Understanding quantum information and computation

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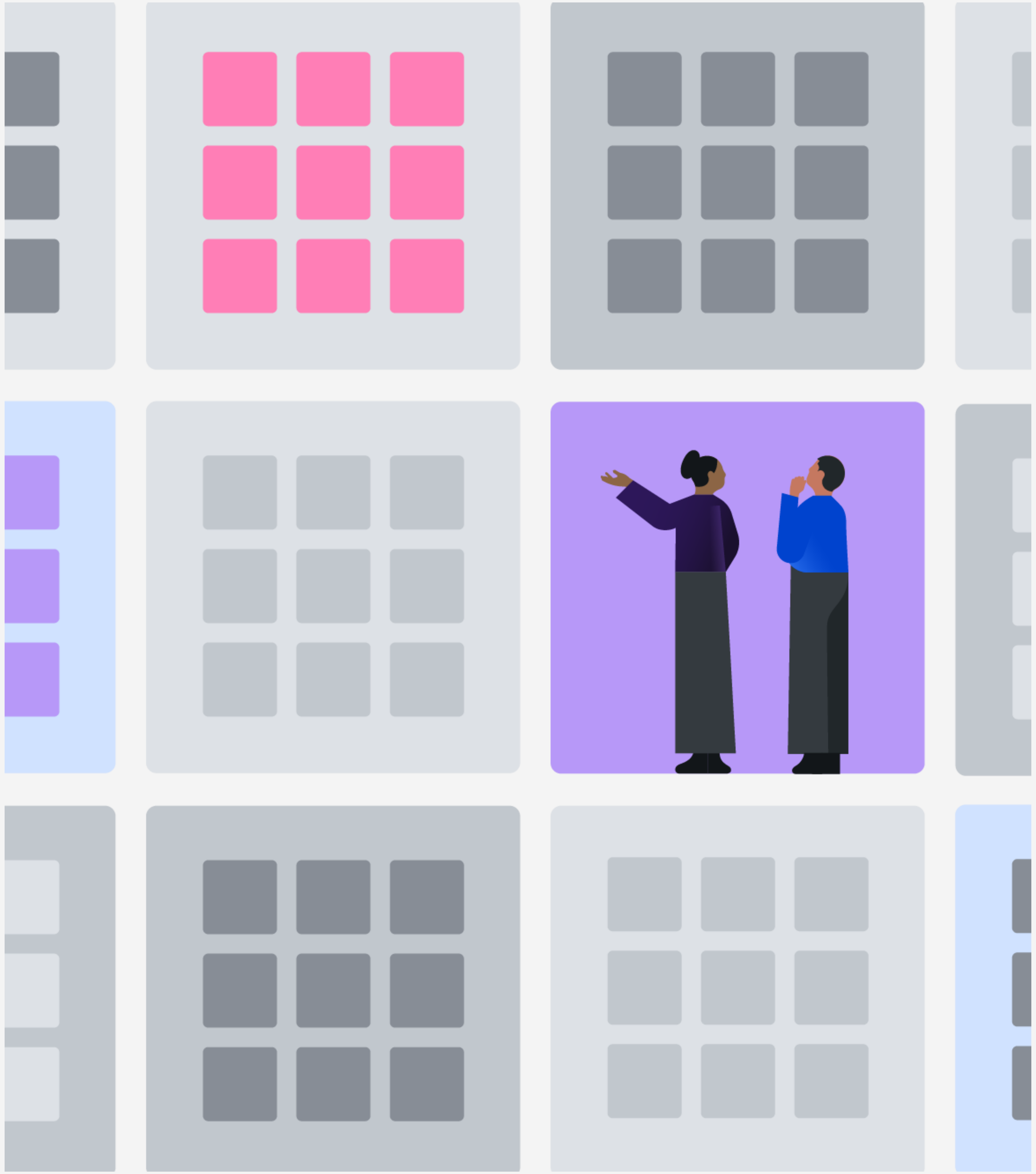
Lesson 12

Purifications and fidelity



Purifications

Definition

A *purification* of a state (represented by a density matrix) is a pure state of a larger, compound system that leaves the original state when the rest of the compound system is traced out.

In mathematical terms: if X is a system in a state ρ, and $|\psi\rangle$ is a quantum state vector of a pair (X, Y) such that

$$\rho = \mathrm{Tr}_{\mathsf{Y}}\big(|\psi\rangle\langle\psi|\big)$$

then $|\psi\rangle$ is a purification of ρ.

Fact

Every density matrix ρ has a purification like this provided that Y has at least as many classical states as X.

This is a critically important notion in quantum information theory.

