MTH9831 HW6

Weiyi Chen

Zhenfeng Liang

Mo Shen

MTH 9831 HW 6.

1. Girsanov theorem.

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{B_1 - \frac{1}{2}} = e^{-\int_0^1 (-1) dB_s - \frac{1}{2}\int_0^1 (-1)^2 \cdot ds}$$

$\mu_s = -1$

So: $\tilde{B}_t = B_t + \int_0^t \mu_s \cdot ds = B_t - t$

We know $\tilde{B}_t$ under $d\tilde{\mathbb{P}}$ is a B.M

$$\tilde{\mathbb{P}}(B_{2/3} < x) = \tilde{\mathbb{P}}(\tilde{B}_{2/3} + \frac{2}{3} < x) = \tilde{\mathbb{P}}(\tilde{B}_{2/3} < x - \frac{3}{2})$$

$$\frac{d\tilde{\mathbb{P}}}{dx} = \frac{1}{\sqrt{2\pi (2/3)}} \cdot e^{-\frac{(x-\frac{2}{3})^2}{2\cdot(\frac{2}{3})}} = \frac{\sqrt{3}}{\sqrt{4\pi}} e^{-\frac{3}{4}(x-\frac{2}{3})^2}$$

which is the density of $B_{2/3}$ with respect to $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$.

(2)

(a) $\check{B}_t = B_t + \mu t = B_t + \int_0^t \mu_s ds$

So. $\mu_s = \mu$

$$\frac{d\check{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^T \mu_s \cdot dB_s - \frac{1}{2}\int_0^T \mu_s^2 \cdot ds} = e^{-\mu B_T - \frac{1}{2}\mu^2 \cdot T}$$

$$d\check{\mathbb{P}} = e^{-\mu B_T - \frac{1}{2}\mu^2 T} \cdot d\mathbb{P}$$

(b) $\mathbb{P}[\check{B}_T \le b, \check{M}_T \ge m] = \int_{\mathbb{1}[\check{B}_T \le b, \check{M}_T \ge m]} d\mathbb{P}$

$$= \int_{\mathbb{1}[\check{B}_T \le b, \check{M}_T \ge m]} \cdot e^{\mu B_T + \frac{1}{2}\mu^2 T} d\check{\mathbb{P}} = \int_{\mathbb{1}[\check{B}_T \le b, \check{M}_T \ge m]} e^{\mu(\check{B}_T - \mu T) + \frac{1}{2}\mu^2 T} d\check{\mathbb{P}}$$

$$= \int_{\mathbb{1}[\check{B}_T \le b, \check{M}_T \ge m]} \cdot e^{\mu \check{B}_T - \frac{1}{2}\mu^2 T} d\check{\mathbb{P}}$$

(Since $\check{B}_t$ is a B.M under $d\check{\mathbb{P}}$, from reflection principle, we get $\check{\mathbb{P}}(\check{B}_T \le b, \check{M}_T \ge m) = \cancel{\check{\mathbb{P}}(2m-b)}\ \check{\mathbb{P}}(\check{B}_T' \ge 2m-b)$

Let $\check{B}_T + \check{B}_T' = 2m$)

$$= \int_{\mathbb{1}[\check{B}_T' \ge 2m-b]} e^{\mu \check{B}_T - \frac{1}{2}\mu^2 T} d\check{\mathbb{P}} \quad \text{(since we change } \check{B}_T \text{ to } \check{B}_T'\text{)}$$

$$= \int_{\mathbb{1}[\check{B}_T' \ge 2m-b]} \cdot e^{\mu(2m - \check{B}_T') - \frac{1}{2}\mu^2 T} \cdot \cancel{d\mathbb{P}}\, d\check{\mathbb{P}}$$

$$= \check{E}[\mathbb{1}_{\{\check{B}_T' \ge 2m-b\}} \cdot e^{\mu(2m - \check{B}_T') - \frac{1}{2}\mu^2 T}]$$ (Because $\check{B}_T'$ and $\check{B}_T$ are the same B.M

