Entanglement in action

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Introduction

In this lesson we'll take a look at three fundamentally important examples. The first two are the *teleportation* and *superdense coding* protocols, which are principally concerned with the transmission of information from a sender to a receiver. The third example is an abstract game, called the *CHSH game,* which illustrates a phenomenon in quantum information that is sometimes referred to as *nonlocality*. (The CHSH game is not always described as a game. It is often described instead as an experiment — specifically, it is an example of a *Bell test* — and is referred to as the *CHSH inequality*.)

Teleportation, superdense coding, and the CHSH game are not merely examples meant to illustrate how quantum information works, although they do serve well in this regard. Rather, they are stones in the foundation of quantum information. Entanglement plays a key role in all three examples, so this lesson provides the first opportunity in this course to see entanglement in action, and to begin to explore what it is that makes entanglement such an interesting and important concept.

Before proceeding to the examples themselves, a few preliminary comments that connect to all three examples are in order.

Alice and Bob

Alice and *Bob* are names traditionally given to hypothetical entities or agents in systems, protocols, games, and other interactions that involve the exchange of information. While these are typically human names, it should be understood that they represent abstractions and not necessarily actual human beings — so Alice and Bob might be expected to perform complex computations, for instance.

These names were first used in this way in the 1970s in the context of cryptography, but the convention has become common more broadly since then. The idea is simply that these are common names (at least in some parts of the world) that start with the letters A and B. It is also quite convenient to refer to Alice with the pronoun "her" and Bob with the pronoun "him" for the sake of brevity.

By default, we imagine that Alice and Bob are in different locations. They may have different goals and behaviors depending on the context in which they arise. For example, in *communication,* meaning the transmission of information, we might decide to use the name Alice to refer to the sender and Bob to refer to the receiver of whatever information is transmitted. In general, it may be that Alice and Bob cooperate, which is typical of a wide range of settings — but in other settings they may be in competition, or they may have different goals that may or may not be consistent or harmonious. These things must be made clear in the situation at hand.

We can also introduce additional characters, such as *Charlie* and *Diane*, as needed. Other names that represent different personas, such as *Eve* for an eavesdropper or *Mallory* for someone behaving maliciously, are also sometimes used.

Alice and Bob appear in all three of the examples to be discussed in this lesson, and we will encounter them from time to time in subsequent lessons.

Entanglement as a resource

Recall this example of an entangled quantum state of two qubits from the Multiple systems lesson:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle. \tag{1}$$

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It is one of the four Bell states, and is often viewed as the archetypal example of an entangled quantum state.

We also encountered this example of a probabilistic state of two bits:

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle. \tag{2}$$

It is, in some sense, analogous to the entangled quantum state (1). It represents a probabilistic state in which two bits are correlated, but it is not entangled. Entanglement is a uniquely quantum phenomenon, essentially by definition: in simplified terms, entanglement refers to *non-classical* quantum correlations.

Unfortunately, defining entanglement as non-classical quantum correlation is somewhat unsatisfying at an intuitive level, because it's a definition of entanglement is in terms of what it is not. It may be for this reason that it's actually rather challenging to explain precisely what entanglement is, and what makes it special, in intuitive terms.

Typical explanations of entanglement often fail to distinguish the two states (1) and (2) in a meaningful way. For example, it is sometimes said that if one of two entangled qubits is measured, then the state of the other qubit is somehow instantaneously affected; or that the state of the two qubits together cannot be described separately; or that the two qubits somehow maintain a memory of each other. These statements are not false, but why are they not also true for the (unentangled) probabilistic state (2) above? The two bits represented by this state are intimately connected: each one has a perfect memory of the other in a literal sense. But the state is nevertheless not entangled.

One way to explain what makes entanglement special, and what makes the quantum state (1) very different from the probabilistic state (2), is to explain what can be done with entanglement, or what we can see happening because of entanglement, that goes beyond the decisions we make about how to represent our knowledge of states using vectors. All three of the examples to be discussed in this lesson have this nature, in that they illustrate things that can be done with the state (1) that cannot be done with *any* classically correlated state, including the state (2).

Indeed, it is typical in the study of quantum information and computation that entanglement is viewed as a resource through which different tasks can be accomplished. When this is done, the state (1) is viewed as representing one *unit* of entanglement, which we refer to as an *e-bit*. The "e" stands for "entangled" or "entanglement." While it is true that the state (1) is a state of two qubits, the quantity of entanglement that it represents is one e-bit.

Incidentally, we can also view the probabilistic state (2) as a resource, which is one bit of *shared randomness*. It can be very useful in cryptography, for instance, to share a random bit with somebody (presuming that nobody else knows what the bit is), so that it can be used as a private key, or part of a private key, for the sake of encryption. But in this lesson the focus is on entanglement and a few things we can do with it.

As a point of clarification regarding terminology, when we say that Alice and Bob *share an e-bit*, what we mean is that Alice has a qubit named A, Bob has a qubit named B, and together the pair (A, B) is in the quantum state (1). Different names could, of course, be chosen for the qubits, but throughout this lesson we will stick with these names in the interest of clarity.

Teleportation

Quantum teleportation, or just teleportation for short, is a protocol where a sender (Alice) transmits a qubit to a receiver (Bob) by making use of a shared entangled quantum state (one e-bit, to be specific) along with two bits of classical communication. The name *teleportation* is meant to be suggestive of the concept in science fiction where matter is transported from one location to another by a futuristic process, but it must be understood that matter is not teleported in quantum teleportation — what is actually teleported is quantum information.

The set-up for teleportation is as follows.

We assume that Alice and Bob share an e-bit: Alice holds a qubit A, Bob holds a qubit B, and together the pair (A, B) is in the state $|\phi^+\rangle$. It could be, for instance, that Alice and Bob were in the same location in the past, they prepared the qubits A and B in the state $|\phi^+\rangle$, and then each went their own way with their qubit in hand. Or, it could be that a different process, such as one involving a third party or a complex distributed process, was used to establish this shared e-bit. These details are not part of the teleportation protocol itself.

Alice then comes into possession of a third qubit Q that she wishes to transmit to Bob. The state of the qubit Q is considered to be *unknown* to Alice and Bob, and no assumptions are made about it. For example, the qubit Q might be entangled with one or more other systems that neither Alice nor Bob can access. To say that Alice wishes to transmit the qubit Q to Bob means that Alice would like Bob to be holding a qubit that is in the same state that Q was in at the start of the protocol, having whatever correlations that Q had with other systems, as if Alice had physically handed Q to Bob.

We could imagine that Alice physically sends the qubit Q to Bob, and if it reaches Bob without being altered or disturbed in transit, then Alice and Bob's task will be accomplished. In the context of teleportation, however, it is our assumption that this is not feasible; Alice cannot send qubits directly to Bob. She may, however, send classical information to Bob.

These are reasonable assumptions in a variety of settings. For example, if Alice doesn't know Bob's exact location, or the distance between them is large, physically sending a qubit using the technology of today, or the foreseeable future, would be challenging to say the least. However, as we know from everyday experiences, classical information transmission under these circumstances is quite straightforward.

At this point, one might ask whether it is possible for Alice and Bob to accomplish their task without even needing to make use of a shared e-bit. In other words, is there any way to transmit a qubit using classical communication alone? The answer is no, it is not possible to transmit quantum information using classical communication alone. This is not too difficult to prove mathematically using basic quantum information theory, but we can alternatively rule out the possibility of transmitting qubits using classical communication alone by thinking about the no-cloning theorem.

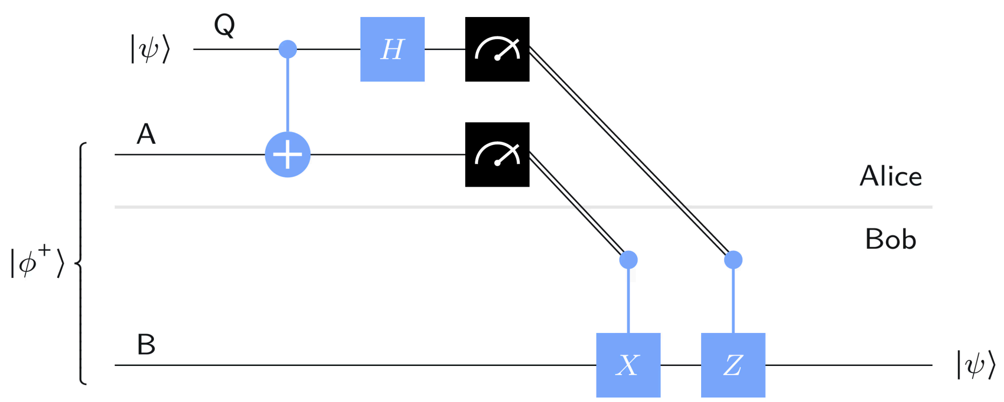
Imagine that there was a way to send quantum information using classical communication alone. Classical information can easily be copied and broadcast, which means that any classical transmission from Alice to

Bob might also potentially be received by a second receiver (Charlie, let us say). But if Charlie receives the same classical communication that Bob received, then would he not also be able to obtain a copy of the qubit Q? This would suggest that Q was cloned, which we already know is impossible from the no-cloning theorem, and so we conclude that there is no way to send quantum information using classical communication alone.

When the assumption that Alice and Bob share an e-bit is in place, however, it is possible for Alice and Bob to accomplish their task. This is precisely what the quantum teleportation protocol does.

Protocol

Here is a quantum circuit diagram that describes the teleportation protocol:



The diagram is slightly stylized in that it depicts the separation between Alice and Bob, with two diagonal wires representing classical bits that are sent from Alice to Bob, but otherwise it is an ordinary quantum circuit diagram. The qubit names are shown above the wires rather than to the left so that the initial states can be shown as well (which we will commonly do when it is convenient).

In words, the teleportation protocol is as follows:

1. Alice performs a controlled-NOT operation on the pair (A,Q), with Q being the control and A being the target, and then performs a Hadamard operation on Q.
2. Alice then measures both A and Q, with respect to a standard basis measurement in both cases, and transmits the classical outcomes to

Bob. Let us refer to the outcome of the measurement of A as a and the outcome of the measurement of Q as b.

3. Bob receives a and b from Alice, and depending on the values of these bits he performs these operations:
   - If $a = 1$, then Bob performs a bit flip (or X gate) on his qubit B.
   - If $b = 1$, then Bob performs a phase flip (or Z gate) on his qubit B.

   That is, conditioned on ab being $00, 01, 10,$ or $11,$ Bob performs one of the operations $\mathbb{I}$, Z, X, or ZX on the qubit B.

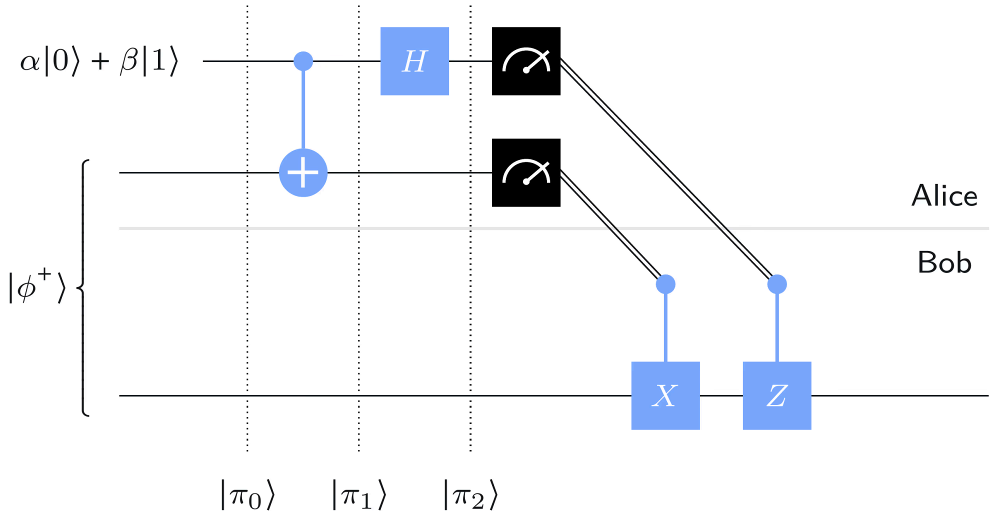
This is the complete description of the teleportation protocol. The analysis that appears below reveals that when it is run, the qubit B will be in whatever state Q was in prior to the protocol being executed, including whatever correlations it had with any other systems — which is to say that the protocol has effectively implemented a perfect qubit communication channel, where the state of Q has been "teleported" into B.

Before proceeding to the analysis, notice that this protocol does not succeed in cloning the state of Q, which we already know is impossible by the no-cloning theorem. Rather, when the protocol is finished, the state of the qubit Q will have changed from its original value to $\vert b\rangle$ as a result of the measurement performed on it. Also notice that the e-bit has effectively been "burned" in the process: the state of A has changed to $\vert a\rangle$ and is no longer entangled with B (or any other system). This is the cost of teleportation.

Analysis

To analyze the teleportation protocol, we'll examine the behavior of the circuit described above, one step at a time, beginning with the situation in which Q is initially in the state $\alpha\vert 0\rangle + \beta\vert 1\rangle$. This is not the most general situation, as it does not capture the possibility that Q is entangled with other systems, but starting with this simpler case will add clarity to the analysis. The more general case is addressed below, following the analysis of the simpler case.

