MTH 9831: Homework 3

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Problem 1

(a)

Since both of pairs X_{t_1}, X_{t_2} and B_{t_1}, B_{t_2} (conditioned on $B_1 = 0$) are jointly normal distributed, the problem boils down to the determination of their expectations and covariance matrices. The expectation of X_t

$$E(X_t) = E(B_t) - E(tB_1) = 0 - t \cdot 0 = 0 \tag{1}$$

The expectation of B_t (conditioned on $B_1 = 0$) is

$$E(B_t|B_1 = 0) = E(tW_{1/t}|1 \cdot W_1 = 0) = tE(W_{1/t} - W_1|W_1 = 0) = 0 \tag{2}$$

The covariance matrix of X_{t_1}, X_{t_2} is

$$\begin{aligned} Cov(X_{t_1}, X_{t_2}) &= Cov(B_{t_1} - t_1B_1, B_{t_2} - t_2B_1) \\ &= E(B_{t_1}B_{t_2}) - t_1E(B_{t_2}B_1) - t_2E(B_{t_1}B_1) + t_1t_2E(B_1^2) \\ &= (t_1 \wedge t_2) - t_1(t_2 \wedge 1) - t_2(t_1 \wedge 1) + t_1t_2(1 \wedge 1) \\ &= t_1(1 - t_2) \end{aligned} \tag{3}$$

The covariance matrix of B_{t_1}, B_{t_2} is

$$\begin{aligned} Cov(B_{t_1}, B_{t_2}|B_1 = 0) &= E(B_{t_1}B_{t_2}|B_1 = 0) \\ &= t_1t_2E(W_{1/t_1}W_{1/t_2}|W_1 = 0) \\ &= t_1t_2E(W_{1/t_1-1}W_{1/t_2-1}) \\ &= t_1t_2(1/t_1 - 1 \wedge 1/t_2 - 1) \\ &= t_1(1 - t_2) \end{aligned} \tag{4}$$

Their expectations and covariance matrices are the same, so their joint distributions are the same.

(b)

The transition density of X_t is

$$P(X_s = y|X_t = x) = \frac{P(X_s = y, X_t = x)}{P(X_t = x)} = \frac{\frac{\partial}{\partial x\partial y}P(X_s \le y, X_t \le x)}{\frac{\partial}{\partial x}P(X_t \le x)} \tag{5}$$

Since the distribution of Brownian bridge is $X_t \sim N(0, t(1-t))$, and from last part we know $Cov(X_s, X_t) = t(1-s)$, then the correlation matrix is

$$\rho_{X_s,X_t} = \frac{Cov(X_s, X_t)}{\sigma_{X_s}\sigma_{X_t}} = \sqrt{\frac{t(1-s)}{s(1-t)}} \tag{6}$$

The joint distribution of X_s, X_t is

$$\begin{aligned} P[X_s \le y, X_t \le x] = &\frac{1}{2\pi s(1-s)t(1-t)}\frac{1}{\sqrt{1-\rho^2}} \\ &\int_{-\infty}^{x}\int_{-\infty}^{y} \exp\{\frac{1}{2(1-\rho^2)}[\frac{u^2}{s(1-s)} + \frac{v^2}{t(1-t)} + \frac{2\rho uv}{\sqrt{st(1-s(1-t))}}]\}dudv \end{aligned} \tag{7}$$

Put ρ into the formula and take the second derivatives, we have

$$\frac{\partial}{\partial x\partial y}P(X_s \le y, X_t \le x) = \frac{1}{2\pi\sqrt{t(1-s)(s-t)}}\exp\{\frac{-s(1-t)}{2(s-t)}[\frac{y^2}{s(1-s)} + \frac{x^2}{t(1-t)} + \frac{2xy}{s(1-t)}]\} \tag{8}$$

$$\frac{\partial}{\partial x}P(X_t \le x) = \frac{1}{\sqrt{2\pi t(1-t)}}\exp\{-\frac{x^2}{t(1-t)}\} \tag{9}$$

Problem 1 [(b)] continued on next page. . .