(c)

$$\mathbb{P}[\tilde B_T \le b, \tilde M_T \le m] = \mathbb{P}[\tilde B_T \le b] - \tilde{\mathbb{P}}[\tilde B_T \le b, \tilde M_T \ge m]$$

$$= \tilde{\mathbb{P}}[\tilde B_T \le b] - \cancel{\int_{\mathbb{1}\{\tilde B_T \ge 2m-b\}}} \int_{\mathbb{1}\{\tilde B_T \le b, \tilde M_T \ge m\}} e^{\mu \tilde B_T - \frac{1}{2}\mu^2 T} \, d\tilde{\mathbb{P}}$$

$$= \mathbb{P}[\tilde B_T \le b] - \int_{\mathbb{1}\{x \ge 2m-b\}} e^{\mu(2m-x) - \frac{1}{2}\mu^2 T} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} \, dx$$

$$= \mathbb{P}[\tilde B_T \le b] - \int_{2m-b}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{2\mu m} \cdot e^{-\frac{x^2 + 2\mu T x + \mu^2 T^2}{2T}} \, dx$$

$$= \mathbb{P}[\tilde B_T \le b] - \int_{2m-b}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{2\mu m} \cdot e^{-\frac{(x+\mu T)^2}{2T}} \, dx \quad (\text{let } y = \frac{x+\mu T}{\sqrt{T}})$$

$$= \mathbb{P}[\tilde B_T \le b] - \int_{\frac{2m-b+\mu T}{\sqrt{T}}}^{\infty} e^{2\mu m} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \, dy$$

$$\cancel{= \mathbb{P}[\tilde B_T \le b] \mp e^{2\mu m} \Phi(\frac{b-2m-}{\sqrt{T}} (1-\Phi(\frac{2m-b+\mu T}{\sqrt{T}}}$$

$$= \mathbb{P}[\tilde B_T \le b] - e^{2\mu m}\left(1 - \Phi\left(\frac{b-2m-\mu T}{\sqrt{T}}\right)\right)$$

$$= \mathbb{P}[\tilde B_T \le b] - e^{2\mu m}\Phi\left(\frac{2m-b+\mu T}{\sqrt{T}}\right)$$

$$\frac{\partial \mathbb{P}}{\partial m} = -\left(2\mu e^{2\mu m}\Phi\left(\frac{2m-b+\mu T}{\sqrt{T}}\right) + e^{2\mu m} \cdot \Phi'\left(\frac{2m-b+\mu T}{\sqrt{T}}\right) \cdot \frac{2}{\sqrt{T}}\right)$$

$$= -2e^{2\mu m}\left(\mu \Phi\left(\frac{2m-b+\mu T}{\sqrt{T}}\right) + \frac{1}{\sqrt{T}}\Phi'\left(\frac{2m-b+\mu T}{\sqrt{T}}\right)\right)$$

$$\frac{\partial \mathbb{P}}{\partial m \partial b} = -2e^{2\mu m}\left(\mu \Phi'\left(\frac{2m-b+\mu T}{\sqrt{T}}\right) \cdot \left(-\frac{1}{\sqrt{T}}\right) + \frac{1}{\sqrt{T}}\Phi''\left(\frac{2m-b+\mu T}{\sqrt{T}}\right) \cdot \left(-\frac{1}{\sqrt{T}}\right)\right)$$

$$= 2\frac{e^{2\mu m}}{\sqrt{T}}\left(\mu \Phi'(\alpha) + \frac{1}{\sqrt{T}}\Phi''(\alpha)\right)$$

where $\alpha = \frac{2m-b+\mu T}{\sqrt{T}}$

page 3 (newest).