Specifically, we will consider the states of the qubits $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ at the times suggested by this figure:



Under the assumption that the qubit Q begins the protocol in the state $\alpha|0\rangle + \beta|1\rangle$, the state of the three qubits $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ together at the start of the protocol is therefore

$$|\pi_0\rangle = |\phi^+\rangle \otimes \bigl(\alpha|0\rangle + \beta|1\rangle\bigr) = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|11}{\sqrt{2}}$$

The first gate that is performed is the controlled-NOT gate, which transforms the state $|\pi_0\rangle$ into

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}}.$$

Then the Hadamard gate is applied, which transforms the state $|\pi_1\rangle$ into

$$\begin{aligned}|\pi_2\rangle &= \frac{\alpha|00\rangle|+\rangle + \alpha|11\rangle|+\rangle + \beta|01\rangle|-\rangle + \beta|10\rangle|-\rangle}{\sqrt{2}}\\ &= \frac{\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle +}{2}\end{aligned}$$

Using the multilinearity of the tensor product, we may alternatively write this state as follows:

$$
\begin{aligned}
\vert \pi_2 \rangle = \quad & \frac{1}{2}\bigl(\alpha\vert 0\rangle + \beta\vert 1\rangle\bigr)\vert 00\rangle\\
& +\frac{1}{2}\bigl(\alpha\vert 0\rangle - \beta\vert 1\rangle\bigr)\vert 01\rangle\\
& +\frac{1}{2}\bigl(\alpha\vert 1\rangle + \beta\vert 0\rangle\bigr)\vert 10\rangle\\
& +\frac{1}{2}\bigl(\alpha\vert 1\rangle - \beta\vert 0\rangle\bigr)\vert 11\rangle.
\end{aligned}
$$

At first glance, it might look like something magical has happened, because the leftmost qubit B now seems to depend on the numbers α and β, even though there has not yet been any communication from Alice to Bob. This is an illusion. Scalars float freely through tensor products, so α and β are neither more nor less associated with the leftmost qubit than they are with the other qubits, and all we have done is to use algebra to express the state in a way that facilitates an analysis of the measurements.

Now let us consider the four possible outcomes of Alice's standard basis measurements, together with the actions that Bob performs as a result.

Possible outcomes

- The outcome of Alice's measurement is $ab = 00$ with probability

  $$
  \left\| \frac{1}{2}\bigl(\alpha\vert 0\rangle + \beta\vert 1\rangle\bigr)\right\|^2 = \frac{\vert\alpha\vert^2 + \vert\beta\vert^2}{4} = \frac{1}{4},
  $$

  in which case the state of $(\mathsf{B},\mathsf{A},\mathsf{Q})$ becomes

  $$
  \bigl(\alpha\vert 0\rangle + \beta\vert 1\rangle\bigr)\vert 00\rangle.
  $$

  Bob does nothing in this case, and so this is the final state of these three qubits.

- The outcome of Alice's measurement is $ab = 01$ with probability

  $$
  \left\| \frac{1}{2}\bigl(\alpha\vert 0\rangle - \beta\vert 1\rangle\bigr)\right\|^2 = \frac{\vert\alpha\vert^2 + \vert -\beta\vert^2}{4} = \frac{1}{4},
  $$

  in which case the state of $(\mathsf{B},\mathsf{A},\mathsf{Q})$ becomes

  $$
  \bigl(\alpha\vert 0\rangle - \beta\vert 1\rangle\bigr)\vert 01\rangle.
  $$

  In this case Bob applies a Z gate to B, leaving $(\mathsf{B},\mathsf{A},\mathsf{Q})$ in the state

$$\big(\alpha|0\rangle + \beta|1\rangle\big)|01\rangle.$$

- The outcome of Alice's measurement is $ab = 10$ with probability

$$\left\|\frac{1}{2}\big(\alpha|1\rangle + \beta|0\rangle\big)\right\|^2 = \frac{|\alpha|^2 + |\beta|^2}{4} = \frac{1}{4},$$

  in which case the state of $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ becomes

$$\big(\alpha|1\rangle + \beta|0\rangle\big)|10\rangle.$$

  In this case, Bob applies an X gate to the qubit B, leaving $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ in the state

$$\big(\alpha|0\rangle + \beta|1\rangle\big)|10\rangle.$$

- The outcome of Alice's measurement is $ab = 11$ with probability

$$\left\|\frac{1}{2}\big(\alpha|1\rangle - \beta|0\rangle\big)\right\|^2 = \frac{|\alpha|^2 + |-\beta|^2}{4} = \frac{1}{4},$$

  in which case the state of $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ becomes

$$\big(\alpha|1\rangle - \beta|0\rangle\big)|11\rangle.$$

  In this case, Bob performs the operation ZX on the qubit B, leaving $(\mathsf{B}, \mathsf{A}, \mathsf{Q})$ in the state

$$\big(\alpha|0\rangle + \beta|1\rangle\big)|11\rangle.$$

We now see, in all four cases, that Bob's qubit B is left in the state $\alpha|0\rangle + \beta|1\rangle$ at the end of the protocol, which is the initial state of the qubit Q. This is what we wanted to show: the teleportation protocol has worked correctly.

We also see that the qubits A and Q are left in one of the four states $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$, each with probability $1/4$, depending upon the measurement outcomes that Alice obtained. Thus, as was already suggested above, at the end of the protocol Alice no longer has the state $\alpha|0\rangle + \beta|1\rangle$, which is consistent with the no-cloning theorem.

Also notice that Alice's measurements yield absolutely no information about the state $\alpha|0\rangle + \beta|1\rangle$. That is, the probability for each of the four possible measurement outcomes is $1/4$, irrespective of α and β. This is also essential for teleportation to work correctly. Extracting information from an unknown quantum state necessarily disturbs it in general, but here Bob obtains the state without it being disturbed.

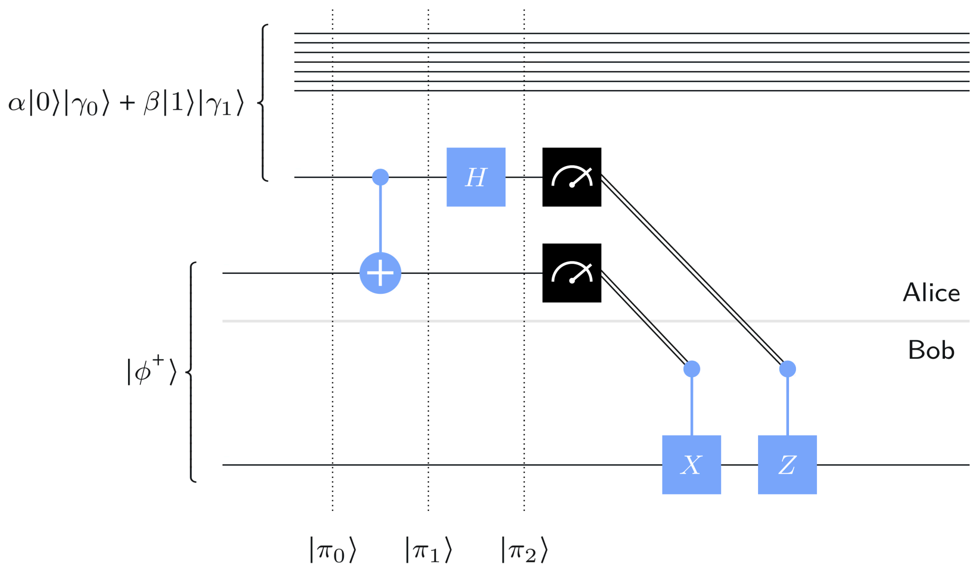
Now let's consider the more general situation in which the qubit Q is initially entangled with another system, which we'll name R. A similar analysis to the one above reveals that the teleportation protocol functions correctly in this more general case: at the end of the protocol, the qubit B held by Bob is entangled with R in the same way that Q was at the start of the protocol, as if Alice had simply handed Q to Bob.

To prove this, let us suppose that the state of the pair (Q,R) is initially given by a quantum state vector of the form

$$\alpha|0\rangle_{\mathsf{Q}}|\gamma_0\rangle_{\mathsf{R}} + \beta|1\rangle_{\mathsf{Q}}|\gamma_1\rangle_{\mathsf{R}},$$

where $|\gamma_0\rangle$ and $|\gamma_1\rangle$ are unit vectors and α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Any quantum state vector of the pair (Q,R) can be expressed in this way.

The following figure depicts the same circuit as before, with the addition of the system R (represented by a collection of qubits on the top of the diagram that nothing happens to).



To analyze what happens when the teleportation protocol is run, it is helpful to permute the systems, along the same lines that was described in the Multiple systems lesson. Specifically, we'll consider the state of the systems in the order $(\mathsf{B},\mathsf{R},\mathsf{A},\mathsf{Q})$ rather than $(\mathsf{B},\mathsf{A},\mathsf{Q},\mathsf{R})$. The names of the various systems are included as subscripts in the expressions that follow for clarity.

At the start of the protocol, the state of these systems is as follows:

$$
\begin{aligned}
|\pi_0\rangle &= |\phi^+\rangle_{\mathsf{BA}} \otimes \bigl(\alpha|0\rangle_{\mathsf{Q}}|\gamma_0\rangle_{\mathsf{R}} + \beta|1\rangle_{\mathsf{Q}}|\gamma_1\rangle_{\mathsf{R}}\bigr)\\
&= \frac{\alpha|0\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}}|00\rangle_{\mathsf{AQ}} + \alpha|1\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}}|10\rangle_{\mathsf{AQ}} + \beta|0\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}|01\rangle_{\mathsf{AQ}}}{\sqrt{2}}
\end{aligned}
$$

First the controlled-NOT gate is applied, which transforms this state to

$$
|\pi_1\rangle = \frac{\alpha|0\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}}|00\rangle_{\mathsf{AQ}} + \alpha|1\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}}|10\rangle_{\mathsf{AQ}} + \beta|0\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}|11\rangle_{\mathsf{AQ}}}{\sqrt{2}}
$$

Then the Hadamard gate is applied. After expanding and simplifying the resulting state, along similar lines to the analysis of the simpler case above, we obtain this expression of the resulting state:

$$
\begin{aligned}
|\pi_2\rangle = \;& \frac{1}{2}\bigl(\alpha|0\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}} + \beta|1\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}\bigr)|00\rangle_{\mathsf{AQ}}\\
&+\frac{1}{2}\bigl(\alpha|0\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}} - \beta|1\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}\bigr)|01\rangle_{\mathsf{AQ}}\\
&+\frac{1}{2}\bigl(\alpha|1\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}} + \beta|0\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}\bigr)|10\rangle_{\mathsf{AQ}}\\
&+\frac{1}{2}\bigl(\alpha|1\rangle_{\mathsf{B}}|\gamma_0\rangle_{\mathsf{R}} - \beta|0\rangle_{\mathsf{B}}|\gamma_1\rangle_{\mathsf{R}}\bigr)|11\rangle_{\mathsf{AQ}}.
\end{aligned}
$$

Proceeding exactly as before, where we consider the four different possible outcomes of Alice's measurements along with the corresponding actions performed by Bob, we find that at the end of the protocol, the state of (B, R) is always

$$
\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle.
$$

Informally speaking, the analysis does not change in a significant way as compared with the simpler case above; $|\gamma_0\rangle$ and $|\gamma_1\rangle$ essentially just "come along for the ride." So, teleportation succeeds in creating a perfect quantum communication channel, effectively transmitting the contents of the qubit Q into B and preserving all correlations with other systems.

This is actually not surprising at all, given the analysis of the simpler case above. As that analysis revealed, we have a physical process that acts like the identity operation on a qubit in an arbitrary quantum state, and there's only one way that can happen: the operation implemented by the protocol must *be* the identity operation. That is, once we know that teleportation works correctly for a single qubit in isolation, we can conclude that the protocol effectively implements a perfect, noiseless quantum channel, and so it must work correctly even if the input qubit is entangled with another system.

Further discussion

Here are a few brief, concluding remarks on teleportation.

First, teleportation is not an *application* of quantum information, it's a *protocol* for performing quantum communication. It is therefore useful only insofar as quantum communication is useful.

Indeed, it is reasonable to speculate that teleportation could one day become a standard way to communicate quantum information, perhaps through a process known as *entanglement distillation*. This is a process that converts a larger number of noisy (or imperfect) e-bits into a smaller number of high-quality e-bits, that could then be used for noiseless or near-noiseless teleportation. The idea is that the process of entanglement distillation is not as delicate as direct quantum communication. We could accept losses, for instance, and if the process doesn't work out, we can just try again. In contrast, the actual qubits we hope to communicate might be much more precious.

Finally, it should be understood that the idea behind teleportation and the way that it works is quite fundamental in quantum information and computation. It really is a cornerstone of quantum information theory, and variations of it arise. For example, quantum gates can be implemented through a closely related process known as *quantum gate teleportation*, which uses teleportation to apply *operations* to qubits rather than communicating them.

