MTH 9831: Homework 4

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Problem 1

Review Ito isometry

In mathematics, the Ito isometry, named after Kiyoshi Ito, is a crucial fact about Ito stochastic integrals. Let $W : [0,T] \times \Omega \to R$ denote the canonical real-valued Wiener process defined up to time $T > 0$, and let $X : [0,T] \times \Omega \to R$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then

$$E\left[\left(\int_0^T X_t \,\mathrm{d}W_t\right)^2\right] = E\left[\int_0^T X_t^2 \,\mathrm{d}t\right] \tag{1}$$

where E denotes expectation with respect to classical Wiener measure γ. In other words, the Ito stochastic integral, as a function, is an isometry of normed vector spaces with respect to the norms induced by the inner products

$$(X,Y)_{L^2(W)} := E\left(\int_0^T X_t \,\mathrm{d}W_t \int_0^T Y_t \,\mathrm{d}W_t\right) = \int_\Omega \left(\int_0^T X_t \,\mathrm{d}W_t \int_0^T Y_t \,\mathrm{d}W_t\right) \mathrm{d}\gamma(\omega) \tag{2}$$

and

$$(A,B)_{L^2(\Omega)} := E(AB) = \int_\Omega A(\omega)B(\omega)\,\mathrm{d}\gamma(\omega). \tag{3}$$

(a)

Since the expectation of Martingale is 0, additionally with Ito isometry and Fubini's theorem,

$$\begin{aligned} cov(X,X) &= E(X^2) - [E(X)]^2 \\ &= E\left(\int_0^1 (\sqrt{t}e^{B_t^2/8})^2 dt\right) - 0 \\ &= \int_\Omega \int_0^1 (\sqrt{t}e^{B(t,w)^2/8})^2 dt dP(w) \\ &= \int_0^1 \int_\Omega (\sqrt{t}e^{B(t,w)^2/8})^2 dP(w) dt \\ &= \int_0^1 tE(e^{B_t^2/4})dt \end{aligned} \tag{4}$$

Now we look at $E(e^{B_t^2/4})$, since $B_t \sim N(0,t)$ let $B_t = \sqrt{t}Z$

$$\begin{aligned} E(e^{B_t^2/4}) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tZ^2/4} e^{-Z^2/2} dZ \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2(\frac{2}{2-t})}} dZ \\ &= \sqrt{\frac{2}{2-t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\frac{2}{2-t}}} e^{-\frac{Z^2}{2(\frac{2}{2-t})}} dZ \\ &= \sqrt{\frac{2}{2-t}} \end{aligned} \tag{5}$$

Here we come back to

$$var(X) = \int_0^1 t\sqrt{\frac{2}{2-t}} dt \tag{6}$$

Let $u = \sqrt{2-t}$, therefore $\frac{du}{dt} = -\frac{1}{2}\frac{1}{\sqrt{2-t}}$, and

$$var(X) = \int_1^{\sqrt{2}} (2-u^2)2\sqrt{2}du = \frac{16}{3} - \frac{10\sqrt{2}}{3} \tag{7}$$

(b)

Since the expectation of Martingale is 0, and with Ito isometry,

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= E\int_a^b f^2(t)(\sin B_t + \cos B_t)^2 dt - 0 \\ &= E\int_a^b f^2(t)(1+\sin 2B_t)dt \\ &= \int_a^b f^2(t)dt + \int_a^b f^2(t)E(\sin 2B_t)dt \end{aligned} \tag{8}$$

We consider $E(\sin 2B_t)$ and find out that

$$E(\sin 2B_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(2z)e^{-z^2/2}dz = 0 \tag{9}$$

since $\sin(2z)$ is an odd function and $e^{-z^2/2}$ is an even function, which makes $\sin(2z)e^{-z^2/2}$ an odd function. Therefore the answer is

$$Var(X) = \int_a^b f^2(t)dt \tag{10}$$

Problem 2

Review of Ito's lemma

We give a sketch of how one can derive Ito's lemma by expanding a Taylor series and applying the rules of stochastic calculus. Assume X_t is a Ito drift-diffusion process that satisfies the stochastic differential equation

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t, \tag{11}$$

where B_t is a Wiener process. If $f(t,x)$ is a twice-differentiable scalar function, its expansion in a Taylor series is

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\, dx^2 + \cdots \tag{12}$$

Substituting X_t for x and $_t dt + \sigma t dB_t$ for dX_t gives

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}(\mu_t\, dt + \sigma_t\, dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu_t^2\, dt^2 + 2\mu_t\sigma_t\, dt\, dB_t + \sigma_t^2\, dB_t^2\right) + \cdots. \tag{13}$$

In the limit as $dt \to 0$, the terms dt^2 and $dtdB_t$ tend to zero faster than dB_2, which is $O(dt)$. Setting the d_t^2 and dtd_B^2 terms to zero, substituting dt for d_B^2, and collecting the dt and dB terms, we obtain

$$df = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}\, dB_t \tag{14}$$

as required.

