COMP90051
Statistical Machine Learning

Workshop Week 2

Xudong Han

https://github.com/HanXudong/COMP90051_2020_S1

About Me

- Xudong Han
- https://hanxudong.github.io/
- xudong.han@unimelb.edu.au
- Slides https://github.com/HanXudong/COMP90051_2020_S1

- Tutorial questions

  Please email me

- All the other questions

  piazza.com/unimelb.edu.au/spring2020/comp90051

- Tutorial Plan
  - Review (5 mins)
  - Go through worksheet briefly by yourselves (10 mins)
  - Let’s do it together (40 mins)

- Bonus Section

  After hour self-learning. For fun 

Running Jupyter Notebook

1. Download worksheet02.ipynb from the LMS
2. Move the downloaded file to a working directory `%WORKDIR%`
3. Start → *Anaconda3 (64-bit)* → *Anaconda Prompt*
4. Type the following command at the prompt:

```
jupyter notebook --notebook-dir=%WORKDIR%
```

5. The Jupyter UI should open in a web browser.
6. Click on worksheet2.ipynb to get started.

Learning Outcomes

At the end of this workshop you should:

Develop intuition about the role of the prior and posterior in Bayesian inference.

Key Points

- Frequentist vs. Bayesian
- Prior, likelihood, posterior
- Probability density function (pdf)
- Bernoulli distribution
- Beta distribution
- Conjugate prior
- Maximum Likelihood Estimation (MLE)
- Maximise A Posterior (MAP)

1. A lucky find

You're interested in determining whether the coin is biased.

Bernoulli Distribution

- Let x_n denote the result of nth flip.
- $x_n = \begin{cases} 1, if\ heads \\ 0, if\ tails \end{cases}$
- $x_n \sim Bernoulli(\theta)$
- $f(x_n|\theta) = \theta^{x_n}(1-\theta)^{(1-x_n)}$
- For a fair coin, $\theta = 0.5$.

```
def toss_coin():
    if bernoulli.rvs(p = (int.from_bytes("coin".encode(), 'little') % 10000)/10000):
        return 1
    return 0
```

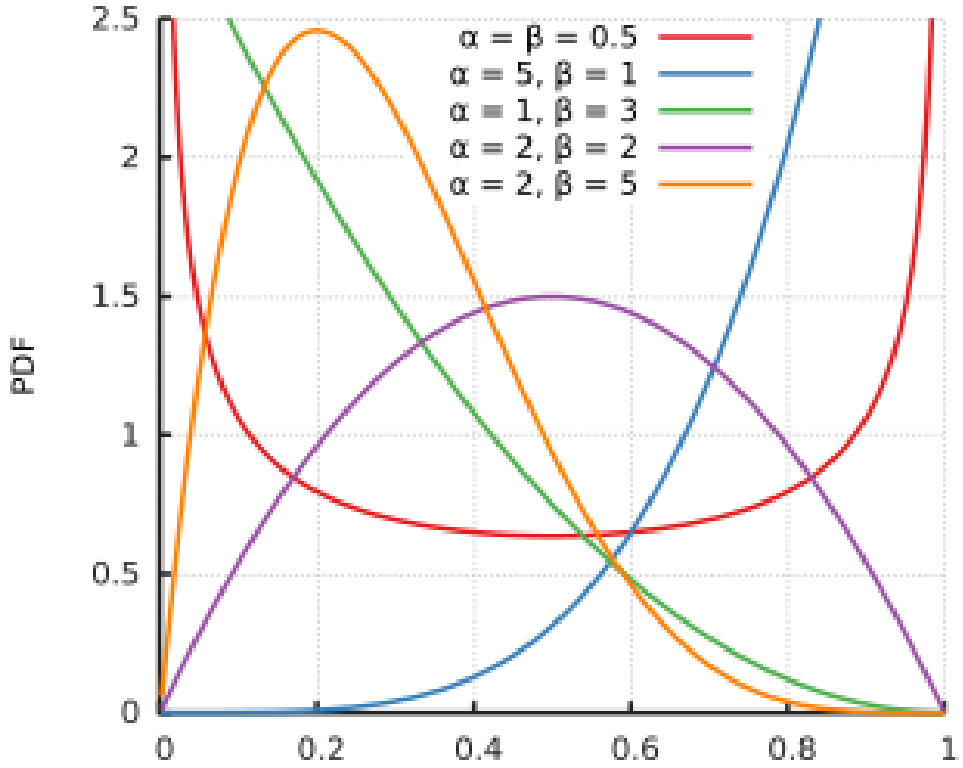
https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.bernoulli.html#scipy.stats.bernoulli