Qiskit implementation

```
from qiskit import __version__
print(__version__)
```

Output:

```
1.3.1
```

```
from qiskit import QuantumCircuit, QuantumRegiste
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram, arra
from qiskit.result import marginal_distribution
```

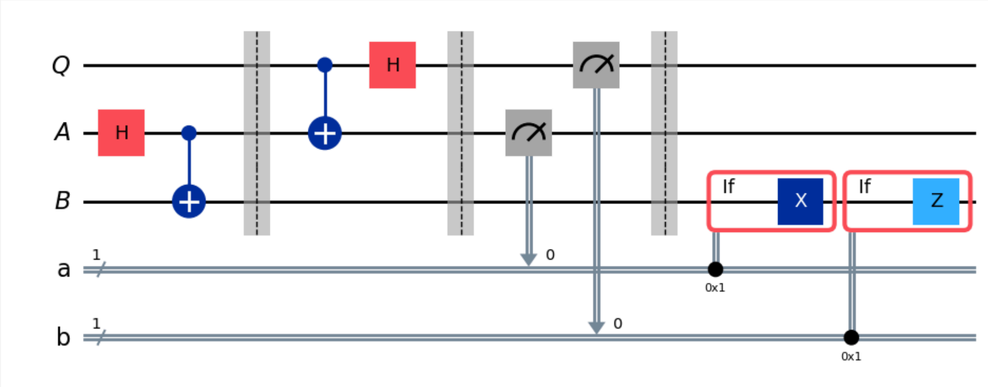
```
from qiskit.circuit.library import UGate
```

No output produced

Here is a quantum circuit implementation of the teleportation protocol.

```
qubit = QuantumRegister(1, "Q")
ebit0 = QuantumRegister(1, "A")
ebit1 = QuantumRegister(1, "B")
a = ClassicalRegister(1, "a")
b = ClassicalRegister(1, "b")

protocol = QuantumCircuit(qubit, ebit0, ebit1, a, b)

# Prepare ebit used for teleportation
protocol.h(ebit0)
protocol.cx(ebit0, ebit1)
protocol.barrier()

# Alice's operations
protocol.cx(qubit, ebit0)
protocol.h(qubit)
protocol.barrier()

# Alice measures and sends classical bits to Bob
protocol.measure(ebit0, a)
protocol.measure(qubit, b)
protocol.barrier()

# Bob uses the classical bits to conditionally apply
with protocol.if_test((a, 1)):
    protocol.x(ebit1)
with protocol.if_test((b, 1)):
    protocol.z(ebit1)

display(protocol.draw(output="mpl"))
```

Output:



The circuit makes use of a few features of Qiskit that we've not yet seen in previous lessons, including the `barrier` and `if_test` functions. The `barrier` function creates a visual separation making the circuit diagram more readable, and it also prevents Qiskit from performing various simplifications and optimizations across the barrier during compilation when circuits are run on real hardware. The `if_test` function applies an operation conditionally depending on a classical bit or register.

The circuit first initializes (A, B) to be in a $|\phi^+\rangle$ state (which is not part of the protocol itself), followed by Alice's operations, then her measurements, and finally Bob's operations. To test that the protocol works correctly, we'll apply a randomly generated single-qubit gate to the initialized $|0\rangle$ state of Q to obtain a random quantum state vector to be teleported. By applying the inverse (i.e., conjugate transpose) of that gate to B after the protocol is run, we can verify that the state was teleported by measuring to see that it has returned to the $|0\rangle$ state.

First we'll randomly choose a unitary qubit gate.

```
random_gate = UGate(
    theta=random.random() * 2 * pi,
    phi=random.random() * 2 * pi,
    lam=random.random() * 2 * pi
)

display(array_to_latex(random_gate.to_matrix()))
```

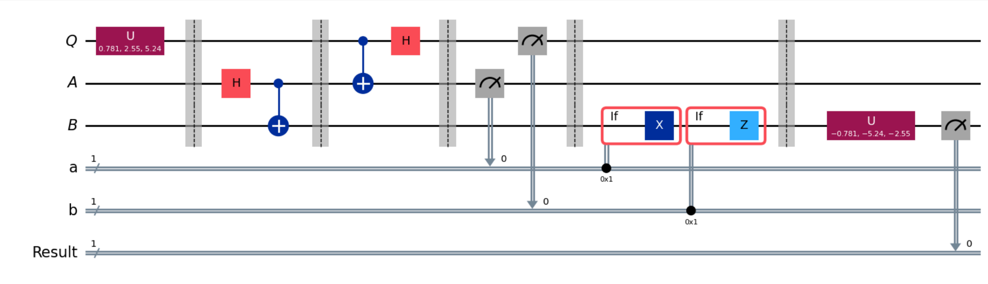
Output:

$$\begin{bmatrix} 0.924700868 & -0.1904078949 + 0.32965 \\ -0.3152211853 + 0.2134570426i & 0.0660168212 + 0.922341 \end{bmatrix}$$

Now we'll create a new testing circuit that first applies our random gate to Q, then runs the teleportation circuit, and finally applies the inverse of our random gate to the qubit B and measures. The outcome should be 0 with certainty.

```
# Create a new circuit including the same bits ar
# teleportation protocol.

test = QuantumCircuit(qubit, ebit0, ebit1, a, b)

# Start with the randomly selected gate on Q

test.append(random_gate, qubit)
test.barrier()

# Append the entire teleportation protocol from above

test = test.compose(protocol)
test.barrier()

# Finally, apply the inverse of the random unitary to

test.append(random_gate.inverse(), ebit1)
result = ClassicalRegister(1, "Result")
test.add_register(result)
test.measure(ebit1, result)

display(test.draw(output="mpl"))
```

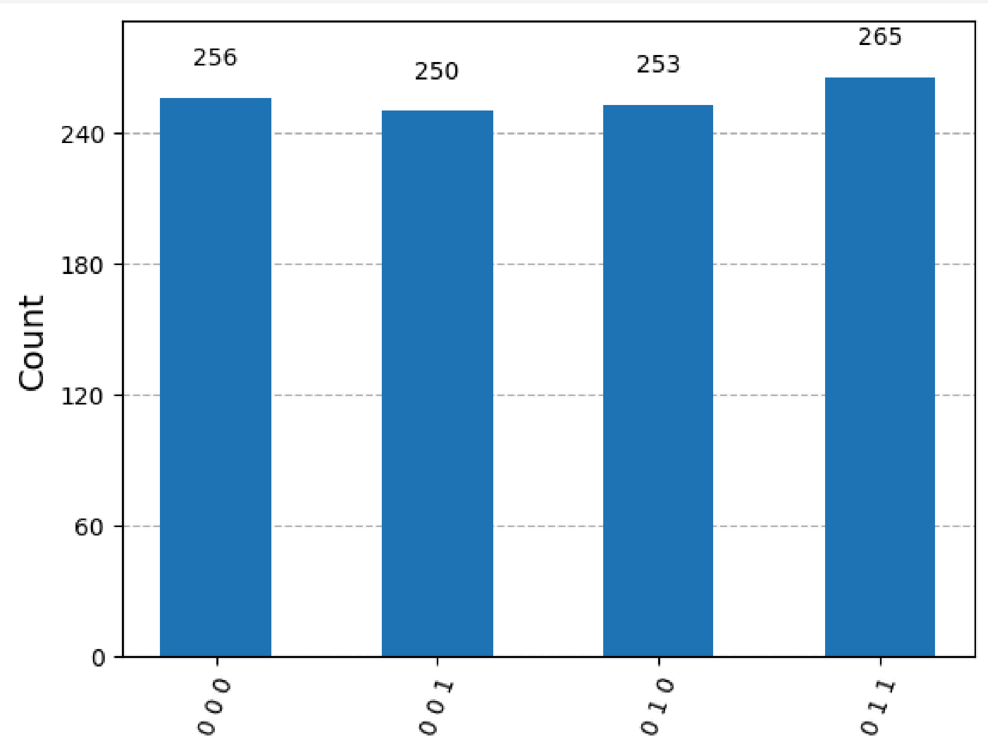
Output:



Finally, let's run the Aer simulator on this circuit and plot a histogram of the outputs. We'll see the statistics for all three classical bits: the bottom/leftmost bit should always be 0, indicating that the qubit Q was successfully teleported into B, while the other two bits should be roughly uniform.

```
result = AerSimulator().run(test).result()
statistics = result.get_counts()
display(plot_histogram(statistics))
```

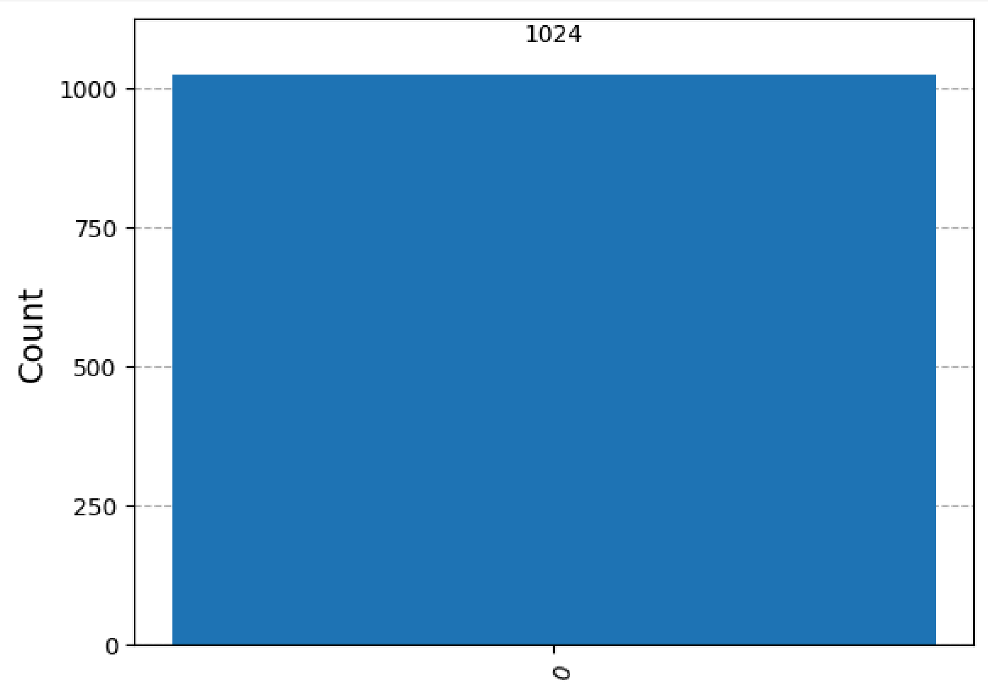
Output:



We can also filter the statistics to focus just on the test result qubit if we wish, like this:

```
filtered_statistics = marginal_distribution(stati
display(plot_histogram(filtered_statistics))
```

Output:



Superdense coding

Superdense coding is a protocol that, in some sense, achieves a complementary aim to teleportation. Rather than allowing for the transmission of one qubit using two classical bits of communication (at the cost of one e-bit of entanglement), it allows for the transmission of two classical bits using one qubit of quantum communication (again, at the cost of one e-bit of entanglement).

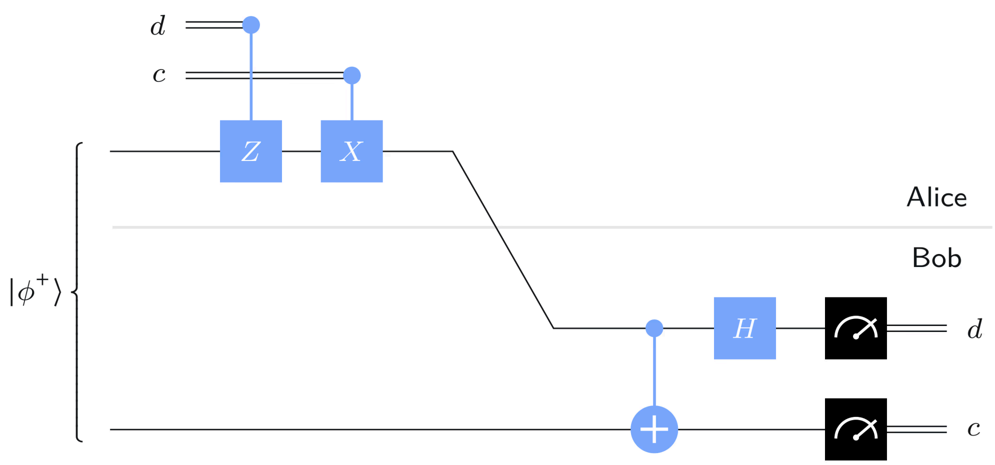
In greater detail, we have a sender (Alice) and a receiver (Bob) that share one e-bit of entanglement. According to the conventions in place for the lesson, this means that Alice holds a qubit A, Bob holds a qubit B, and together the pair (A, B) is in the state $|\phi^+\rangle$. Alice wishes to transmit two classical bits to Bob, which we'll denoted by c and d, and she will accomplish this by sending him one qubit.

It is reasonable to view this feat as being less interesting than the one that teleportation accomplishes. Sending qubits is likely to be so much more difficult than sending classical bits for the foreseeable future that trading one qubit of quantum communication for two bits of classical communication, at the cost of an e-bit no less, hardly seems worth it. However, this does not imply that superdense coding is not interesting, for it most certainly is.

Fitting the theme of the lesson, one reason why superdense coding is interesting is that it demonstrates a concrete and (in the context of information theory) rather striking use of entanglement. A famous theorem in quantum information theory, known as *Holevo's theorem*, implies that without the use of a shared entangled state, it is impossible to communicate more than one bit of classical information by sending a single qubit. (Holevo's theorem is more general than this. Its precise statement is technical and requires explanation, but this is one consequence of it.) So, through superdense coding, shared entanglement effectively allows for the *doubling* of the classical information-carrying capacity of sending qubits.

