\mathbb{1}_{\mathsf{A}} \otimes \langle b \vert \otimes \mathbb{1}_{\mathsf{C}}\bigr)\rho\bigl(\mathbb{1}_{\mathsf{A}} \otimes \vert b \rangle \otimes \mathbb{1}_{\mathsf{C}}\bigr)$$

$$\rho_{\mathsf{C}} = \sum_{a \in \Sigma} \sum_{b \in \Gamma} \bigl(\langle a \vert \otimes \langle b \vert \otimes \mathbb{1}_{\mathsf{C}}\bigr)\rho\bigl(\vert a \rangle \otimes \vert b \rangle \otimes \mathbb{1}_{\mathsf{C}}\bigr)$$

The partial trace

Definition

$$\mathrm{Tr}_B(\rho) = \sum_{b\in\Gamma} (\mathbb{1}_A \otimes \langle b|)\rho(\mathbb{1}_A \otimes |b\rangle)$$

$$\mathrm{Tr}_A(\rho) = \sum_{a\in\Sigma} (\langle a| \otimes \mathbb{1}_B)\rho(|a\rangle \otimes \mathbb{1}_B)$$

Equivalent definition

Tr_A and Tr_B are the unique linear mappings for which these equations are always true:

$$\mathrm{Tr}_A(M \otimes N) = \mathrm{Tr}(M)N$$

$$\mathrm{Tr}_B(M \otimes N) = \mathrm{Tr}(N)M$$

Example

Consider this state of a pair of qubits (A, B):

$$\rho = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |+\rangle\langle +|$$

$$\rho_A = \mathrm{Tr}_B(\rho) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \qquad \rho_B = \mathrm{Tr}_A(\rho) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$