Existence of purifications

Suppose X is a system and ρ is a density matrix representing a state of X. Consider any expression of ρ as a convex combination of pure states.

$$\rho = \sum_{a=0}^{n-1} p_a |\phi_a\rangle\langle\phi_a|$$

In this expression $(p_0, \ldots, p_{n-1})$ is a probability vector and $|\phi_0\rangle, \ldots, |\phi_{n-1}\rangle$ are quantum state vectors.

Here's a purification of ρ:

$$|\psi\rangle = \sum_{a=0}^{n-1} \sqrt{p_a}\, |\phi_a\rangle \otimes |a\rangle$$

(We're assume for simplicity that the classical states of Y include $0, \ldots, n-1$.)

$$\operatorname{Tr}_{\mathsf{Y}}\bigl(|\psi\rangle\langle\psi|\bigr) = \sum_{a,b=0}^{n-1} \sqrt{p_a}\sqrt{p_b}\, |\phi_a\rangle\langle\phi_b|\, \operatorname{Tr}\bigl(|a\rangle\langle b|\bigr) = \sum_{a=0}^{n-1} p_a\, |\phi_a\rangle\langle\phi_a| = \rho$$

Existence of purifications

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Example

$$\rho = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|+\rangle \otimes |1\rangle$$

Schmidt decompositions

Every quantum state vector $|\psi\rangle$ of a pair of systems (X, Y) can be expressed in a special form known as a *Schmidt decomposition:*

$$|\psi\rangle = \sum_{a=0}^{r-1} \sqrt{p_a}\, |x_a\rangle \otimes |y_a\rangle \qquad (p_0, \ldots, p_{r-1} > 0)$$

Both of the sets $\{|x_0\rangle, \ldots, |x_{r-1}\rangle\}$ and $\{|y_0\rangle, \ldots, |y_{r-1}\rangle\}$ must be *orthonormal.*

Finding a Schmidt decomposition

1. Compute a spectral decomposition of the reduced state $\rho = \mathrm{Tr}_{\mathsf{Y}}(|\psi\rangle\langle\psi|)$:

$$\rho = \sum_{a=0}^{r-1} p_a |x_a\rangle\langle x_a| \qquad (p_0, \ldots, p_{r-1} > 0)$$

2. For each $a = 0, \ldots, r-1$ define $|y_a\rangle$ as follows:

$$|y_a\rangle = \frac{(\langle x_a| \otimes \mathbb{1})|\psi\rangle}{\sqrt{p_a}}$$

Schmidt decompositions

Example

Consider this state of a pair of qubits (X, Y):

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|+\rangle \otimes |1\rangle$$

First compute a spectral decomposition of the reduced state of X:

$$\rho = \cos^2(\pi/8)\,|\psi_{\pi/8}\rangle\langle\psi_{\pi/8}| + \sin^2(\pi/8)\,|\psi_{5\pi/8}\rangle\langle\psi_{5\pi/8}|$$

We will make these selections:

$$p_0 = \cos^2(\pi/8), \quad p_1 = \sin^2(\pi/8), \quad |x_0\rangle = |\psi_{\pi/8}\rangle, \quad |x_1\rangle = |\psi_{5\pi/8}\rangle.$$

It remains to compute $|y_0\rangle$ and $|y_1\rangle$:

$$|y_0\rangle = \frac{(\langle x_0| \otimes \mathbb{1})|\psi\rangle}{\sqrt{p_0}} = |+\rangle \qquad |y_1\rangle = \frac{(\langle x_1| \otimes \mathbb{1})|\psi\rangle}{\sqrt{p_1}} = -|-\rangle$$

Schmidt decompositions

Example

Consider this state of a pair of qubits (X, Y):

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|+\rangle \otimes |1\rangle$$

$$p_0 = \cos^2(\pi/8), \quad p_1 = \sin^2(\pi/8), \quad |x_0\rangle = |\psi_{\pi/8}\rangle, \quad |x_1\rangle = |\psi_{5\pi/8}\rangle.$$

$$|y_0\rangle = \frac{(\langle x_0| \otimes \mathbb{1})|\psi\rangle}{\sqrt{p_0}} = |+\rangle \qquad |y_1\rangle = \frac{(\langle x_1| \otimes \mathbb{1})|\psi\rangle}{\sqrt{p_1}} = -|-\rangle$$

We obtain the following Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \cos(\pi/8)\,|\psi_{\pi/8}\rangle \otimes |+\rangle \; - \; \sin(\pi/8)\,|\psi_{5\pi/8}\rangle \otimes |-\rangle$$