Protocol

The following quantum circuit diagram describes the superdense coding protocol:



In words, here is what Alice does:

1. If $d = 1$, Alice performs a Z gate on her qubit A (and if $d = 0$ she does not).
2. If $c = 1$, Alice performs an X gate on her qubit A (and if $c = 0$ she does not).

Alice then sends her qubit A to Bob.

What Bob does when he receives the qubit A is to first perform a controlled-NOT gate, with A being the control and B being the target, and then he applies a Hadamard gate to A. He then measures B to obtain c and A to obtain d, with standard basis measurements in both cases.

Analysis

The idea behind this protocol is pretty simple: Alice effectively chooses which Bell state she would like to be sharing with Bob, she sends Bob her qubit, and Bob measures to determine which Bell state Alice chose.

That is, they initially share $|\phi^+\rangle$, and depending upon the bits c and d, Alice either leaves this state alone or shifts it to one of the other Bell states by applying $\mathbb{I}$, X, Z, or XZ to her qubit A.

$$
\begin{aligned}
(\mathbb{I} \otimes \mathbb{I})|\phi^+\rangle &= |\phi^+\rangle\\
(\mathbb{I} \otimes Z)|\phi^+\rangle &= |\phi^-\rangle\\
(\mathbb{I} \otimes X)|\phi^+\rangle &= |\psi^+\rangle\\
(\mathbb{I} \otimes XZ)|\phi^+\rangle &= |\psi^-\rangle
\end{aligned}
$$

Bob's actions have the following effects on the four Bell states:

$$
\begin{aligned}
|\phi^+\rangle &\mapsto |00\rangle\\
|\phi^-\rangle &\mapsto |01\rangle\\
|\psi^+\rangle &\mapsto |10\rangle\\
|\psi^-\rangle &\mapsto -|11\rangle
\end{aligned}
$$

This can be checked directly, by computing the results of Bob's operations on these states one at a time.

So, when Bob performs his measurements, he is able to determine which Bell state Alice chose. To verify that the protocol works correctly is a matter of checking each case:

- If $cd = 00$, then the state of (B, A) when Bob receives A is $|\phi^+\rangle$. He transforms this state into $|00\rangle$ and obtains $cd = 00$.

- If $cd = 01$, then the state of (B, A) when Bob receives A is $|\phi^-\rangle$. He transforms this state into $|01\rangle$ and obtains $cd = 01$.
- If $cd = 10$, then the state of (B, A) when Bob receives A is $|\psi^+\rangle$. He transforms this state into $|10\rangle$ and obtains $cd = 10$.
- If $cd = 11$, then the state of (B, A) when Bob receives A is $|\psi^-\rangle$. He transforms this state into $-|11\rangle$ and obtains $cd = 11$. (The negative-one phase factor has no effect here.)

Qiskit implementation

Here is a simple implementation of superdense coding where we specify the circuit itself depending on the bits to be transmitted. First we'll choose two bits to be transmitted. (Later we'll choose them randomly, but for now we'll just make an arbitrary choice.)

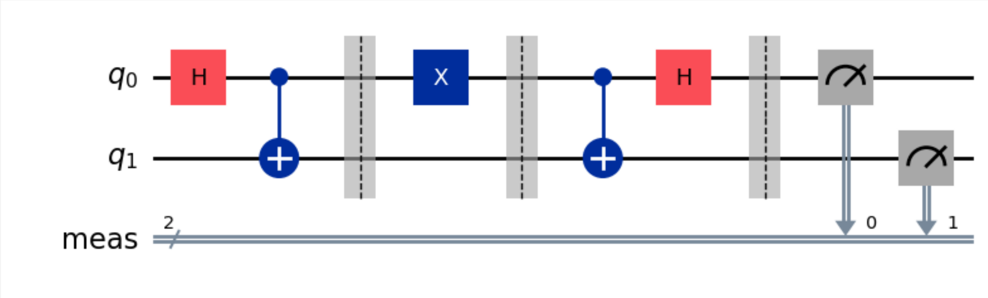
```
c = "1"
d = "0"
```

No output produced

Now we'll build the circuit accordingly. Here we'll just allow Qiskit to use the default names for the qubits: q_0 for the top qubit and q_1 for the bottom one.

```
protocol = QuantumCircuit(2)

# Prepare ebit used for superdense coding
protocol.h(0)
protocol.cx(0, 1)
protocol.barrier()

# Alice's operations
if d == "1":
    protocol.z(0)
if c == "1":
    protocol.x(0)
protocol.barrier()

# Bob's actions
protocol.cx(0, 1)
protocol.h(0)
protocol.measure_all()
```

Output:



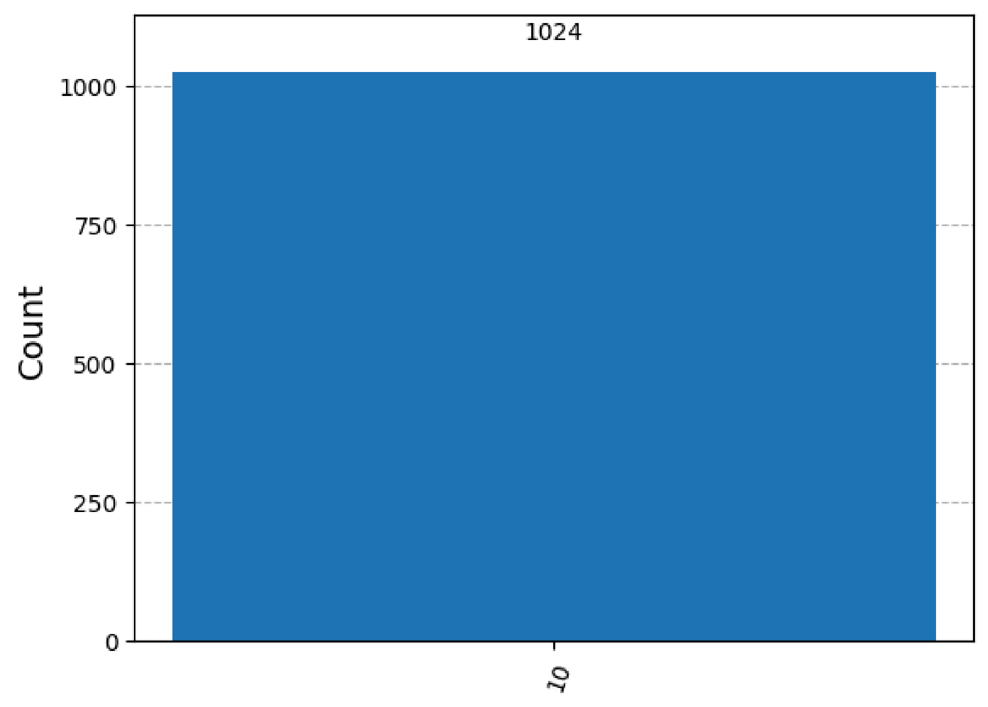
Not much is new here, except the `measure_all` function, which measures all of the qubits and puts the results into a single classical register (therefore having two bits in this case).

Running the Aer simulator produces the expected output.

```
result = AerSimulator().run(protocol).result()
statistics = result.get_counts()

for outcome, frequency in statistics.items():
    print(f"Measured {outcome} with frequency {frequen

display(plot_histogram(statistics))
```

Output:

```
Measured 10 with frequency 1024
```

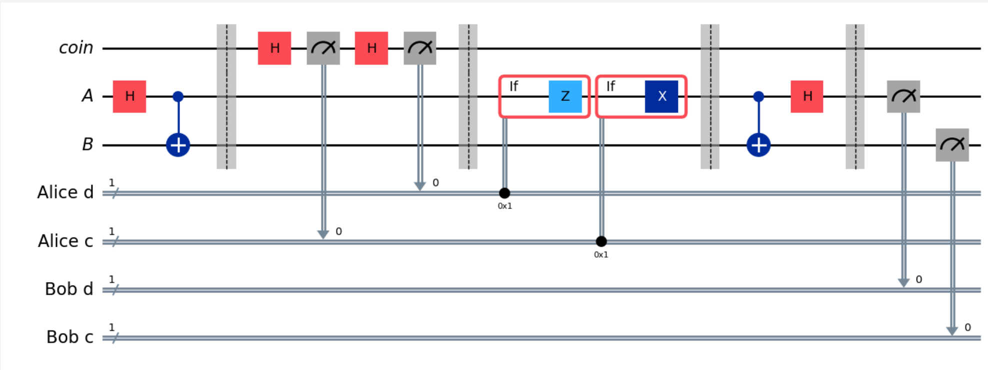


Now let's use an additional qubit as a random bit generator — essentially to flip fair coins. We'll use it to randomly choose c and d, and then run the superdense coding protocol.

```
rbg = QuantumRegister(1, "coin")
ebit0 = QuantumRegister(1, "A")
ebit1 = QuantumRegister(1, "B")

Alice_c = ClassicalRegister(1, "Alice c")
Alice_d = ClassicalRegister(1, "Alice d")

test = QuantumCircuit(rbg, ebit0, ebit1, Alice_d, Ali

# Initialize the ebit
test.h(ebit0)
test.cx(ebit0, ebit1)
test.barrier()

# Use the 'coin' qubit twice to generate Alice's bits
test.h(rbg)
test.measure(rbg, Alice_c)
test.h(rbg)
test.measure(rbg, Alice_d)
test.barrier()

# Now the protocol runs, starting with Alice's actior
# on her bits.
```

```
with test.if_test((Alice_d, 1), label="Z"):
    test.z(ebit0)
with test.if_test((Alice_c, 1), label="X"):
    test.x(ebit0)
test.barrier()

# Bob's actions
test.cx(ebit0, ebit1)
test.h(ebit0)
test.barrier()

Bob_c = ClassicalRegister(1, "Bob c")
Bob_d = ClassicalRegister(1, "Bob d")
test.add_register(Bob_d)
test.add_register(Bob_c)
test.measure(ebit0, Bob_d)
test.measure(ebit1, Bob_c)

display(test.draw(output="mpl"))
```

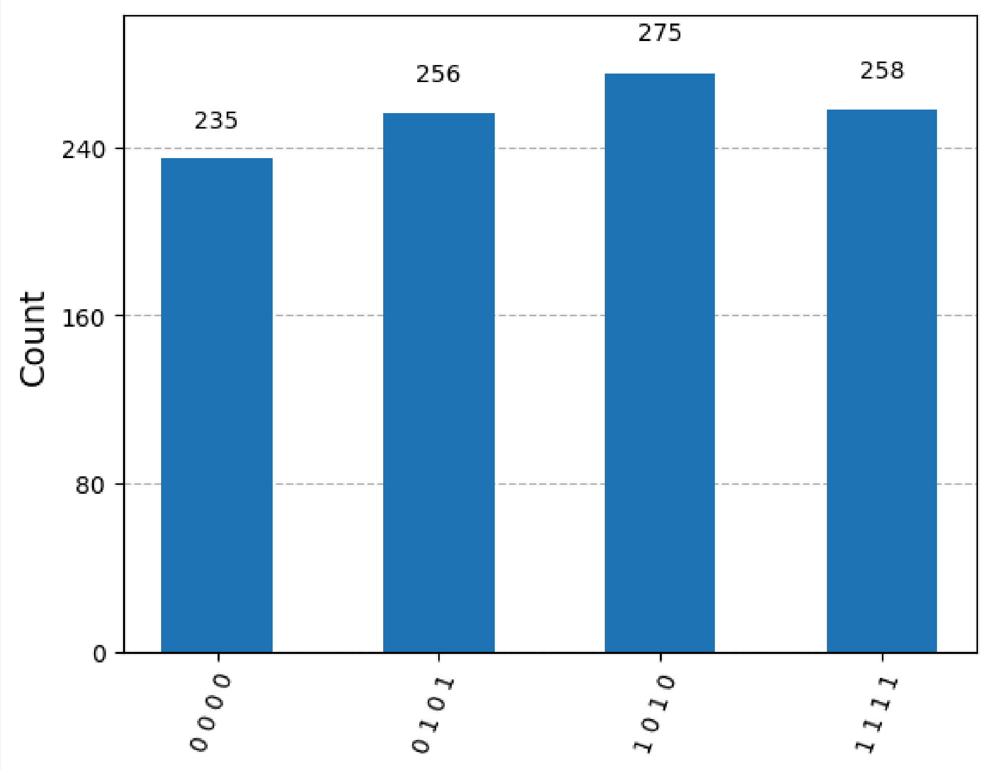
Output:



Running the Aer simulator shows the results: Alice and Bob's classical bits always agree.

```
result = AerSimulator().run(test).result()
statistics = result.get_counts()
display(plot_histogram(statistics))
```

Output:



The CHSH game

The last example to be discussed in this lesson is not a protocol, but a *game* known as the *CHSH game*.

When we speak of a game in this context, we're not talking about something that's meant to be played for fun or sport, but rather a mathematical abstraction in the sense of *game theory*. Mathematical abstractions of games are studied in economics and computer science, for instance, and they are both fascinating and useful.

The letters CHSH refer to the authors — John Clauser, Michael Horne, Abner Shimony, and Richard Holt — of a 1969 paper where the example was first described. They did not describe the example as a game, but rather as an experiment. Its description as a game, however, is both natural and intuitive.

The CHSH game falls within a class of games known as *nonlocal games*. Nonlocal games are incredibly interesting and have deep connections to physics, computer science, and mathematics — holding mysteries that still remain unsolved. We'll begin the section by explaining what nonlocal games are, and then we'll focus in on the CHSH game and what makes it interesting.