Therefore we derive

$$P(X_s = y|X_t = x) = \frac{\sqrt{1-t}}{\sqrt{2\pi(1-s)(s-t)}} \exp\{\frac{-(1-t)y^2}{2(s-t)(1-s)} - \frac{sx^2}{2t(s-t)} + \frac{xy}{s-t} - \frac{x^2}{t(1-t)}\} \tag{10}$$

Problem 2

(a)

The infinitesimal generator is

$$Lf(x) = \lim_{h\to 0} \frac{1}{h} E[f(X_{t+h})|X_t = a] - f(a) \tag{11}$$

where

$$E[f(X_{t+h})|X_t = a] = E[a + \sigma B_h + \mu h] = \int_{-\infty}^{\infty} \frac{exp[-\frac{(y-\mu h)^2}{2\sigma^2 h}]}{\sqrt{2\pi h\sigma^2}} f(a+y)dy \tag{12}$$

where $y = \sigma B_h + \mu h$ with $E(y) = \mu h$ and $Var(y) = \sigma^2 h$. We continue to let $y = \sigma\sqrt{h}$z, we have

$$E[f(X_{t+h})|X_t = a] = \int_{-\infty}^{\infty} \frac{\exp[-\frac{1}{2}(z - \frac{\mu}{\sigma}\sqrt{h})^2]}{\sqrt{2\pi}} f(a + \sigma\sqrt{h}z)dz \tag{13}$$

Using taylor expansion we derive

$$E[f(X_{t+h})|X_t = a] = f(a) + f'(a)\sigma\sqrt{h}\frac{\mu h}{\sigma} + \frac{\sigma^2 h}{2} f''(a)(1 + \frac{\mu^2 h}{\sigma^2}) + O(h^{3/2}) \tag{14}$$

Then

$$\begin{aligned} Lf(x) &= \lim_{h\to 0} \frac{1}{h} E[f'(a)\sigma\sqrt{h}\frac{\mu\sqrt{h}}{\sigma} + \frac{\sigma^2 h}{2} f''(a)(1 + \frac{\mu^2 h}{\sigma^2}) + O(h^{3/2})] \\ &= \mu f'(a) + \frac{\sigma^2}{2} f''(a) \end{aligned} \tag{15}$$

(b)

Following the same step in part(a), we have

$$E[f(X_{t+h})|X_t = a] = E[f(X_h + a)] \tag{16}$$

where

$$f(X_h + a) = f(a) + f'(a)X_h + \frac{1}{2} f''(a)X_h^2 + \dots \tag{17}$$

Then

$$\begin{aligned} E[f(X_h + a)] &= E[f(a) + f'(a)X_h + \frac{1}{2} f''(a)X_h^2 + \dots] \\ &= f(a) + f'(a)E(B_{\tau(h)}) + \frac{1}{2} f''(a)E(B^2_{\tau(h)}) \\ &= f(a) + \frac{1}{2}\tau(h) + O(h^2) \end{aligned} \tag{18}$$

The infinitesimal generator is

$$Lf(x) = \lim_{h\to 0} \frac{1}{2} f''(a) \frac{\int_0^h \theta(s)ds}{h} \tag{19}$$

Using L'Hopital rule, we derive

$$Lf(x) = \frac{1}{2} f''(a) \lim_{h\to 0} \frac{\theta(h)}{1} = \frac{1}{2} f''(a)\theta(0) \tag{20}$$

Problem 3

According to reflection principle, we have

$$P[m_t \le m] = 2P[B_t \le m] \tag{21}$$
$$P[m_t \le m, B_t \ge B] = P[B_t \le 2m - B] \tag{22}$$

Then

$$\begin{aligned} P[m_t \le m, B_t \le B] &= P[m_t \le m] - P[m_t \le m, B_t \ge B] \\ &= 2P[B_t \le m] - P[B_t \le 2m - B] \\ &= 2N(\frac{m}{\sqrt{t}}) - N(\frac{2m-B}{\sqrt{t}}) \end{aligned} \tag{23}$$

where $N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx$. Therefore,

$$\frac{\partial}{\partial B} P[m_t \le m, B_t \le B] = \frac{1}{\sqrt{t}} N'(\frac{2m-B}{\sqrt{t}}) \tag{24}$$

and

$$\frac{\partial}{\partial B \partial m} P[m_t \le m, B_t \le B] = \frac{2}{t} N''(\frac{2m-B}{\sqrt{t}}) \tag{25}$$

Problem 4

4. $m_t = \min_{0 \le s \le T} S_s$

(a) $\varphi(x) = \mathbb{1}_{[K,\infty)}(T)$. $V(T) = \begin{cases} 1 & \text{if } m_t \geq L, S_T \geq k. \\ 0 & \text{otherwise}. \end{cases}$

