

Correlation and Convolution

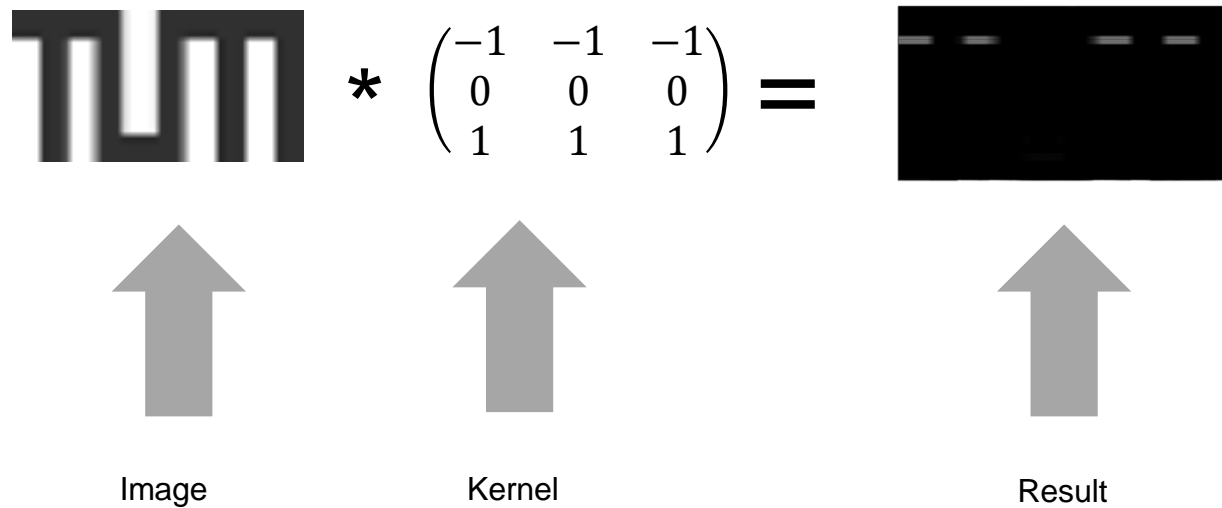
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Clemens Schmid

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Motivation

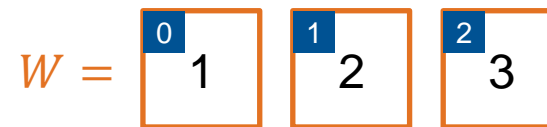
- Convolutions allow us to filter an image
- Example: Extract horizontal lines:



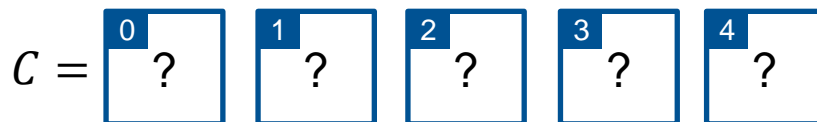
Notation

- Let's start with the one dimensional case
- Given an Image I and a kernel W :

Blue squares indicate the array element position (Spatial Index)



- We want to calculate the elements of an output vector C using correlation:



Correlation

$$I = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \quad W = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Correlation:

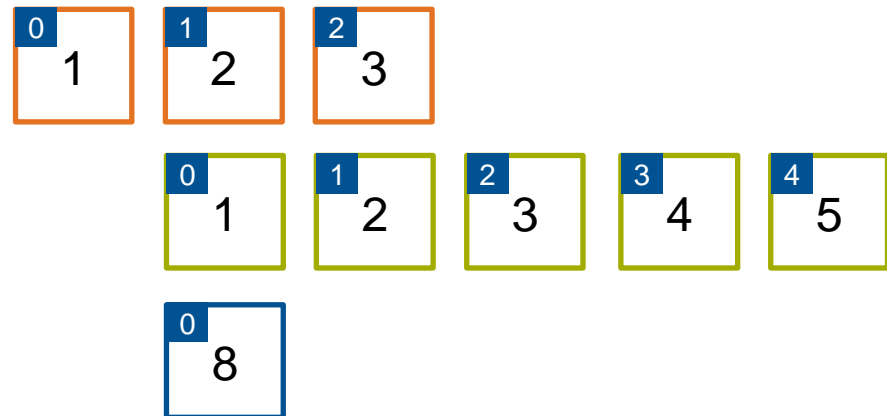
Example:

$$C_i = \sum_j I_{i+j-k} W_j \quad C_2 = \sum_{j=0}^2 I_{2+j-1} W_j = I_1 W_0 + I_2 W_1 + I_3 W_2 = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 = 20$$

- j : spatial index of W
- k : center of kernel (origin)

Illustration:

- Move the kernel over the image I
- Multiply vertically
- Add horizontally
- Write result into C at the position of the origin



Correlation

$$I = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \quad W = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Correlation:

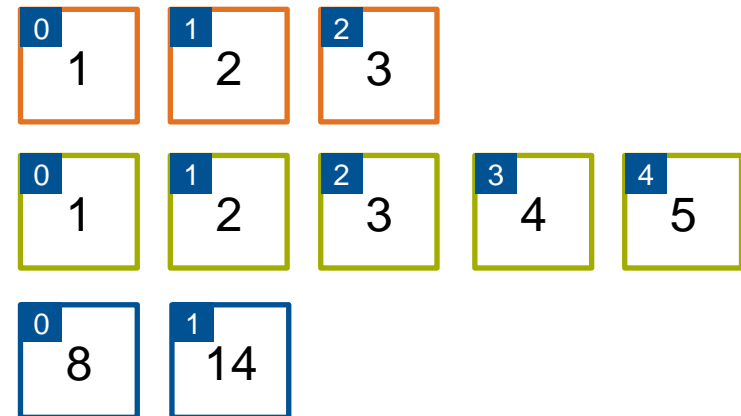
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Correlation:

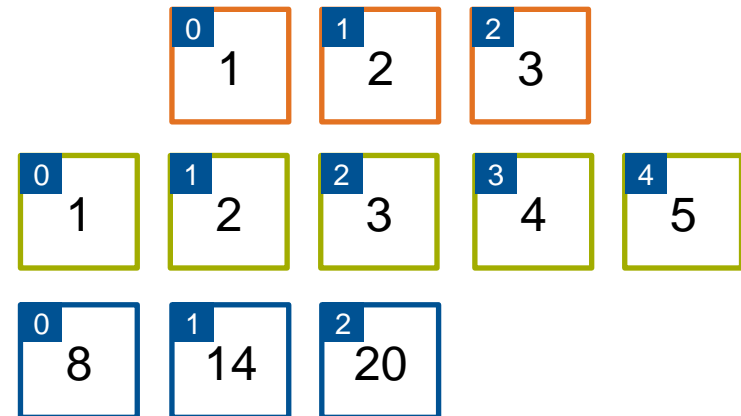
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Correlation

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		$\begin{array}{ c } \hline 0 \\ \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline 3 \\ \hline \end{array}$	
$\begin{array}{ c } \hline 0 \\ \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline 3 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 4 \\ \hline \end{array}$	$\begin{array}{ c } \hline 4 \\ \hline 5 \\ \hline \end{array}$	
$\begin{array}{ c } \hline 0 \\ \hline 8 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline 14 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline 20 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 26 \\ \hline \end{array}$		

Correlation

$$I = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \quad W = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Correlation:

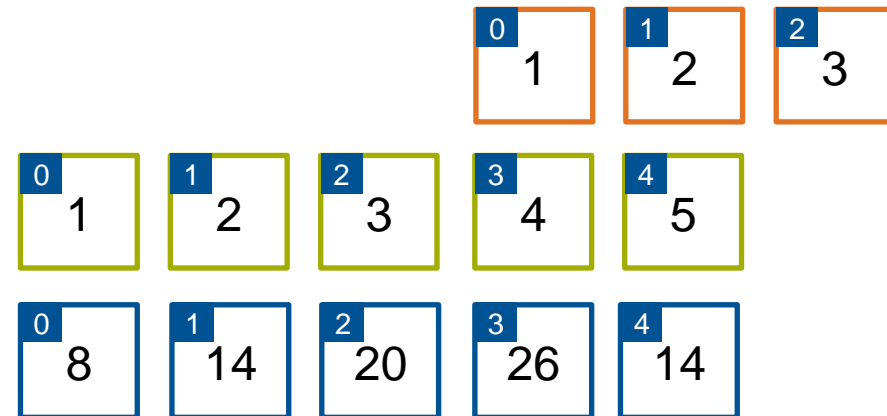
Example:

$$C_i = \sum_j I_{i+j-k} W_j \quad C_2 = \sum_{j=0}^2 I_{2+j-1} W_j = I_1 W_0 + I_2 W_1 + I_3 W_2 = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 = 20$$

- j : spatial index of W
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- Move the kernel over the image I
- Multiply vertically
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Convolution

$$I = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \quad W = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Convolution:

Example:

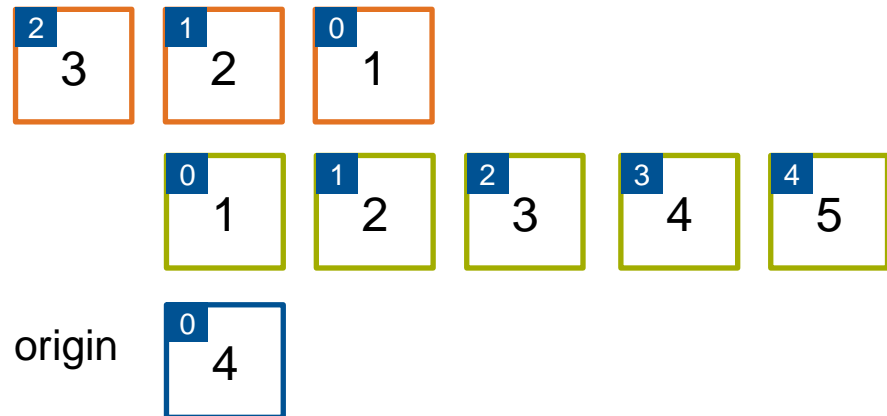
$$C_i = \sum_j I_{i-j+k} W_j$$

$$C_2 = \sum_{j=0}^2 I_{2-j+1} W_j = I_3 W_0 + I_2 W_1 + I_1 W_2 = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16$$

- j : spatial index of W
- k : center of kernel (origin)

Illustration:

- *Flip the kernel W*
- Move the kernel over the image I
- Multiply vertically
- Add horizontally
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Convolution

$$I = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Convolution:

Example:

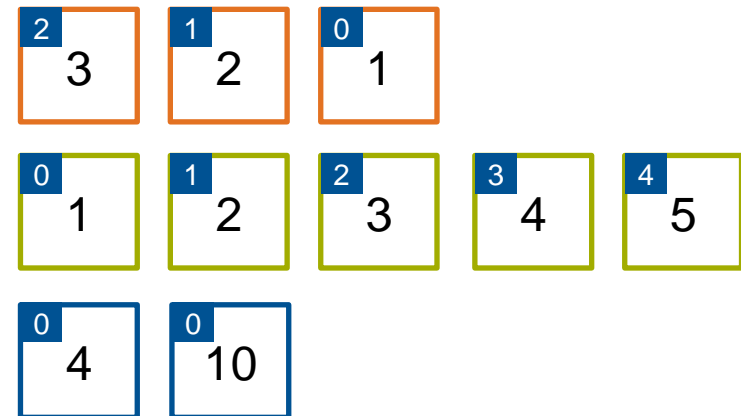
$$C_i = \sum_j I_{i-j+k} W_j$$

$$C_2 = \sum_{j=0}^2 I_{2-j+1} W_j = I_3 W_0 + I_2 W_1 + I_1 W_2 = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16$$