2. Prior belief

- How could we estimate the parameterθ?
  θ = the number of heads / total number
- In a Bayesian way:
  1) Prior belief in θ encoded by prior distribution $P(\theta)$.
  2) Beliefs (prior) + New observations (likelihood) = New beliefs (posterior)

Parameter θ and Beta distribution

- θ is a probability, thus $0 \leq \theta \leq 1$
- https://en.wikipedia.org/wiki/Beta_distribution



Try

a = 2 b =4

a = 4 b = 2

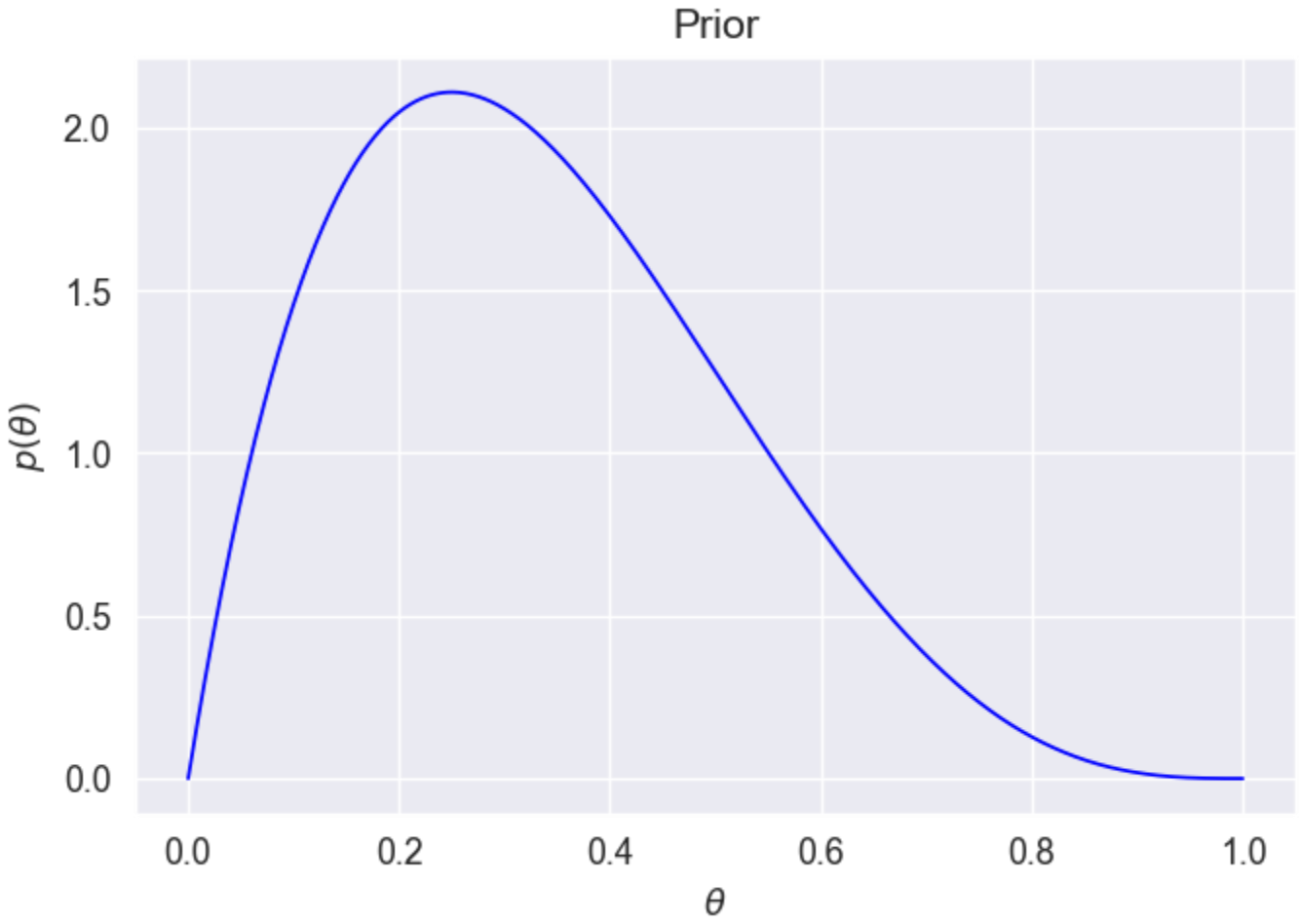
a = 100 b = 200

```
a = ... # fill in
b = ... # fill in
theta = np.linspace(0, 1, 1000)

FIGURE_RESOLUTION = 128
plt.rcParams['figure.dpi'] = FIGURE_RESOLUTION

plt.plot(theta, beta.pdf(theta, a, b), 'b-', lw=1)
plt.title('Prior')
plt.xlabel(r'$\theta$')
plt.ylabel(r'$p(\theta)$')
plt.show()
```