$$V(T) = \mathbb{P}[m_t \geq L, S_T \geq k]$$
$$= \mathbb{P}(S_T \geq k) - \mathbb{P}[m_t \leq L, S_T \geq k]$$
$$(\because \mathbb{P}[m_t = L, S_T \geq k] = 0)$$
$$= 1 - \mathbb{P}(S_T \leq k) - \mathbb{P}[m_t \leq L, S_T \geq k]$$
$(\because \mathbb{P}(S_T = k) = 0$, and Reflection principle II)
$$= 1 - \mathbb{P}(S_T \leq k) - \mathbb{P}(S_T \leq 2L - k]$$
$$= 1 - \mathbb{P}\left(B_T \leq \frac{k - S_0}{\sigma}\right) - \mathbb{P}\left(B_T \leq \frac{2L - k - S_0}{\sigma}\right)$$
$$= 1 - \mathbb{P}\left(Z \leq \frac{k - S_0}{\sigma\sqrt{T}}\right) - \mathbb{P}\left(Z \leq \frac{2L - k - S_0}{\sigma\sqrt{T}}\right)$$
$$= 1 - N\left(\frac{k - S_0}{\sigma\sqrt{T}}\right) - N\left(\frac{2L - k - S_0}{\sigma\sqrt{T}}\right)$$

where ~~$N(x)$~~
$$N(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$= N\left(\frac{S_0 - k}{\sigma\sqrt{T}}\right) - N\left(\frac{2L - k - S_0}{\sigma\sqrt{T}}\right)$$

4. ①

(b) Call Option $\min_{0\le s\le T} S_s = m_t$

$\varphi(x) = (x-k)^+$ $\quad V_c(T) = \begin{cases} S_T - k & \text{if } S_T \geq k,\ m_t \geq L. \\ 0 & \text{otherwise}. \end{cases}$

Let A = ~~A~~ all the paths go from S_0 to S_T, and $S_T \geq k$, and ~~the~~ never go below L.

$$C = E(V_c(T))\cdot e^{-\cancel{r}\,0\cdot T} = E(V_c(T))$$

$$= \int_G \frac{1}{\sqrt{2\pi}} \cancel{(x-k}(S_T - k)e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_G (S_0 + \sigma\sqrt{T}z - k)e^{-\frac{z^2}{2}}dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_G (S_0 - k)e^{-\frac{z^2}{2}}dz + \frac{1}{\sqrt{2\pi}}\int_G \sigma\sqrt{T}\, z\, e^{-\frac{z^2}{2}}dz.$$

$\cancel{\frac{1}{\sqrt{2\pi}}}$ From part (a)

$$\frac{1}{\sqrt{2\pi}}(S_0 - k)\int_G e^{-\frac{z^2}{2}}dz = \frac{S_0 - k}{\cancel{\sqrt{2\pi}}}\left(N\left(\frac{S_0 - k}{\sigma\sqrt{T}}\right) - N\left(\frac{2L - k - S_0}{\sigma\sqrt{T}}\right)\right)$$

$\cancel{\frac{1}{\sqrt{2\pi}}}$ Let A = all paths start from S_0 to $S_T \geq k$. $\{S_T \geq k\}$

B = all paths start from S_0 to $S_T \geq k$, and ever go below L. $\{m_t < L, S_T \geq k\}$

$\hookrightarrow \{S_T \leq 2L - k\}$

$$\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\int_G z e^{-\frac{z^2}{2}}dz$$

$$= \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\int_A z e^{-\frac{z^2}{2}}dz - \int_B z e^{-\frac{z^2}{2}}dz\right]$$

$$= \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\int_{\frac{k-S_0}{\sigma\sqrt{T}}}^{\infty} z e^{-\frac{z^2}{2}}dz - \int_{-\infty}^{\frac{2L-k-S_0}{\sigma\sqrt{T}}} z e^{-\frac{z^2}{2}}dz\right]$$

$$= \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\exp\left(-\frac{(k-S_0)^2}{2\sigma^2 T}\right) + \exp\left(-\frac{(2L-k-S_0)^2}{2\sigma^2 T}\right)\right]$$

Problem 4 (b) Cont.

$$E[V_c(T)] = (S_0-K)\left[N\left(\frac{S_0-K}{\sigma\sqrt{T}}\right) - N\left(\frac{2L-K-S_0}{\sigma\sqrt{T}}\right)\right] + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\exp\left(-\frac{(K-S_0)^2}{2\sigma^2 T}\right) + \exp\left(-\frac{(2L-K-S_0)^2}{2\sigma^2 T}\right)\right]$$

② ~~P~~ Put Option

$\varphi(x) = (K-x)^+$ $\quad V_p(T) = \begin{cases} K-S_T & \text{if } m_t \geq L, \text{ and } S_T \leq K. \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{P}[m_t \geq L, S_T \leq K] = \mathbb{P}[S_T \leq K] - \mathbb{P}[m_t \leq L, S_T \leq K]$$
$$= \mathbb{P}\left[Z \leq \frac{K-S_0}{\sigma\sqrt{T}}\right] - \underbrace{\mathbb{P}\left[\min_{0\leq s\leq T} B_s \leq \frac{L-S_0}{\sigma}, B_T \leq \frac{K-S_0}{\sigma}\right]}$$

From problem 3.

$$= N\left(\frac{K-S_0}{\sigma\sqrt{T}}\right) - 2N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right) + N\left(\frac{2L-S_0-K}{\sigma\sqrt{T}}\right)$$

~~C = All the paths starting from $S_0 \to S_T \leq K$. and never go below~~ L.