Unitary equivalence of purifications

Unitary equivalence of purifications

Suppose that $|\psi\rangle$ and $|\phi\rangle$ are pure states of a pair of systems (X, Y) that satisfy

$$\mathrm{Tr}_{\mathsf{Y}}(|\psi\rangle\langle\psi|) = \rho = \mathrm{Tr}_{\mathsf{Y}}(|\phi\rangle\langle\phi|)$$

There exists a unitary operation U on Y alone that transforms $|\psi\rangle$ into $|\phi\rangle$:

$$(\mathbb{1}_{\mathsf{X}} \otimes U)|\psi\rangle = |\phi\rangle$$

Consider a spectral decomposition of ρ.

$$\rho = \sum_{a=0}^{r-1} p_a |x_a\rangle\langle x_a|$$

Compute Schmidt decompositions for both $|\psi\rangle$ and $|\phi\rangle$:

$$|\psi\rangle = \sum_{a=0}^{r-1} \sqrt{p_a}\, |x_a\rangle \otimes |y_a\rangle \qquad |\phi\rangle = \sum_{a=0}^{r-1} \sqrt{p_a}\, |x_a\rangle \otimes |z_a\rangle$$

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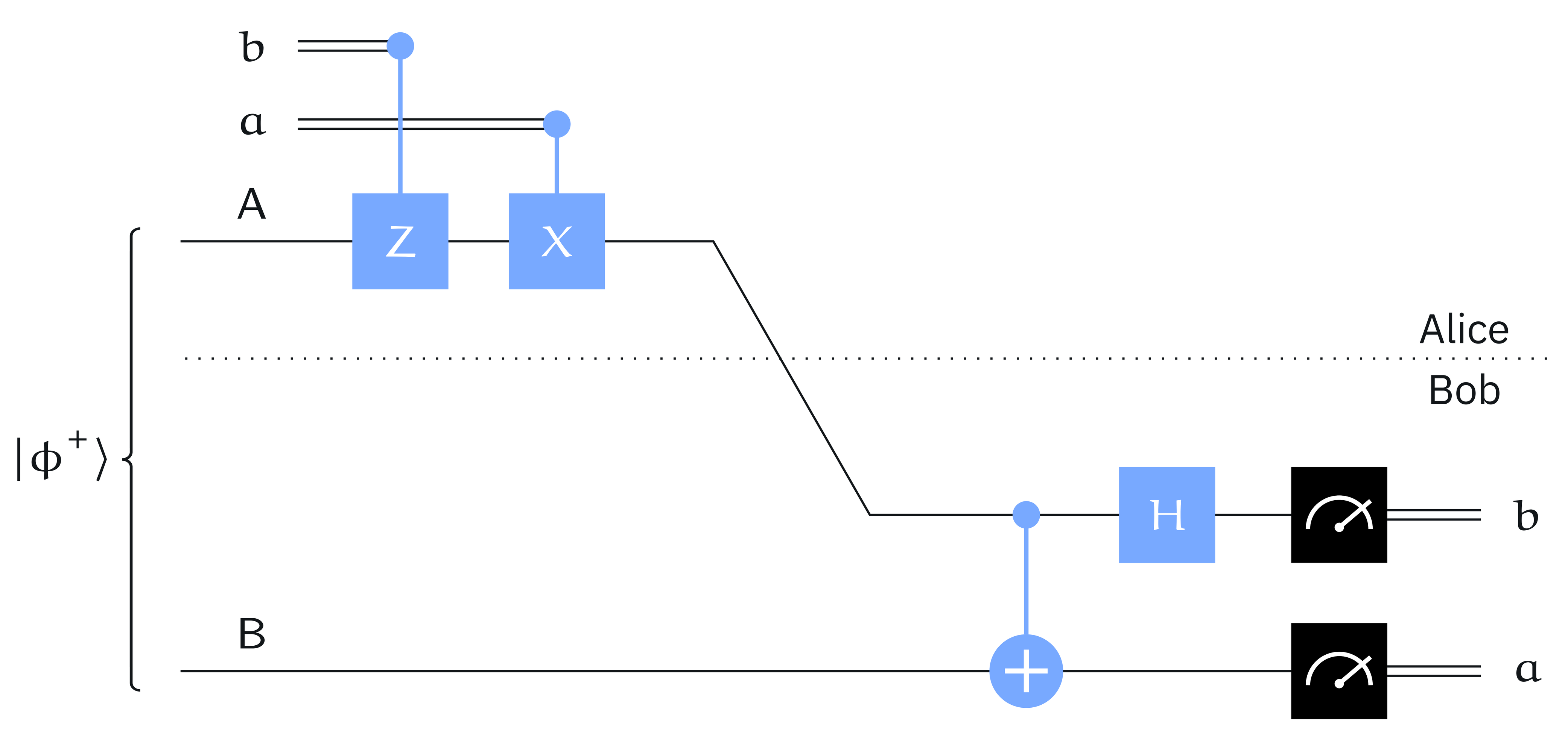
Choose U to be any unitary matrix satisfying $U|y_a\rangle = |z_a\rangle$ for $a = 0, \ldots, r-1$.