Nonlocal games

A nonlocal game is a *cooperative game* where two players, Alice and Bob, work together to achieve a particular outcome. The game is run by a *referee*, who behaves according to strict guidelines that are known to Alice and Bob.

Alice and Bob can prepare for the game however they choose, but once the game starts they are *forbidden from communicating*. We might imagine the game taking place in a secure facility of some sort — as if the referee is playing the role of a detective and Alice and Bob are suspects being interrogated in different rooms. But another way to think about the set-up is that Alice and Bob are separated by a vast distance, and communication is prohibited because the speed of light doesn't allow for it within the running time of the game. That is to say, if Alice tries to send a message to Bob, the game will be over by the time he receives it, and vice versa.

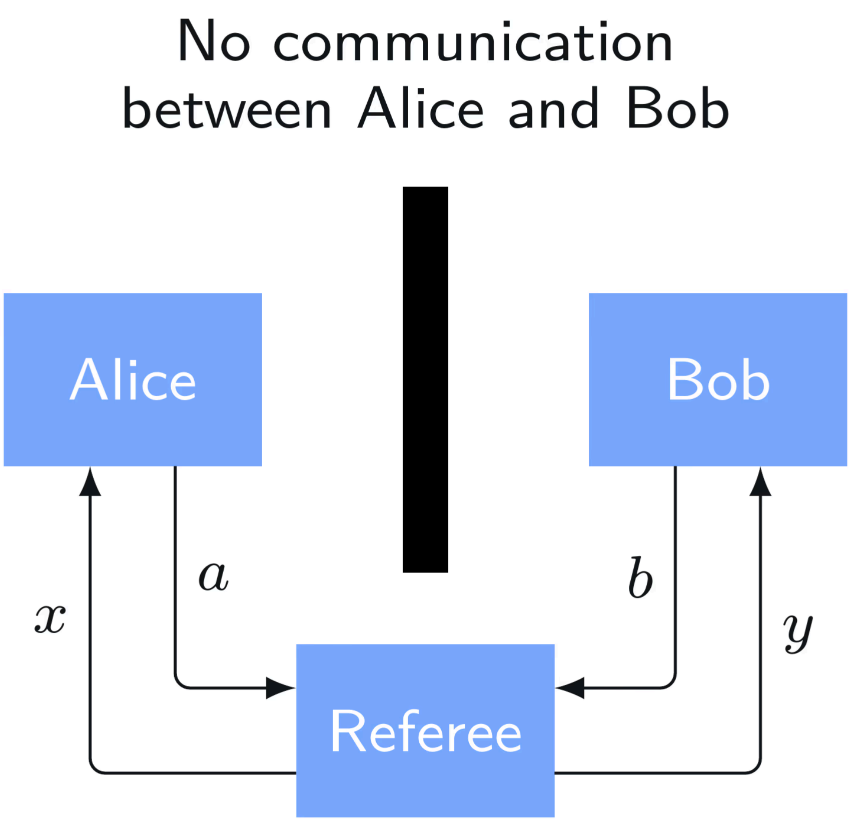
The way a nonlocal game works is that the referee first asks each of Alice and Bob a question. We'll use the letter x to refer to Alice's question and y to refer to Bob's question. Here we're thinking of x and y as being classical states, and in the CHSH game x and y are bits.

The referee uses *randomness* to select these questions. To be precise, there is some probability $p(x, y)$ associated with each possible pair (x, y) of questions, and the referee has vowed to choose the questions randomly, at the time of the game, in this way. Everyone, including Alice and Bob, knows these probabilities — but nobody knows specifically which pair (x, y) will be chosen until the game begins.

After Alice and Bob receive their questions, they must then provide answers: Alice's answer is a and Bob's answer is b. Again, these are classical states in general, and bits in the CHSH game.

At this point the referee makes a decision: Alice and Bob either *win* or *lose* depending on whether or not the pair of answers (a, b) is deemed correct for the pair of questions (x, y) according to some fixed set of rules. Different rules mean different games, and the rules for the CHSH game specifically are described in the section following this one. As was already suggested, the rules are known to everyone.

The following diagram provides a graphic representation of the interactions.



It is the uncertainty about which questions will be asked, and specifically the fact that each player doesn't know the other player's question, that makes nonlocal games challenging for Alice and Bob — just like colluding suspects in different rooms trying to keep their story straight.

A precise description of the referee defines an instance of a nonlocal game. This includes a specification of the probabilities $p(x, y)$ for each question pair along with the rules that determine whether each pair of answers (a, b) wins or loses for each possible question pair (x, y).

We'll take a look at the CHSH game momentarily, but before that let us briefly acknowledge that it's also interesting to consider other nonlocal games. In fact, it's extremely interesting; there are some pretty simple nonlocal games for which it's currently not known how well Alice and Bob can play using entanglement. The set-up is simple, but there's complexity at work — and for some games it can be impossibly difficult to compute best or near-best strategies for Alice and Bob. This is the mind-blowing nature of the non-local games model.

CHSH game description

Here is the precise description of the CHSH game, where (as above) x is Alice's question, y is Bob's question, a is Alice's answer, and b is Bob's answer:

- The questions and answers are all bits: $x, y, a, b \in \{0, 1\}$.
- The referee chooses the questions (x, y) *uniformly at random*. That is, each of the four possibilities, $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, is selected with probability $1/4$.
- The answers (a, b) *win* for the questions (x, y) if $a \oplus b = x \wedge y$ and *lose* otherwise. The following table expresses this rule by listing the winning and losing conditions on the answers (a, b) for each pair of questions (x, y).

(x, y)	win	lose
$(0, 0)$	$a = b$	$a \neq b$
$(0, 1)$	$a = b$	$a \neq b$
$(1, 0)$	$a = b$	$a \neq b$
$(1, 1)$	$a \neq b$	$a = b$

Limitation of classical strategies

Now let's consider strategies for Alice and Bob in the CHSH game, beginning with *classical* strategies.

Deterministic strategies

We'll start with *deterministic* strategies, where Alice's answer a is a function of the question x that she receives, and likewise Bob's answer b is a function of the question y he receives. So, for instance, we may write $a(0)$ to represent Alice's answer when her question is 0, and $a(1)$ to represent Alice's answer when her question is 1.

No deterministic strategy can possibly win the CHSH game every time. One way to reason this is simply to go one-by-one through all of the possible deterministic strategies and check that every one of them loses for at least one of the four possible question pairs. Alice and Bob can each choose from four possible functions from one bit to one bit — which we encountered in the lesson on Single systems — and so there are 16 different deterministic strategies in total to check.

We can also reason this analytically. If Alice and Bob's strategy wins when $(x, y) = (0, 0)$, then it must be that $a(0) = b(0)$; if their strategy wins when $(x, y) = (0, 1)$, then $a(0) = b(1)$; and similarly, if the strategy wins for $(x, y) = (1, 0)$ then $a(1) = b(0)$. So, if their strategy wins for all three possibilities, then

$$b(1) = a(0) = b(0) = a(1).$$

This implies that the strategy loses in the final case $(x, y) = (1, 1)$, for here winning requires that $a(1) \neq b(1)$. Thus, there can be no deterministic strategy that wins every time.

On the other hand, it is easy to find deterministic strategies that win in three of the four cases, such as $a(0) = a(1) = b(0) = b(1) = 0$. From this we conclude that the maximum probability for Alice and Bob to win using a deterministic strategy is $3/4$.

Probabilistic strategies

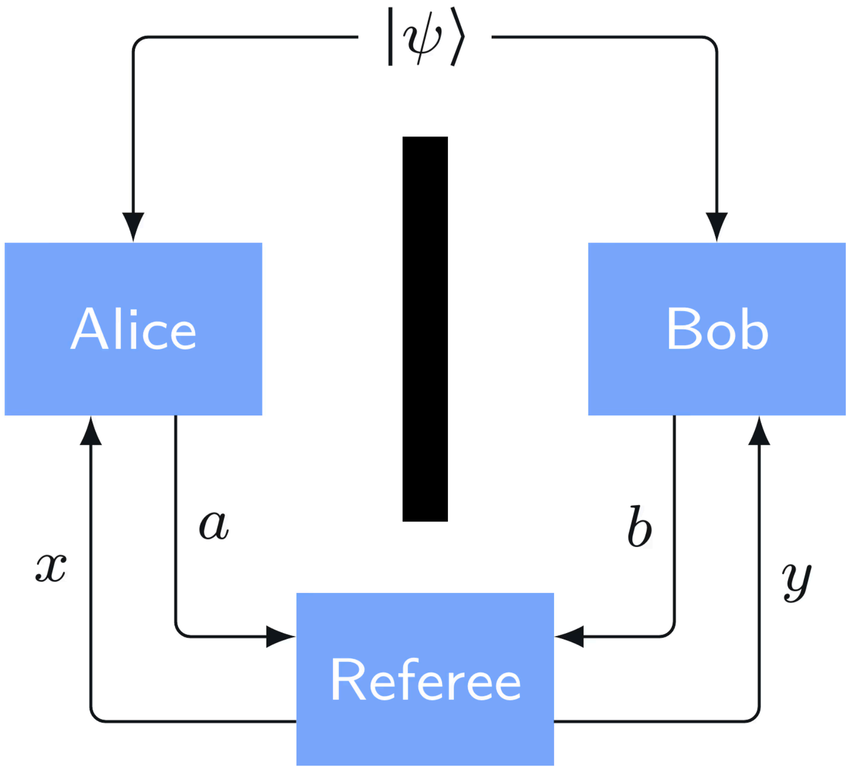
As we just concluded, Alice and Bob cannot do better than winning the CHSH game 75% of the time using a deterministic strategy. But what about a probabilistic strategy? Could it help Alice and Bob to use randomness — including the possibility of *shared randomness*, where their random choices are correlated?

It turns out that probabilistic strategies don't help at all to increase the probability that Alice and Bob win. This is because every probabilistic strategy can alternatively be viewed as a random selection of a deterministic strategy, just like (as was mentioned in the Single systems lesson) probabilistic operations can be viewed as random selections of deterministic operations. The average is never larger than the maximum, and so it follows that probabilistic strategies don't offer any advantage in terms of their overall winning probability.

Thus, winning with probability $3/4$ is the best that Alice and Bob can do using any classical strategy, whether deterministic or probabilistic.

CHSH game strategy

A natural question to ask at this point is whether Alice and Bob can do any better using a *quantum* strategy. In particular, if they share an entangled quantum state as the following figure suggests, which they could have prepared prior to playing the game, can they increase their winning probability?



The answer is yes, and this is the main point of the example and why it's so interesting. So let's see exactly how Alice and Bob can do better in this game using entanglement.

Required vectors and matrices

The first thing we need to do is to define a qubit state vector $|\psi_\theta\rangle$, for each real number θ (which we'll think of as an angle measured in radians) as follows.

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

Here are some simple examples:

$$\begin{aligned}
|\psi_0\rangle &= |0\rangle \\
|\psi_{\pi/2}\rangle &= |1\rangle \\
|\psi_{\pi/4}\rangle &= |+\rangle \\
|\psi_{-\pi/4}\rangle &= |-\rangle
\end{aligned}$$

We also have the following examples, which arise in the analysis below:

$$
\begin{aligned}
|\psi_{-\pi/8}\rangle &= \frac{\sqrt{2+\sqrt{2}}}{2}|0\rangle - \frac{\sqrt{2-\sqrt{2}}}{2}|1\rangle \\
|\psi_{\pi/8}\rangle &= \frac{\sqrt{2+\sqrt{2}}}{2}|0\rangle + \frac{\sqrt{2-\sqrt{2}}}{2}|1\rangle \\
|\psi_{3\pi/8}\rangle &= \frac{\sqrt{2-\sqrt{2}}}{2}|0\rangle + \frac{\sqrt{2+\sqrt{2}}}{2}|1\rangle \\
|\psi_{5\pi/8}\rangle &= -\frac{\sqrt{2-\sqrt{2}}}{2}|0\rangle + \frac{\sqrt{2+\sqrt{2}}}{2}|1\rangle
\end{aligned}
$$

Looking at the general form, we see that the inner product between any two of these vectors has this formula:

$$
\langle\psi_\alpha|\psi_\beta\rangle = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha-\beta). \quad (3)
$$

In detail, there are only real number entries in these vectors, so there are no complex conjugates to worry about: the inner product is the product of the cosines plus the product of the sines. Using one of the *angle addition formulas* from trigonometry leads to the simplification above. This formula reveals the geometric interpretation of the inner product between real unit vectors as the cosine of the angle between them.

If we compute the inner product of the *tensor product* of any two of these vectors with the $|\phi^+\rangle$ state, we obtain a similar expression, except that it has a $\sqrt{2}$ in the denominator:

$$
\langle\psi_\alpha\otimes\psi_\beta|\phi^+\rangle = \frac{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}{\sqrt{2}} = \frac{\cos(\alpha-\beta)}{\sqrt{2}} \quad (4)
$$

Our interest in this particular inner product will become clear shortly, but for now we're simply observing this as a formula.