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$$I = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \quad W = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Convolution:

Example:

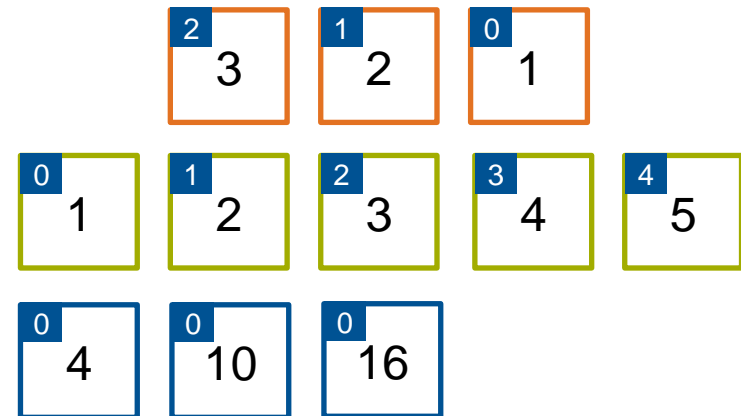
$$C_i = \sum_j I_{i-j+k} W_j$$

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Convolution

$$I = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 4 \\ \hline & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 4 & 5 \\ \hline & 5 \\ \hline \end{array} \quad W = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline & 3 \\ \hline \end{array}$$

Convolution:

Example:

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$$\begin{array}{ccccc} & & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline & 3 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline & 1 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 0 & 1 \\ \hline & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline & 3 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 3 & 4 \\ \hline & 4 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 4 & 5 \\ \hline & 5 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 0 & 4 \\ \hline & 4 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 10 \\ \hline & 10 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 16 \\ \hline & 16 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 22 \\ \hline & 22 \\ \hline \end{array} & \end{array}$$

Convolution

$$I = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Convolution:

Example:

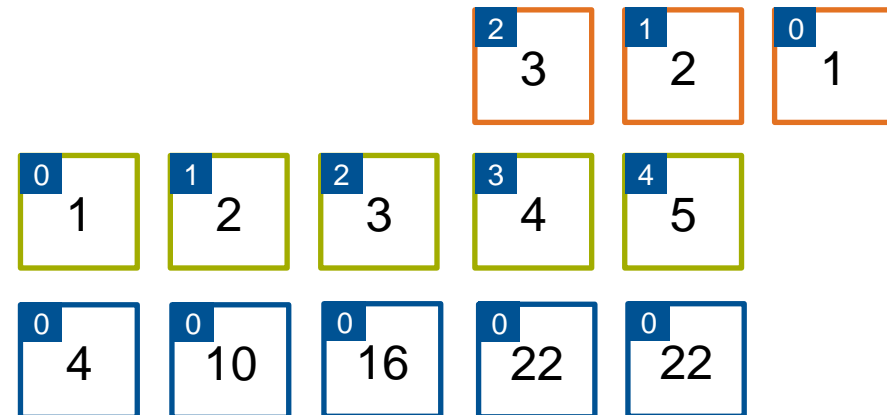
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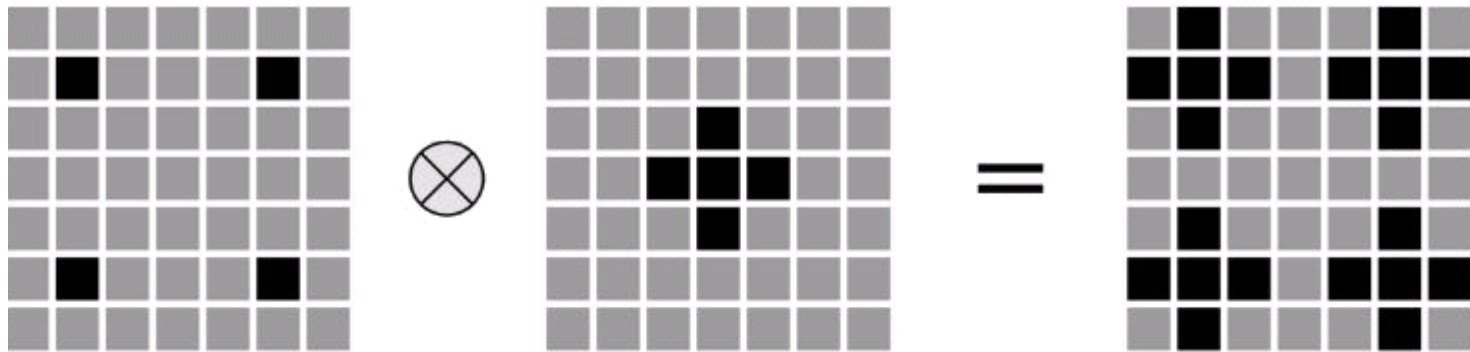
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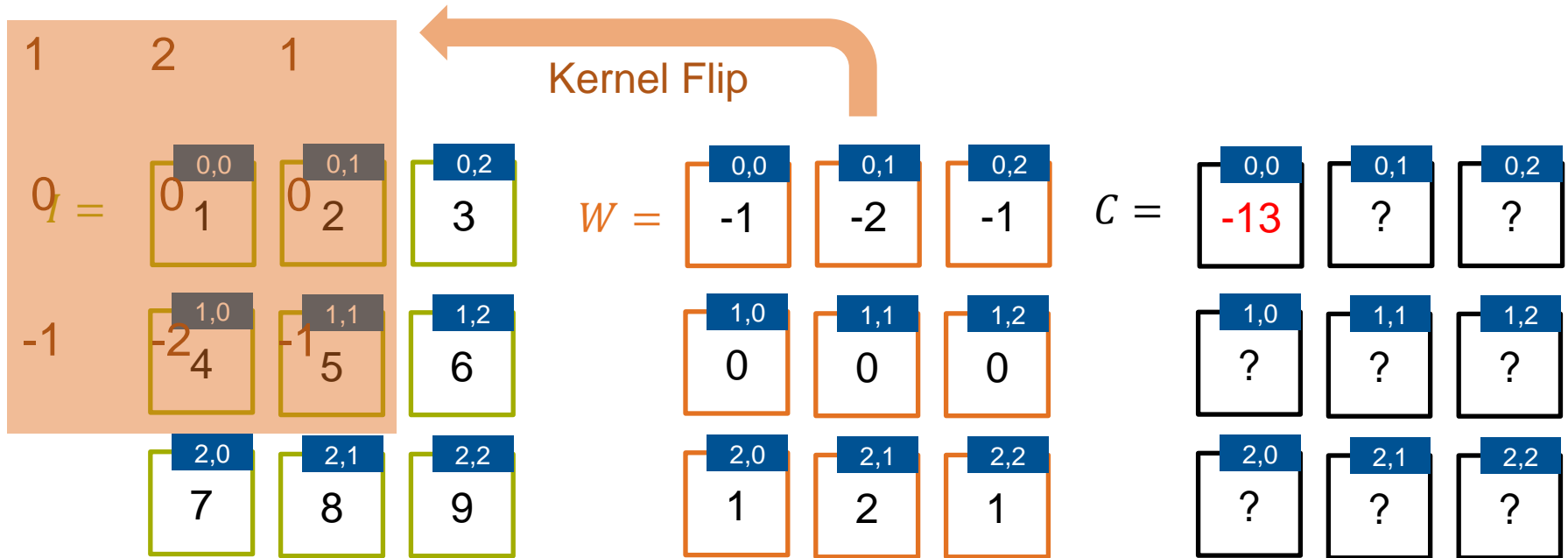
2D Convolution