Suppose we expect that tails are more likely.



3. Posterior updates

- Now toss the coin once and denote the outcome by x_1
- Update beliefs:

$$p(\theta|x_1) = \frac{p(\theta, x_1)}{p(x_1)} = \frac{p(x_1|\theta)p(\theta)}{p(x_1)} \propto p(x_1|\theta)p(\theta)$$

- Now the likelihood:

$$p(x_1|\theta) = \theta^{x_1}(1-\theta)^{(1-x_1)}$$

- $p(\theta)$ is our prior distribution. Beta(a, b)

$$p(\theta) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}$$

Exercise: Show (on paper) that the posterior takes the form of a Beta distribution.

$$\theta|x_1 \sim \text{Beta}[x_1 + a, (1 - x_1) + b]$$

If the posterior and prior have the same function form, we say that the prior (in this case Beta) is *conjugate* to the likelihood function (in this case Bernoulli). This is convenient because the prior and posterior distributions have the same functional form, which admits a closed form expression for all posterior updates. This also allows us to ignore constant factors in intermediate calculations, as we may recover the normalizing factors by comparison with the standard form of the relevant density for the posterior.

[Hint: a similar calculation was performed in L2, slide 20 for the case of a normal prior and posterior.]

Exercise: Show that for N coin tosses, the posterior $p(\theta|x_1, \ldots, x_N) = \text{Beta}[N_H + a, N - N_H + b]$ where $N_H = \sum_{n=1}^{N} x_n$ is the number of heads observed.

- We assume the tosses are independent.
- Likelihood now is the product of all the pdf of each x.

$$p(x_1, x_2, \ldots, x_n|\theta) = \prod f(x_i|\theta)$$

- Remember

$$p(x_1|\theta) = \theta^{x_1}(1-\theta)^{(1-x_1)}$$

4. MAP estimator and MLE estimator

The posterior $\theta|x_1, \ldots, x_N$ contains all the information we know about θ after observing N coin tosses. One way of obtaining a point estimate of θ from the posterior, is to take the value with the maximum a posteriori probability (MAP):

Mode

$$\hat{\theta}_{MAP} = \arg\max_{\theta} p(\theta|x_1, \ldots, x_N)$$
$$= \frac{N_H + a - 1}{N + a + b - 2}$$

In general, the MAP estimator gives a different result to the maximum likelihood estimator (MLE) for θ:

$$\hat{\theta}_{MLE} = \arg\max_{\theta} p(x_1, \ldots, x_N|\theta)$$
$$= \frac{N_H}{N}$$