(c) $\mathbb{P}[\hat{B}_T \le b, \hat{M}_T \le m] = \mathbb{P}[\hat{B}_T \le b] - \mathbb{P}[\hat{B}_T \le b, \hat{M}_T \ge m]$

$$= \mathbb{P}[\hat{B}_T \le b] - \int_{\mathbb{I}\{\hat{B}_T \ge 2m-b\}} e^{\mu(2m-\hat{B}_T)-\frac{1}{2}\mu^2 T}\cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\cdot dx$$

$$= \mathbb{P}[\hat{B}_T \le b] - \int_{2m-b}^{\infty} e^{\mu(2m-x)-\frac{1}{2}\mu^2 T}\cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx$$

$$\frac{\partial \mathbb{P}}{\partial m} = e^{\mu(2m-(2m-b))-\frac{1}{2}\mu^2 T}\cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2m-b)^2}{2T}}\cdot 2$$

$$= \frac{2}{\sqrt{2\pi T}} e^{\mu b-\frac{1}{2}\mu^2 T-\frac{(2m-b)^2}{2T}}$$

$$\frac{\partial \mathbb{P}}{\partial m\,\partial b} = \frac{2}{\sqrt{2\pi T}} e^{\mu b-\frac{1}{2}\mu^2 T-\frac{(2m-b)^2}{2T}}\left(\mu - \frac{1}{2T}\cdot 2(2m-b)\cdot(-1)\right)$$

$$= \frac{2}{\sqrt{2\pi T}} e^{\mu b-\frac{1}{2}\mu^2 T-\frac{(2m-b)^2}{2T}}\left(\mu + \frac{2m-b}{T}\right)$$

(d) $\mathbb{P}[\tau_m \le T] = \mathbb{P}(\hat{M}_T \ge m)$

$$= \hat{\mathbb{P}}(\hat{B}_T \ge m, \hat{M}_T \ge m) + \hat{\mathbb{P}}(\hat{B}_T \le m, \hat{M}_T \ge m)$$

$$= \hat{\mathbb{P}}(\hat{B}_T \ge m) + \hat{\mathbb{P}}(\hat{B}_T \le m, \hat{M}_T \ge m)$$

① $\mathbb{P}(\hat{B}_T \ge m) = \int_{\mathbb{I}\{\hat{B}_T \ge m\}} e^{\mu \hat{B}_T - \frac{1}{2}\mu^2 T}\, d\hat{\mathbb{P}}$ (From (a))

$$= 1 - \int_{-\infty}^{m} e^{\mu x - \frac{1}{2}\mu^2 T}\cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx$$

$$= 1 - \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{m} e^{-\frac{(x-\mu T)^2}{2T}} dx \quad \left(\text{Let } y = \frac{x-\mu T}{\sqrt{T}}\right)$$

page 3 $= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-\mu T}{\sqrt{T}}} e^{-\frac{y^2}{2}} dy = 1 - \Phi\left(\frac{m-\mu T}{\sqrt{T}}\right)$ (Φ is cdf of standard normal)

② From part (b)

$$\mathbb{P}[\hat{B}_T \le m, \hat{M}_T \ge m] = \hat{\mathbb{E}}[\mathbb{1}_{\{\hat{B}_T \ge 2m\}} \cdot e^{\mu(2m-\hat{B}_T)-\frac{1}{2}\mu^2 T}]$$

$$= 1 - \int_{-\infty}^{m} e^{2\mu(2m-x)-\frac{1}{2}\mu^2 T} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx$$

$$= 1 - e^{2\mu m} \cdot \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{m} e^{-\frac{x^2+2\mu T x+\mu^2 T^2}{2T}} dx$$

$$= 1 - e^{2\mu m} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{m} e^{-\frac{(x+\mu T)^2}{2T}} dx \quad (\text{Let } y = \frac{x+\mu T}{\sqrt{T}})$$

$$= 1 - e^{2\mu m} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m+\mu T}{\sqrt{T}}} e^{-\frac{y^2}{2}} \cdot dy$$

$$= 1 - e^{2\mu m} \cdot \Phi\left(\frac{m+\mu T}{\sqrt{T}}\right)$$

$$\mathbb{P}[\tau_m \le T] = \hat{\mathbb{P}}[\hat{B}_T \ge m] + \hat{\mathbb{P}}[\hat{B}_T \le m, \hat{M}_T \ge m]$$

$$= 1 - \Phi\left(\frac{m-\mu T}{\sqrt{T}}\right) + 1 - e^{2\mu m} \cdot \Phi\left(\frac{m+\mu T}{\sqrt{T}}\right)$$

$$= 2 - \Phi\left(\frac{m-\mu T}{\sqrt{T}}\right) - e^{2\mu m} \Phi\left(\frac{m+\mu T}{\sqrt{T}}\right)$$

where $\Phi(x)$ is the cdf of standard normal dist.