2D Convolution

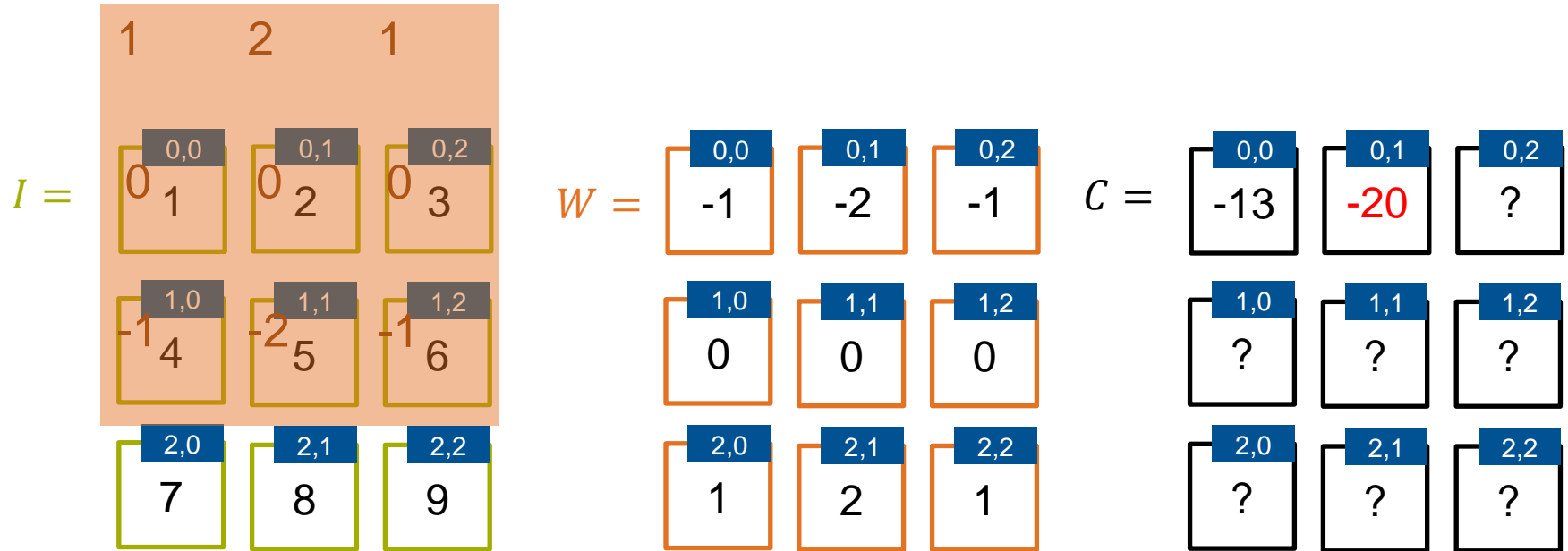
$$I = \begin{array}{|c|c|c|} \hline \begin{smallmatrix} 0,0 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 0,1 \\ 2 \end{smallmatrix} & \begin{smallmatrix} 0,2 \\ 3 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1,0 \\ 4 \end{smallmatrix} & \begin{smallmatrix} 1,1 \\ 5 \end{smallmatrix} & \begin{smallmatrix} 1,2 \\ 6 \end{smallmatrix} \\ \hline \begin{smallmatrix} 2,0 \\ 7 \end{smallmatrix} & \begin{smallmatrix} 2,1 \\ 8 \end{smallmatrix} & \begin{smallmatrix} 2,2 \\ 9 \end{smallmatrix} \\ \hline \end{array} \quad W = \begin{array}{|c|c|c|} \hline \begin{smallmatrix} 0,0 \\ -1 \end{smallmatrix} & \begin{smallmatrix} 0,1 \\ -2 \end{smallmatrix} & \begin{smallmatrix} 0,2 \\ -1 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1,0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 1,1 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 1,2 \\ 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 2,0 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 2,1 \\ 2 \end{smallmatrix} & \begin{smallmatrix} 2,2 \\ 1 \end{smallmatrix} \\ \hline \end{array} \quad C = \begin{array}{|c|c|c|} \hline \begin{smallmatrix} 0,0 \\ ? \end{smallmatrix} & \begin{smallmatrix} 0,1 \\ ? \end{smallmatrix} & \begin{smallmatrix} 0,2 \\ ? \end{smallmatrix} \\ \hline \begin{smallmatrix} 1,0 \\ ? \end{smallmatrix} & \begin{smallmatrix} 1,1 \\ ? \end{smallmatrix} & \begin{smallmatrix} 1,2 \\ ? \end{smallmatrix} \\ \hline \begin{smallmatrix} 2,0 \\ ? \end{smallmatrix} & \begin{smallmatrix} 2,1 \\ ? \end{smallmatrix} & \begin{smallmatrix} 2,2 \\ ? \end{smallmatrix} \\ \hline \end{array}$$

