

Environmental Remote Sensing

GEOG 2021

Lecture 7

Mathematical Modelling in Geography II

Last time....

- Different types of model
 - statistical, empirical, physically-based, combinations..
- This time some examples
- Simple population growth model
- Require:
 - model of population Q as a function of time t i.e. $Q(t)$
- Theory:
 - in a ‘closed’ community, population change given by:
 - increase due to births
 - decrease due to deaths
 - over some given time period δt

‘Physically-based’ models

- population change given by:
 - Old population + increase due to births - decrease due to deaths

$$Q(t + \delta t) = Q(t) + \textit{births}(t) - \textit{deaths}(t) \quad [1]$$

‘Physically-based’ models

- For period δt
 - rate of births per head of population is B
 - rate of deaths per head of population is D
- So...

$$\text{births}(t) = B * Q(t) * \delta t$$

$$\text{deaths}(t) = D * Q(t) * \delta t \quad [2]$$

- Implicit assumption that B, D are constant over time δt
- So, IF our assumptions are correct, and we can ignore other things (big IF??)

$$\delta Q(t) = Q(t + \delta t) - Q(t)$$

so :

$$\frac{\delta Q}{\delta t} = \frac{Q(t + \delta t) - Q(t)}{\delta t}$$

$$= \frac{births(t) - deaths(t)}{\delta t}$$

$$= (B - D)Q(t) \quad [3]$$

‘Physically-based’ models

- As time period considered decreases, can write eqn [3] as a differential equation:

$$\frac{dQ}{dt} = (B - D)Q$$

- i.e. rate of change of population with time equal to birth-rate minus death-rate multiplied by current population
- Solution is ...

‘Physically-based’ models

$$Q(t) = Q_0 e^{(B-D)t}$$

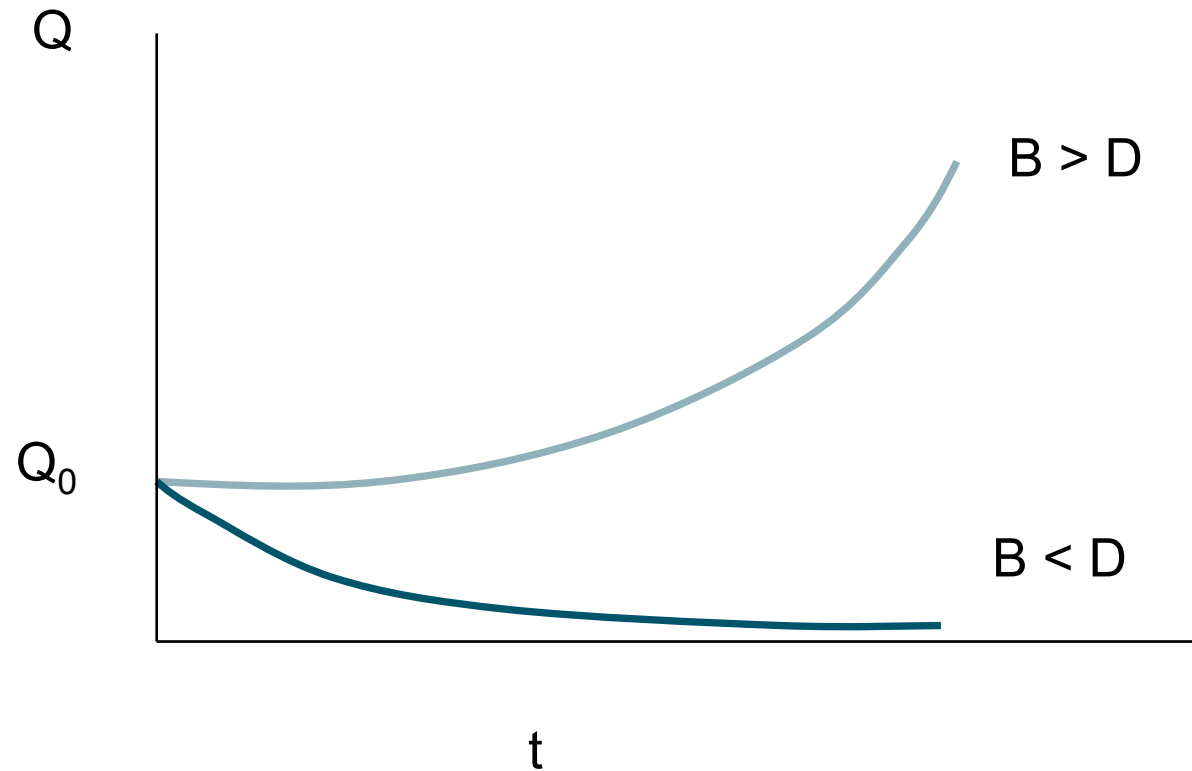
- Consider the following:
 - what does Q_0 mean?
 - does the model work if the population is small?
 - What happens if $B > D$ (and *vice-versa*)?
 - How might you ‘calibrate’ the model parameters?

