Ito Integral

Treacher

Abstract

Just some notes about the Ito integral, along with some preliminary information about Riemann and Riemann-Stieltjes integration. Summarised from various places online, including but not limited to:

- https://www.youtube.com/watch?v=vHn1M6pUAWg
- http://www.columbia.edu/ ks20/FE-Notes/4700-07-Notes-Ito.pdf

1 Preliminaries

1.1 Riemann integration

For a regular non-stochastic function (black line in figure 1), you can use Riemann integration to approximate the integral. This approach involves:

- Divide range of integration into set of subintervals $P = \{t_0, t_1, t_2, ..., t_n\}$
- Over all subintervals, calculate the lower and upper Riemann sums which represent the areas of the rectangles of width equal to the subinterval, and heights equal to the minimum and maximum function value in that interval (see figure 1):
  - Lower sum: $L = \sum_{i=1}^{n} \min\{f(t_i)\} \cdot (t_i - t_{i-1})$
  - Upper sum: $U = \sum_{i=1}^{n} \max\{f(t_i)\} \cdot (t_i - t_{i-1})$
- For a given function and a small number of partitions, the lower sum will be smaller than the upper sum.
- As the number of partitions increases, the values of the two sums will converge toward eachother (if the integral exists).

- When the values of the sums intersect, that value is taken to be the Riemann integral of that function over the relevant limits.

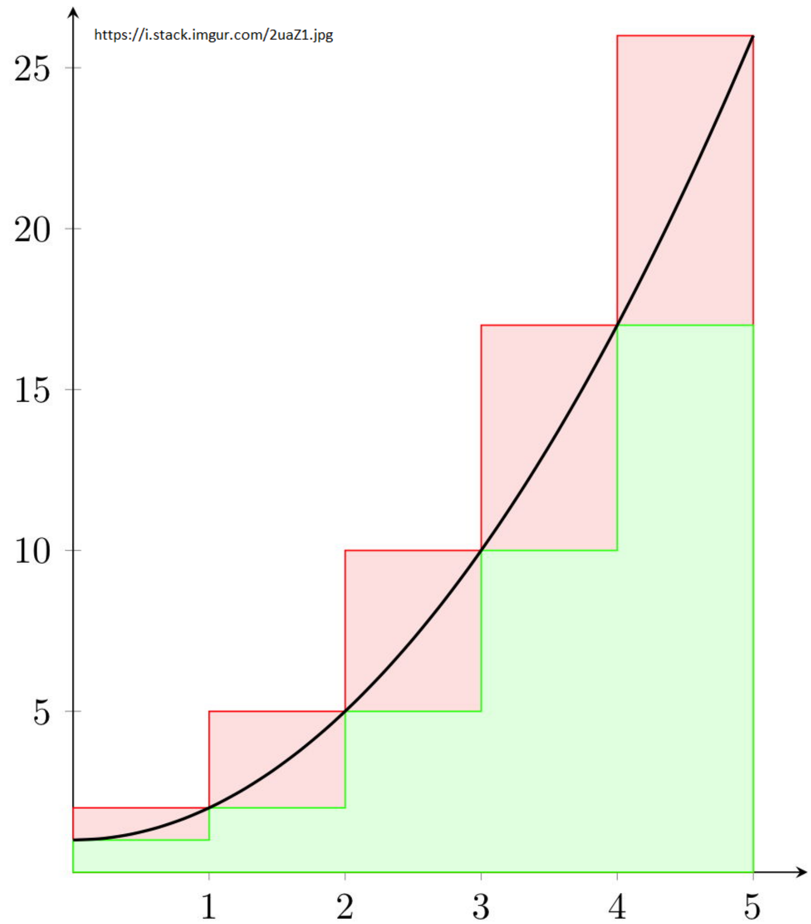


Figure 1: Lower (green) and upper (red) Riemann sums of a function (black)

1.2 Riemann-Stieltjes integration

This is a generalisation of Riemann integration, but where Riemann uses fixed subintervals of $(t_i - t_{i-1})$ as the integrator $(\mathrm{d}t)$, the Stieltjes approach uses a monotonic function instead.

Now the analogous lower and upper sums are:

$$L = \sum_{i=1}^{n} \min\{f(t_i)\} \cdot (g(t_i) - g(t_{i-1})) \tag{1}$$
$$U = \sum_{i=1}^{n} \max\{f(t_i)\} \cdot (g(t_i) - g(t_{i-1})) \tag{2}$$

where $g(t_i) > g(t_{i-1})$ when $t_i > t_{i-1}$

Imagine a 3D plot of $f(t)$ vs $g(t)$ vs t such as in figure 2.

