MTH9831

HW9

Weiyi Chen

Zhenfeng Liang

Mo Shen

MTH 9831

(1) $\cdot N_t = \min\{n \geq 0; S_{n+1} > t\}$.

We need to solve $\mathbb{P}(N_t \leq n)$

$\mathbb{P}(N_t \leq n) = \mathbb{P}\{\min\{m \geq 0; S_{m+1} > t\} \leq n\}$

Let $B = \{m \geq 0, S_{m+1} > t\}$. $B \in [\alpha, +\infty])$ ~~fo~~

α is the lower bound of B, upper bound is infinity.

So: $\min\{m \geq 0; S_{m+1} > t\} \leq n$

is equivalent to $n \in B$. from which we know

$S_{n+1} > t$.

So, what we need to do now is to calculate

$\mathbb{P}(N_t \leq n) = \mathbb{P}(\min\{m \geq 0: S_{m+1} > t\} \leq n) = \mathbb{P}(n \in B)$

$= \mathbb{P}(S_{n+1} > t)$.

Recall: $S_{n+1} = \sum_{k=1}^{n+1} T_k \sim \Gamma(n+1, \lambda)$ where T_i's are ~~i.i.e~~ i.i.d

exponential random ~~ra~~ variable.

And, $f_x(x) \doteq \frac{\lambda^{n+1}}{\Gamma(n+1)} \cdot x^n e^{-\lambda x} \cdot \mathbb{1}_{[0,+\infty)}(x)$

~~So:~~

So:

$$\mathbb{P}(N_t \le n) = \mathbb{P}(S_{n+1} > t) = \int_t^{\infty} \frac{\lambda^{n+1}}{\Gamma(n+1)} x^n \cdot e^{-\lambda x} \cancel{\lambda} dx$$

$$= \frac{\lambda^{n+1}}{n!} \int_t^{\infty} x^n e^{-\lambda x} dx = \frac{\lambda^{n+1}}{n!} f_n(x)$$

where $f_n(x) = \int_t^{\infty} x^n e^{-\lambda x} dx$.

$$f_n(x) = -\frac{1}{\lambda}\int_t^{\infty} x^n de^{-\lambda x} = -\frac{1}{\lambda}\left[x^n e^{-\lambda x}\Big|_t^{\infty} - n\int_t^{\infty} e^{-\lambda x} \cdot x^{n-1} dx\right]$$

$$= \frac{1}{\lambda} t^n e^{-\lambda t} + \frac{1}{\lambda} n f_{n-1}(x)$$

So. $\mathbb{P}(N_t \le n) = \frac{\lambda^{n+1}}{n!}\left(\frac{1}{\lambda} t^n e^{-\lambda t} + \frac{1}{\lambda} n f_{n-1}(x)\right)$ ~~=λ~~

$$\mathbb{P}(N_t \le n-1) = \frac{\lambda^n}{(n-1)!} f_{n-1}(x)$$

$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t \le n) - \mathbb{P}(N_t \le n-1)$$

$$= \frac{\lambda^n}{n!}\left(t^n e^{-\lambda t} + n f_{n-1}(x)\right) - \frac{\lambda^n}{(n-1)!} f_{n-1}(x)$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

which is the possion ~~r~~ random variable's PMF.

(2) We firist find the charastic function of Possion Process.

$$E[e^{i\xi X}] = \sum_{k=0}^{\infty} e^{i\xi X}\mathbb{P}(X=k) = \sum_{k=0}^{\infty} e^{i\xi k}\cdot\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

$$= e^{-\lambda t}\cdot\sum_{k=0}^{\infty}\frac{(e^{i\xi}\cdot\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\cdot e^{e^{i\xi}\cdot\lambda t} = e^{\lambda t(e^{i\xi}-1)}$$

$$Q_1(t) = \sum_{j=1}^{N_t^1} Y_{1j} = \sum_{j=1}^{N_t^1}\cdot y_1 = N_t^1\cdot y_1$$

$$Q_2(t) = N_t^2\cdot y_2 \quad \cdots$$

$$Q_3(t) = N_t^3\cdot y_3$$

$$\vdots$$

$$Q_m(t) = N_t^m\cdot y_m$$

So. $Q(t) = \sum_{i=1}^{m} Q_i(t) = \sum_{i=1}^{m} y_i\cdot N_t^i$

$$E[e^{i\xi Q(t)}] = E[e^{i\xi\cdot\sum_{i=1}^{m} y_i N_t^i}] = \prod_{i=1}^{m} E[e^{i\xi\cdot y_i N_t^i}]$$

$$= \prod_{i=1}^{m} e^{\lambda_i t(e^{i\xi y_i}-1)} = e^{\sum_{i=1}^{m}\lambda_i t(e^{i\xi y_i}-1)}$$