~~A = All the paths~~

$$\mathbb{P}[m_t \geq L, S_T \leq K] = \mathbb{P}[S_T \leq K] - \mathbb{P}[m_t < L, S_T \leq K]$$
$$= \mathbb{P}[S_T \leq K] - \mathbb{P}[m_t < L] + \mathbb{P}[m_t \leq L, S_T \geq K]$$
$$= \mathbb{P}[S_T \leq K] - 2\mathbb{P}[S_T \leq L] + \mathbb{P}[S_T \leq 2L-K]$$

Same trick as for call option.

$$E(V_p(T)) = (K-S_0-\sigma B_T)\int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$
$$= (K-S_0)\left[N\left(\frac{K-S_0}{\sigma\sqrt{T}}\right) - 2N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right) + N\left(\frac{2L-S_0-K}{\sigma\sqrt{T}}\right)\right]$$
$$-\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\int_{-\infty}^{\frac{K-S_0}{\sigma\sqrt{T}}} z e^{-\frac{z^2}{2}} dz - 2\int_{-\infty}^{\frac{L-S_0}{\sigma\sqrt{T}}} z e^{-\frac{z^2}{2}} dz + \int_{-\infty}^{\frac{(2L-K-S_0)}{\sigma\sqrt{T}}} z e^{-\frac{z^2}{2}} dz\right.$$

Problem 4 (b) Cont.

$$E(V_P(T)) = (K-S_0)\left[N\left(\frac{K-S_0}{\sigma\sqrt{T}}\right) - 2N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right) + N\left(\frac{2L-S_0-K}{\sigma\sqrt{T}}\right)\right]$$
$$- \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[-\exp\left(-\frac{(K-S_0)^2}{2\sigma^2 T}\right) + 2\exp\left(\frac{-(L-S_0)^2}{2\sigma^2 T}\right) - \exp\left(-\frac{(2L-K-S_0)^2}{2\sigma^2 T}\right)\right]$$

Problem 4 (c).

Put-Call Parity $C-P = S_0-K$. given $r=0$.

$$C-P = (S_0-K)\left[N\left(\frac{S_0-K}{\sigma\sqrt{T}}\right) - \cancel{N\left(\frac{2L-K-S_0}{\sigma\sqrt{T}}\right)}\right] + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[\cancel{\exp\left(-\frac{(K-S_0)^2}{2\sigma^2 T}\right)}\right.$$
$$\left.+ \cancel{\exp\left(-\frac{(2L-K-S_0)^2}{2\sigma^2 T}\right)}\right] + (S_0-K)\left[N\left(\frac{K-S_0}{\sigma\sqrt{T}}\right) - 2N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right)\right.$$
$$\left.+ \cancel{N\left(\frac{2L-S_0-K}{\sigma\sqrt{T}}\right)}\right] + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left[2\exp\left(\frac{-(L-S_0)^2}{2\sigma^2 T}\right) - \cancel{\exp\left(-\frac{(K-S_0)^2}{2\sigma^2 T}\right)}\right.$$
$$\left.- \cancel{\exp\left(-\frac{(2L-K-S_0)^2}{2\sigma^2 T}\right)}\right]$$
$$= (S_0-K)\left[1 - 2N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right)\right] + 2\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\exp\left(-\frac{(L-S_0)^2}{2\sigma^2 T}\right)$$

As we can see, put-call parity doesn't hold because of the non-touch feature, this term depends on L. As $L \to -\infty$, $N\left(\frac{L-S_0}{\sigma\sqrt{T}}\right) \to 0$, $\exp\left(-\frac{(L-S_0)^2}{2\sigma^2 T}\right) \to 0$.

$C-P = S_0 - K$, so the put-call parity holds when $L \to -\infty$, which is the case without non-touch feature.

