Enginyeria Física Q4 **Dept. Física (UPC)**
Numerical and Computational Methods II (2015-2016) **Name:**

Final Exam, June 2016
Mobile phones forbidden.

1. Exercise (3.5 p.): Recall the general s-step linear multistep formula (LMSF)

$$\sum_{j=0}^{s} \alpha_j v^{n+j} = k \sum_{j=0}^{s} \beta_j f^{n+j}$$

to discretize the ODE $u_t = f(t, u)$.

In this exercise you are asked to obtain the *most accurate* s-step LMSF with

$$\beta_0 = \beta_1 = \beta_2 = \cdots = \beta_{s-1} = 0 \text{ and } \beta_s = 1.$$

In other words, you have to compute the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s$ that make the *local truncation error* of the *highest* possible order in k. In this case, we will **not** use the $\alpha_s = 1$ convention.

(a) For $s = 4$, write down the system of equations for the α_j

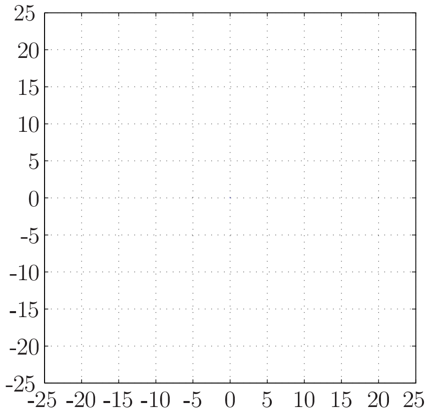
(0.5 p.)

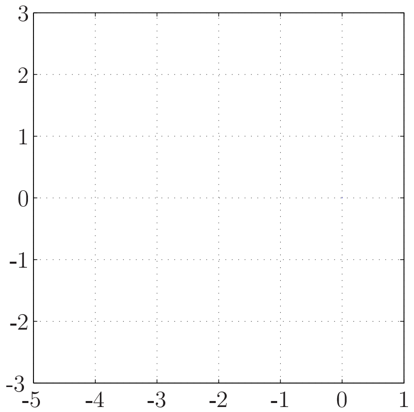
(b) For $s = 6$, provide the *exact* values of the α_j:

$\alpha_0 =$	$\alpha_1 =$	$\alpha_2 =$	$\alpha_3 =$
$\alpha_4 =$	$\alpha_5 =$	$\alpha_6 =$	

(1.5 p.)

(c) For $s = 6$, sketch by hand the stability boundary of the formula on the complex $\bar{k}$-plane below on the left. Indicate the stable/unstable regions. You can also use the graph on the right if you need to zoom the boundary to give more detail for part **(d)**.





(1.0 p.)

(d) What type of stability has this formula? Justify your answer based on your plots above.

(0.5 p.)

1. Problem (3.5 p.): Consider the function:

$$V(x,y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2.$$

(a) This function has many critical points of different type (minima, maxima or saddles) within the domain $(x,y) \in [-6,6] \times [-6,6]$. Find 2 minima, 1 maximum and 2 saddles. Provide the coordinates of these points (with at least 6 exact digits) in the table below and classify their type according to the classical Hessian criteria [1]

x	y	Type	
			(0.5 p.)
			(0.5 p.)
			(0.5 p.)
			(0.5 p.)
			(0.5 p.)

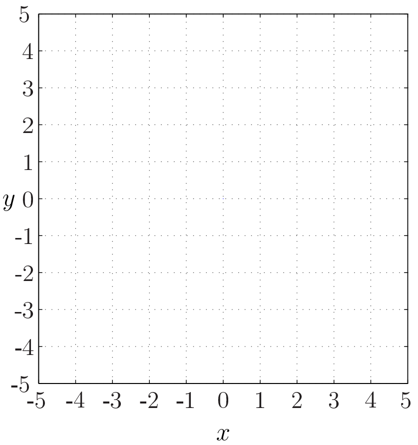
(b) On the right, sketch by hand the curve (or curves) given by the equation

$$(x^2 + y - 11)^2 + (x + y^2 - 7)^2 - 100 = 0.$$

Hint: some orientative points close to the curve(s):

$$(-0.1, 1.8), \quad (-2.3, -1.6).$$

(1.0 p.)



