Q1) i) $\log_x\left(\dfrac{8-\log_5(x)}{\log_5(x)}\right)^{\log_3(x)} = 1$

as $\log_b(x^r) = r\log_b(x)$, exponent law

$$\log_3(x)\log_x\left(\frac{8-\log_5(x)}{\log_5(x)}\right) = 1$$

as $\log_3(x) = \dfrac{\log_x(x)}{\log_x(3)}$, change of base rule

and $\log_x(x) = 1$

so $$\frac{\log_x\left(\frac{8-\log_5(x)}{\log_5(x)}\right)}{\log_x(3)} = 1$$

$$\log_x\left(\frac{8-\log_5(x)}{\log_5(x)}\right) = \log_x(3)$$

cancel $\log_x$ by taking exponent

$$x^{\log_x\left(\frac{8-\log_5(x)}{\log_5(x)}\right)} = x^{\log_x(3)}$$

$$\frac{8-\log_5(x)}{\log_5(x)} = 3$$

$$8-\log_5(x) = 3\log_5(x)$$

$$8 = 3\log_5(x) + \log_5(x)$$

$$8 = 4\log_5(x)$$

$$2 = \log_5(x)$$

$$5^2 = x$$

$$x = 25$$

to confirm, substitute $x = 25$ into

$$\log_x\left(\frac{8-\log_5(x)}{\log_5(x)}\right)^{\log_3(x)} = 1 \quad , \quad x=25$$

$$\log_{25}\left(\frac{8-\log_5(25)}{\log_5(25)}\right)^{\log_3(25)} = 1$$

$$\log_{25}\left(\frac{8-2}{2}\right)^{\log_3(25)} = 1$$

$$\log_3(25) * \log_{25}(3) = 1$$

using change of base $\log_3(25) = \frac{\log_{25}(25)}{\log_{25}(3)}$

so, $= \frac{1}{\log_{25}(3)}$

$$\frac{\log_{25}(3)}{\log_{25}(3)} = 1$$

This is a true statement so we have shown that $x=25$ satisfies the original eqn.

---

ii) A log function $f(x) = \log_b(x)$ is only defined where $b > 0$ and $b \neq 1$. Additionally the argument x must be >0.

Additionally $\log_5(x)$ must not equal 0 as this would give a 0 in the dendominator. This occurs at $x = 1$

as $\log_a(x)$ is only defined for $x > 0$

$$8 - \log_5(x) > 0$$

$$8 > \log_5(x)$$

$$5^8 > x$$

$$\therefore x < 390625$$

Additionally we need to calculate when the argument is equal to 1.

$$\frac{8 - \log_5(x)}{\log_5(x)} = 1$$

$$8 - \log_5(x) = \log_5(x)$$

$$\underset{\div 2}{8} = \underset{\div 2}{2\log_5(x)}$$

$$4 = \log_5(x)$$

$$x = 5^4$$

$$= 625$$

From all of the above we see that $f(x)$ is not defined for $x = 0, 1, 625$ and $0 < x < 390625$

We can therefore express $f(x)$ as

$$f(x) = \log_x\left(\frac{8 - \log_5(x)}{\log_5(x)}\right)^{\log_3(x)} \quad \{x \mid 0 < x < 5^8, x \neq 1, 62$$

Q2) $f(x) = |x^2 - 3x + 2| + |x - 3|$

i) to find the derivative using piece-wise fuction we must first determine when the function is defined for different x values.

I will first graph $g(x) = |x^2 - 3x + 2|$ and $h(x) = |x - 3|$

$g(x) = |(x-1)(x-2)|$

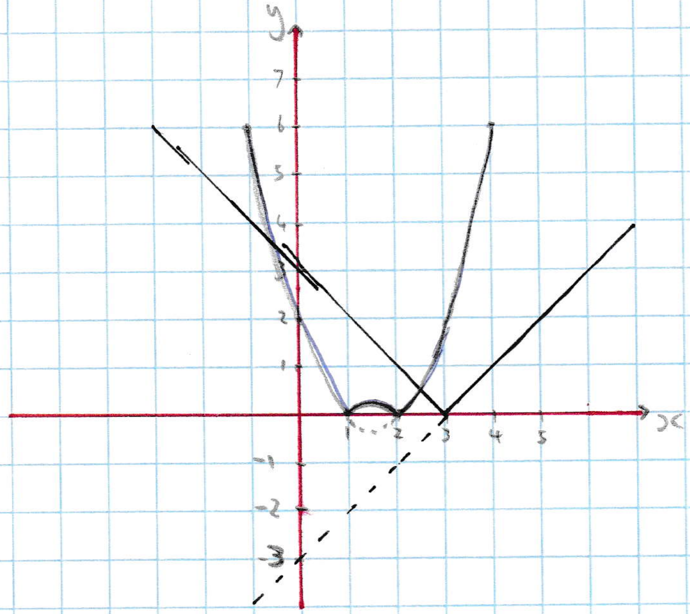
$= 0$ when $x = 1, 2$

$g(0) = 2$

$g(3) = 2$

$g(4) = 16 - 12 + 2 = 6$

$g(-1) = 6$



We can see that when we sum these functions together we can see that there are 3 points where the function will not be differentiable.