$$\mathbb{P}[\tau_m < \infty] = 2 - \Phi(-\infty) - e^{2\mu m} \Phi(+\infty)$$

$$= 2 - 0 - e^{2\mu m}$$

$$= 2 - e^{2\mu m}$$

(e) Let $\eta < b$

$$\mathbb{P}[\tilde{B}_T \le b, \tilde{m}_T \le \eta] = \mathbb{P}[\tilde{m}_T \le \eta] - \mathbb{P}[\tilde{B}_T \ge b, \tilde{m}_T \le \eta]$$

$$\int_{\mathbb{1}\{\tilde{B}_T \ge b, \tilde{m}_T \le \eta\}} e^{\mu \tilde{B}_T - \frac{1}{2}\mu^2 T} \cdot d\tilde{\mathbb{P}} \quad (\text{Let } \tilde{B}_T + \tilde{B}_T' = 2\eta).$$

$$= \int_{\mathbb{1}\{\tilde{B}_T' \le 2\eta - b\}} e^{\mu(2\eta - \tilde{B}_T') - \frac{1}{2}\mu^2 T} \cdot d\tilde{\mathbb{P}}$$

$$= \int_{-\infty}^{2\eta - b} e^{\mu(2\eta - x) - \frac{1}{2}\mu^2 T} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} \cdot dx$$

$$= e^{2\mu\eta} \cdot \cancel{\Phi(2\eta - b)}\ \Phi\left(\frac{2\eta - b + \mu T}{\sqrt{T}}\right) \qquad e^{2\mu\eta}\Phi\left(\frac{2\eta - b + \mu T}{\sqrt{T}}\right)$$

Therefore: $\mathbb{P}[\tilde{B}_T \le b, \tilde{m}_T \le \eta] = \mathbb{P}[\tilde{m}_T \le \eta] - \cancel{e^{2\mu\eta}\Phi(2\eta - b)}$.

$$\cancel{\frac{\partial \mathbb{P}}{\partial b} = -e^{2\mu\eta}\Phi'(2\eta - b) \cdot (-1)\left(\frac{1}{\sqrt{T}}\right) = e^{2\mu\eta} \cdot \Phi'(2\eta - b)}$$

$$\cancel{\frac{\partial^2 \mathbb{P}}{\partial b \cdot \partial \eta} = 2\mu \cdot e^{2\mu\eta}\Phi'(2\eta - b) + e^{2\mu\eta} \cdot \Phi''(2\eta - b) \cdot 2}$$

$$\cancel{= 2e^{2\mu\eta}(\mu\Phi'(2\eta - b) + \Phi''(2\eta - b))}$$

$$\frac{\partial \mathbb{P}}{\partial b} = \frac{1}{\sqrt{T}} e^{2\mu\eta}\Phi'\left(\frac{2\eta - b + \mu T}{\sqrt{T}}\right)$$

$$\frac{\partial^2 \mathbb{P}}{\partial b \partial \eta} = \frac{1}{\sqrt{T}} \cdot \left(2\mu e^{2\mu\eta} \cdot \Phi'\left(\frac{2\eta - b + \mu T}{\sqrt{T}}\right) + e^{2\mu\eta}\Phi''\left(\frac{2\eta - b + \mu T}{\sqrt{T}}\right) \cdot \left(\frac{2}{\sqrt{T}}\right)\right)$$

$$= \frac{2}{\sqrt{T}} e^{2\mu\eta}\left(\mu\Phi'(\beta) + \frac{1}{\sqrt{T}}\Phi''(\beta)\right)$$

where $\beta = \frac{2\eta - b + \mu T}{\sqrt{T}}$

page 5 (Newest).

