Quadratic Variation

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Abstract

Just some notes about quadratic variation, summarised from https://benjaminwhiteside.com/2017/01/26/quadratic-variation/

1 Definition

Quadratic variation is defined as the sum of squared changes in a process X_t:

$$[X]_t = \lim_{\max(\Delta t)\to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2 \tag{1}$$

where $\Delta t = t_i - t_{i-1}$ for any t_i which is the i^{th} partition of the full time space $t \in [0, T]$. The limit is over the maximum time increment going to zero (implying the different increments can tend to zero at different rates?)

2 Using a Wiener process

Properties of the Wiener process:

- $W_0 = N(0, 0)$
- $W_t - W_s = N(0, t - s)$
- $\text{Var}(W_t - W_s) = t - s$
- $W_t, W_{t+1}, ...W_{t+n} \overset{\text{iid}}{\sim}$

We can therefore write the variance of a Wiener process in terms of the expectation value knowing that $\mathrm{Var}(x) = \mathbb{E}[(x - \mathbb{E}[x])^2]$, and the expectation of a Wiener process is always 0 to say:

$$\mathrm{Var}(W_t - W_s) = \mathbb{E}[(\{W_t - W_s\} - \mathbb{E}[W_t - W_s])^2] = \mathbb{E}[(W_t - W_s)^2] = t - s \quad (2)$$

Now consider the sum of squared differences in a Wiener process denoted by Q_p

$$Q_p = \sum_{i=1}^{n} \left(W_{t_{i+1}} - W_{t_i}\right)^2 \quad (3)$$

and take the expected value to recover something similar to the variance expression:

$$\mathbb{E}[Q_p] = \mathbb{E}\left[\sum_{i=1}^{n} \left(W_{t_{i+1}} - W_{t_i}\right)^2\right] = \sum_{i=1}^{n} t_{i+1} - t_i \quad (4)$$

having changed the order of the expectation and summation, and observing the expression at the end of equation 2.

The sum of the independent time increments is by definition equal to the to the final value of the time, such that

$$\mathbb{E}[Q_p] = t. \quad (5)$$

From this we conclude that “Brownian motion accumulates quadratic variation at a rate of 1 per unit of time.”

3 Ito integral

The definition of an Ito integral of a stochastic process $\xi(t)$ is (do separate summary of where this comes from at some point):

$$\int_0^t \xi(t) \mathrm{d}W_s = \lim_{h \to 0} \sum_{k=1}^{i_t} \xi(t_{k-1}) \cdot \left(W_{t_k} - W_{t_{k-1}}\right) \quad (6)$$

If we take the stochastic process $\xi(t) = 1$ and square both sides then the right hand side of the above is just $\mathbb{E}[Q_p]$ which we know is equal to t. This allows us to say

$$\int_0^t \mathrm{d}W_s^2 = t \tag{7}$$

or by differentiating,

$$\mathrm{d}W_s^2 = \mathrm{d}t. \tag{8}$$

4 Application to dt^2

Consider the quadratic variation as above but applied to a time process t instead of a stochastic process X:

$$[t]_t = \lim_{n\to\infty} \sum_{i=1}^{n} (t_{i+1} - t_i)^2 \tag{9}$$

We know there are n time increments, which must have equal length of t/n which allows the above to be written as

$$[t]_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{t}{n}\right)^2 = \lim_{n\to\infty} \left(\frac{t}{n}\right)^2 \sum_{i=1}^{n} = \lim_{n\to\infty} \frac{t^2}{n} = 0 \tag{10}$$

which reveals the quadratic variation of time $[t]_t = 0$. Using the Ito integral in equation 6 with respect to t, we can then say

$$\int_0^t \mathrm{d}s \cdot \mathrm{d}s = [t]_t = 0 \tag{11}$$

or equivalently via differentiation:

$$\mathrm{d}t^2 = 0 \tag{12}$$

5 Physical interpretation

Quadratic variation is like an internal clock for a random process that describes the accumulation of randomness over time.

In regular calculus, the majority of functions are continuously differentiable, and these functions have the property that their quadratic variation is 0.

As a result, in a Taylor expansion of a non-stochastic function, you typically only need consider the first order terms. This is not the case with stochastic functions, where a second order correction term is required. This second order correction represents the effect of the non-zero quadratic variation present in stochastic functions.