We will need to define the function across the distinc values of x within the domain.

$$f(x) \begin{cases} -\infty < x < 1 \\ 1 \leqslant x < 2 \\ 2 \leqslant x < 3 \\ 3 \leqslant x < \infty \end{cases}$$

$$g(x) = \begin{cases} x^2-3x+2 & -\infty < x < 1 \\ -x^2+3x-2 & 1 \leq x < 2 \\ x^2-3x+2 & 2 \leq x < \infty \end{cases}$$

$$h(x) = \begin{cases} -x+3 & -\infty < x < 3 \\ x-3 & 3 \leq x < \infty \end{cases}$$

$$f(x) = \begin{cases} x^2-3x+2-x+3 & x^2-4x+5 & -\infty < x < 1 \\ -x^2+3x-2-x+3 & -x^2+2x+1 & 1 \leq x < 2 \\ x^2-3x+2-x+3 & x^2-4x+5 & 2 \leq x < 3 \\ x^2-3x+2+x-3 & x^2-2x-1 & 3 \leq x < \infty \end{cases}$$

$$f'(x) = \begin{cases} 2x-4 & -\infty < x < 1 \\ -2x+2 & 1 < x < 2 \\ 2x-4 & 2 < x < 3 \\ 2x-2 & 3 < x < 0 \end{cases}$$

ii) We can see that summing $g(x)$ and $h(x)$ the function $f(x)$ is continuous. To determine if the function is differentiable at the points $x = 1, 2, 3$ we must confirm that the derivatives of each side are equal.

test

$x = 1$ $\quad 2x - 4 = -2x + 2$

$2(1) - 4 = -2(1) + 2$

$-2 \neq 0$

LHS $\neq$ RHS so the function is not differentiable at $x = 1$

$x = 2$ Test $-2x + 2 = 2x - 4$

$-2(2) + 2 = 2(2) - 4$

$-4 + 2 \neq 0$

LHS $\neq$ RHS so the function is not differentiab

at $x = 2$

$x = 3$ Test $2x - 4 = 2x - 2$

$2(3) - 4 = 2(3) - 2$

$6 - 4 = 6 - 2$

$2 \neq 4$

LHS $\neq$ RHS so the function is not

differentiable at $x = 3$

3i) $f(x) = \sqrt{x} + \sqrt{a-2x}, \{x \in \mathbb{R}: 0 \leq x \leq \frac{a}{2}\}$

Range of this function is $x \geq 0$, $x \leq \frac{a}{2}$

$f(x)$ is minimum at $x = 0$

$f(0) = 0 + \sqrt{a}$

$f(x)$ is maximum at $x = 2a$

$f(\frac{a}{2}) = \sqrt{\frac{a}{2}} + \sqrt{a-a}$

$= \sqrt{\frac{a}{2}}$

$\sqrt{12} = \sqrt{a} \times \sqrt{\frac{a}{2}}$

$\sqrt{12} = \sqrt{\frac{a^2}{2}}$

$\sqrt{12} = \frac{a}{\sqrt{2}}$

$a = \sqrt{24}$

$a = \sqrt{4 \times 6}$

$a = 2\sqrt{6}$

ii) fix $a = 4$, $f(x) = \sqrt{x} + \sqrt{4-2x}, \{x \in \mathbb{R}: 0 \leq x \leq 2\}$

y intercepts occur at $x = 0$

$f(0) = 0 + \sqrt{4}$

$= 2$

x intercepts occur at $f(x) = 0$

$0 = \sqrt{x} + \sqrt{4-2x}$

$-\sqrt{x} = \sqrt{4-2x}$

$$-\sqrt{x}^2 = \sqrt{4-2x}^2$$

$$x = 4-2x$$

$$+2x \qquad +2x$$

$$3x = 4$$

$$x = \frac{4}{3}$$

however substituting $x = \frac{4}{3}$ into $f(x)$ gives

$$f\left(\tfrac{4}{3}\right) = \sqrt{\frac{4}{3}} + \sqrt{4 - \frac{8}{3}}$$

$$= \sqrt{\frac{4}{3}} + \sqrt{\frac{4}{3}}$$

as $2\sqrt{\frac{4}{3}} \neq 0$ there are no solutions to $f(x)=0$.

iii) Stationary points occur at $f'(x) = 0$. These may be local minima, maxima or points of inflection.

