Stochastic calculus notes

Treacher

Abstract

Just some notes about important derivations and maths relevant to quant work

1 Preliminaries

1.1 Brownian motion

A random walk from time $t = t$ to $t = T$ can be defined as:

$$Z_T - Z_t = \sum_{j=t}^{j=T} \epsilon_j \tag{1}$$

where $\epsilon_j \overset{\text{iid}}{\sim} \mathrm{N}(0, 1)$ are the independent and identically distributed (iid) normal random variable(s)

Given that they're iid, the expectation and variance are found as follows:

$$\mathbb{E}\left[Z_T - Z_t\right] = \mathbb{E}\left[\sum_{j=t}^{j=T} \epsilon_j\right] = (T - t) \cdot 0 = 0 \tag{2}$$

$$\mathrm{Var}(Z_T - Z_t) = \mathrm{Var}\left(\sum_{j=t}^{j=T} \epsilon_j\right) = (T - t) \cdot \mathrm{Var}(\epsilon_j) = \Delta t. \tag{3}$$

Generalising to a time interval of $t \to (t + \Delta)$ we find that as $\Delta \to 0$

$$\lim_{\Delta \to 0} (Z_{t+\Delta} - Z_t) = dZ_t \tag{4}$$

Combining equations 3 with 4, we see that in the limit of $\Delta \to 0$, the variance of Brownian motion tends to the corresponding infinitesimal time increment:

$$\mathrm{Var}(dZ_t) \to dt \tag{5}$$
$$\sigma \sim \sqrt{dt} \tag{6}$$

where σ is the standard deviation of the Brownian process dZ_t.

1.1.1 Key properties of Brownian motion

1. $\mathbb{E}[dZ_t] = 0$ hence $\mathrm{Var}(dZ_t) = \mathbb{E}[dZ_t^2]$
2. $\mathrm{Var}(dZ_t) = dt = dZ_t^2$, or the temporal variation scales with the square of the spatial variation.
3. $\mathrm{Cov}(dZ_s, dZ_t) = 0 \quad \forall\, s \neq t$

2 From a Taylor expansion to Ito's lemma

2.1 The Taylor expansion

Consider a function of two variables, $f(x,t)$. This has a two dimensional Taylor expansion about the point(s) (x_0, t_0) of:

$$\begin{aligned} f(x,t) = f + f_x \cdot (x - x_0) + f_t \cdot (t - t_0) + \\ \frac{1}{2}\left[f_{xx} \cdot (x - x_0)^2 + f_{tt} \cdot (t - t_0)^2 + 2f_{xt} \cdot (x - x_0)(t - t_0)\right] \end{aligned} \tag{7}$$

where the right hand sign terms are evaluated at (x_0, t_0), and f_t denotes $\frac{\partial f}{\partial t}$.

We rewrite $(x - x_0) = dx$ and $(t - t_0) = dt$ in the above and move the first term to the right hand side to get

$$f(x,t) - f(x - x_0, t - t_0) = f_x\, dx + f_t\, dt + \frac{1}{2}\left[f_{xx}\, dx^2 + f_{tt}\, dt^2 + 2f_{xt}\, dx\, dt\right]. \tag{8}$$

Now we recognise that the left hand side of equation 8 is the infinitesimal change in f. Further, the only quadratic variation of significance on the right hand side is dx^2. This is because (as per point 2 in subsection 1.1.1) the other two quadratic terms, dt^2 and $dx\,dt$ have temporal variations of dt raised to powers of 2 and $\frac{3}{2}$ respectively. These are higher, and therefore decay faster than that of dx^2 which is equivalent to dt^1. We therefore disregard the other terms and continue with the following:

$$df = f_x\,dx + f_t\,dt + \frac{1}{2} f_{xx}\,dx^2. \tag{9}$$

This is the 2D Taylor expansion of a stochastic differential equation to 'first' order. Note that we have terms that vary with dx and dt only (because $dx^2 = dt$), but have to carry a second partial derivative which wouldn't be the case in a regular non-stochastic function expansion.