[1] Let $H = \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix}$, then:
Saddle: $\det(H) < 0$ | Max: $\det(H) > 0$ and $V_{xx} < 0$ and $V_{yy} < 0$ | Min: $\det(H) > 0$ and $V_{xx} > 0$ and $V_{yy} > 0$

2. Problem (3.0 p.): Consider the two-dimensional Hamiltonian:

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + (x^2 + y - 11)^2 + (x + y^2 - 7)^2,$$

and its corresponding Hamilton ordinary differential equations:

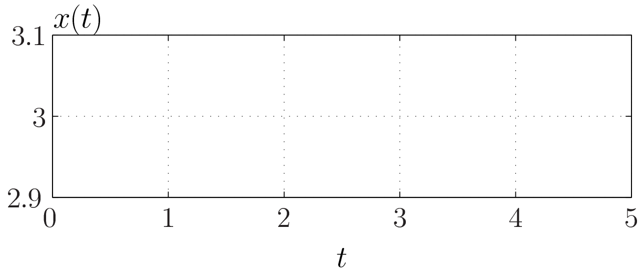
$$\dot{x} = \frac{\partial H}{\partial p_x}$$
$$\dot{p_x} = -\frac{\partial H}{\partial x}$$
$$\dot{y} = \frac{\partial H}{\partial p_y}$$
$$\dot{p_y} = -\frac{\partial H}{\partial y}$$

Use the fourth-order Runge-Kutta scheme given in the lectures (RK4) to integrate the ODE system above. Start the integration from the initial condition

$$x = 3.0,\ y = 2.1,\ p_x = 0,\ p_y = 0,$$

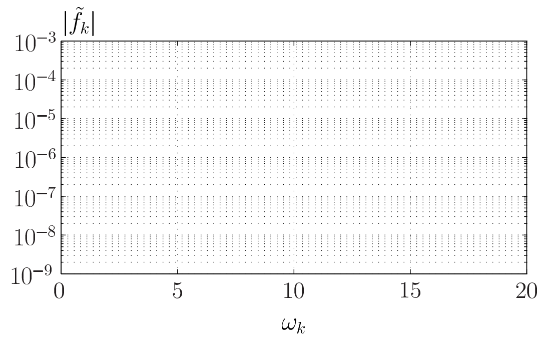
and using a time step $k = 0.01$. Integrate the equations within the time interval $t \in [0, 60]$.

(a) Plot the time evolution of the x coordinate within the initial time interval $t \in [0, 5]$. Sketch it qualitatively on the graph on the right. **(0.5 p.)**

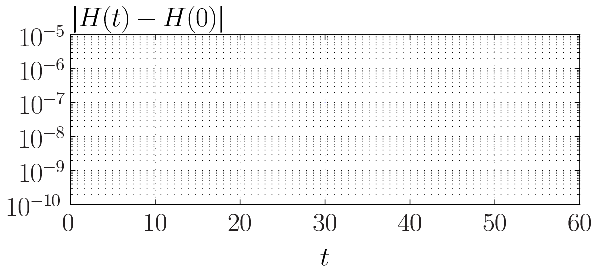


(b) Carry out a Fourier (DFT) analysis of $x(t)$ for the whole time integration $t \in [0, 60]$. Show on the semilogarithmic plot on the right the absolute amplitudes $|\tilde{f}_k|$ of your DFT and identify the two leading frequencies (the approximate values of ω_k with largest $|\tilde{f}_k|$). Provide 2 digits for each frequency. **(1.0 p.)**

ω_1	ω_2



(c) Plot the absolute variation of the hamiltonian $|H(t) - H(0)|$ as a function of time on the semilogarithmic graph on the right. **(0.5 p.)**



(d) Provide more accurate frequencies of the normal modes approximated in **(b)** by linearizing the hamiltonian system at the minimum potential $(x, y) = (3, 2)$. Provide the exact matrix of the linearization (use `format rat`) and its eigenvalues or frequencies (5 or 6 digits):

ω_1	ω_2

(0.5 p.)

(0.5 p.)