[hint - think logarithmically]

‘Physically-based’ models

$$Q(t) = Q_0 e^{(B-D)t}$$

- So B , D and Q_0 all have ‘physical’ meanings in this system



Hydrological (catchment) models

- Want to know water volume in/out of catchment
- Simplest e.g. time-area method not dependent on space (1D) (lumped model)
- next level of complexity - semi-distributed models e.g. sub-basin division
- spatially distributed i.e. need 2D representation of catchment area
- More complex still, use full 3D representation of catchment area, topography, soil types etc.

Simplest catchment models: time-area hydrograph

- Schematic model
 - very simplified, no physics
 - Empirical/statistical
 - predicts discharge, Q (m^3s^{-1}), based on rainfall intensity, i (mm hr^{-1}), and catchment area, A (m^2) i.e. $Q = ciA$
 - c is (empirical) runoff coefficient i.e. fraction of rainfall which becomes runoff
 - c is particular to a given catchment (limitation of model)
 - more than one area? Divide drainage basins into isochrones (lines of equal travel time along channel), and add up....
 - $Q(t) = c_1A_1i_{(t-1)} + c_2A_2i_{(t-2)} + \dots + c_nA_ni_{(t-n)}$
 - e.g. TIMAREA model in Kirkby et al.

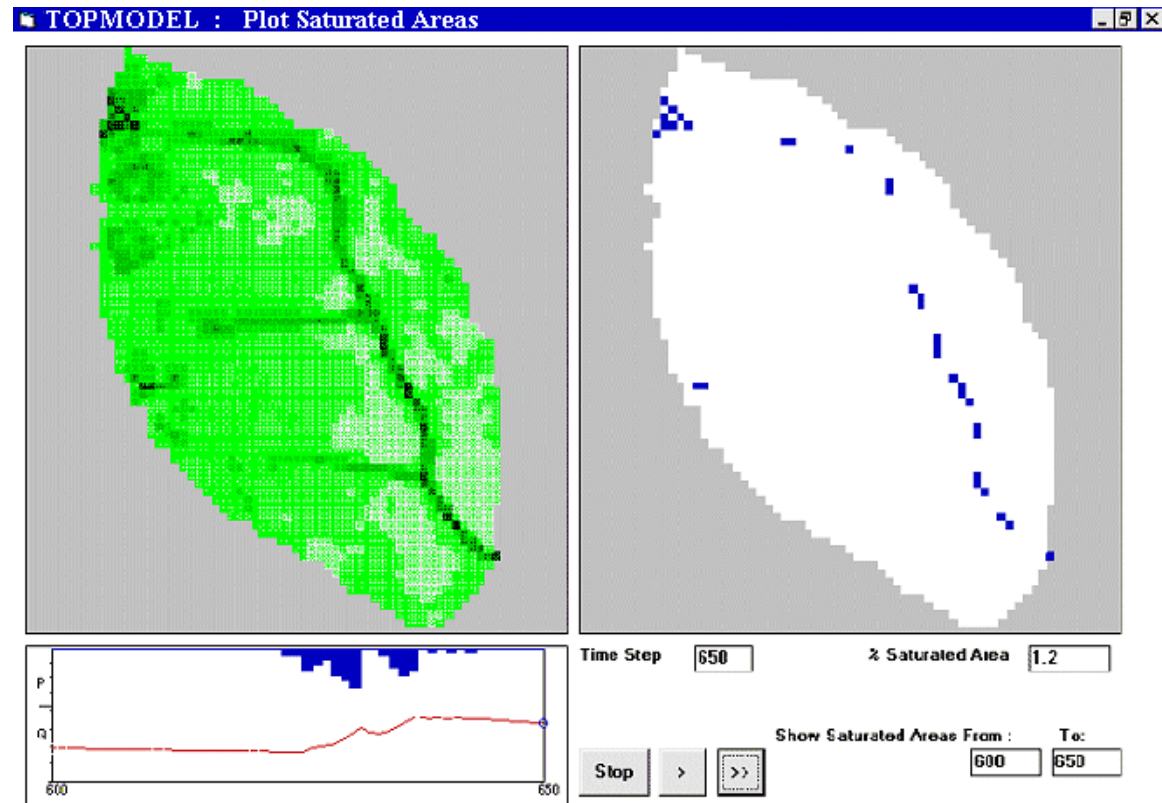
Process-type catchment models

- **More complex – some physics**
 - precipitation, evapotranspiration, infiltration
 - soil moisture conditions (saturation, interflow, groundwater flow, throughflow, overland flow, runoff etc.)
- **From conservation of “stuff” - water balance equation**
 - $dS/dt = R - E - Q$
 - i.e. rate of change of storage of moisture in the catchment system, S , with time t , is equal to inflow (rainfall, R), minus outflow (runoff, Q , plus evapotranspiration, E)
 - E.g. STORFLO model (in Kirkby et al.)

More complex?

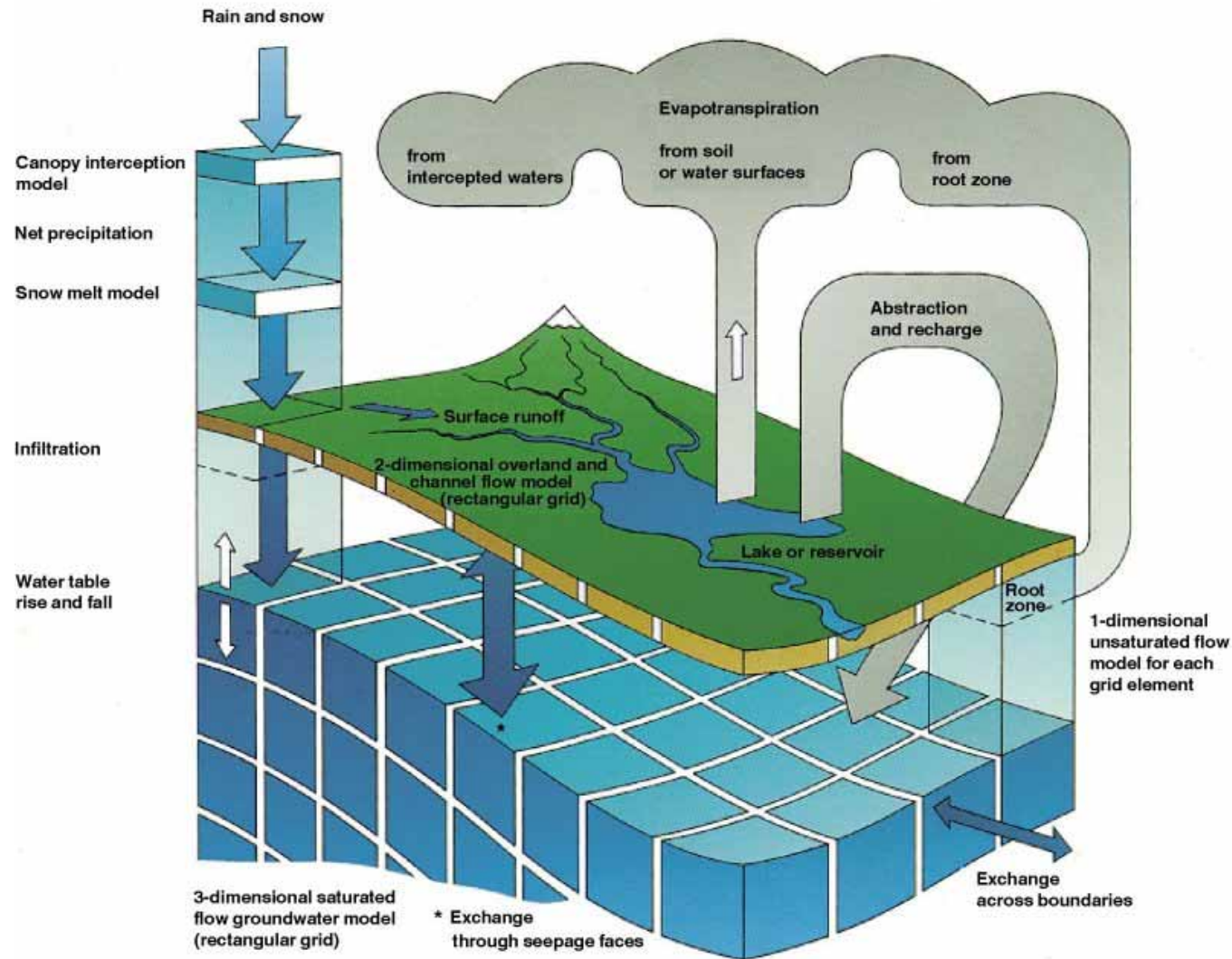
- Consider basin morphometry (shape) on runoff
 - Slope, area, shape, density of drainage networks
 - Consider 2D/3D elements, soil types and hydraulic properties
- How to divide catchment area?
 - Lumped models
 - Consider all flow at once... Over whole area
 - Semi-distributed
 - isochrone division, sub-basin division
 - Distributed models
 - finite difference grid mesh, finite element (regular, irregular)
 - Use GIS to represent - vector overlay of network?
 - Time and space representation?