Next, define a unitary matrix U_θ for each angle θ as follows.

$$
U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}|
$$

Intuitively speaking, this matrix transforms $|\psi_\theta\rangle$ into $|0\rangle$ and $|\psi_{\theta+\pi/2}\rangle$ into $|1\rangle$. To check that this is a unitary matrix, a key observation is that the vectors $|\psi_\theta\rangle$ and $|\psi_{\theta+\pi/2}\rangle$ are orthogonal for every angle θ:

$$
\langle\psi_\theta|\psi_{\theta+\pi/2}\rangle = \cos(\pi/2) = 0.
$$

Thus, we find that

$$
\begin{aligned}
U_\theta U_\theta^\dagger &= \bigl(|0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}|\bigr)\bigl(|\psi_\theta\rangle\langle 0| + |\psi_{\theta+\pi/2}\rangle\langle 1|\bigr)\\
&= |0\rangle\langle\psi_\theta|\psi_\theta\rangle\langle 0| + |0\rangle\langle\psi_\theta|\psi_{\theta+\pi/2}\rangle\langle 1| + |1\rangle\langle\psi_{\theta+\pi/2}|\psi_\theta\rangle\langle 0| + |\\
&= |0\rangle\langle 0| + |1\rangle\langle 1|\\
&= \mathbb{I}.
\end{aligned}
$$

We may alternatively write this matrix explicitly as

$$
U_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \cos(\theta+\pi/2) & \sin(\theta+\pi/2) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.
$$

This is an example of a *rotation matrix*, and specifically it rotates two-dimensional vectors with real number entries by an angle of $-\theta$ about the origin. If we follow a standard convention for naming and parameterizing rotations of various forms, we have $U_\theta = R_y(-2\theta)$ where

$$
R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
$$

Strategy description

Now we can describe the quantum strategy.

- **Set-up:** Alice and Bob start the game sharing an e-bit: Alice holds a qubit A, Bob holds a qubit B, and together the two qubits (X,Y) are in the $|\phi^+\rangle$ state.
- **Alice's actions:**
  - If Alice receives the question $x = 0$, she applies U_0 to her qubit A.
  - If Alice receives the question $x = 1$, she applies $U_{\pi/4}$ to her qubit A.

  The operation Alice performs on A may alternatively be described like this:

  $$
  \begin{cases} U_0 & \text{if } x = 0 \\ U_{\pi/4} & \text{if } x = 1 \end{cases}
  $$

  After Alice applies this operation, she measures A with a standard basis measurement and sets her answer a to be the measurement outcome.
- **Bob's actions:**

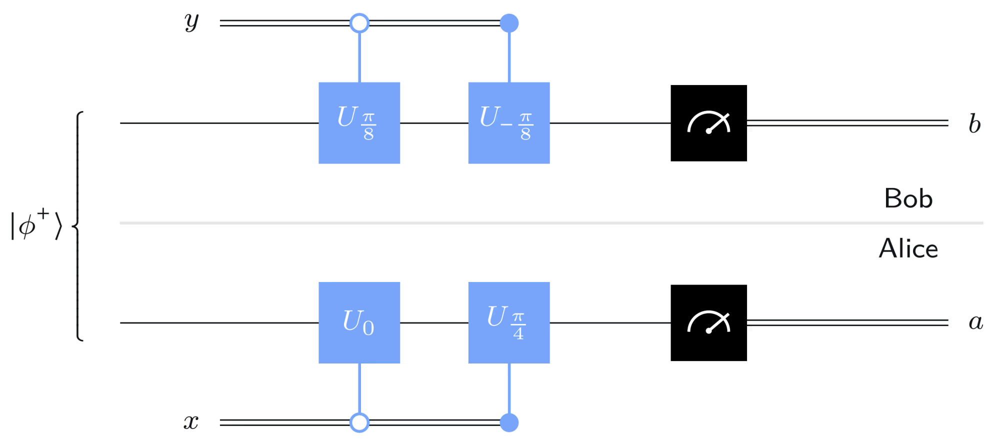
- If Bob receives the question $y = 0$, he applies $U_{\pi/8}$ to his qubit B.
- If Bob receives the question $y = 1$, he applies $U_{-\pi/8}$ to his qubit B.

Like we did for Alice, we can express Bob's operation on B like this:

$$\begin{cases} U_{\pi/8} & \text{if } y = 0 \\ U_{-\pi/8} & \text{if } y = 1 \end{cases}$$

After Bob applies this operation, he measures B with a standard basis measurement and sets his answer b to be the measurement outcome.

Here is a quantum circuit diagram that describes this strategy:



In this diagram we see two ordinary controlled gates, one for $U_{-\pi/8}$ on the top and one for $U_{\pi/4}$ on the bottom. We also have two gates that look like controlled gates, one for $U_{\pi/8}$ on the top and one for U_0 on the bottom, except that the circle representing the control is not filled in. This denotes a different type of controlled gate where the gate is performed if the control is set to 0 (rather than 1 like an ordinary controlled gate). So, effectively, Bob performs $U_{\pi/8}$ on his qubit if $y = 0$ and $U_{-\pi/8}$ if $y = 1$; and Alice performs U_0 on her qubit if $x = 0$ and $U_{\pi/4}$ if $x = 1$, which is consistent with the description of the protocol in words above.

It remains to figure out how well this strategy for Alice and Bob works. We'll do this by going through the four possible question pairs individually.

Case-by-case analysis

- Case 1: $(x, y) = (0, 0)$.

In this case Alice performs U_0 on her qubit and Bob performs $U_{\pi/8}$ on his, so the state of the two qubits (A,B) after they perform their operations is

$$\begin{aligned}\big(U_0 \otimes U_{\pi/8}\big)\vert\phi^+\rangle &= \vert 00\rangle\langle\psi_0 \otimes \psi_{\pi/8}\vert\phi^+\rangle + \vert 01\rangle\langle\psi_0 \otimes \psi_{5\pi/8}\vert\phi^+\rangle \\ &\quad + \vert 10\rangle\langle\psi_{\pi/2} \otimes \psi_{\pi/8}\vert\phi^+\rangle + \vert 11\rangle\langle\psi_{\pi/2} \otimes \psi_{5} \\ &= \frac{\cos\big(-\frac{\pi}{8}\big)\vert 00\rangle + \cos\big(-\frac{5\pi}{8}\big)\vert 01\rangle + \cos\big(\frac{3\pi}{8}\big)\vert 10}{\sqrt{2}}\end{aligned}$$

The probabilities for the four possible answer pairs (a,b) are therefore as follows.

$$\begin{aligned}\operatorname{Pr}\big((a,b)=(0,0)\big) &= \frac{1}{2}\cos^2\Big(-\frac{\pi}{8}\Big) = \frac{2+\sqrt{2}}{8}\\ \operatorname{Pr}\big((a,b)=(0,1)\big) &= \frac{1}{2}\cos^2\Big(-\frac{5\pi}{8}\Big) = \frac{2-\sqrt{2}}{8}\\ \operatorname{Pr}\big((a,b)=(1,0)\big) &= \frac{1}{2}\cos^2\Big(\frac{3\pi}{8}\Big) = \frac{2-\sqrt{2}}{8}\\ \operatorname{Pr}\big((a,b)=(1,1)\big) &= \frac{1}{2}\cos^2\Big(-\frac{\pi}{8}\Big) = \frac{2+\sqrt{2}}{8}\end{aligned}$$

We can then obtain the probabilities that $a=b$ and $a\neq b$ by summing.

$$\begin{aligned}\operatorname{Pr}(a=b) &= \frac{2+\sqrt{2}}{4}\\ \operatorname{Pr}(a\neq b) &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

For the question pair $(0,0)$, Alice and Bob win if $a=b$, and therefore they win in this case with probability

$$\frac{2+\sqrt{2}}{4}.$$

- Case 2: $(x,y)=(0,1)$.
  In this case Alice performs U_0 on her qubit and Bob performs $U_{-\pi/8}$ on his, so the state of the two qubits (A,B) after they perform their operations is

$$\begin{aligned}(U_0 \otimes U_{-\pi/8})|\phi^+\rangle = |00\rangle\langle\psi_0 \otimes \psi_{-\pi/8}|\phi^+\rangle + |01\rangle\langle\psi_0 \otimes \psi_{3\pi/8}|\phi^+\\ + |10\rangle\langle\psi_{\pi/2} \otimes \psi_{-\pi/8}|\phi^+\rangle + |11\rangle\langle\psi_{\pi/2} \otimes\\ = \frac{\cos\left(\frac{\pi}{8}\right)|00\rangle + \cos\left(-\frac{3\pi}{8}\right)|01\rangle + \cos\left(\frac{5\pi}{8}\right)|10}{\sqrt{2}}\end{aligned}$$

The probabilities for the four possible answer pairs (a, b) are therefore as follows.

$$\begin{aligned}\Pr\big((a,b) = (0,0)\big) &= \frac{1}{2}\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{8}\\ \Pr\big((a,b) = (0,1)\big) &= \frac{1}{2}\cos^2\left(-\frac{3\pi}{8}\right) = \frac{2-\sqrt{2}}{8}\\ \Pr\big((a,b) = (1,0)\big) &= \frac{1}{2}\cos^2\left(\frac{5\pi}{8}\right) = \frac{2-\sqrt{2}}{8}\\ \Pr\big((a,b) = (1,1)\big) &= \frac{1}{2}\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{8}\end{aligned}$$

Again, we can obtain the probabilities that $a = b$ and $a \neq b$ by summing.

$$\begin{aligned}\Pr(a = b) &= \frac{2+\sqrt{2}}{4}\\ \Pr(a \neq b) &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

For the question pair $(0, 1)$, Alice and Bob win if $a = b$, and therefore they win in this case with probability

$$\frac{2+\sqrt{2}}{4}.$$

- Case 3: $(x, y) = (1, 0)$.
  In this case Alice performs $U_{\pi/4}$ on her qubit and Bob performs $U_{\pi/8}$ on his, so the state of the two qubits (A, B) after they perform their operations is

$$\begin{aligned}(U_{\pi/4} \otimes U_{\pi/8})|\phi^+\rangle = |00\rangle\langle\psi_{\pi/4} \otimes \psi_{\pi/8}|\phi^+\rangle + |01\rangle\langle\psi_{\pi/4} \otimes \psi_{5\pi/8}\\ + |10\rangle\langle\psi_{3\pi/4} \otimes \psi_{\pi/8}|\phi^+\rangle + |11\rangle\langle\psi_{3\pi/4} \otimes\\ = \frac{\cos\left(\frac{\pi}{8}\right)|00\rangle + \cos\left(-\frac{3\pi}{8}\right)|01\rangle + \cos\left(\frac{5\pi}{8}\right)|10}{\sqrt{2}}\end{aligned}$$

The probabilities for the four possible answer pairs (a, b) are therefore as follows.

$$\begin{aligned}
\operatorname{Pr}\bigl((a,b)=(0,0)\bigr) &= \frac{1}{2}\cos^2\Bigl(\frac{\pi}{8}\Bigr) = \frac{2+\sqrt{2}}{8}\\
\operatorname{Pr}\bigl((a,b)=(0,1)\bigr) &= \frac{1}{2}\cos^2\Bigl(-\frac{3\pi}{8}\Bigr) = \frac{2-\sqrt{2}}{8}\\
\operatorname{Pr}\bigl((a,b)=(1,0)\bigr) &= \frac{1}{2}\cos^2\Bigl(\frac{5\pi}{8}\Bigr) = \frac{2-\sqrt{2}}{8}\\
\operatorname{Pr}\bigl((a,b)=(1,1)\bigr) &= \frac{1}{2}\cos^2\Bigl(\frac{\pi}{8}\Bigr) = \frac{2+\sqrt{2}}{8}
\end{aligned}$$

We find, once again, that probabilities that $a = b$ and $a \neq b$ are as follows.

$$\begin{aligned}
\operatorname{Pr}(a=b) &= \frac{2+\sqrt{2}}{4}\\
\operatorname{Pr}(a\neq b) &= \frac{2-\sqrt{2}}{4}
\end{aligned}$$

For the question pair $(1, 0)$, Alice and Bob win if $a = b$, so they win in this case with probability

$$\frac{2+\sqrt{2}}{4}.$$

- Case 4: $(x, y) = (1, 1)$.

