MTH 9831: Homework 2

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Problem 1

'Only if' part:

According to the tower rule,

$$E\left[\sum_{t=1}^{N} H_t \Delta X_t\right] = E\left[E\left[\sum_{t=1}^{N} H_t \Delta X_t | \mathcal{F}_{t-1}\right]\right] \tag{1}$$

according to linearity of expectation,

$$E\left[E\left[\sum_{t=1}^{N} H_t \Delta X_t | \mathcal{F}_{t-1}\right]\right] = E\left[\sum_{t=1}^{N} E\left[H_t \Delta X_t | \mathcal{F}_{t-1}\right]\right] \tag{2}$$

since H_t is predictable,

$$E\left[\sum_{t=1}^{N} E\left[H_t \Delta X_t | \mathcal{F}_{t-1}\right]\right] = E\left[\sum_{t=1}^{N} H_t E\left[X_t - X_{t-1} | \mathcal{F}_{t-1}\right]\right] \tag{3}$$

since X_t is an adapted stochastic process

$$E\left[\sum_{t=1}^{N} H_t E\left[X_t - X_{t-1} | \mathcal{F}_{t-1}\right]\right] = E\left[\sum_{t=1}^{N} H_t (X_{t-1} - X_{t-1})\right] = 0 \tag{4}$$

'If' part:

Pick an event $A_s \in \mathcal{F}_{s-1}$ for A where $s \in \{1, 2, \ldots, N\}$, then let

$$H_t = \begin{cases} 0, & \text{if } s \neq t \\ 1_{A_s}, & \text{if } s = t \end{cases} \tag{5}$$

then

$$\sum_{t=1}^{N} H_t \Delta X_t = 1_{A_s}(X_s - X_{s-1}) \tag{6}$$

We know $E(1_{A_s}(X_s - X_{s-1})) = 0$, therefore

$$E(1_{A_s} X_s) = E(1_{A_s} X_{s-1}) \tag{7}$$

that is,

$$\int_{A_s} X_s dP = \int_{A_s} X_{s-1} dP \tag{8}$$

Since $A_s \in \mathcal{F}_{s-1}$ is arbitrary, then

$$E(X_s | \mathcal{F}_{s-1}) = E(X_{s-1} | F_{s-1}) = X_{s-1} \tag{9}$$

s is also arbitrage, therefore X_t is a martingale.

Problem 2

(a)

According to the definition,

$$\{S \wedge T \le t\} = \{S \le t \text{ or } T \le t\} = \{S \le t\} \cup \{T \le t\} \in \mathcal{F}_t \tag{10}$$

so $S \wedge T$ is stopping time.
Similarly,

$$\{S \vee T \le t\} = \{S \le t \text{ and } T \le t\} = \{S \le t\} \cap \{T \le t\} \in \mathcal{F}_t \tag{11}$$

so $S \vee T$ is stopping time.

(b)

For each fixed ω, I claim $S(\omega) + T(\omega) < t$ iff there are positive rationals p, q with $p + q \le t$ and $S(\omega) \le p, T(\omega) \le q$. Suppose $S(\omega) + T(\omega) \le t$; we can find a rational r with $S(\omega) + T(\omega) \le r \le t$. Then $S(\omega) \le r - T(\omega)$, so we can find p with $S(\omega) \le p \le r - T(\omega)$. Setting $q = r - p$ we see that we have $T(\omega) \le q$. The reverse implication is obvious.
Thus we have,

$$\{S + T \le t\} = \bigcup_{p,q \in Q^+, p+q \le t}^{t} (\{S \le p\}) \cap \{T \le q\}) \tag{12}$$

Since $\{S \le p\} \in \mathcal{F}_p \subseteq F_t$ and $\{T \le q\} \in \mathcal{F}_q \subseteq F_t$. Thus $\{S + T < t\}$ is a countable union of events from $\mathcal{F}_t$, and so it itself in $\mathcal{F}_t$.
Similarly for $S \cdot T$, we have (only in decrete case)

$$\{S \cdot T \le t\} = \bigcup_{p,q \in Q^+, pq \le t}^{t} (\{S \le p\}) \cap \{T \le q\}) \tag{13}$$

Same reason $\{S \cdot T < t\}$ is a countable union of events from $\mathcal{F}_t$, and so it itself in $\mathcal{F}_t$.