Example: superdense coding



$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

The reduced state of B for all four Bell states is the completely mixed state.

$$\mathrm{Tr}_A(|\phi^+\rangle\langle\phi^+|) = \mathrm{Tr}_A(|\phi^-\rangle\langle\phi^-|) = \mathrm{Tr}_A(|\psi^+\rangle\langle\psi^+|) = \mathrm{Tr}_A(|\psi^-\rangle\langle\psi^-|) = \frac{\mathbb{1}}{2}$$

By the *unitary equivalence of purifications,* we conclude that Alice can transform $|\phi^+\rangle$ to any of the four Bell states by applying a unitary operation to A alone.

Cryptographic implications

The unitary equivalence of purifications has implications to *quantum cryptography.*
For instance, it rules out an unconditionally secure quantum protocol for *bit commitment.*

Bit commitment

Bit commitment is a cryptographic primitive allowing Alice to *commit* to a selection of a bit $b \in \{0, 1\}$, which remains hidden until she chooses to *reveal* it to Bob.

- Binding property: Alice cannot change her mind once she's committed to b.
- Concealing property: Bob cannot determine b until Alice chooses to reveal it.

Let A and B be Alice's and Bob's systems in a *purified* version of a hypothetical protocol and let $|\psi_0\rangle$ and $|\psi_1\rangle$ be the states of (A, B) after Alice commits but before she reveals.

If the protocol is perfectly concealing, then

$$\mathrm{Tr}_A\big(|\psi_0\rangle\langle\psi_0|\big) = \mathrm{Tr}_A\big(|\psi_1\rangle\langle\psi_1|\big)$$

This implies that the protocol is not binding: Alice can change her commitment by performing a unitary operation on A alone.

HJW theorem

Hughston-Jozsa-Wootters theorem

Suppose X and Y are systems and $|\phi\rangle$ is a quantum state vector of (X, Y).

Let N be a positive integer, let $(p_0, \ldots, p_{N-1})$ be a probability vector, and let $|\psi_0\rangle, \ldots, |\psi_{N-1}\rangle$ be quantum state vectors of X such that

$$\operatorname{Tr}_{\mathsf{Y}}(|\phi\rangle\langle\phi|) = \sum_{a=0}^{N-1} p_a |\psi_a\rangle\langle\psi_a|$$

There exists a measurement $\{P_0, \ldots, P_{N-1}\}$ of Y such that these statements are true when Y is measured when (X, Y) is in the state $|\phi\rangle$:

- Each measurement outcome $a \in \{0, \ldots, N-1\}$ appears with probability p_a.
- Conditioned on obtaining the outcome a, the state of X becomes $|\psi_a\rangle$.

HJW theorem

Proof sketch. We have the following relationship:

$$\sum_{a=0}^{N-1} p_a |\psi_a\rangle\langle\psi_a| = \rho = \mathrm{Tr}_Y\big(|\phi\rangle\langle\phi|\big)$$

Introduce a new system Z having classical states $\{0, \ldots, N-1\}$. These two state vectors of (X, Y, Z) are both purifications of ρ:

$$\begin{aligned} |\gamma_0\rangle &= |\phi\rangle_{XY} \otimes |0\rangle_Z \\ |\gamma_1\rangle &= \sum_{a=0}^{N-1} \sqrt{p_a}\, |\psi_a\rangle_X \otimes |0\rangle_Y \otimes |a\rangle_Z \end{aligned}$$

By the unitary equivalence of purifications, there is a unitary operation on (Y, Z) that transforms $|\gamma_0\rangle$ into $|\gamma_1\rangle$:

$$(\mathbb{1}_X \otimes U)|\gamma_0\rangle = |\gamma_1\rangle$$

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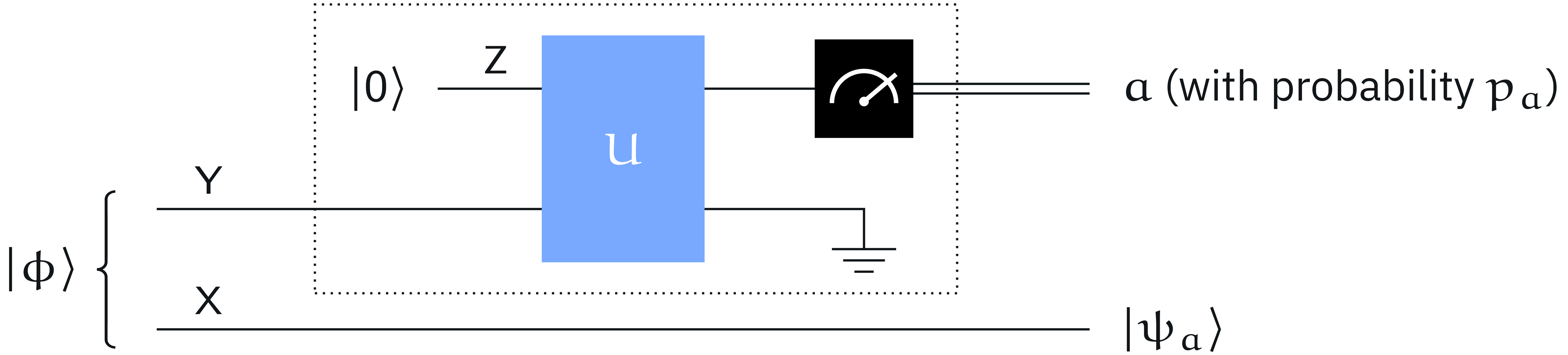

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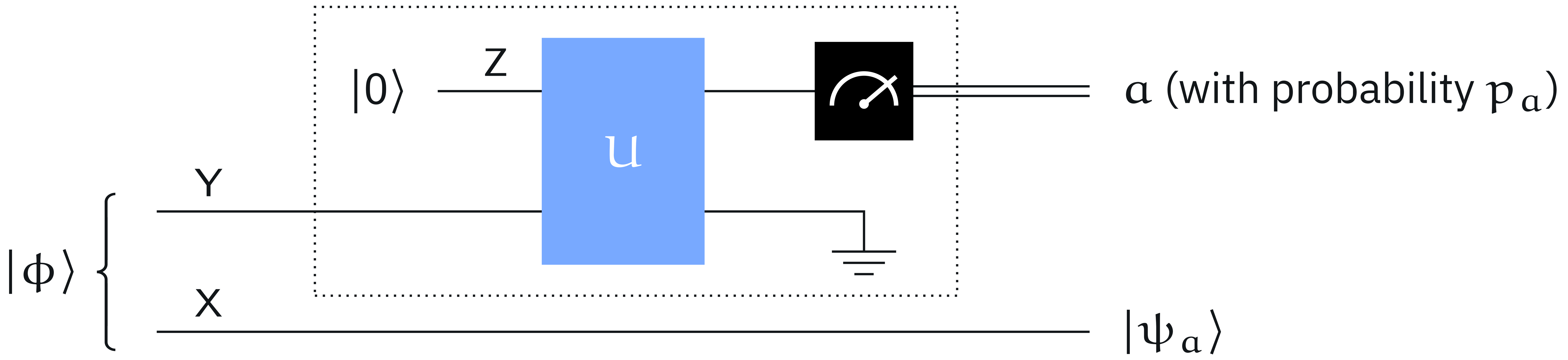

This measurement is described by matrices $\{P_0, \ldots, P_{N-1}\}$ defined as follows:

$$P_a = (\mathbb{1}_{\mathsf{Y}} \otimes \langle 0|)U^{\dagger}(\mathbb{1}_{\mathsf{Y}} \otimes |a\rangle\langle a|)U(\mathbb{1}_{\mathsf{Y}} \otimes |0\rangle)$$

Definition of fidelity

The fidelity between two quantum states measures their *similarity* or *overlap.*

For two states represented by density matrices ρ and σ it is defined as follows:

$$F(\rho, \sigma) = \operatorname{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$$

The matrix $\sqrt{\rho}\,\sigma\sqrt{\rho}$ is positive semidefinite: $\sqrt{\rho}\,\sigma\sqrt{\rho} = M^\dagger M$ for $M = \sqrt{\sigma}\sqrt{\rho}$. We can therefore take the square root of this matrix:

$$\sqrt{\rho}\,\sigma\sqrt{\rho} = \sum_{k=0}^{n-1} \lambda_k |\phi_k\rangle\langle\phi_k| \quad \Rightarrow \quad \sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} = \sum_{k=0}^{n-1} \sqrt{\lambda_k} |\phi_k\rangle\langle\phi_k|$$

$$F(\rho, \sigma) = \sum_{k=0}^{n-1} \sqrt{\lambda_k}$$

An equivalent formula in terms of the trace norm $\|M\|_1 = \operatorname{Tr}\sqrt{MM^\dagger} = \operatorname{Tr}\sqrt{M^\dagger M}$:

$$F(\rho, \sigma) = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 = \left\|\sqrt{\sigma}\sqrt{\rho}\right\|_1$$