(3)

(a) Apply Itô

$$d\check{S}_t = dD_tS_t = D_t\cdot dS_t + S_t\cdot dD_t + \underbrace{d[S_t, D_t]}_{\to 0}$$

$$= D_t(\mu S_t\cdot dt + \sigma S_t\cdot dB_t) \cancel{*} - rD_t\cdot S_t\cdot dt$$

$$= \check{S}_t((\mu - r)\cdot dt + \sigma dB_t)$$

We need to find $\check{\mathbb{P}}$, under which $\cancel{\sigma}\sigma d\check{B}_t = \sigma\cancel{B}dB_t + (\mu-r)dt$ holds

$$\check{B}_t = \cancel{\int} B_t + \int_0^T \frac{\mu - r}{\sigma}\cdot dt \quad (\mu_{\cancel{s}t} = \frac{\mu-r}{\sigma}, \text{ is a const.})$$

$$\frac{d\check{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^T \mu_s\cdot dB_s - \frac{1}{2}\int_0^T \mu_s^2\cdot ds}$$

$$= e^{-\int_0^T \frac{\mu-r}{\sigma}\cdot dB_s - \frac{1}{2}\int_0^T (\frac{\mu-r}{\sigma})^2\cdot ds}$$

$$= e^{-\frac{\mu-r}{\sigma}\cdot B_T - \frac{1}{2}(\frac{\mu-r}{\sigma})^2\cdot T} \cancel{= e}$$

$$= e^{-\frac{\mu-r}{\sigma}(B_T + \frac{1}{2}\cdot\frac{\mu-r}{\sigma}\cdot T)}$$

$$d\check{\mathbb{P}} = e^{-\frac{\mu-r}{\sigma}(B_T + \frac{1}{2}\frac{\mu-r}{\sigma}\cdot T)}\cdot d\mathbb{P}$$

under measure $\check{\mathbb{P}}$, $\check{S}_t$ is a martingale.

(b) $\frac{dD_t}{D_t} = -r\cdot dt$. $\int \frac{1}{D_t} dD_t = \int -r\cdot dt + \ln C$

$\ln D_t = -rt + \ln C$ $D_t = C\cdot e^{-rt}$.

$D_0 = C\cdot e^{-0} = 1$ $C = 1$

$D_t = \cancel{c}e^{-rt}$.

But $\tilde{S}_t$ is a stochastic process, we cannot integrate ~~it~~ like this.

$d\tilde{S}_t = \tilde{S}_t\cdot\sigma\cdot d\tilde{B}_t$

$d\ln\tilde{S}_t = \frac{1}{\tilde{S}_t}\cdot d\tilde{S}_t + \frac{1}{2}\cdot\left(-\frac{1}{\tilde{S}_t^2}\right)\cdot d[\tilde{S}_t]_t$

$= \frac{1}{\tilde{S}_t} d\tilde{S}_t - \frac{1}{2}\sigma^2\cdot dt$.

So $\frac{d\tilde{S}_t}{\tilde{S}_t} = d\ln\tilde{S}_t + \frac{1}{2}\sigma^2\cdot dt$.

$\int\frac{d\tilde{S}_t}{\tilde{S}_t} = \int d\ln\tilde{S}_t + \int\frac{1}{2}\sigma^2 dt = \int \cancel{S_t}\cdot\sigma\cdot d\tilde{B}_t + \ln C$

$\ln\tilde{S}_t + \frac{1}{2}\sigma^2 t = \cancel{S_t}\,\sigma\tilde{B}_t + \ln C$.

$\tilde{S}_t = C\cdot e^{\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t}$ $\tilde{S}_t = \tilde{S}_0\cdot e^{\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t}$

$S_t = \tilde{S}_t\cdot D_t^{-1} = \tilde{S}_0\cdot e^{\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t + rt}$

$S_0 = \tilde{S}_0\cdot e^0 = x$ $\tilde{S}_0 = x$

So; under measure $\tilde{\mathbb{P}}$.