Problem 3

(a)

Since $X_0 < a$ and $\tau_a = \inf\{t \ge 0 : X_t \ge a\}$, then

$$\{\tau_a = n\} = \{X_0, X_1, \ldots, X_{n-1} < a, X_n \ge a\} = \bigcap_{j=0}^{n-1} \{X_j < a\} \cap \{X_n \ge a\} \tag{14}$$

by 2(a), it is a countable intersection of events from $\mathcal{F}_t$, and so it itself in $\mathcal{F}_t$.

(b)

Since $X_0 > b$ and $\tau_b = \inf\{t \ge 0 : X_t \le b\}$, then

$$\{\tau_b = n\} = \{X_0, X_1, \ldots, X_{n-1} > b, X_n \le b\} = \bigcap_{j=0}^{n-1} \{X_j > b\} \cap \{X_n \le b\} \tag{15}$$

by 2(a), it is a countable intersection of events from $\mathcal{F}_t$, and so it itself in $\mathcal{F}_t$.

Problem 4

(a)

$X_t = -B_t$ is a Brownian motion by the following facts:

- $X_0 = -B_0 = 0$
- X_t is almost surely continuous since B_t is almost surely continuous.
- X_t has independent increments since B_t has independent increments.
- $X_t - X_s = B_s - B_t \sim N(0, s-t)$ since $B_t - X_s \sim N(0, t-s)$.

(b)

$Y_t = \frac{1}{c}B_{c^2t}$ is a Brownian motion by the following facts:

- Clearly $Y_0 = \frac{1}{c}B_0 = 0$
- If $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, then $0 \leq c^2 t_1 \leq \cdots \leq c^2 t_n$. By the definition of Brownian motion,

$$B(c^2t_n) - B(c^2t_{n-1}), \ldots, B(c^2t_2) - B(c^2t_1) \tag{16}$$

  are independent random variables. It then follows that

$$\frac{1}{c}B(c^2t_n) - \frac{1}{c}B(c^2t_{n-1}), \ldots, \frac{1}{c}B(c^2t_2) - \frac{1}{c}B(c^2t_1) \tag{17}$$

  are independent random variables, i.e. the stochastic process $\{Y_t | 0 \leq t\}$ has independent increments.

- The fact that $Y_{t+h} - Y_t$ is normally distributed follows immediately from the fact that $B(c^2t + c^2h) - B(c^2t)$ is normally distributed. Furthermore,

$$E[Y_{t+h} - Y_t] = E[\frac{1}{c}B(c^2t + c^2h) - \frac{1}{c}B(c^2t)] = \frac{1}{c}E[B(c^2t + c^2h) - B(c^2t)] = 0 \tag{18}$$

  where the last equality follows from the definition of Brownian Motion. To show that the variance equals h, observe that

$$var[Y_{t+h} - Y_t] = var[\frac{1}{c}B(c^2t + c^2h) - \frac{1}{c}B(c^2t)] = \frac{1}{c^2}[B(c^2t + c^2h) - B(c^2t)] = \frac{1}{c^2}c^2h = h \tag{19}$$

- Because the function $t \to B(t)$ is almost surely continuous, the function $t \to Y_t = \frac{1}{c}B(c^2h)$ is the composition of (almost surely) continuous functions and is therefore almost surely continuous.

(c)

$Z_t = tB_{1/t}$ is a Brownian motion by the following facts:

- $Z_0 = 0$ given
- For Brownian motions,

$$Cov(B_t, B_{t+s}) = Cov(B_t, B_{t+s}B_t) + Cov(B_t, B_t) = t \tag{20}$$

  for all $t, s \geq 0$. For our process Z_t, we compute the Covariance function for $s < t$,

$$Cov[Z_t, Z_{t+s}] = Cov[tB_{1/t}, (t+s)B(t_{1/(t+s)}))] = t(t+s)Cov(B_{1/t}, B_{1/(t+s)}) = t \tag{21}$$

So

$$Cov[Z_t, Z_{t+s} - Z_t] = t - t = 0 \tag{22}$$

Because the random variables Z_{t+s} and Z_t are normal, $Cov(Z_t, Z_{t+s}Z_t) = 0$ implies that $Z_{t+s} - Z_t$ and Z_t are independent.