(a)

Let $f(t,x) = e^{\sigma x - \frac{\sigma^2}{2}t}$, then

$$f_t(t,x) = -\frac{\sigma^2}{2} e^{\sigma x - \frac{\sigma^2}{2}t} \tag{15}$$
$$f_x(t,x) = \sigma e^{\sigma x - \frac{\sigma^2}{2}t} \tag{16}$$
$$f_{xx}(t,x) = \sigma^2 e^{\sigma x - \frac{\sigma^2}{2}t} \tag{17}$$

After putting formulas above into

$$df(t,X_t) = f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2} f_{xx}(t,X_t)d[X]_t \tag{18}$$

we fortunately have

$$df(t,B_t) = \sigma e^{\sigma B_s - \frac{\sigma^2 t}{2}} \tag{19}$$

By writing the integration form, we find our goal

$$\frac{1}{\sigma}(f(t,B_t) - f(0,B_0)) = \int_0^t e^{\sigma B_s - \frac{\sigma^2 s}{2}} dB_s \tag{20}$$

where $f(0,B_0) = 1$, therefore the answer is

$$\int_0^t e^{\sigma B_s - \frac{\sigma^2 s}{2}} dB_s = \frac{1}{\sigma}(f(t,B_t) - 1) \tag{21}$$

(b)

Apply the last formula in Review of Ito's lemma, let

$$f_x = \frac{e^{-s}}{1+B_s^2} \tag{22}$$

Obviously we have

$$f = e^{-s} \arctan B_s \tag{23}$$
$$f_{xx} = -2\frac{e^{-s}B_s}{(1+B_s^2)^2} \tag{24}$$
$$f_s = -e^{-s} \arctan B_s \tag{25}$$

Apply Ito's lemma, we have

$$df = (f_s + \frac{\sigma^2}{2} f_{xx})ds + f_x dx = \left[-e^{-s} \arctan B_s + \frac{\sigma^2}{2} \frac{-2e^{-s}B_s}{(1+B_s^2)^2} \right] ds + (\frac{e^{-s}}{1+B_s^2})dB_s \tag{26}$$

Write it in integral form, we have

$$f_t - f_0 = \int_0^t (\frac{e^{-s}}{1+B_s^2})dB_s - \int_0^t e^{-s}(\arctan B_s + \frac{B_s}{(1+B_s^2)^2})ds \tag{27}$$

that is

$$\int_0^t (\frac{e^{-s}}{1+B_s^2})dB_s = e^{-t} \arctan B_t + \int_0^t e^{-s}(\arctan B_s + \frac{B_s}{(1+B_s^2)^2})ds \tag{28}$$

Problem 3

(a)

Since we are given

$$dX_t = \lambda(m - X_t)dt + \sigma\sqrt{X_t}dB_t \tag{29}$$

then

$$E(X_t) = X_0 + E\int_0^t \lambda(m - X_s)ds + E\int_0^t \sqrt{X_s}dB_s = X_0 + \lambda mt - \lambda\int_0^t E(X_s)ds \tag{30}$$

where we used Fubini's theorem. Then we let $f(t) = E(X_t)$, we have

$$f(t) = X_0 + \lambda mt - \lambda\int_0^t f(s)ds \tag{31}$$

in differential form,

$$\lambda f(t) + f'(t) = \lambda m \tag{32}$$

To solve this first order differential equation, we multiply $e^{\lambda t}$ to both sides, we are able to derive

$$(e^{\lambda t}f(t))' = \lambda m e^{\lambda t} \tag{33}$$

$$e^{\lambda t}f(t) = me^{\lambda t} + c \tag{34}$$

We solved as

$$E(X_t) = f(t) = m - ce^{-\lambda t} \tag{35}$$

With the condition of $X_0 = x$, we can derive c and the equation above becomes

$$E(X_t) = m - (m - x)e^{-\lambda t} \tag{36}$$

(b)

Our problem is to derive $E(X_t^2)$. So

$$d(X_t^2) = d(\phi(X_t)) \tag{37}$$

where $\phi(x) = x^2$, then

$$\phi_t = 0, \phi_x = 2x, \phi_{xx} = 2 \tag{38}$$

which makes $d(X_t^2)$ become

$$\begin{aligned} d(X_t^2) &= 0 + 2X_t dX_t + \frac{1}{2}\cdot 2d[X]_t \\ &= 2X_t dX_t + (\lambda(m - X_t)dt + \sigma\sqrt{X_t}dB_t)^2 \\ &= 2X_t dX_t + \sigma^2 X_t dt \\ &= (2\lambda X_t(m - X_t) + \sigma^2 X_t)dt + 2\sigma X_t^{3/2}dB_t \end{aligned} \tag{39}$$

In integration form,

$$\begin{aligned} \int_0^t d(X_t^2) = X_t^2 - X_0^2 &= \int_0^t (2\lambda X_t(m - X_t) + \sigma^2 X_t)dt + \int_0^t 2\sigma X_t^{3/2}dB_t \\ E(X_t^2) &= X_0^2 + \int_0^t (2\lambda m + \sigma^2)E(X_s) - 2\lambda E(X_s^2)ds \end{aligned} \tag{40}$$

Problem 3 [(b)] continued on next page. . .