Let $\lambda = \sum_{i=1}^{m}\lambda_i$

So. $E[e^{i\xi Q(t)}] = e^{\lambda t\sum_{i=1}^{m}\frac{\lambda_i}{\lambda}(e^{i\xi y_i}-1)} = e^{\lambda t\cdot(\sum_{i=1}^{m}\frac{\lambda_i}{\lambda}\cdot e^{i\xi y_i} - \sum_{i=1}^{m}\frac{\lambda_i}{\lambda})}$

$$= e^{\lambda t\cdot(\sum_{i=1}^{m}\frac{\lambda_i}{\lambda}e^{i\xi y_i} - 1)}$$

So. we ~~know~~ complete the proof Q_t is a compound Poisson process with intensity $\lambda = \sum_{i=1}^{m}\lambda_i$. and jump size distribution Y given by

$$\mathbb{P}[Y = y_i] = \frac{\lambda_i}{\lambda} = \frac{\lambda}{\sum_{i=1}^{m}\lambda_i} \quad i = 1, 2 \cdots m$$

(3).

(a).

Method 1: (Definition)

$$\int_0^t N_{s-}dN_s = \sum_{0<s\le t} N_{s-}\cdot\Delta N_s = \cancel{N_s}\, N_{S_1-}\cdot\Delta N_{S_1} + N_{S_2-}\Delta N_{S_2} + \cdots + N_{S_{N_t}-}\cdot\Delta N_{S_{N_t}}$$

(since $\Delta N_{S_i} = 1$)

So: $\int_0^t N_{s-}dN_s = N_{S_1-} + N_{S_2-} + \cdots + N_{S_{N_t}-}$

$$= 0 + 1 + \cdots + (N_t - 1) = \frac{(1+N_t-1)}{2}(N_t-1) = \frac{N_t(N_t-1)}{2}$$

Method 2: (Apply Itô)

Recall for jump process, Itô formula is:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)d[X^c]_s + \sum_{0<s\le t}(f(X_s) - f(X_{s-}))$$

In this case $X_t = N_t$, $f(x) = x^2$, $X^c = 0$.

So: $$N_t^2 - N_0^2 = \sum_{0<s\le t}(N_s^2 - N_{s-}^2) = \sum_{0<s\le t}((N_{s-}+\Delta N_s)^2 - N_{s-}^2)$$

$$= \sum_{0<s\le t}(2N_{s-}\cdot\Delta N_s + \Delta N_s^2 \cancel{-N})$$

$$= \sum_{0<s\le t}(2N_{s-} - \Delta N_s)\cdot\Delta N_s$$

$$= 2\sum_{0<s\le t} N_{s-}\Delta N_s - \sum_{0<s\le t}\Delta N_s \qquad (\text{Since } \Delta N_s^2 = \Delta N_s)$$

$$= 2\int_0^t N_{s-}dN_s - N_t$$

So: $$\int_0^t N_{s-}dN_s = \frac{N_t^2 - N_t}{2} = \frac{N_t(N_t-1)}{2}$$

which is the same as the result gotten by definition (b). $f(x) = x^3$, $X = M_t = N_t - \lambda t$ $\quad M_t^c = -\lambda t$ $\quad$ ~~dE~~

$dM_t^c = -\lambda dt \quad d[M^c]_t = 0$.