- And

$$Var(Z_{t+s} - Z_t) = Var(Z_{t+s}) + Var(Z_t) - 2Cov(Z_{t+s}, Z_t) = (t+s) + t - 2t = s \tag{23}$$

so our increments are independent and have the right variances.

- Continuity is clear for $t > 0$. We know that Z_t has the distribution of a Brownian motion on Q, so

$$0 = \lim_{n\to\infty} Z(\frac{1}{n}) = \lim_{t\to 0} Z_t \tag{24}$$

and we conclude that Z_t is continuous at $t = 0$, so Z_t satisfies the properties of a standard Brownian motion.

Problem 5

(1)

To prove Z_t is a martingale, we need to show

- Integrability:

$$E|Z_t| = E|e^{\sigma B_t - \frac{\sigma^2 t}{2}}| \leq E|e^{\sigma B_t}|E|e^{-\frac{\sigma^2 t}{2}}| \leq ME|e^{-\frac{\sigma^2 t}{2}}| < \infty \tag{25}$$

- Adapted: Since B_t and t are adapted, so is their combination Z_t.

- The conditional expected value of the next generation:

$$E(Z_t|\mathcal{F}_s) = E(e^{\sigma(B_t - B_s)}e^{\sigma B_s}|\mathcal{F}_s)E(e^{-\frac{\sigma^2(t-s)}{2}}e^{-\frac{\sigma^2 s}{2}}|\mathcal{F}_s) \tag{26}$$

since $B_t - B_s$ is independent of $\mathcal{F}_s$ and B_s is measurable by $\mathcal{F}_s$, the equation above becomes

$$e^{\sigma B_s}E(e^{\sigma(B_t - B_s)})e^{-\frac{\sigma^2(t-s)}{2}}e^{-\frac{\sigma^2 s}{2}} \tag{27}$$

Since $B_t - B_s \sim N(0, t-s)$, let $B_t - B_s = x$, we can show that

$$\begin{aligned} E(e^{\sigma x}) &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} e^{\sigma x}e^{-\frac{x^2}{2(t-s)}} \\ &= e^{\frac{\sigma^2(t-s)}{2}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-\sigma(t-s))^2}{2(t-s)}} \\ &= e^{\frac{\sigma^2(t-s)}{2}} \end{aligned} \tag{28}$$

Therefore,

$$E(Z_t|\mathcal{F}_s) = e^{\sigma B_s}e^{\frac{\sigma^2(t-s)}{2}}e^{-\frac{\sigma^2(t-s)}{2}}e^{-\frac{\sigma^2 s}{2}} = e^{\sigma B_s - \frac{\sigma^2 s}{2}} = Z_s \tag{29}$$

Z_t is a martingale.

Problem 5 continued on next page...

(2)

To prove X_t is a martingale, we need to show

- Integrability:
$$E|X_t| = E|B_t^2 - t| \le E|B_t^2| + E(t) < \infty \tag{30}$$
- Adapted: since B_t is adapted then so is X_t.
- The conditional expected value of the next generation:
$$\begin{aligned} E[B_t^2 - t|\mathcal{F}_s] &= E[((B_t - B_s) + B_s)]^2 - ((t-s)+s)|F_s) \\ &= E[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2|F_s] - [(t-s)+s] \end{aligned} \tag{31}$$
since $B_t - B_s$ is independent of $\mathcal{F}_s$ and B_s is measurable by $\mathcal{F}_s$, the equation above becomes
$$\begin{aligned} E[B_t^2 - t|\mathcal{F}_s] &= E[(B_t - B_s)^2) + 2B_s E(B_t - B_s) + B_s^2 - [(t-s)+s] \\ &= [t-s] + B_s^2 - [(t-s)+s] \\ &= B_s^2 - s \\ &= X_s \end{aligned} \tag{32}$$

X_t is a martingale.