5. Convergence of the estimates

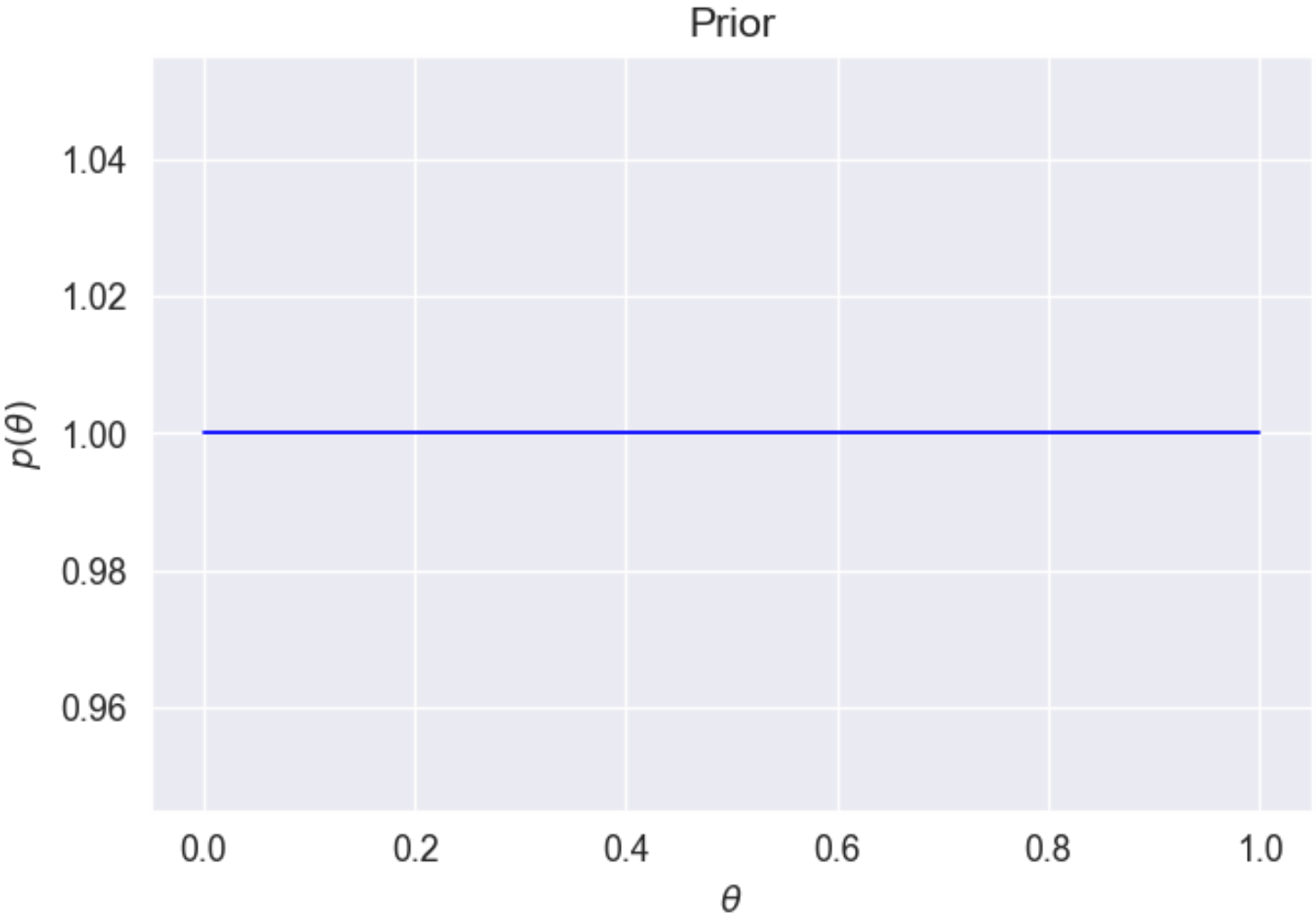
```
extra_tosses = 48
num_tosses = 2 + extra_tosses
num_heads = 0
theta_map = np.zeros(num_tosses)
theta_mle = np.zeros(num_tosses)
for i in range(1, num_tosses):
    if i == 1:
        num_heads += x1
    elif i == 2:
        num_heads += x2
    else:
        num_heads += toss_coin()
    theta_map[i] = ... # fill in
    theta_mle[i] = ... # fill in

theta_map = theta_map[1:]
theta_mle = theta_mle[1:]
```

$$\hat{\theta}_{MAP} = \arg\max_{\theta} p(\theta|x_1, \ldots, x_N)$$
$$= \frac{N_H + a - 1}{N + a + b - 2}$$

$$\hat{\theta}_{MLE} = \arg\max_{\theta} p(x_1, \ldots, x_N|\theta)$$
$$= \frac{N_H}{N}$$

What happens if you set a=b=1?



What happens if you set a=b=1?

$$\hat{\theta}_{MAP} = \arg\max_{\theta} p(\theta|x_1, \ldots, x_N) = \frac{N_H + a - 1}{N + a + b - 2}$$

$$\hat{\theta}_{MLE} = \arg\max_{\theta} p(x_1, \ldots, x_N|\theta) = \frac{N_H}{N}$$