$S_t = xe^{\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t + rt}$

$\tilde{S}_t = xe^{\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t +}$

(c)

$$\check{S}_t = x e^{\sigma \check{B}_t - \frac{1}{2}\sigma^2 t} = x e^{\sigma \hat{B}_t} \quad \text{where } \sigma\hat{B}_t = \sigma\check{B}_t - \frac{1}{2}\sigma^2 t$$

$$\check{\mathbb{P}}[\check{S}_t \geq b, m_t \geq a] = \check{\mathbb{P}}[\check{S}_t \geq b] - \check{\mathbb{P}}[\check{S}_t \geq b, m_t \leq a]$$

$$\hat{B}_t = \check{B}_t + \int_0^t \cancel{(-\tfrac{1}{2}\sigma)} (-\tfrac{1}{2}\sigma)\cdot ds \qquad (\mu_s = -\tfrac{\sigma}{2})$$

$$\frac{d\hat{\mathbb{P}}}{d\check{\mathbb{P}}} = e^{-\int_0^T (-\frac{1}{2}\sigma)\cdot d\check{B}_s - \frac{1}{2}\int_0^T (-\frac{1}{2}\sigma)^2 ds} = e^{\frac{1}{2}\sigma\check{B}_T - \frac{1}{8}\sigma^2 \cdot T}$$

① $$\check{\mathbb{P}}[\check{S}_t \geq b] = \int_{\mathbb{1}\{\check{S}_t \geq b\}} \cdot d\check{\mathbb{P}} = \int_{\mathbb{1}\{\hat{B}_t \geq \frac{1}{\sigma}\ln\frac{b}{x}\}} \cdot e^{\frac{1}{8}\sigma^2 t - \frac{1}{2}\sigma\check{B}_t} \cdot d\hat{\mathbb{P}}$$

Let $\frac{1}{\sigma}\ln\frac{b}{x} = \alpha$.

$$= \int_\alpha^{+\infty} e^{\frac{1}{8}\sigma^2 t - (\frac{1}{2}\sigma\hat{B}_t + \frac{1}{4}\sigma^2 t)} \cdot d\hat{\mathbb{P}}$$

$$= \int_\alpha^{+\infty} e^{-\frac{1}{2}\sigma\hat{B}_t - \frac{1}{8}\sigma^2 t} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{\hat{B}_t^2}{2t}} \cdot d\hat{B}_t$$

$$= \int_\alpha^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{\hat{B}_t^2 + \sigma t \hat{B}_t + \frac{1}{4}\sigma^2 t^2}{2t}} d\hat{B}_t$$

$$= \int_\alpha^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\hat{B}_t + \frac{1}{2}\sigma t)^2}{2t}} \cdot d\hat{B}_t \qquad \left(\text{Let } y = \frac{\hat{B}_t + \frac{1}{2}\sigma t}{\sqrt{t}}\right)$$

$$= \int_{\frac{\alpha + \frac{1}{2}\sigma t}{\sqrt{t}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1 - \Phi(\hat{\alpha}) = \Phi(-\hat{\alpha}) = \Phi(\hat{\alpha}')$$

where $\hat{\alpha} = \dfrac{\frac{1}{\sigma}\ln\frac{b}{x} + \frac{1}{2}\sigma t}{\sqrt{t}}$

$$\hat{\alpha}' = -\hat{\alpha} = \frac{\ln\frac{x}{b}}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}$$

② $\hat{\mathbb{P}}[\hat{S}_t \geq b, m_t \leq a]$

$= \int_{\mathbb{1}\{\hat{B}_t \geq \alpha, \eta_t \leq \beta\}} e^{-\frac{1}{2}\sigma \hat{B}_t - \frac{1}{8}\sigma^2 t} \cdot d\hat{\mathbb{P}}$

where $\beta = \frac{1}{\sigma}\ln\frac{a}{x}$ $\eta_t = \min_{0<u<t}\{\hat{B}_u\}$ $\alpha = \frac{1}{\sigma}\ln\frac{b}{x}$

$= \int_{\mathbb{1}\{\hat{B}_t' \leq 2\beta - \alpha\}} e^{-\frac{1}{2}\sigma(2\beta - \hat{B}_t') - \frac{1}{8}\sigma^2 t} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{\hat{B}_t'^2}{2t}} d\hat{B}_t'$

$= \int_{-\infty}^{2\beta-\alpha} \frac{1}{\sqrt{2\pi t}} \cdot e^{-\sigma\beta} \cdot e^{-\frac{x^2 - \sigma t \cdot x + \frac{1}{4}\sigma^2 t^2}{2t}} dx$. (Let $y = \frac{(x - \frac{1}{2}\sigma t)}{\sqrt{t}}$)