Problem 6

Since f is of finite quadratic variation, then

$$V_f^2(T) < \infty \tag{33}$$

g is of finite variation and continuous, then

$$V_g(T) < \infty \tag{34}$$

Furthermore,

$$V_f^p(T) = \lim_{||\pi||\to 0} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|^p < \infty \tag{35}$$

where

$$||\pi_n|| = \max_{j=1,\ldots,n} \{t_j - t_{j-1}\}, \pi_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = T\} \tag{36}$$

The proof that the covariation of continuous finite variation process and finite quadratic variation is zero follows from the following inequality. Here, π_n is a partition of the interval [0,t], and $V_t(X)$ is the variation of X over $[0,t]$. Using Cauchy-Schwarz inequality,

$$\begin{aligned} [f,g](T) &= \lim_{||\pi||\to 0} \sum_{j=1}^{n} \left(f(X_{t_j}) - f(X_{t_{j-1}})\right)\left(g(X_{t_j}) - g(X_{t_{j-1}})\right) \\ &\le \lim_{||\pi||\to 0} \sqrt{\sum_{j=1}^{n} \left(f(X_{t_j}) - f(X_{t_{j-1}})\right)^2 \sum_{j=1}^{n} \left(g(X_{t_j}) - g(X_{t_{j-1}})\right)^2} \\ &\le \lim_{||\pi||\to 0} \sqrt{\sum_{j=1}^{n} \left(f(X_{t_j}) - f(X_{t_{j-1}})\right)^2} \sqrt{\sum_{j=1}^{n} \left(g(X_{t_j}) - g(X_{t_{j-1}})\right)^2} \end{aligned} \tag{37}$$

By the continuity and finite variation of g,

$$\lim_{||\pi||\to 0} \sqrt{\sum_{j=1}^{n} \left(g(X_{t_j}) - g(X_{t_{j-1}})\right)^2} = 0 \tag{38}$$

and f is of finite quadratic variation,

$$\lim_{||\pi||\to 0} \sqrt{\sum_{j=1}^{n} \left(f(X_{t_j}) - f(X_{t_{j-1}})\right)^2} < \infty \tag{39}$$

So $[f,g](T)$ vanishes in the limit as $||\pi_n||$ goes to zero, i.e.,

$$[f,g](T) = \lim_{||\pi||\to 0} \sum_{j=1}^{n} \left(f(X_{t_j}) - f(X_{t_{j-1}})\right)\left(g(X_{t_j}) - g(X_{t_{j-1}})\right) = 0 \tag{40}$$

Problem 7

(a)

Using Ito's lemma, in general suppose we have $X_t = f(t, B_t)$, for some function $f(t,x) \in C^{1,2}$, then

$$dX_t = f_t dt + f_x dB_t + \frac{1}{2} f_{xx} d[B,B]_t = (f_t + \frac{1}{2} f_{xx})dt + f_x dB_t \tag{41}$$

In our case,

$$X_t = B_t^6 = f(t, B_t), i.e. f(t,x) = x^6 \tag{42}$$

then

$$f_t = 0, f_x = 6x^5, f_{xx} = 30x^4 \tag{43}$$

therefore with $X_t - X_0 = \int_0^t dX_s$ and $dX_t = 6B_t^5 dB_t + 15B_t^4 dt$

$$X_t - X_0 = 6\int_0^t B_s^5 dB_s(*) + 15\int_0^t B_s^4 ds \tag{44}$$

since B_s is Martingale. so (*) is also a martingale, which says it's 0.
Finally,

$$E(X_t) = 15\int (\int_0^t B_s^4 ds)dP = 15\int_0^t (\int B_s^4 dP)ds = 15\int_0^t E(B_s^4)ds = 15t^3 \tag{45}$$

where we use Fubini's theorem and the kurtosis $E(B_s^4) = 3s^2$. Similarly, we can apply to the other one

$$E(|X|^3) = \sigma^3 (2)!! \sqrt{\frac{2}{\pi}} = 2\sqrt{\frac{2}{\pi}} t^{3/2} \tag{46}$$

(b)