  The last case is a little bit different, as we might expect because the winning condition is different in this case. When x and y are both 1, Alice and Bob win when a and b are *different*. In this case Alice performs $U_{\pi/4}$ on her qubit and Bob performs $U_{-\pi/8}$ on his, so the state of the two qubits (A, B) after they perform their operations is

$$\begin{aligned}
\bigl(U_{\pi/4}\otimes U_{-\pi/8}\bigr)\vert\phi^+\rangle &= \vert 00\rangle\langle\psi_{\pi/4}\otimes\psi_{-\pi/8}\vert\phi^+\rangle + \vert 01\rangle\langle\psi_{\pi/4}\otimes\psi_{3\pi
\\
&\quad + \vert 10\rangle\langle\psi_{3\pi/4}\otimes\psi_{-\pi/8}\vert\phi^+\rangle + \vert 11\rangle\langle\psi_{3\pi/4}
\\
&= \frac{\cos\bigl(\frac{3\pi}{8}\bigr)\vert 00\rangle + \cos\bigl(-\frac{\pi}{8}\bigr)\vert 01\rangle + \cos\bigl(\frac{7\pi}{8}\bigr)\vert 1}{\sqrt{2}}
\end{aligned}$$

The probabilities for the four possible answer pairs (a, b) are therefore as follows.

$$\operatorname{Pr}\big((a,b)=(0,0)\big) = \frac{1}{2}\cos^2\Big(\frac{3\pi}{8}\Big) = \frac{2-\sqrt{2}}{8}$$
$$\operatorname{Pr}\big((a,b)=(0,1)\big) = \frac{1}{2}\cos^2\Big(-\frac{\pi}{8}\Big) = \frac{2+\sqrt{2}}{8}$$
$$\operatorname{Pr}\big((a,b)=(1,0)\big) = \frac{1}{2}\cos^2\Big(\frac{7\pi}{8}\Big) = \frac{2+\sqrt{2}}{8}$$
$$\operatorname{Pr}\big((a,b)=(1,1)\big) = \frac{1}{2}\cos^2\Big(\frac{3\pi}{8}\Big) = \frac{2-\sqrt{2}}{8}$$

The probabilities have effectively swapped places from in the three other cases. We obtain the probabilities that $a = b$ and $a \neq b$ by summing.

$$\operatorname{Pr}(a=b) = \frac{2-\sqrt{2}}{4}$$
$$\operatorname{Pr}(a\neq b) = \frac{2+\sqrt{2}}{4}$$

For the question pair $(1,1)$, Alice and Bob win if $a \neq b$, and therefore they win in this case with probability

$$\frac{2+\sqrt{2}}{4}.$$

They win in every case with the same probability:

$$\frac{2+\sqrt{2}}{4} \approx 0.85.$$

This is therefore the probability that they win overall. That's significantly better than any classical strategy can do for this game; classical strategies have winning probability bounded by $3/4$. And that makes this a very interesting example.

This happens to be the *optimal* winning probability for quantum strategies; we can't do any better than this, no matter what entangled state or measurements we choose. This fact is known as *Tsirelson's inequality*, named for Boris Tsirelson who first proved it — and who first described the CHSH experiment as a game.

Geometric picture

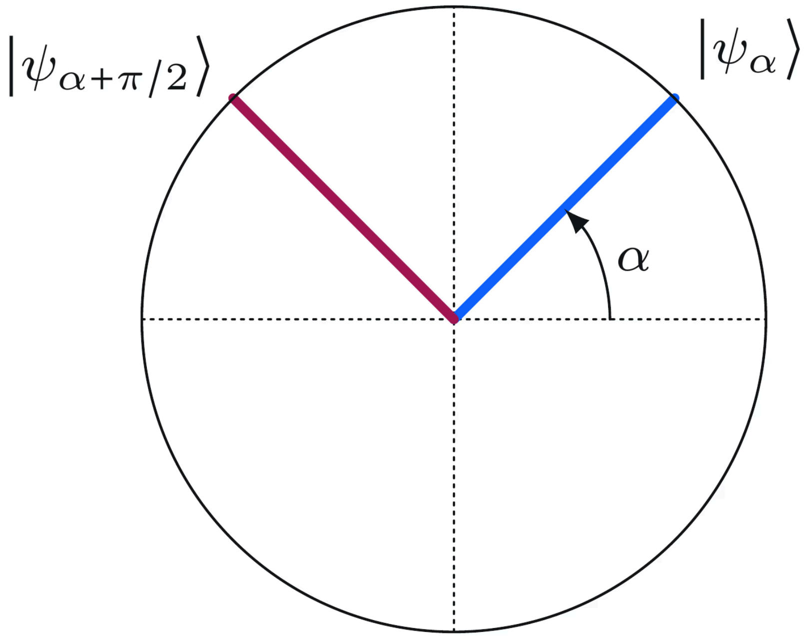
It is possible to think about the strategy described above geometrically, which may be helpful for understanding the relationships among the various angles chosen for Alice and Bob's operations.

What Alice effectively does is to choose an angle α, depending on her question x, and then to apply U_α to her qubit and measure. Similarly, Bob chooses an angle β, depending on y, and then he applies U_β to his qubit and measures. We've chosen α and β like so.

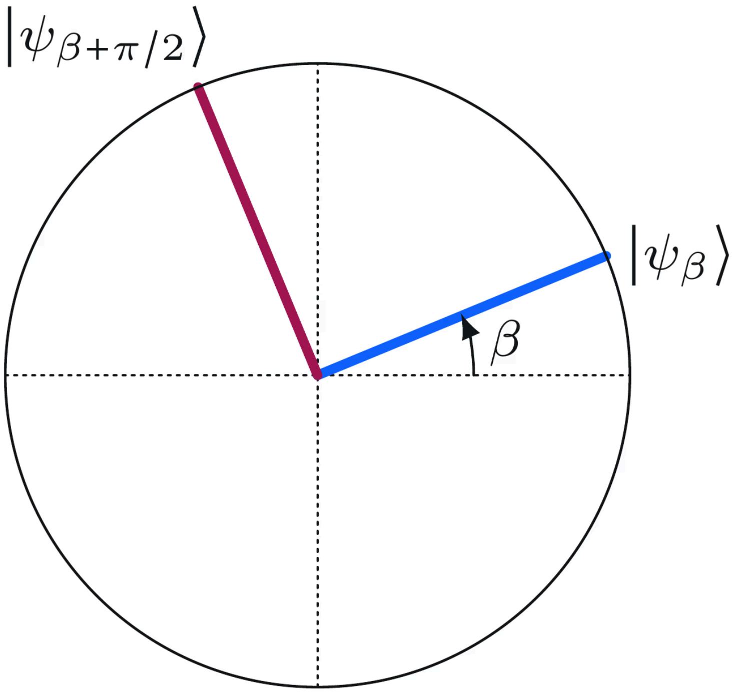
$$\alpha = \begin{cases} 0 & x = 0 \\ \pi/4 & x = 1 \end{cases}$$

$$\beta = \begin{cases} \pi/8 & y = 0 \\ -\pi/8 & y = 1 \end{cases}$$

For the moment, though, let's take α and β to be arbitrary. By choosing α, Alice effectively defines an orthonormal basis of vectors that looks like this:



Bob does likewise, except that his angle is β:



The colors of the vectors correspond to Alice and Bob's answers: blue for 0 and red for 1.

Now, if we combine together (3) and (4) we get the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle;$$

this works for all real numbers α and β.

Following the same sort of analysis that we went through above, but with α and β being variables, we find this:

$$\begin{aligned}
\bigl(U_\alpha \otimes U_\beta\bigr) |\phi^+\rangle \\
&= |00\rangle \langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle + |01\rangle \langle \psi_\alpha \otimes \psi_{\beta+\pi/2} | \phi^+ \rangle \\
&\qquad + |10\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_\beta | \phi^+ \rangle + |11\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_{\beta+\pi/2} | \phi^+ \rangle \\
&= \frac{\langle \psi_\alpha | \psi_\beta \rangle |00\rangle + \langle \psi_\alpha | \psi_{\beta+\pi/2} \rangle |01\rangle + \langle \psi_{\alpha+\pi/2} | \psi_\beta \rangle |10\rangle + \langle \psi_{\alpha-}}{\sqrt{2}}
\end{aligned}$$

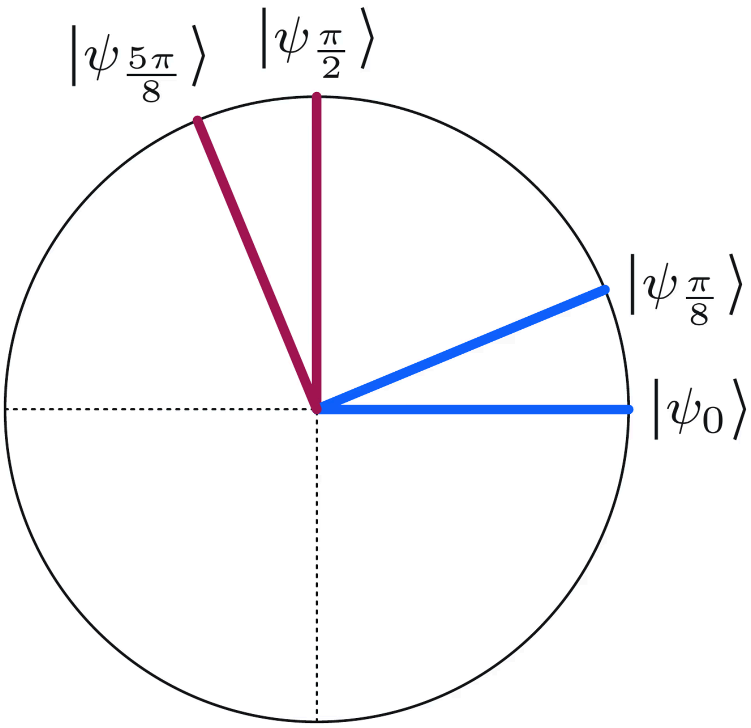
We conclude these two formulas:

$$\Pr(a = b) = \frac{1}{2}|\langle\psi_\alpha|\psi_\beta\rangle|^2 + \frac{1}{2}|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|^2 = \cos^2(\alpha - \beta)$$
$$\Pr(a \neq b) = \frac{1}{2}|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|^2 + \frac{1}{2}|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|^2 = \sin^2(\alpha - \beta).$$

These equations can be connected to the figures above by imagining that we superimpose the bases chosen by Alice and Bob.

Exploring the strategy

When $(x, y) = (0, 0)$, Alice and Bob choose $\alpha = 0$ and $\beta = \pi/8$, and by superimposing their bases we obtain this figure:



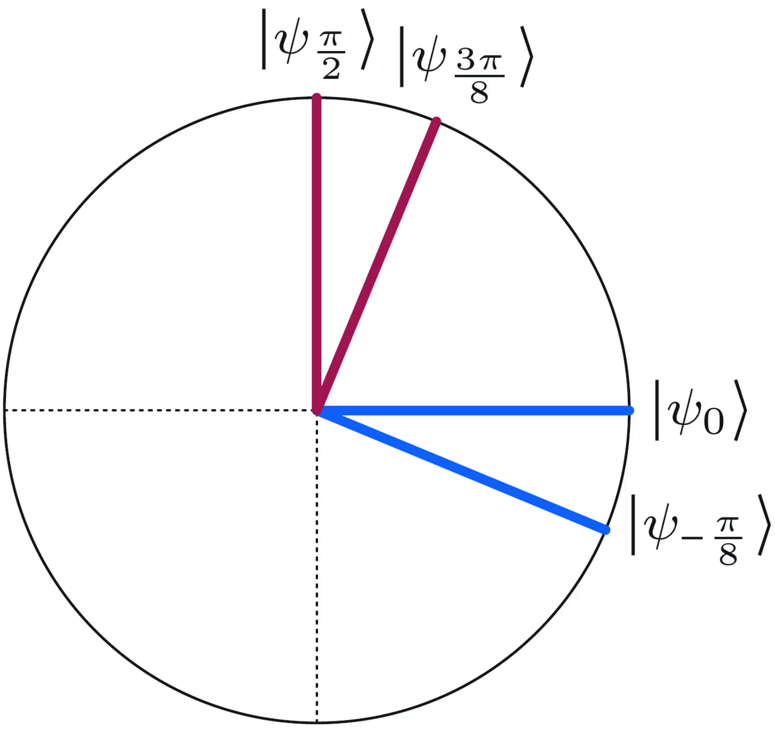
The angle between the red vectors is $\pi/8$, which is the same as the angle between the two blue vectors. The probability that Alice and Bob's outcomes agree is the cosine-squared of this angle,

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2 + \sqrt{2}}{4},$$

while the probability they disagree is the sine-squared of this angle,

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{2 - \sqrt{2}}{4}.$$

When $(x, y) = (0, 1)$, Alice and Bob choose $\alpha = 0$ and $\beta = -\pi/8$, and by superimposing their bases we obtain this figure:



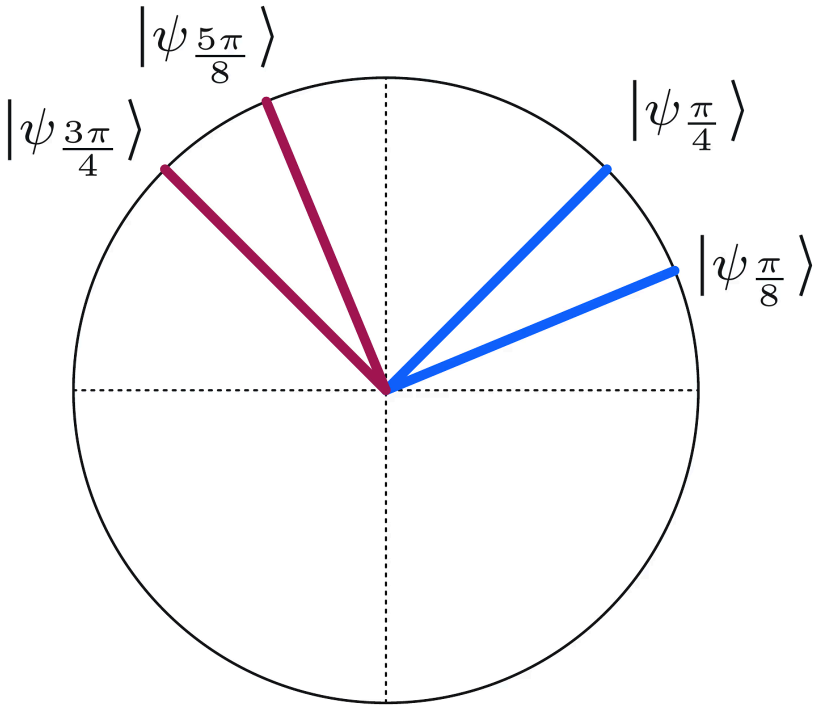
The angle between the red vectors is again $\pi/8$, as is the angle between the blue vectors. The probability that Alice and Bob's outcomes agree is again the cosine-squared of this angle,