Problem 5

(a)

X_t is normal distribution with expectation as of

$$E(X_t) = \int_0^t r(s)ds \tag{26}$$

and variance as of

$$Var(X_t) = Var(\int_0^t \sigma(s)dB_s) = \int_0^t \sigma^2(s)ds \tag{27}$$

Therefore $X_t \sim N(\int_0^t r(s)ds, \int_0^t \sigma^2(s)ds)$.
S_t is log-normal distribution with expectation as of

$$E(\ln(S_t)) = \ln(S_0) - \frac{1}{2}\int_0^t \sigma^2(s)ds + E(X_t) = \ln(S_0) + \int_0^t [r(s) - \frac{\sigma^2(s)}{2}]ds \tag{28}$$

and variance as of

$$Var(\ln(S_t)) = Var(X_t) = \int_0^t \sigma^2(s)ds \tag{29}$$

Therefore $\ln(S_t) \sim N(\ln(S_0) + \int_0^t [r(s) - \frac{\sigma^2(s)}{2}]ds, \int_0^t \sigma^2(s)ds)$.

(b)

The price of the call is

$$\begin{aligned} C &= \exp[-\int_0^T r(t)dt]E[(S_T - K)^+] \\ &= \exp[-\int_0^T r(t)dt][E(S_T I_{S_T \geq K}) - KE(I_{S_T \geq K})] \end{aligned} \tag{30}$$

Now we want to see when $S_t \geq K$, i.e. (let $\mu_t = \int_0^t r(s)ds, \Sigma_t = \int_0^t \sigma^2(s)dt$),

$$\begin{aligned} S_t &= S_0 \exp[X_t - \frac{1}{2}\int_0^t \sigma^2(s)ds] > K \\ &\Rightarrow X_t > \ln(K/S_0) + \frac{1}{2}\int_0^t \sigma^2(s)ds \\ &\Rightarrow \mu_t + \sqrt{\Sigma_t}Z > \ln(K/S_0) + \frac{1}{2}\Sigma_t \\ &\Rightarrow Z > \frac{\ln(K/S_0) + \frac{1}{2}\Sigma_t - \mu_t}{\sqrt{\Sigma_t}} \end{aligned} \tag{31}$$

Therefore we derive

$$E(I_{S_T \geq K}) = N(-\frac{\ln(K/S_0) + \frac{1}{2}\Sigma_T - \mu_T}{\sqrt{\Sigma_T}}) \tag{32}$$

In similar way, we are able to solve $E(S_T I_{S_T \geq K})$ as of

$$E(S_T I_{S_T \geq K}) = \frac{S_0 e^{\mu_T}}{\sqrt{2\pi}} \int_{-\infty}^{-w-\sqrt{\Sigma_T}} e^{-t^2/2}dt = \frac{S_0 e^{\mu_0}}{\sqrt{2\pi}} N(-w - \sqrt{\Sigma_T}) \tag{33}$$

where

$$w = \frac{\ln(K/S_0) + \frac{1}{2}\Sigma_t - \mu_t}{\sqrt{\Sigma_t}} \tag{34}$$

Problem 5 [(b)] continued on next page...

Finally we put all these stuffs into the formula of pricing call, we derive

$$C = \exp[-\int_0^T r(t)dt][E(S_T I_{S_T \geq K}) - KE(I_{S_T \geq K})] = S_0 N(-w - \sqrt{\Sigma_T}) - Ke^{-\mu_T} N(-w) \tag{35}$$

where

$$N(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \tag{36}$$

and w, Σ_t, μ_t are given above.

Problem 6

According to the definition,

$$\int_0^t B_s^2 dB_s = \lim_{||\pi_n|| \to 0} \sum_{t=1}^{n} B_{t_{k-1}}^2 \Delta B_{t_k} \tag{37}$$

From our last homework we have the cubic variation as 0, therefore,

$$\begin{aligned}
3\int_0^t B_s^2 dB_s &= \lim_{||\pi_n|| \to 0} \sum_{t=1}^{n} 3B_{t_{k-1}}^2 \Delta B_{t_k} + (B_{t_k} - B_{t_{k-1}})^3 \\
&= \lim_{||\pi_n|| \to 0} \sum_{t=1}^{n} [(B_{t_k}^3 - B_{t_{k-1}}^3) - 3B_{t_k - 1}(B_{t_k} - B_{t_{k-1}})^2] \\
&= B_T^3 - 3 \lim_{||\pi_n|| \to 0} \sum_{t=1}^{n} B_{t_k - 1}(B_{t_k} - B_{t_{k-1}})^2 \\
&= B_T^3 - 3\int_0^T B_t (dB_t)^2 \\
&= B_T^3 - 3\int_0^T B_t dt
\end{aligned} \tag{38}$$

Therefore,

$$\int_0^t B_s^2 dB_s = 1/3 B_T^3 - \int_0^T B_t dt \tag{39}$$