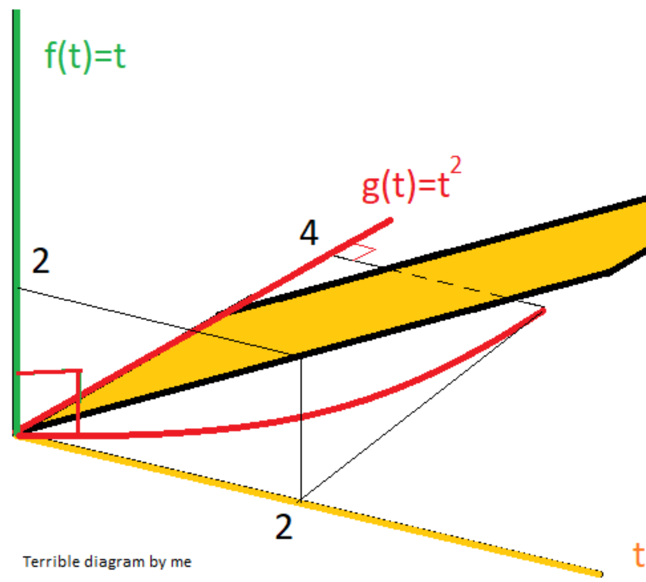


Figure 2: Stieltjes framework with the integrand $f(t) = t$ and integrator $g(t) = t^2$.

The lower sum will be the value of $f(t = 0) = 0$ multiplied by the corresponding change in value of $g(t)$. This will be zero. The upper sum will be the max value of $f(t)$ over the range of $[0, 2]$ which is $f(2) = 2$ multiplied by the change in value of integrator: $g(t_i) - g(t_{i-1}) = 2^2 - 0^2 = 4$, giving a final result of 8.

Same logic as above applies when increasing the number of partitions until the upper and lower sums intersect, which represents the value of the integral (if it exists). Note that when adding more partitions, the should be placed at equally spaced values of the integrator $g(t)$. In this example with $g(t) = t^2$, the half way point would be half the max value of $g(t)$ which is 4, evaluated at the corresponding value of t. This would be $t = \sqrt{2}$.

2 Brownian motion as integrator

2.1 Riemann-Stieltjes approach

The above Stieltjes approach doesn't translate to using Brownian motion as the integrator because of its fractal and zig-zagy nature. These factors make proofs of convergence difficult and motivate the choice for a different kind of integration: One using 'simple functions'.

More formally, if you were to try and integrate standard brownian motion as

$$\int_0^t B_s \mathrm{d}B_s \tag{3}$$

you would partition the integral into the n relevant subintervals and try to find the Riemann-Stieltjes sums:

$$S_n^1(t) = \sum_{i=1}^n B(t_{i-1}) \cdot (B(t_i) - B(t_{i-1})) \tag{4}$$

$$S_n^2(t) = \sum_{i=1}^n B(t_i) \cdot (B(t_i) - B(t_{i-1})) \tag{5}$$

Now if the Riemann-Stieltjes integral exists, $S_n^1(t) - S_n^2(t) \to 0$ as the largest subinterval shrank to zero: $\max\{t_i - t_{i-1}\} \to 0$. But multiplying out the argument of the sums above, and observing that disjoint time interval brownian terms are independent hence multiply to zero, we see

$$S_n^1(t) - S_n^2(t) = \sum_{i=1}^n \left(B_{t_i} - B_{t_{i-1}}\right)^2 > 0 \tag{6}$$

and as shown in the quadraticVariation folder, the expectation of $B_{t_i} - B_{t_{i-1}}$ is $t_i - t_{i-1}$, the sum of which is just equal to the time horizon t:

$$\mathbb{E}\left[S_n^1(t) - S_n^2(t)\right] = \sum_{i=1}^n t_i - t_{i-1} = t \neq 0 \tag{7}$$

meaning that the sums don't converge, and the Riemann-Stieltjes integral does not exist for a Brownian motion process.

2.2 Simple functions approach

Basic steps:

- Replace the function f with a constant. Set the value as the average of f between the relevant limits
- Now the integral of the new constant function with respect to the Brownian integrator is just the constant value multiplied by the change in the Brownian over that the interval.
- Now make subintervals by splitting the constant valued function into multiple regions of constant values (each constant is just the average of the function in that region)
- As the number of subintervals becomes large, you end up with a series of step functions that approximate the level of the function (see figure 3).
- The limiting value as the number of subintervals $\to \infty$ is the Ito's integral value.