TOPMODEL: Rainfall runoff



From: <http://www.es.lancs.ac.uk/hfdg/topmodel.html>

Very complex: MIKE-SHE



- Name
- Combination of physical, empirical and black-box...
- Can “simulate all major processes in land phase of hydrological cycle” !!

From: http://www.dhisoftware.com/mikeshe/Key_features/

Other distinctions

- Analytical
 - resolution of statement of model as combination of mathematical variables and ‘analytical functions’
 - i.e. “something we can actually write down”
 - Very handy & (usually) very unlikely.
 - e.g. biomass = $a + b \cdot \text{NDVI}$
 - e.g.

$$\frac{dQ}{dt} = aQ \longrightarrow Q = Q_0 e^{at}$$

Other Distinctions

- Numerical
 - solution to model statement found e.g. by calculating various model components over discrete intervals
 - e.g. for integration / differentiation

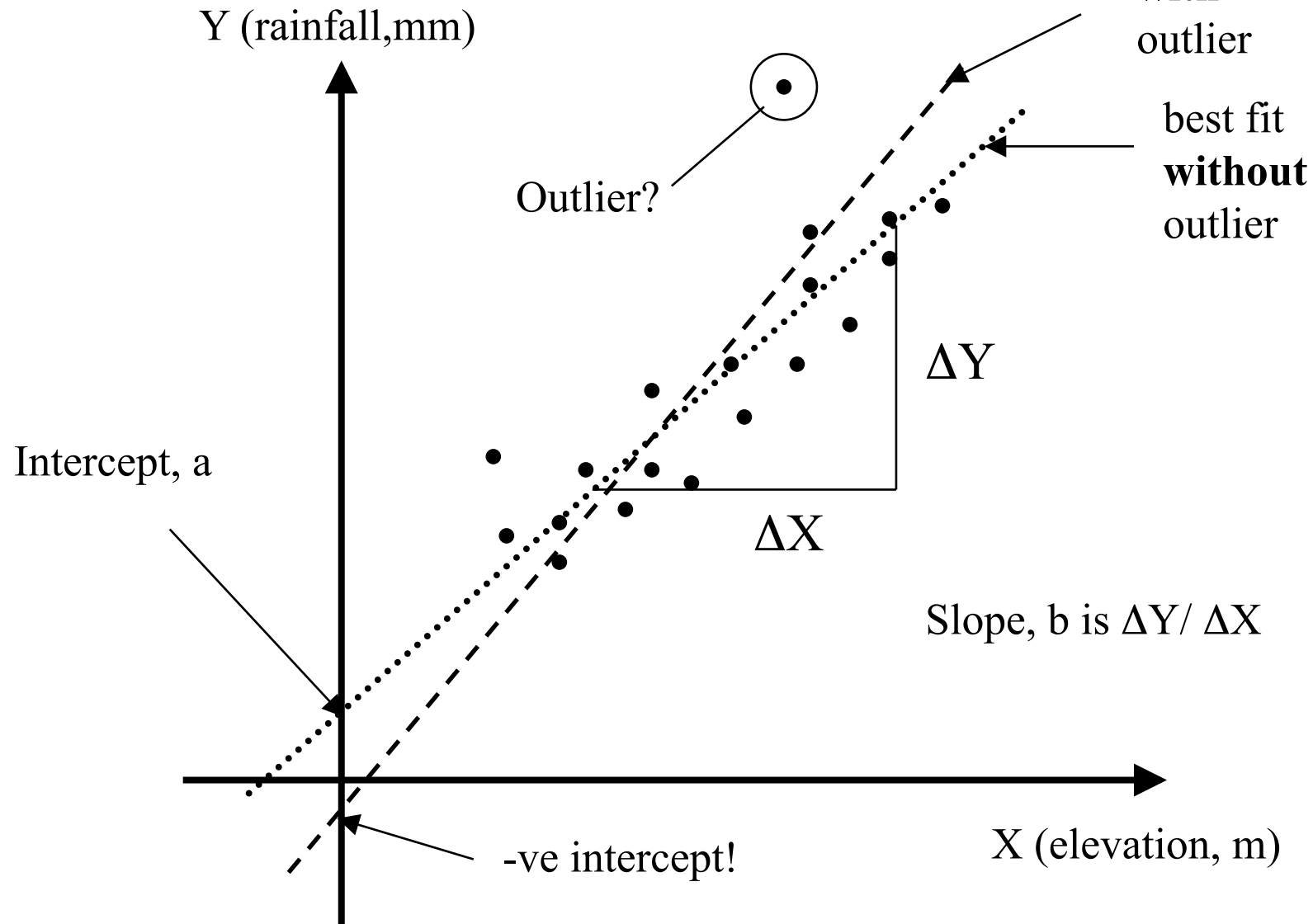
Inversion

- Remember - turn model around i.e. use model to explain observations (rather than make predictions)
 - Estimate value of model parameters
 - E.g. physical model: canopy reflectance, ρ_{canopy} as a function of leaf area index, LAI
 - $\rho_{canopy} = f(LAI, \dots)$
 - Measure reflectance and use to estimate LAI
 - $LAI = f^{-1}(\text{measured } \rho_{canopy})$

Inversion example: linear regression

- E.g. model of rainfall with altitude
 - $Y = a + bX + \varepsilon$
 - Y is predicted rainfall, X is elevation, a and b are constants of regression (slope and intercept), ε is residual error
 - ε arises because our observations contain error
 - (and also if our model does not explain observed data perfectly e.g. there may be dependence on time of day, say....)
 - fit line to measured rainfall data, correlate with elevation