So:

$$M_t^3 - M_0^3 = 3\int_0^t M_{s-}^2 \cdot dM_s^c + 3\int_0^t M_s \, d[M^c]_s + \sum_{0<s\le t} (M_s^3 - M_{s-}^3)$$

$$= -3\lambda \int_0^t M_{s-}^2 \cdot dt + \sum_{0<s\le t} (M_s^3 - M_{s-}^3)$$

(Because $N_s = N_{s-} + \Delta N_s$. $M_s = N_s - \lambda S = N_{s-} - \lambda S_- + \Delta N_s$

$M_s = M_{s-} + \Delta N_s$)

We get $M_t^3 - M_0^3 = -3\lambda \int_0^t (M_{s-} + \Delta N_s)^2 \, ds + \sum_{0<s\le t} ((M_{s-} + \Delta N_s)^3 - M_{s-}^3)$

$$= -3\lambda \int_0^t M_{s-}^2 + 2M_{s-}\Delta N_s + \Delta N_s^2 \cdot ds + \sum_{0<s\le t} (3M_{s-}^2 \Delta N_s + 3 \cdot M_{s-} \Delta N_s^2 + \Delta N_s^3)$$

And also. $\Delta N_s^3 = \Delta N_s^2 = \Delta N_s \quad \int_0^t \Delta N_s ds = 0$.

So: $M_t^3 - M_0^3 = -3\lambda \int_0^t M_{s-}^2 ds + 3\sum_{0<s\le t} M_{s-}^2 \Delta N_s + 3\sum_{0<s\le t} M_{s-} \cdot \Delta N_s + \sum_{0<s\le t} \Delta N_s$

$$= 3\left(-\lambda \int_0^t M_{s-}^2 ds + 3\int_0^t M_{s-}^2 \cdot dN_s\right) + 3\sum_{0<s\le t} M_s \cdot \Delta N_s + N_t$$

$$= 3\int_0^t M_{s-}^2 dM_s + 3\sum_{0<s\le t} M_{s-} \Delta N_s + N_t$$

We need to calculate $\sum_{0<s\le t} M_{s-} \Delta N_s$.

$\sum_{0<s\le t} M_{s-} \Delta N_s = \sum_{0<s\le t} \Delta N_s (N_{s-} - \lambda S^-) = \sum_{0<s\le t} N_{s-} \Delta N_s - \lambda \sum_{0<s\le t} S \cdot \Delta N_s$

We know from part (a) $\sum_{0<s\le t} N_{s-} \Delta N_s = \frac{N_t (N_t - 1)}{2}$

We need to calculate $\sum_{0<s\le t} S \cdot \Delta N_s$ now.

From note, we calculate $\int_0^t M_{s-} dM_s = \frac{1}{2}(M_t^2 - N_t)$

So. $\int_0^t M_{s-} dM_s = \int_0^t (N_{s-} - \lambda s) dN_s - \lambda \int_0^t (N_{s-} - \lambda s) ds$

$= \int_0^t N_{s-} dN_s - \lambda \int_0^t s dN_s + \lambda^2 \int_0^t s \cdot ds - \lambda \int_0^t N_{s-} ds$

$= \frac{N_t(N_t - 1)}{2} + \lambda^2 \cdot \frac{t^2}{2} - \lambda \int_0^t N_{s-} ds - \lambda \int_0^t s \cdot dN_s$

So $\cancel{\lambda \int_0}$ $-\lambda \sum_{0<s\le t} s \cdot \Delta N_s = -\lambda \int_0^t s \cdot dN_s = \frac{M_t^2 - N_t}{2} + \lambda \int_0^t N_s ds - \frac{\lambda^2 t^2}{2} - \frac{N_t(N_t - 1)}{2}$

So: $\sum_{0<s\le t} M_{s-} \Delta N_s = \frac{M_t^2 - N_t}{2} + \lambda \int_0^t N_s ds - \frac{(\lambda t)^2}{2}$

So. $\int_0^t M_{s-}^2 dM_s = \frac{M_t^3 - M_0^3}{3} - \sum_{0<s\le t} M_s \Delta N_s - \frac{1}{3} N_t$

$= \frac{M_t^3}{3} - \frac{M_t^2 - N_t}{2} - \lambda \int_0^t N_s ds + \frac{(\lambda t)^2}{2} - \frac{1}{3} N_t$

$= \frac{M_t^3}{3} - \frac{M_t^2}{2} - \frac{5}{6} N_t - \lambda \int_0^t N_s ds + \frac{(\lambda t)^2}{2}$

(4) $S_t = S_0 e^{-\lambda\sigma t}(1+\sigma)^{N_t}$

Apply Itô: $(N_s^c = 0)$

$$S_t - S_0 = -\lambda\sigma\int_0^t S_0 e^{-\lambda\sigma s}(1+\sigma)^{N_s}ds + \sum_{0<s\le t}(S_s - S_{s-})$$

$$= -\lambda\sigma\int_0^t S_s\,ds + \sum_{0<s\le t}(S_s - S_{s-})$$

In the continuous integral. $-\lambda\sigma\int_0^t S_s\,ds = -\lambda\sigma\int_0^t S_{s-}ds$

So. $S_t = S_0 - \lambda\sigma\int_0^t S_{\tau-}d\tau + \sum_{0<s\le t}(S_s - S_{s-})$.