2.2 Geometric Brownian Motion

We now introduce Geometric Brownian Motion (GBM) as

$$\frac{dS}{S} = \mu dt + \sigma dB_t \tag{10}$$

where S denotes the spot price of some financial instrument (spatial variation), μ characterises a deterministic drift term (temporal variation), and σ scales the stochastic component (random variation) characterised by the Brownian motion / Weiner process dB_t.

We substitute equation 10 into that of the expansion in equation 9, observing that the variation in the underlying dS takes the place of the spatial variation dx, and the variation in time dt will be switched in and out with that of the Brownian motion dB_t:

$$df = \frac{\partial f}{\partial S} S\left(\mu dt + \sigma dB_t\right) + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} S^2 \left(\mu dt + \sigma dB_t\right)^2. \tag{11}$$

If we expand the squared bracket we'll get three terms, one with dt^2, one with $dt\,dB_t$ and one with dB_t^2. Of those three, the one with the slowest temporal variation (most important) is $dB_t^2 (= dt)$ so we disregard the other two. Rearranging the above then gives:

$$df = \left(\frac{\partial f}{\partial S} S\mu + \frac{\partial f}{\partial t} + \frac{S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right) dt + \frac{\partial f}{\partial S} S\sigma\, dB_t \tag{12}$$

which is Ito's lemma as applied to GBM.

Note that if instead of using equation 10 above, we used a general ito diffusion process:

$$dX_t = \mu\, dt + \sigma\, dB_t \tag{13}$$

we'd end up with pretty much the same thing but without all the extra S dependencies.

2.2.1 Key takeaways

1. Taylor expansion of SDE requires some additional terms compared to regular deterministic function.
2. Geometric brownian motion comprises a deterministic drift and a random stochastic variation.
3. Taylor expanding GBM gives you a version of Ito's lemma describing how a function f varies given a change in the spot price of the underlying S.

3 From a Ito's lemma to Black Scholes PDE

3.1 Black Scholes as an expression of the replicating portfolio

3.1.1 Replicating portfolio

The replicating portfolio is a principal that says the payout of an option or derivative can be replicated by holding certain positions in lending / borrowing and the underlying:

- Call option = (long Δ units of the underlying) - (B amount of borrowed cash)
- Put option = (short Δ units of the underlying) + (B amount of borrowed cash)

so when looking at a binomial tree describing option prices as certain intermediary dates, to calculate the fair price, you can work backwards from the terminal nodes solving

$$\Delta S_T^{\text{up}} - B(1+r) = V_T^{\text{up}} \tag{14}$$
$$\Delta S_T^{\text{down}} + B(1+r) = V_T^{\text{down}} \tag{15}$$
$$\tag{16}$$

where r is the risk free rate, $S_t^{u/d}$ are the underlying spot prices at time t in the higher/lower price state and $V_t^{u/d}$ are the corresponding option prices. In that framework you would only know the terminal option prices, hence the need to work backwards from the end.

Jumping forward, the Black Scholes FORMULA for the price of a call option can be written as

$$C(S,t) = S_t\Phi(d_1) - k\mathrm{e}^{-rt}\Phi(d_2) \tag{17}$$

where $S_t\Phi(d_1)$ represents the long underlying position and the $k\mathrm{e}^{-rt}\Phi(d_2)$ represents the borrowing of cash (discounted back to today's value). Hence the Black Scholes formula is an expression of the replicating portfolio.

3.1.2 Delta hedging portfolio

Now denoting an option as V, consider a portfolio consisting of going short one option $(-V)$ and long some shares in the underlying $(+\frac{\partial V}{\partial S})$:

$$\Pi = -V + \frac{\partial S}{\partial V} S \tag{18}$$

which has an infinitesimal value change over time $\Delta\, t$ of

$$d\Pi = -dV + \frac{\partial S}{\partial V}\, dS \tag{19}$$