MATH 9831 - Homework 1

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Problem 1

Let

$$Y = \sum_{k=1}^{n} \frac{\int_{E_k} X dP}{P[E_k]} \mathbf{1}_{E_k}(w) \tag{1}$$

Y is the conditional expectation of X if it meets 2 criteria:

- Y is $\mathcal{G}$-measurable.
  We know that Y is $\mathcal{G}$-measurable since Y is constant in E_k for $k = 1 : n$, and the only term containing ω is $\mathbf{1}_{E_k}(\omega)$, which is clearly $\mathcal{G}$-measurable.

- $\int_A Y dP = \int_A X dP$ for all $A \in \mathcal{G}$ where $Y = E(X|\mathcal{G})$.
  According to linearity of expectation,

$$\int_A Y dP = \int_A \sum_{k=1}^{n} \frac{\int_{E_k} X dP}{P[E_k]} \mathbf{1}_{E_k}(w) dP = \sum_{k=1}^{n} \int_A \frac{\int_{E_k} X dP}{P[E_k]} \mathbf{1}_{E_k}(w) dP = \sum_{k=1}^{n} \int_{A \cap E_k} \frac{\int_{E_k} X dP}{P[E_k]} dP \tag{2}$$

  Since $A \in \mathcal{G}$, then A is some union of E_i's, say $A = \cup_{i \in I} E_i$ where $I \subseteq \{1, 2, ..., n\}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \int_{A \cap E_k} \frac{\int_{E_k} X dP}{P[E_k]} dP &= \sum_{E_k \subseteq A} \int_{E_k} \frac{\int_{E_k} X dP}{P[E_k]} dP && (3) \\
&= \sum_{E_k \subseteq A} \frac{\int_{E_k} X dP}{P[E_k]} \int_{E_k} dP && (4) \\
&= \sum_{E_k \subseteq A} \frac{\int_{E_k} X dP}{P[E_k]} P[E_k] && (5) \\
&= \sum_{E_k \subseteq A} \int_{E_k} X dP && (6) \\
&= \int_A X dP && (7)
\end{aligned}$$

Y meets both criteria. Therefore, by definition, $Y = E(X|G)(w)$.

Problem 2

- Integrability: we require $E|X_t| < \infty$. Since the sum of ξ_i's is between $-t$ and t, therefore

$$E|X_t| = E\left|\sum_{i=1}^{t} \xi_i\right| \leq t < \infty \tag{8}$$

- Adapted: X_t is obviously adapted since we are considering natural filtration. That is, X_t is only composed of elements ξ_i for $i \leq t$ which are $\mathcal{F}_i$-measurable, X_t is obviously adapted.

- Expected value of next step: the expected value of X_{t+1} under current filtration is

$$
\begin{aligned}
E(X_{t+1}|\mathcal{F}_t) &= E(\xi_{t+1} + \sum_{i=1}^{t} \xi_i|\mathcal{F}_t) && (9)\\
&= E(\xi_{t+1}|\mathcal{F}_t) + E(\sum_{i=1}^{t} \xi_i|\mathcal{F}_t) && (10)\\
&= E(\xi_{t+1}) + E(X_t|\mathcal{F}_t) && (11)\\
&= 1p - 1(1-p) + X_t && (12)\\
&= X_t + 2p - 1 && (13)
\end{aligned}
$$

  If $p = 1/2$, then $E(X_{t+1}|\mathcal{F}_t) = X_t$, so X_t is a martingale.
  If $p > 1/2$, then $E(X_{t+1}|\mathcal{F}_t) > X_t$, so X_t is a submartingale.
  If $p < 1/2$, then $E(X_{t+1}|\mathcal{F}_t) < X_t$, so X_t is a supermartingale.

Problem 3

- Integrable
  We know that S_n is a martingale and therefore must be integrable, i.e., $E|S_n| < \infty$. Then we have

$$
E|C_n| = E|(S_n - K)^+| \leq E|S_n - K| \leq E|S_n| + E\,|-K| = E|S_n| + K < \infty \tag{14}
$$

  Similarly,

$$
E|P_n| = E|(K - S_n)^+| \leq E|K - S_n| \leq E|K| + E|-S_n| = K + E|S_n| < \infty \tag{15}
$$

  So C_n and P_n are integrable.

- Adapted
  Since K is constant, both C_n and P_n are adapted since their only other component is S_n, which is adapted.

- Expectation of next step
  We verify both cases for the option price,

$$
E(C_{n+1}|\mathcal{F}_t) = E((S_{n+1} - K)^+|\mathcal{F}_t) \geq E(S_{n+1} - K|\mathcal{F}_t) = E(S_{n+1}|\mathcal{F}_t)) - K = S_n - K \tag{16}
$$

  and

$$
E(C_{n+1}|\mathcal{F}_t) = E((S_{n+1} - K)^+|\mathcal{F}_t) \geq E(0|\mathcal{F}_t) = 0 \tag{17}
$$

  Therefore $E(C_{n+1}|\mathcal{F}_t) \geq (S_n - K)^+ = C_n$, which means C_n is a submartingale. Similarly, we can show that P_n is a submartingale by

$$
E(P_{n+1}|\mathcal{F}_t) = E((K - S_{n+1})^+|\mathcal{F}_t) \geq E(K - S_{n+1}|\mathcal{F}_t) = K - E(S_{n+1}|\mathcal{F}_t)) = K - S_n \tag{18}
$$

  and

$$
E(P_{n+1}|\mathcal{F}_t) = E((K - S_{n+1})^+|\mathcal{F}_t) \geq E(0|\mathcal{F}_t) = 0 \tag{19}
$$

  Therefore $E(P_{n+1}|\mathcal{F}_t) \geq (K - S_n)^+ = P_n$, which means P_n is a submartingale.