2D Convolution



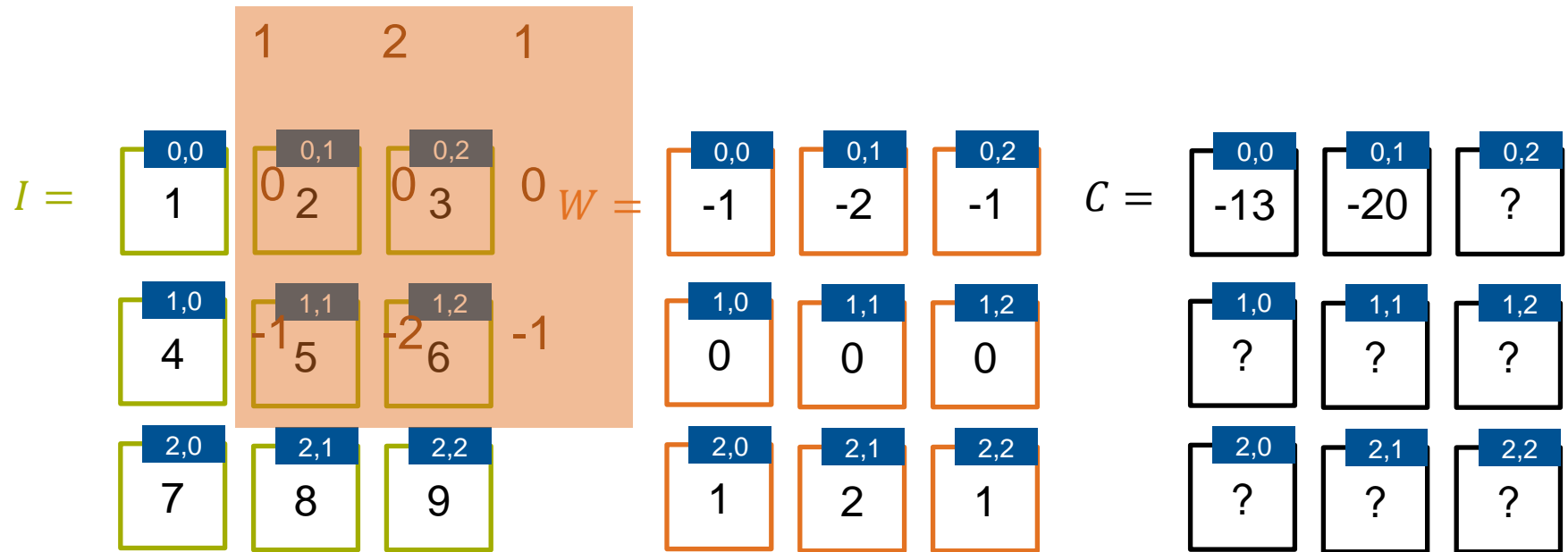
$$C_{0,0} = (0 * 1) + (0 * 2) + (-2 * 4) + (-1 * 5) = -13$$

2D Convolution



$$C_{0,1} = (0 * 1) + (0 * 2) + (0 * 3) + (-1 * 4) + (-2 * 5) + (-1 * 6) = -20$$

2D Convolution



$$C_{0,2} = \text{????}$$

Quiz: (a) 17 (b) 12 (c) 13 (d) -17