Proofs of convergence of the Ito integral depend on convergence in the mean squared sense which for a stochastic process X_n and terminal value X is defined as

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^2\right] = 0 \tag{8}$$

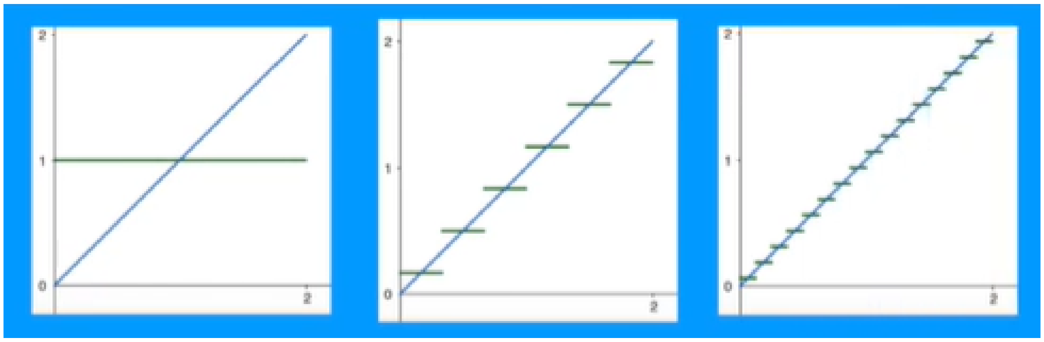
To test the convergence of the Ito integral approach, you use the Cauchy convergence or Cauchy criteria test: For every $\epsilon > 0$, $\exists\, N$ such that $\forall\, n > N, n > M : |X_n - X_m| < \epsilon$. This is essentially saying that for large enough values of n, m, terms n, m of the process will converge to within a distance of ϵ, allowing the mean squared convergence to be recast as

$$\lim_{n,m\to\infty} \mathbb{E}\left[|X_n - X_m|^2\right] = 0 \tag{9}$$

Useful because we can prove convergence without knowing the limit X.

Now the expectation is just a probability weighted sum, allowing us to write the above in terms of the L_2 norm:

$$\lim_{n,m\to\infty} ||X_n - X_m||_{L_2}^2 = 0 \tag{10}$$



https://www.youtube.com/watch?v=vHn1M6pUAWg

Figure 3: Simple functions (green) approximations to a function (blue) with an increasing number of subintervals (left to right).

The sequence of subintervals can be written as

$$S(t) = \sum_{i=1}^{n} c_i \cdot \mathbb{1}_{t_{i-1},t_i} \tag{11}$$

where c_i are the simple function values and $\mathbb{1}_{t_{i-1},t_i}$ is the indicator variable that is $= 1$ between t_{i-1}, t_i and 0 everywhere else.

The c_is in the above are there to approximate the function f. As a technicality, we say that the c_is are 'adapted to the filtration F' which roughly means that at a given time t, only the values of c_i with $i \leq t_{i-1}$ are available implying we can't see into the future.

From equation 11, we can write the ito integral as the sum of simple (constant) functions over subintervals:

$$I(t) = \sum_{i=1}^{n} c_i \cdot \left(B_{t_i} - B_{t_{i-1}}\right) \tag{12}$$

3 Summary

So we started with trying to calculate the integral of some function with respect to a stochastic Brownian integrator:

$$I(f) = \int_0^t f(u, B_u)\,\mathrm{d}B_u. \tag{13}$$

Regular Riemann or Riemann-Stieltjes approached don't work because of the details of how the Brownian motion is put together.

Next step was to approximate the integral as a limit of a sum of simple functions c_i which locally approximate the value(s) of f:

$$S(t) = \sum_{i=1}^{n} c_i \cdot \mathbb{1}_{t_{i-1},t_i} \tag{14}$$

From there, we define the Ito integral as the limiting value of a sum:

$$I(t) = \sum_{i=1}^{n} c_i \cdot \left(B_{t_i} - B_{t_{i-1}}\right) \tag{15}$$

so for completeness, the **Ito integral** is:

$$\int_0^t f(u, B_u)\,\mathrm{d}B_u \approx \sum_{i=1}^{n} f_i \cdot \left(B_{t_i} - B_{t_{i-1}}\right) \tag{16}$$

where I've replaced c_i with f_i to emphasize the approximation.

The proof of convergence is done via the Cauchy criteria and is somewhat involved. Check the youtube link in the abstract for a decent walkthrough.