In the same way as part(a), let $g(t) = E(X_t^2)$, we have differential equation

$$g'(t) = (2\lambda m + \sigma^2)f(t) - 2\lambda g(t) \tag{41}$$

Solving it will generate

$$e^{2\lambda t}g(t) = \int e^{2\lambda s}(2\lambda m + \sigma^2)f(s)ds + C \tag{42}$$

By solving $g(t)$ and the constant, we are able to derive

$$\begin{aligned} Var(X_t) &= E(X_t^2) - [E(X_t)]^2 \\ &= \frac{\sigma^2}{\lambda}X_0(e^{-\lambda t} - e^{-2\lambda t}) + \frac{m\sigma^2}{2\lambda}(1 - 2e^{-\lambda t} + e^{-2\lambda t}) \end{aligned} \tag{43}$$

Problem 4

According to the hint, since

$$\frac{1}{1-x^2} = \frac{1}{2}(\frac{1}{1+x} + \frac{1}{1-x}) \tag{44}$$

Then its antiderivative is

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = f(x) \tag{45}$$

therefore,

$$f_t = 0 \tag{46}$$

$$f_x = \frac{1}{1-x^2} \tag{47}$$

$$f_{xx} = \frac{2x}{(1-x^2)^2} \tag{48}$$

Write the original SDE in the problem as

$$\frac{dX_t}{1-X_t^2} = -\beta^2 X_t dt + \beta dB_t \tag{49}$$

its left part turns to be

$$df(X_t) = \frac{1}{1-X_t^2}dX_t + \frac{X_t}{(1-X_t^2)^2}d[X]_t \tag{50}$$

where $d[X]_t = \beta^2(1-X_t^2)^2 dB_t dB_t = \beta^2(1-X_t^2)^2 dt$. Then

$$df(X_t) = -\beta^2 X_t dt + \beta dB_t + \frac{X_t}{(1-X_t^2)^2}\beta^2(1-X_t^2)^2 dt \tag{51}$$

that is

$$df(X_t) = \beta dB_t \tag{52}$$

Write it in integration form, we have

$$\frac{1}{2}\ln\left|\frac{1+X_t}{1-X_t}\right| = \beta B_t + f(X_0) \tag{53}$$

i.e.

$$\frac{1+X_t}{1-X_t} = \exp[2(\beta B_t + f(X_0))] \tag{54}$$

The formula above is solved as

$$X_t = \frac{\exp[2(\beta B_t + f(X_0))] - 1}{\exp[2(\beta B_t + f(X_0))] + 1} \tag{55}$$

Problem 5

(a)

Since Brownian Motion itself is a Martingale,

$$E(B_\tau) = E(B_0) = 0 \tag{56}$$

We can also write $E(B_\tau)$ as

$$E(B_\tau) = E(B_\tau|\tau = \tau_1)P(\tau = \tau_1) + E(B_\tau|\tau = \tau_{-1})P(\tau = \tau_{-1}) = P(\tau = \tau_1) - P(\tau = \tau_{-1}) \tag{57}$$

resulting in

$$P(\tau = \tau_1) = P(\tau = \tau_{-1}) = \frac{1}{2} \tag{58}$$

(b)

Since the Brownian martingale is a martingale, then

$$E(X_\tau) = E(X_0) = 1 \tag{59}$$

where $X_\tau = e^{\theta B_\tau - \frac{\theta^2}{2}\tau}$.

(c)

Continue to solve the identity given, we figure out $B_1 = 1$ when $\tau = \tau_1$ and $B_1 = -1$ when $\tau = \tau_{-1}$,

$$1 = E(X_\tau) = \frac{1}{2}\left[E[e^{\theta - \frac{\theta^2}{2}\tau}|\tau = \tau_1] + E[e^{\theta - \frac{\theta^2}{2}\tau}|\tau = \tau_{-1}]\right] \tag{60}$$

Consider employing a similar identity but replacing θ by $-\theta$, we have following two fomulas

$$2 = e^{\theta}E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_1] + e^{-\theta}E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_{-1}] \tag{61}$$

$$2 = e^{-\theta}E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_1] + e^{\theta}E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_{-1}] \tag{62}$$

which can be solved as

$$E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_1] = E[e^{-\frac{\theta^2}{2}\tau}|\tau = \tau_{-1}] = \frac{2}{e^\theta + e^{-\theta}} \tag{63}$$

(d)

All the elements of the formula are calculated in former parts, therefore

$$E[e^{-\frac{\theta^2}{2}\tau}] = \frac{2}{e^\theta + e^{-\theta}} \cdot \frac{1}{2} + \frac{2}{e^\theta + e^{-\theta}} \cdot \frac{1}{2} = \frac{2}{e^\theta + e^{-\theta}} \tag{64}$$

(e)

Replace $-\frac{\theta^2}{2}$ by λ in part(d), then $\theta = \sqrt{2|\lambda|}$, we obtain

$$E[e^{\lambda\tau}] = \frac{2}{e^{\sqrt{2|\lambda|}} + e^{-\sqrt{2|\lambda|}}} = \frac{1}{\cosh\sqrt{2|\lambda|}} \tag{65}$$