$$f(x) = \sqrt{x} + \sqrt{4-2x}$$

$$f(x) = x^{\frac{1}{2}} + (4-2x)^{\frac{1}{2}}$$

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + \frac{d}{dx}(4-2x)^{\frac{1}{2}}$$

$$= \frac{x^{-\frac{1}{2}}}{2} + \frac{d}{dx}(4-2x)^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}} + \frac{d}{dx}(4-2x)^{\frac{1}{2}}$$

$$\frac{d}{dx}(4-2x)^{\frac{1}{2}}$$

using the chain rule with $u = 4-2x$

$$\frac{d\sqrt{u}}{du} \quad \frac{du}{dx} \qquad , \frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{4-2x}}$$

$$= \frac{\frac{d}{dx}(4-2x)}{2\sqrt{4-2x}}$$

$$= \frac{0-2}{2\sqrt{4-2x}}$$

$$= -\frac{1}{\sqrt{4-2x}}$$

so $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{4-2x}}$

As stationary points occur at $f'(x) = 0$ we need to solve for 0.

$$0 = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{4-2x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{4-2x}}$$

take reciprocal of both sides

$$2\sqrt{x} = \sqrt{4-2x}$$

square both sides

$$4x = 4-2x$$

$$+2x \qquad +2x$$

$$6x = 4 \Rightarrow x = \frac{2}{3}$$

substituting $x = \frac{2}{3}$ into $f'(x)$

$$f'(\tfrac{2}{3}) = \frac{1}{2\sqrt{\frac{2}{3}}} - \frac{1}{\sqrt{4-\frac{4}{3}}}$$

$$= \frac{1}{2\sqrt{\frac{2}{3}}} - \frac{1}{\sqrt{\frac{8}{3}}}$$

$$= \frac{1}{2\sqrt{\frac{2}{3}}} - \frac{1}{\sqrt{4\frac{2}{3}}}$$

$$= \frac{1}{2\sqrt{\frac{2}{3}}} - \frac{1}{2\sqrt{\frac{2}{3}}}$$

$$= 0.$$

thus we have shown that $x = \frac{2}{3}$ is a stationary point of $f(x)$.

To classify wether the critical point is a maxima, minima or inflection we take the second derivative.

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right) - \frac{d}{dx}\left(\frac{1}{\sqrt{4-2x}}\right)$$

$$= \frac{d}{dx}\frac{x^{-\frac{1}{2}}}{2} - \frac{d}{dx}(4-2x)^{-\frac{1}{2}}$$

$$= -\frac{x^{-\frac{3}{2}}}{4} - \frac{d}{dx}(4-2x)^{-\frac{1}{2}}$$

Using the chain rule with $u = 4-2x$

$$\frac{du^{-\frac{1}{2}}}{du}\,\frac{du}{dx} = -\frac{1}{2}u^{-\frac{3}{2}}\frac{d}{dx}(4-2x)$$

$$= -\frac{1}{2(4-2x)^{\frac{3}{2}}} \frac{d}{dx}(4-2x)$$

$$= \frac{2}{2(4-2x)^{\frac{3}{2}}}$$

$$= \frac{1}{(4-2x)^{\frac{3}{2}}}$$

$$f''(x) = -\frac{1}{4x^{3/2}} - \frac{1}{(4-2x)^{\frac{3}{2}}}$$

substitute $x = \frac{2}{3}$ into $f''(x)$

$$f''\left(\tfrac{2}{3}\right) = -\frac{1}{4\left(\frac{2}{3}\right)^{\frac{3}{2}}} - \frac{1}{\left(4-\frac{4}{3}\right)^{\frac{3}{2}}}$$

$$\simeq -\frac{1}{2.2} - \frac{1}{4.4}$$

$$\simeq -\frac{1}{6.6}$$

As $f''\left(\frac{2}{3}\right)$ is < 0 we can determine that the critical point corresponds to a local maxima of $f(x)$ at $x = \frac{2}{3}$

iv) To determine when the function is increasing and decreasing we can make a sign chart of $f'(x)$ at critical points and test points. As we know there is only one critical point and we have determined it corresponds to a local Maxima we know to the left of that point the function is increasing and to the right of that point decreasing. We will show this below.

$$f'(1) = \frac{1}{2} - \frac{1}{\sqrt{2}}$$
$$\simeq -0.2$$
decreasing

$$f'(\tfrac{1}{3}) = \frac{1}{2\sqrt{\frac{1}{3}}} - \frac{1}{\sqrt{4-\frac{2}{3}}}$$
$$\simeq 0.87 - 0.55$$
$$\simeq 0.32$$
increasing

$$f'(\tfrac{2}{3}) = 0$$

x	$\frac{1}{3}$	$\frac{2}{3}$	1
$f'(x)$	$+^{ve}$	0	$-^{ve}$

v) To find the value of $f(x)$ at the critical point we substitute $x=\frac{2}{3}$ into $f(x)$

$$f\left(\tfrac{2}{3}\right) = \sqrt{\tfrac{2}{3}} + \sqrt{4-\tfrac{4}{3}}$$
$$= \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{8}{3}}$$
$$= \sqrt{\tfrac{2}{3}} + \sqrt{4\tfrac{2}{3}}$$
$$= \sqrt{\tfrac{2}{3}} + 2\sqrt{\tfrac{2}{3}}$$
$$= 3\sqrt{\tfrac{2}{3}}$$
$$\simeq 2.45$$

$$f(2) = \sqrt{2} - 0$$
$$\simeq 1.4$$