Given the expression of $|C_B^n|^2$, we have its expectation

$$E|C_B^n|^2 = E\left(\sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right|^6\right) + E\left(\sum_{i\neq j} \left|B_{t_i} - B_{t_{i-1}}\right|^3 \left|B_{t_j} - B_{t_{j-1}}\right|^3\right) \tag{47}$$

Since $B_{t_i} - B_{t_{i-1}}$ and $B_{t_j} - B_{t_{j-1}}$ is independent for all $i \neq j$, therefore,

$$E|C_B^n|^2 = \sum_{i=1}^{n} E\left(\left|B_{t_i} - B_{t_{i-1}}\right|^6\right) + \sum_{i\neq j} E\left|B_{t_i} - B_{t_{i-1}}\right|^3 E\left|B_{t_j} - B_{t_{j-1}}\right|^3 \tag{48}$$

since $B_{t_i} - B_{t_{i-1}} \sim B(0, t_i - t_{i-1})$ and using conclusion of (a),

$$E|C_B^n|^2 = 15\sum_{i=1}^{n}(t_i - t_{i-1})^3 + \frac{8}{\pi}\sum_{i\neq j}(t_i - t_{i-1})^{3/2}(t_j - t_{j-1})^{3/2} \tag{49}$$

(c)

Since

$$||\Pi_n|| = \max_{1\le i\le n}\{t_i - t_{i-1}\} \tag{50}$$

then

$$\begin{aligned}
E|C_B^n|^2 &\le 15||\Pi_n||^2\sum_{i=1}^{n}(t_i - t_{i-1}) + \frac{8}{\pi}||\Pi_n||\sum_{i\neq j}(t_i - t_{i-1})(t_j - t_{j-1}) \\
&\le 15||\Pi_n||^2\sum_{i=1}^{n}(t_i - t_{i-1}) + \frac{8}{\pi}||\Pi_n||\sum_{i}(t_i - t_{i-1})\sum_{j}(t_j - t_{j-1}) \\
&\le 15||\Pi_n||^2 T + \frac{8}{\pi}||\Pi_n||T^2
\end{aligned} \tag{51}$$

where obviously $T = \sum_{i=1}^{n}(t_i - t_{i-1})$, then

$$\lim_{||\Pi_n||\to 0} E|C_B^n|^2 \le 15||\Pi_n||^2 T + \frac{8}{\pi}||\Pi_n||T^2 = 0 \tag{52}$$

which implies that $\lim_{||\Pi_n||\to 0} E|C_B^n|^2 = 0$. We can conclude that the cubic variation of Brownian motion in $[0, T]$ is 0.

Problem 8

According to the given expression,

$$\frac{S_{t_i}}{S_{t_{i-1}}} = e^{\sigma(B_{t_i} - B_{t_{i-1}}) + (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1})} \tag{53}$$

Therefore,

$$\begin{aligned}
(\log\frac{S_{t_i}}{S_{t_{i-1}}})^2 &= \left[\sigma(B_{t_i} - B_{t_{i-1}}) + (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1})\right]^2 \\
&= \sigma^2(B_{t_i} - B_{t_{i-1}})^2 + 2\sigma(B_{t_i} - B_{t_{i-1}})(\mu - \frac{\sigma^2}{2})(t_i - t_{i-1}) + (\mu - \frac{\sigma^2}{2})^2(t_i - t_{i-1})^2
\end{aligned} \tag{54}$$

Furthermore it is easy to check that

$$\lim_{||\Pi_n||\to 0}(\log\frac{S_{t_i}}{S_{t_{i-1}}})^2 = \sigma^2[B, B](T) + 2\sigma(\mu - \frac{\sigma^2}{2})[B, t](T) + (\mu - \frac{\sigma^2}{2})^2[t, t](T) \tag{55}$$

given the previous expression. Since B is of finite quadratic variation and t is of finite variation and continuous. According to the conclusion of problem 6, we have

$$\lim_{||\Pi_n||\to 0}(\log\frac{S_{t_i}}{S_{t_{i-1}}})^2 = \sigma^2[B, B](T) = \sigma^2 T \tag{56}$$