9831 Real Analysis and Probability

Homework 3 - Baruch MFE, Fall 2014

Assigned: Sep-22-2014, **Due**: Sep-29-2014, 6:05pm

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and B_t a Brownian motion defined over Ω.

(1) Recall that the Brownian bridge X_t, for $t \in [0,1]$, is defined as $X_t = B_t - tB_1$.

(a) Show that, for any $0 < t_1 < t_2 < 1$, the joint distribution of X_{t_1} and X_{t_2} is the same as the joint distribution of B_{t_1} and B_{t_2} conditioned on $B_1 = 0$. In other words,

$$\mathbb{P}\left[X_{t_1} \leq x_1, X_{t_2} \leq x_2\right] = \mathbb{P}\left[B_{t_1} \leq x_1, B_{t_2} \leq x_2 | B_1 = 0\right].$$

Hint: Since both of pairs X_{t_1}, X_{t_2} and B_{t_1}, B_{t_2} (conditioned on $B_1 = 0$) are jointly normal distributed, the problem boils down to the determination of their expectations and covariance matrices.

(Note: Indeed, one can show that X_t has the same finite dimensional distributions as those of the Brownian motion B_t conditioned on $B_1 = 0$.)

(b) Determine the transition density of X_t, i.e., $p(s, y|t, x) = \mathbb{P}\left[X_s = y | X_t = x\right]$, for $0 < t < s < 1$.

(2) Derive the infinitesimal generator for the process X_t.

(a) $X_t = \sigma B_t + \mu t$, where μ, σ are constants and B_t is a standard Brownian motion.

(b) $X_t = B_{\tau(t)}$, where $\tau(t) = \int_0^t \theta(s) ds$, θ is positive. In other words, X_t is a time-changed Brownian motion.

(3) Let $m_t = \min_{0 \leq s \leq t} B_s$ be the running minimum of Brownian motion B_t. Compute the joint density for the random variables (B_t, m_t). Note that the pair of random

variables (B_t, m_t) is supported in the region $\{(B,m) \in \mathbb{R}^2 | m \leq 0, B \geq m\}$ in the (B,m)-plane.

(Hint: Alter reflection principle to calculate, for any pair of numbers m, B such that $m \leq 0$ and $B \geq m$, the probability $\mathbb{P}[m_t \leq m, B_t \geq B]$)

(4) Assume the underlying asset S follows the Bachelier model $S_t = S_0 + \sigma B_t$, $S_0 > 0$. A contingent claim pays $\varphi(S_T)$ at expiry T if the underlying never goes below $L > 0$ before expiry, otherwise it is worthless, i.e., a no-touch option. Assume interest rate is zero. Compute the price for the claims.

   (a) Binary/digital call struck at $K > L$, i.e., $\varphi(x) = \mathbb{1}_{[K,\infty)}(x)$.

   (b) Call option struck at $K > L$, i.e., $\varphi(x) = (x-K)^+$ and the corresponding put option.

   (c) Do we have put-call parity in this case? Explain your answer.

(5) Let $\sigma(t)$ and $r(t)$ be positive deterministic functions of t. Define

$$X_t = \int_0^t r(s)ds + \int_0^t \sigma(s)dB_s, \quad S_t = s_0 e^{X_t - \frac{1}{2}\int_0^t \sigma^2(s)ds}.$$

   (a) What are the distributions of X_t and S_t?

   (b) Assume S_t is the price of a stock under risk neutral measure, what is the price of a call on S_t struck at K, expired at T? Notice that the discount factor in this case is $e^{-\int_0^T r(t)dt}$.

(6) Calculate the stochastic integral $\int_0^t B_s^2 dB_s$ by definition.