~~$= e^{-\sigma\beta} \int_{-\infty}^{2\beta-\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$~~

$= e^{-\sigma\beta} \int_{-\infty}^{\hat{\beta}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ $\left(\hat{\beta} = \frac{2\beta - \alpha - \frac{1}{2}\sigma t}{\sqrt{t}}\right)$

$= e^{-\sigma\beta} \Phi(\hat{\beta})$ $\left(\hat{\beta} = \frac{\frac{1}{\sigma}\ln\frac{a}{x} + \frac{1}{\sigma}\ln\frac{a}{b} - \frac{1}{2}\sigma t}{\sqrt{t}}\right)$

Therefore:

$\hat{\mathbb{P}}[\hat{S}_t \geq b, m_t \geq a] = \hat{\mathbb{P}}[\hat{S}_t \geq b] - \hat{\mathbb{P}}[\hat{S}_t \geq b, m_t \leq a]$

$= \Phi(\hat{\alpha}') - e^{-\sigma\beta}\Phi(\hat{\beta})$

where $\hat{\alpha}' = \frac{\ln\frac{x}{b}}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}$

$\hat{\beta} = \frac{\frac{1}{\sigma}\ln\frac{a}{x} + \frac{1}{\sigma}\ln\frac{a}{b}}{\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}$.

(d) Let $\eta_t = \min\limits_{0<u<t}\{S_u\}$

$$\breve{\mathbb{E}}(\mathbb{1}_{\{\eta_t>a\}}) = \int_{\mathbb{1}_{\{\eta_t>a\}}} \cdot d\breve{\mathbb{P}}$$

~~Find~~ $S_t = X\cdot e^{\sigma\breve{B}_t - \frac{1}{2}\sigma^2 t + rt} = X e^{\sigma\hat{\hat{B}}_t}$

Find measur $d\hat{\mathbb{P}}$, under which $\sigma\hat{B}_t = \sigma\breve{B}_t - \frac{1}{2}\sigma^2 t + rt$ holds

$$\hat{B}_t = \breve{B}_t + \left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)t = \breve{B}_t + \int_0^t\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)ds \quad \left(u_s = \frac{r}{\sigma} - \frac{1}{2}\sigma\right)$$

$$\frac{d\hat{\mathbb{P}}}{d\breve{\mathbb{P}}} = e^{-\int_0^t \frac{r}{\sigma} - \frac{1}{2}\sigma \, d\breve{B}_s - \frac{1}{2}\int_0^t \left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)^2 ds}$$

$$= e^{-\mu\breve{B}_t - \frac{1}{2}\mu^2 \cdot t} \qquad \text{where } \mu = \frac{r}{\sigma} - \frac{1}{2}\sigma$$

$$\breve{\mathbb{E}}(\mathbb{1}_{\{\eta_t>a\}}) = 1 - \breve{\mathbb{P}}(\eta_t \le a)$$

$$\breve{\mathbb{P}}(\eta_t \le a) = \breve{\mathbb{P}}(S_u \le a, \eta_t \le a) + \breve{\mathbb{P}}(S_u \ge a, \eta_t \le a)$$

$$= \breve{\mathbb{P}}(S_u \le a) + \breve{\mathbb{P}}(S_u \ge a, \eta_t \le a)$$

$$\breve{\mathbb{P}}(S_u \ge a, \eta_t \le a) = \int_{\mathbb{1}_{\{S_u \ge a, \eta_t \le a\}}} \cdot d\breve{\mathbb{P}} = \int_{\mathbb{1}_{\{\hat{B}_t \ge \lambda, \min \hat{B}_t \le \lambda\}}} \cdot e^{\mu\breve{B}_t + \frac{1}{2}\mu^2 t} d\hat{\mathbb{P}}$$

where $\lambda = \frac{1}{\sigma}\ln\frac{a}{X}$

$$= \int_{\mathbb{1}_{\{\hat{B}_t \le \lambda\}}} e^{\mu(\hat{B}_t - \mu t) + \frac{1}{2}\mu^2 t} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{\hat{B}_t^2}{2t}} d\hat{B}_t$$