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The trace norm can also be defined as $\|M\|_1 = \max_U \left|\operatorname{Tr}(MU)\right|$ (maximum over all unitary U).

$$F(\rho, \sigma) = \max_{U \text{ unitary}} \left|\operatorname{Tr}\left(\sqrt{\rho}\sqrt{\sigma}\,U\right)\right|$$

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$$F(\rho, \sigma) = \max_{U \text{ unitary}} \left| \operatorname{Tr}\left(\sqrt{\rho}\sqrt{\sigma}\, U\right) \right|$$

There are simpler formulas when at least one of the states is pure:

$$F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) = |\langle\phi|\psi\rangle|$$

$$F(|\phi\rangle\langle\phi|, \sigma) = \sqrt{\langle\phi|\sigma|\phi\rangle}$$

Properties of fidelity

1. For any two density matrices ρ and σ we have $0 \leq F(\rho, \sigma) \leq 1$.
   - $F(\rho, \sigma) = 0$ if and only if ρ and σ have orthogonal images.
   - $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$.
2. The fidelity is symmetric: $F(\rho, \sigma) = F(\sigma, \rho)$.
3. The fidelity is multiplicative for product states:

$$F(\rho_1 \otimes \cdots \otimes \rho_m, \sigma_1 \otimes \cdots \otimes \sigma_m) = F(\rho_1, \sigma_1) \cdots F(\rho_m, \sigma_m)$$

4. For any two density matrices ρ and σ and any channel Φ we have

$$F(\rho, \sigma) \leq F(\Phi(\rho), \Phi(\sigma))$$

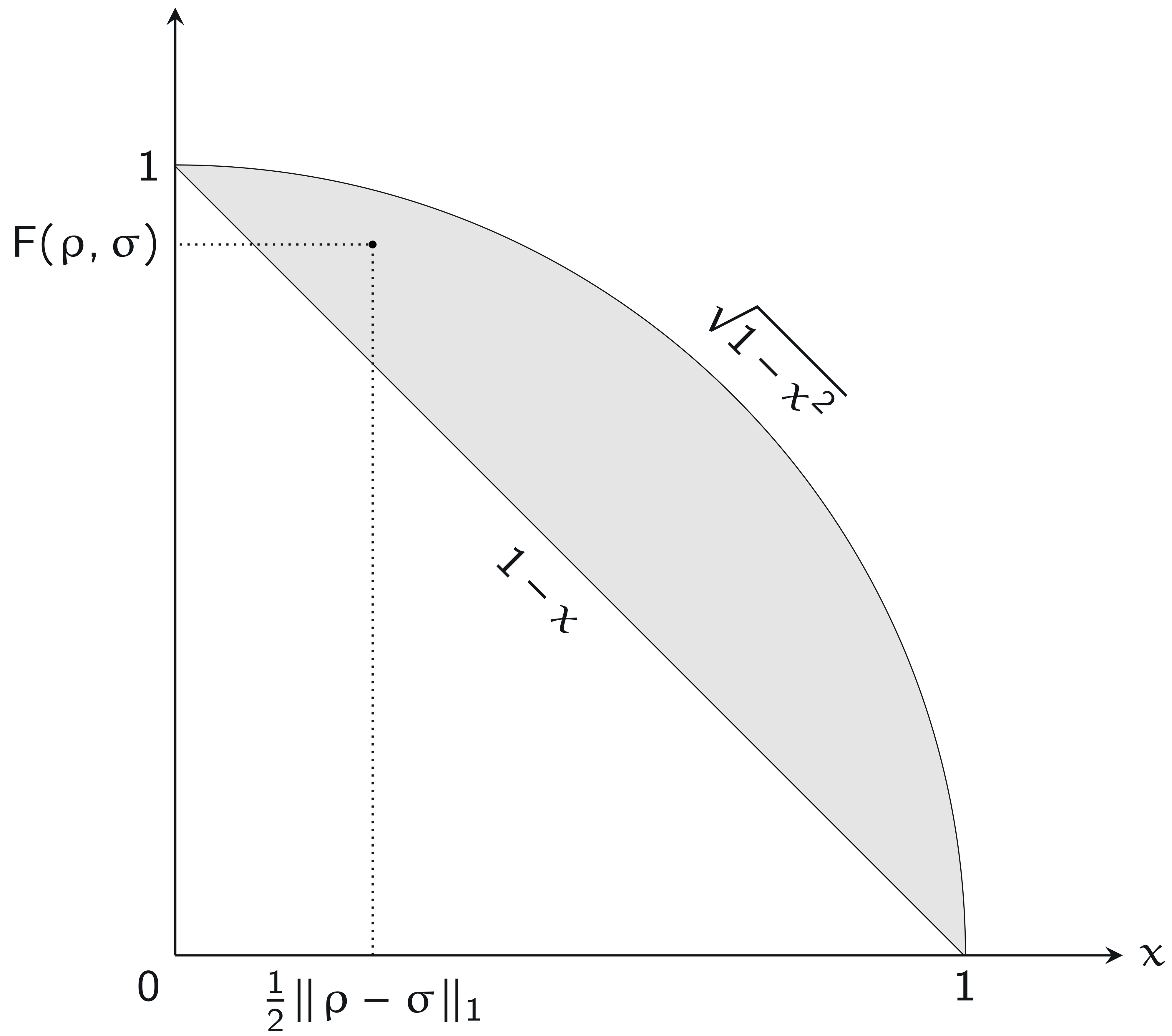
5. There is a close relationship between fidelity and trace distance:

$$1 - \frac{1}{2}\|\rho - \sigma\|_1 \leq F(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}$$

Properties of fidelity

5. There is a close relationship between fidelity and trace distance:

$$1 - \frac{1}{2}\|\rho - \sigma\|_1 \leq \mathrm{F}(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}$$



Gentle measurement lemma

Let X be a system, let ρ be a state of X, and let $\{P_0, \dots, P_{m-1}\}$ be a measurement. Suppose that one of the measurement outcomes is very likely to appear.

$$\mathrm{Tr}(P_0\rho) > 1 - \varepsilon$$

A *non-destructive* implementation of this measurements (through Naimark's theorem) works like this:

$$\begin{tikzcd}
 & \rho \arrow[dl, "\text{probability } \mathrm{Tr}(P_0\rho)"'] \arrow[dr, "\text{probability } \mathrm{Tr}(P_{m-1}\rho)"] & \\
\dfrac{\sqrt{P_0}\,\rho\sqrt{P_0}}{\mathrm{Tr}(P_0\rho)} & \cdots & \dfrac{\sqrt{P_{m-1}}\,\rho\sqrt{P_{m-1}}}{\mathrm{Tr}(P_{m-1}\rho)}
\end{tikzcd}$$

The gentle measurement lemma implies that only a *small disturbance* occurs when the likely measurement outcome appears.

$$\mathrm{F}\left(\rho, \frac{\sqrt{P_0}\,\rho\sqrt{P_0}}{\mathrm{Tr}(P_0\rho)}\right)^2 > 1 - \varepsilon$$

Gentle measurement lemma

Let X be a system, let ρ be a state of X, and let $\{P_0, \ldots, P_{m-1}\}$ be a measurement. Suppose that one of the measurement outcomes is very likely to appear.

$$\operatorname{Tr}(P_0 \rho) > 1 - \varepsilon$$

We can evaluate the fidelity between the pre- and post-measurement states:

$$F\left(\rho, \frac{\sqrt{P_0}\rho\sqrt{P_0}}{\operatorname{Tr}(P_0\rho)}\right) = \operatorname{Tr}\sqrt{\frac{\sqrt{\rho}\sqrt{P_0}\rho\sqrt{P_0}\sqrt{\rho}}{\operatorname{Tr}(P_0\rho)}} = \operatorname{Tr}\sqrt{\left(\frac{\sqrt{\rho}\sqrt{P_0}\sqrt{\rho}}{\sqrt{\operatorname{Tr}(P_0\rho)}}\right)^2}$$

$$= \operatorname{Tr}\left(\frac{\sqrt{\rho}\sqrt{P_0}\sqrt{\rho}}{\sqrt{\operatorname{Tr}(P_0\rho)}}\right) = \frac{\operatorname{Tr}\left(\sqrt{P_0}\rho\right)}{\sqrt{\operatorname{Tr}(P_0\rho)}} \geq \frac{\operatorname{Tr}(P_0\rho)}{\sqrt{\operatorname{Tr}(P_0\rho)}}$$

$$P_0 = \sum_{k=0}^{n-1} \lambda_k |\psi_k\rangle\langle\psi_k|$$

$$\operatorname{Tr}\left(\sqrt{P_0}\rho\right) = \sum_{k=0}^{n-1} \sqrt{\lambda_k}\langle\psi_k|\rho|\psi_k\rangle \geq \sum_{k=0}^{n-1} \lambda_k\langle\psi_k|\rho|\psi_k\rangle = \operatorname{Tr}(P_0\rho)$$