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4},$$

while the probability they disagree is the sine-squared of this angle,

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}.$$

When $(x, y) = (1, 0)$, Alice and Bob choose $\alpha = \pi/4$ and $\beta = \pi/8$, and by superimposing their bases we obtain this figure:



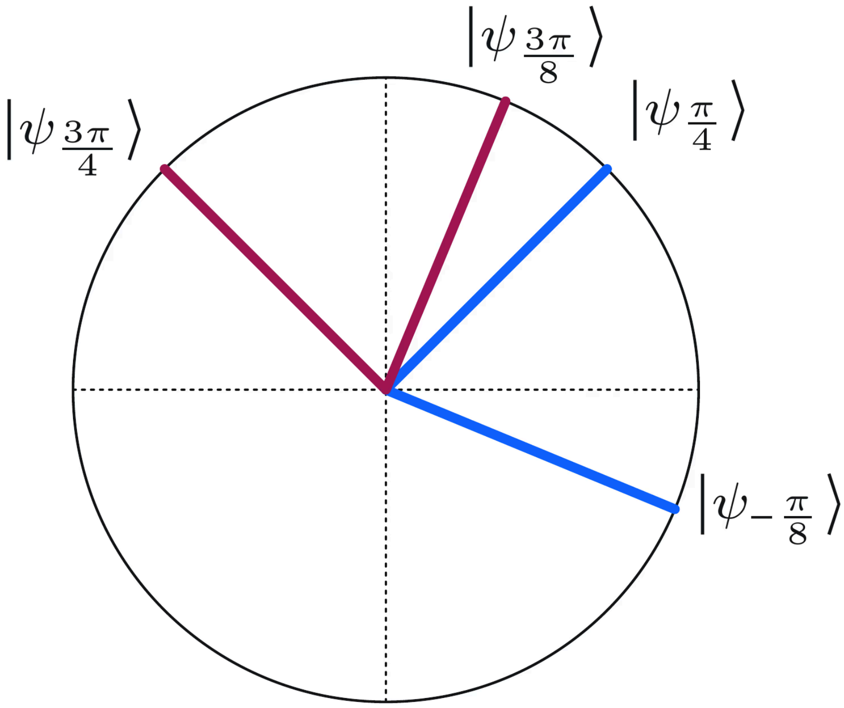
The bases have changed but the angles haven't — once again the angle between vectors of the same color is $\pi/8$. The probability that Alice and Bob's outcomes agree is

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4},$$

and the probability they disagree is

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}.$$

When $(x, y) = (1, 1)$, Alice and Bob choose $\alpha = \pi/4$ and $\beta = -\pi/8$. When we superimpose their bases, we see that something different has happened:



By the way the angles were chosen, this time the angle between vectors having the same color is $3\pi/8$ rather than $\pi/8$. The probability that Alice and Bob's outcomes agree is still the cosine-squared of this angle, but this time the value is

$$\cos^2\left(\frac{3\pi}{8}\right) = \frac{2-\sqrt{2}}{4}.$$

The probability the outcomes disagree is the sine-squared of this angle, which in this case is this:

$$\sin^2\left(\frac{3\pi}{8}\right) = \frac{2+\sqrt{2}}{4}.$$

Remarks

The basic idea of an experiment like the CHSH game, where entanglement leads to statistical results that are inconsistent with purely classical reasoning, is due to John Bell, the namesake of the Bell states. For this reason, people often refer to experiments of this sort as *Bell tests*. Sometimes people also refer to *Bell's theorem*, which can be formulated in different ways — but the essence of it is that quantum mechanics is not compatible with so-called *local hidden variable* *theories*. The CHSH game is a particularly clean and simple example of a Bell test, and can be viewed as a proof, or demonstration, of Bell's theorem.

The CHSH game offers a way to experimentally test the theory of quantum information. Experiments can be performed that implement the CHSH game, and test the sorts of strategies based on entanglement described above. This provides us with a high degree of confidence that entanglement is real — and unlike the sometimes vague or poetic ways that we come up with to explain entanglement, the CHSH game gives us a concrete and testable way to *observe* entanglement. The 2022 Nobel Prize in Physics acknowledges the importance of this line of work: the prize was awarded to Alain Aspect, John Clauser (the C in CHSH) and Anton Zeilinger, for observing entanglement through Bell tests on entangled photons.

Qiskit implementation

We can implement the CHSH game, together with the quantum strategy defined above, in Qiskit as follows.

First, here's the definition of the game itself, which allows an arbitrary strategy to be plugged in as an argument.

```
def chsh_game(strategy):
    # This function runs the CHSH game, using the str
    # from two bits to two bits), returning 1 for a w

    # Choose x and y randomly
    x, y = random.randint(0, 2), random.randint(0, 2)

    # Use the strategy to determine a and b
    a, b = strategy(x, y)

    # Decide if the strategy wins or loses
    if (a != b) == (x & y):
        return 1  # Win
    return 0  # Lose
```

No output produced

Now we'll create a function that outputs a circuit depending on the questions for Alice and Bob. We'll let the qubits have their default names for simplicity, and we'll use the built-in $R_y(\theta)$ gate for Alice and Bob's actions.

```python
def chsh_circuit(x, y):
    # This function creates a `QuantumCircuit` impleme
    # strategy described above (including the e-bit p

    qc = QuantumCircuit(2, 2)

    # Prepare an e-bit
    qc.h(0)
    qc.cx(0, 1)
    qc.barrier()

    # Alice's actions
    if x == 0:
        qc.ry(0, 0)
    else:
        qc.ry(-pi / 2, 0)
    qc.measure(0, 0)

    # Bob's actions
    if y == 0:
        qc.ry(-pi / 4, 1)
    else:
        qc.ry(pi / 4, 1)
    qc.measure(1, 1)

    return qc
```

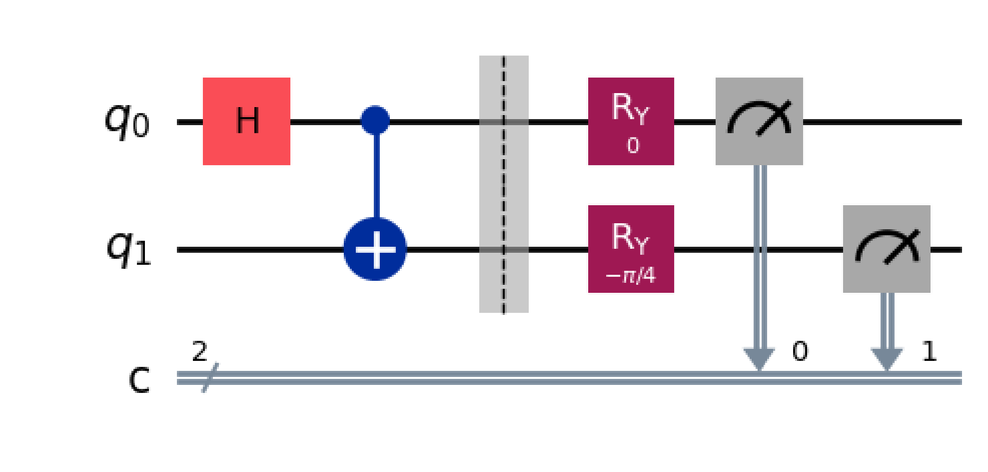
No output produced

Here are the four possible circuits, depending on which questions are asked.

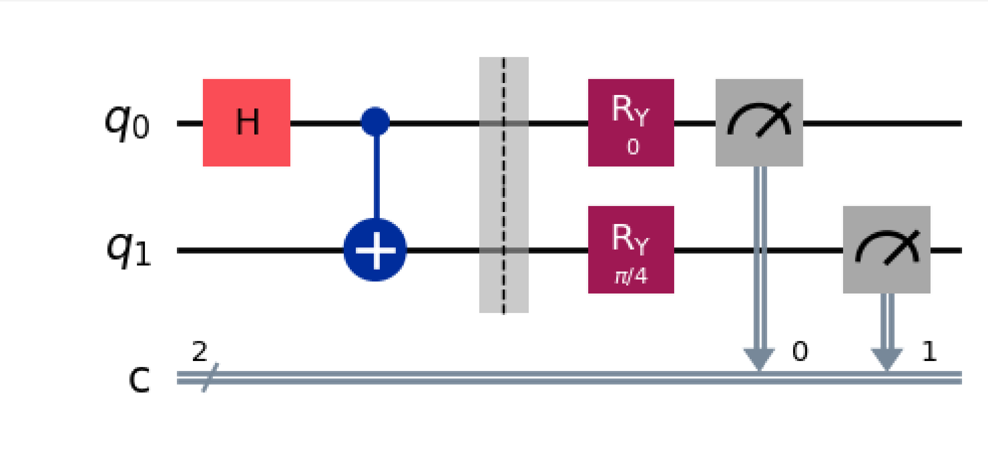
```python
# Draw the four possible circuits

print("(x,y) = (0,0)")
display(chsh_circuit(0, 0).draw(output="mpl"))

print("(x,y) = (0,1)")
display(chsh_circuit(0, 1).draw(output="mpl"))

print("(x,y) = (1,0)")
display(chsh_circuit(1, 0).draw(output="mpl"))

print("(x,y) = (1,1)")
display(chsh_circuit(1, 1).draw(output="mpl"))
```

Output:

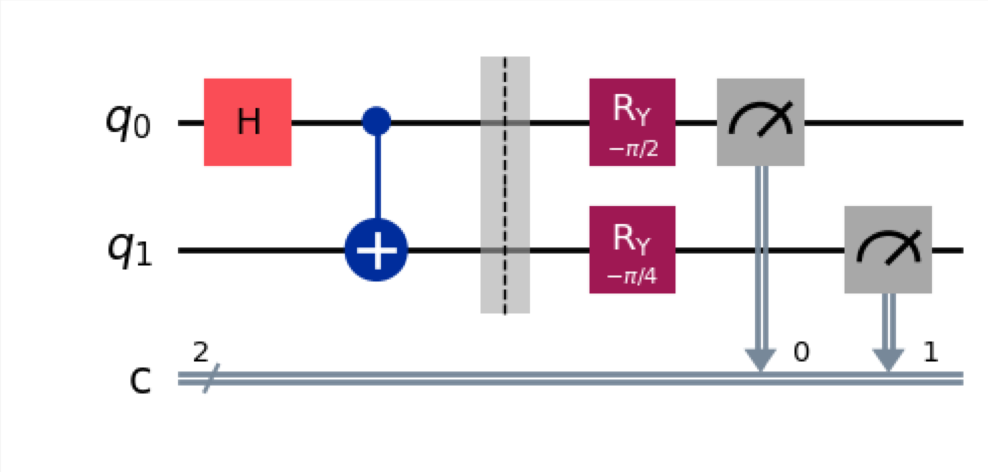
```
(x,y) = (0,0)
```



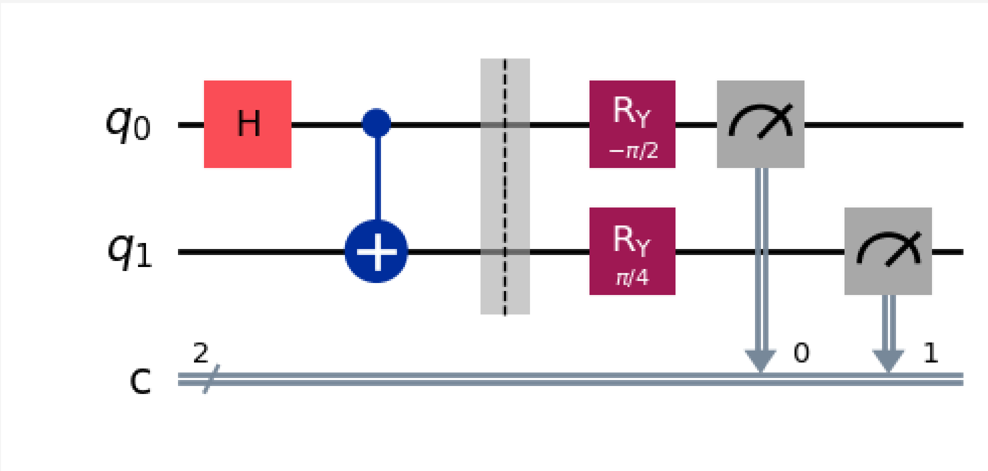

```
(x,y) = (0,1)
```




```
(x,y) = (1,0)
```




```
(x,y) = (1,1)
```



Now we'll create a job using the Aer simulator that runs the circuit a single time for a given input pair (x, y).

```
def quantum_strategy(x, y):
    # This function runs the appropriate quantum circ
    # one time and returns the measurement results

    # Setting `shots=1` to run the circuit once
    result = AerSimulator().run(chsh_circuit(x, y), s
    statistics = result.get_counts()

    # Determine the output bits and return them
    bits = list(statistics.keys())[0]
    a, b = bits[0], bits[1]
```

No output produced

Finally, we'll play the game 1,000 times and compute the fraction of them that the strategy wins.

```
NUM_GAMES = 1000
TOTAL_SCORE = 0
```

Complete

Quantum circuits

Take exam →