Inversion example: linear regression



Inversion

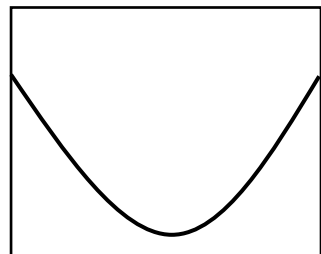
- Seek to find “best fit” of model to observations somehow
 - most basic is linear least-squares (regression)
 - “Best fit” - find parameter values which minimise some error function e.g. RMSE (root mean square error)

$$RMSE = \sqrt{\sum_{n=1}^{n=N} \frac{(measured - modelled)^2}{N - 1}}$$

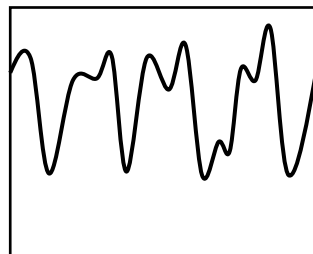
- Easy if we can write inverse model down (analytical)
- If we can't ?

Inversion

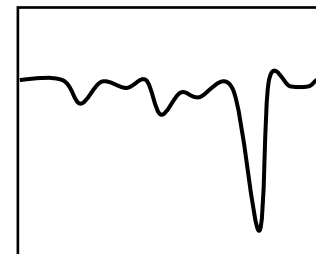
- Have to solve numerically i.e. using sophisticated trial and error methods
 - Generally more than 1 parameter, mostly non-linear
 - Have an error surface (in several dimensions) and want to find lowest points (minima)
 - Ideally want global minimum (very lowest) but can be problematic if problem has many near equivalent minima



'Easy'



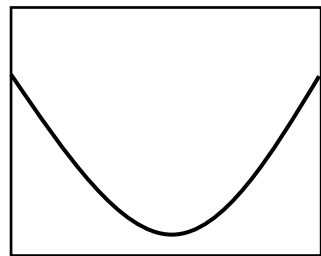
Hard



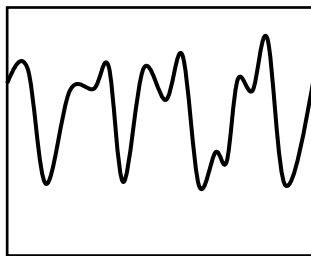
'Easy'

Inversion

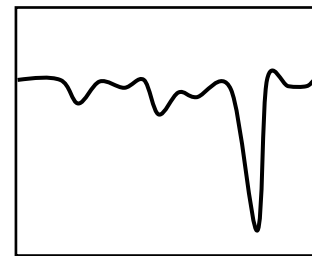
- E.g. of aircraft over airport and organising them to land in the right place at the right time
 - Many parameters (each aircraft location, velocity, time, runway availability, fuel loads etc.)
 - Few aircraft? 1 solution i.e. global minimum & easy to find
 - Many aircraft? Many nearly equivalent solutions
 - Somewhere in middle? Many solutions but only one good one



'Easy'



'Easy'



'Hard'

Which type of model to use?

- Statistical
 - **advantages**
 - simple to formulate & (generally) quick to calculate
 - require little / no knowledge of underlying (e.g. physical) principles
 - (often) easy to invert as have simple analytical formulation
 - **disadvantages**
 - may only be appropriate to limited range of parameter
 - may only be applicable under limited observation conditions
 - validity in extrapolation difficult to justify
 - does not improve general understanding of process

Which type of model to use?

- Physical/Theoretical/Mechanistic
 - **advantages**
 - if based on fundamental principles, more widely applicable
 - may help to understand processes e.g. examine role of different assumptions
 - **disadvantages**
 - more complex models require more time to calculate
 - Need to know about all important processes and variables AND write mathematical equations for processes
 - often difficult to obtain analytical solution & tricky to invert

Summary

- Empirical (regression) vs theoretical (understanding)
- uncertainties
- validation
 - *Computerised Environmental Modelling: A Practical Introduction Using Excel*, Jack Hardisty, D. M. Taylor, S. E. Metcalfe, 1993 (Wiley)
 - *Computer Simulation in Physical Geography*, M. J. Kirkby, P. S. Naden, T. P. Burt, D. P. Butcher, 1993 (Wiley)
 - <http://www.sportsci.org/resource/stats/models.html>

Summary

- No one trusts a model except the person who wrote it
- Everyone trusts an observation except the person who made it