Gentle measurement lemma

Let X be a system, let ρ be a state of X, and let $\{P_0, \ldots, P_{m-1}\}$ be a measurement. Suppose that one of the measurement outcomes is very likely to appear.

$$\operatorname{Tr}(P_0 \rho) > 1 - \varepsilon$$

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$$\begin{aligned}
\mathrm{F}\left(\rho, \frac{\sqrt{P_0}\rho\sqrt{P_0}}{\operatorname{Tr}(P_0\rho)}\right) &= \operatorname{Tr}\sqrt{\frac{\sqrt{\rho}\sqrt{P_0}\rho\sqrt{P_0}\sqrt{\rho}}{\operatorname{Tr}(P_0\rho)}} = \operatorname{Tr}\sqrt{\left(\frac{\sqrt{\rho}\sqrt{P_0}\sqrt{\rho}}{\sqrt{\operatorname{Tr}(P_0\rho)}}\right)^2} \\
&= \operatorname{Tr}\left(\frac{\sqrt{\rho}\sqrt{P_0}\sqrt{\rho}}{\sqrt{\operatorname{Tr}(P_0\rho)}}\right) = \frac{\operatorname{Tr}(\sqrt{P_0}\rho)}{\sqrt{\operatorname{Tr}(P_0\rho)}} \geq \frac{\operatorname{Tr}(P_0\rho)}{\sqrt{\operatorname{Tr}(P_0\rho)}} = \sqrt{\operatorname{Tr}(P_0\rho)}
\end{aligned}$$

$$\mathrm{F}\left(\rho, \frac{\sqrt{P_0}\rho\sqrt{P_0}}{\operatorname{Tr}(P_0\rho)}\right)^2 \geq \operatorname{Tr}(P_0\rho) > 1 - \varepsilon$$

Uhlmann's theorem

Uhlmann's theorem is a fundamentally important fact connecting fidelity with purifications.

Uhlmann's theorem

The fidelity between two quantum states equals the *maximum inner product* (in absolute value) between two purifications of these states.

In greater detail...

Suppose ρ and σ are density matrices representing states of a system X, and let Y be a system with at least as many classical states as X.

$$\mathrm{F}(\rho, \sigma) = \max\Big\{ \big|\langle \phi | \psi \rangle\big| \;:\; \mathrm{Tr}_{\mathsf{Y}}(|\phi\rangle\langle\phi|) = \rho,\; \mathrm{Tr}_{\mathsf{Y}}(|\psi\rangle\langle\psi|) = \sigma \Big\}$$

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Consider spectral decompositions of ρ and σ:

$$\rho = \sum_{a=0}^{n-1} p_a |u_a\rangle\langle u_a| \qquad \text{and} \qquad \sigma = \sum_{b=0}^{n-1} q_b |v_b\rangle\langle v_b|$$

These state vectors purify ρ and σ:

$$\sum_{a=0}^{n-1} \sqrt{p_a}\, |u_a\rangle \otimes |\overline{u_a}\rangle \qquad \text{and} \qquad \sum_{b=0}^{n-1} \sqrt{q_b}\, |v_b\rangle \otimes |\overline{v_b}\rangle$$

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By the unitary equivalence of purifications, all purifications of ρ and σ to (X,Y) take these forms (for U and V unitary):

$$|\phi\rangle = \sum_{a=0}^{n-1} \sqrt{p_a}\,|u_a\rangle \otimes U\,|\overline{u_a}\rangle \qquad \text{and} \qquad |\psi\rangle = \sum_{b=0}^{n-1} \sqrt{q_b}\,|v_b\rangle \otimes V\,|\overline{v_b}\rangle$$

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$$\begin{aligned}
&\max\Big\{ \big|\langle\phi|\psi\rangle\big| \,:\, \mathrm{Tr}_{\mathsf{Y}}(|\phi\rangle\langle\phi|) = \rho,\; \mathrm{Tr}_{\mathsf{Y}}(|\psi\rangle\langle\psi|) = \sigma \Big\}\\
&\qquad = \max_{U,V\text{ unitary}} \left| \sum_{a,b=0}^{n-1} \sqrt{p_a}\sqrt{q_b}\,\langle u_a|v_b\rangle\,\langle v_b|V^{\mathsf{T}}\overline{U}|u_a\rangle \right|\\
&\qquad = \max_{U,V\text{ unitary}} \left|\mathrm{Tr}\Big(\sqrt{\rho}\sqrt{\sigma}\,V^{\mathsf{T}}\overline{U}\Big)\right| = \Big\|\sqrt{\rho}\sqrt{\sigma}\Big\|_1 = \mathrm{F}(\rho,\sigma)
\end{aligned}$$