Let's calculate $S_s - S_{s-}$.

$$S_s - S_{s-} = S_0 e^{-\lambda\sigma s}(1+\sigma)^{N_s} - S_0 e^{-\lambda\sigma s-}(1+\sigma)^{N_{s-}}$$

$$= S_0 e^{-\lambda\sigma s}\left((1+\sigma)^{(N_{s-}+\Delta N_s)} - (1+\sigma)^{N_{s-}}\right)$$

$$= S_0 e^{-\lambda\sigma s}\cdot(1+\sigma)^{N_{s-}}\left((1+\sigma)^{\Delta N_s} - 1\right)$$

$$= S_0 e^{-\lambda\sigma s}(1+\sigma)^{N_{s-}}\left((1+\sigma)^1 - 1\right)\Delta N_s.$$

$$= \sigma S_0 e^{-\lambda\sigma s}(1+\sigma)^{N_{s-}}\cdot\Delta N_s$$

$$\sum_{0<s\le t}(S_s - S_{s-}) = \sigma\sum_{0<s\le t} S_{s-}\Delta N_s.$$

So:

$$S_t = S_0 - \lambda\sigma\int_0^t S_{\tau-}d\tau + \sigma\sum_{0<\tau\le t} S_{\tau-}\Delta N_\tau$$

which is what we need to prove.

(5) $Z_t = Z_0 e^{(\lambda-\check{\lambda})t}\left(\frac{\check{\lambda}}{\lambda}\right)^{N_t}$.

Apply Itô.

$$Z_t - Z_0 = \int_0^t (\lambda-\check{\lambda})\cdot Z_0 e^{(\lambda-\check{\lambda})\tau}\left(\frac{\check{\lambda}}{\lambda}\right)^{N_\tau} d\tau + \sum_{0<\tau\le t}(Z_\tau - Z_{\tau-})$$

$$= \cancel{(\lambda-\check{\lambda})}\cdot \frac{\check{\lambda}-\lambda}{\lambda}\int_0^t Z_{\tau-}\, d(\lambda\tau) + \sum_{0<\tau\le t}(Z_\tau - Z_{\tau-})$$

$$Z_\tau - Z_{\tau-} = Z_0 e^{(\lambda-\check{\lambda})\tau}\left(\frac{\check{\lambda}}{\lambda}\right)^{N_\tau} - Z_0 e^{(\lambda-\check{\lambda})\tau}\left(\frac{\check{\lambda}}{\lambda}\right)^{N_{\tau-}}$$

$$= Z_0 e^{(\lambda-\check{\lambda})\tau}\left(\frac{\check{\lambda}}{\lambda}\right)^{N_{\tau-}}\left(\left(\frac{\check{\lambda}}{\lambda}\right)^{\Delta N_\tau} - 1\right)$$

$$\cancel{= Z_0 e^{(\lambda-\check{\lambda})\tau}}$$

$$= Z_{\tau-}\cdot\left(\frac{\check{\lambda}-\lambda}{\lambda}\right)\cdot \Delta N_\tau$$

So. $\sum_{0<\tau\le t}(Z_\tau - Z_{\tau-}) = \frac{\check{\lambda}-\lambda}{\lambda}\sum_{0<\tau\le t} Z_{\tau-}\Delta N_\tau = \frac{\check{\lambda}-\lambda}{\lambda}\int_0^t Z_{\tau-}\, dN_\tau$.

So: $Z_t - Z_0 = \frac{\check{\lambda}-\lambda}{\lambda}\int_0^t Z_{\tau-}\, d(N_\tau - \lambda\tau)$

$$= \frac{\check{\lambda}-\lambda}{\lambda}\int_0^t Z_{\tau-}\, dM_\tau$$

So $dZ_t = \frac{\check{\lambda}-\lambda}{\lambda} Z_{t-}\, dM_t$

Because $M_t = N_t - \lambda t$ is a compensated Possion process. i.e, a, m.g and Z_{t-} is predictable, Z_t is a martingale