which is also $\breve{\mathbb{P}}(S_u \le a)$

$$= \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\hat{B} - \mu t)^2}{2t}} d\hat{B} \quad \left(\text{Let } y = \frac{\hat{B} - \mu t}{\sqrt{t}}\right)$$

$$= \int_{-\infty}^{\frac{\lambda - \mu t}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \Phi(\hat{\beta})$$

 where $\hat{\beta} = \frac{\lambda - \mu t}{\sqrt{t}}$ ~~$\lambda = \frac{1}{\sigma}\ln$~~

So: $\check{\mathbb{P}}(\eta_t \le a) = 2\check{\mathbb{P}}(S_u \le a) = 2\Phi(\hat{\beta})$

$\check{\mathbb{E}}(\cancel{\Psi_t}\, \mathbb{I}_{\{\eta_t \ge a\}}) = 1 - 2\Phi(\hat{\beta})$

Value of the option

$= e^{-rT} \cdot (1 - 2\Phi(\hat{\beta}))$

where $\hat{\beta} = \frac{\lambda - \mu t}{\sqrt{t}}$ $\quad \lambda = \frac{1}{\sigma} \ln \frac{a}{x}$ $\cancel{\#}$ $\mu = \frac{r}{\sigma} - \frac{1}{2}\sigma$

---

(4)

(a) $Y_t = \int_0^t B_s \cdot 1 \cdot ds$

$|Y_t|^2 \le \int_0^t |B_s|^2 ds \cdot \int_0^t 1\, ds = t \int_0^t |B_s|^2 ds.$

$E|Y_t|^2 \le E(t \int_0^t |B_s|^2 ds) = t \int_0^t E(B_s^2) ds$

$= t \int_0^t s\, ds = \frac{1}{2} t^3$

which is finite.

$Var(Y_t) = E(Y_t^2) - E(Y_t)^2$

$E(Y_t) = \int_0^t E(B_s) ds = 0$

$\cancel{E(Y}$ $f = Y^2$. $\frac{\partial f}{\partial Y} = Y$ $\quad \frac{\partial f}{\partial t} = 0$ $\quad \frac{\partial^2 f}{\partial Y^2} = 1$

$df = Y_t \cdot dY + \frac{1}{2} \cdot d[Y_t]_t = Y_t dY = Y_t B_t dt.$

(4)

(a) $E(Y_t) = \int_0^t E(B_s)\,ds = 0.$

$E(Y_t^2) = E(\int_0^t B_s ds \int_0^t B_u du) = \int_0^t \int_0^t E(B_s B_u) ds\, du$

$= \int_0^t \int_0^u E(B_s B_u) ds\,du + \int_0^t \int_u^t E(B_s B_u) ds\,du$

$= \int_0^t \int_0^u s\, ds\,du + \int_0^t \int_u^t u\, ds\,du$

$= \int_0^t \frac{1}{2}u^2 du + \int_0^t u(t-u)\,du$

$= \frac{1}{6}t^3 + (\frac{1}{2}tu^2 - \frac{1}{3}u^3)\Big|_0^t$

$= \frac{1}{6}t^3 + \frac{1}{2}t^3 - \frac{1}{3}t^3$

$= \frac{1}{3}t^3$

$Var(Y_t) = \frac{1}{3}t^3$

Since $E(Y_t^2)$ is finite, Y_t is square integrable.

(b). $dM_t = \theta_t \cdot dB_t$.

$M_t = E[Y_T | F_t) = E(\int_0^t B_s ds | F_t) + E(\int_t^T B_s ds | F_t)$

$= \int_0^t B_s ds + \int_t^T E(B_s | F_t) ds$

$= \int_0^t B_s ds + \int_t^T B_t ds = Y_t + B_t(T-t)$

$dM_t = B_t dt + d(B_t(T-t)) = B_t dt + (T-t)dB_t - B_t dt$

$= (T-t)dB_t = \theta_t dB_t$

$\theta_s = T - S.$