MTH9831

HW8

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(1). $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$. Let $f(X_t, t)$

$$df(X_t, t) = f_t dt + f_x dx + \frac{1}{2} f_{xx} \cdot d[X]_t$$

$$= [\overset{0}{\cancel{f_t}} + \mu f_x + \frac{\sigma^2}{2} f_{xx}] dt + \sigma(X_t, t) f_x \cdot dW_t$$

So $\mathcal{L} = \mu f_x + \frac{\sigma^2}{2} f_{xx}$.

In this case $\mu(X_t, t) = r \quad \sigma(X_t, t) = \alpha X_t$

$$\mathcal{L} = r \cdot f_x + \frac{\alpha^2}{2} X_t^2 f_{xx}.$$

(2) From (1) we know $\mathcal{L} = \mu f_x + \frac{\sigma^2}{2} f_{xx}$

If $\mathcal{L} = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}$ So $\mu = 1$, $\frac{\sigma^2}{2} = 1$ $\sigma = \sqrt{2}$.

So. $dX_t = dt + \sqrt{2}\, dW_t$

(3).

(a) Take $Y_t = e^{\int_0^t \frac{1}{1-s} ds} \cdot X_t = e^{-\ln|1-t|} X_t = \frac{X_t}{1-t}$ since $0 \le t < 1$

$$dY_t = \frac{1}{1-t} Y_t dt + \frac{1}{1-t} dX_t + \frac{1}{2} \times 0 \cdot d[X]_t$$

$$= \frac{1}{1-t} [Y_t dt - \frac{X_t}{1-t} dt + dW_t]$$

$$= \frac{1}{1-t} dW_t$$

$$Y_T - Y_0 = \int_0^T \frac{1}{1-t} dW_t = \frac{X_T}{1-T} - X_0$$

So: $X_t = (1-t)X_0 + (1-t)\int_0^t \frac{1}{1-s} dW_s$

(b) $X_t = (1-t)X_0 + (1-t)\int_0^t f(s)\, dW_s$, where $f(s)$ is a deterministic function, so X_t is a Gaussian Process.

Suppose $0 \le s \le t < 1$

$$Cov(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t) = (1-t)(1-s)X_0^2 + (1-t)(1-s) \cdot \int_0^s \frac{1}{(1-\tau)^2} d\tau$$

$$- (1-t)(1-s)X_0^2 = (1-t)(1-s) \left. \frac{1}{1-\tau} \right|_0^s$$

$$= (1-t)(1-s)\left(\frac{1}{1-s} - 1\right) = (1-t) - (1-t)(1-s)$$

$$= s(1-t)$$

with the right autocovariance function.

(4) Take $Y_t = e^{-\int_0^t V(X_s,s)ds} u(X_t,t) - \int_0^t e^{-\int_0^r V(X_s,s)ds} f(X_r, r)\, dr$

Let $g(t) = e^{-\int_0^t V(X_s,s)ds}$. where $dX_t = \mu dt + \sigma dW_t$

$$dY_t = (u_t \cdot g(t) + g'(t)u - g(t)f(X_t,t))dt + g(t)\, u_x \cdot dX + \tfrac{1}{2} g(t) u_{xx} d[X]_t$$

$$= g(t)\underbrace{\left(u_t - Vu - f(X_t,t) + \mu u_x \cancel{dt} + \tfrac{1}{2}\sigma^2 u_{xx}\right]}_{0}dt + g(t)\, u_x \cdot \sigma dW_t$$

$$= \sigma \cdot g(t)\, u_x\, dW_t.$$

$$Y_T - Y_t = e^{-\int_0^T V(X_s,s)ds} u(X_T,T) - \int_0^T e^{-\int_0^r V(X_s,s)ds} \cdot f(X_r,r)\, dr$$

$$- e^{-\int_0^t V(X_s,s)ds} u(X_t,t) + \int_0^t e^{-\int_0^r V(X_s,s)ds} f(X_r,r)\, dr$$

$$= \sigma \int_t^T u_x\, g(s)\, dW_s.$$

$$u(X_t,t) = e^{-\int_t^T V(X_s,s)ds} u(X_T,T) - \int_t^T e^{-\int_0^r V(X_s,s)ds} \cdot f(X_r,r)\, dr$$

$$- \sigma \int_t^T u_x \cdot e^{-\int_t^s V(X_r,r)dr} \cdot dW_s.$$

Take conditional expectation.

$$E_{t,x}(u(X_t,t)) = u(X_t,t) = E_{t,x}\left[e^{-\int_t^T V(X_s,s)ds} h(x) - \int_t^T e^{-\int_0^r V(X_s,s)ds} f(X_r,r)\,dr \right.$$

(5) Take $v = e^u$ $\quad u = \ln v$ $\qquad dX_t = \mu dt + \sigma dW_t$.

$u_x = \frac{1}{v} v_x \quad u_t = \frac{1}{v} v_t \quad u_{xx} = \frac{v_{xx}}{v} - \frac{v_x^2}{v^2}$

So. ~~the~~ we change the original PDE:

$$u_t + \frac{\sigma^2}{2} u_{xx} + \frac{\sigma^2}{2} u_x^2 + \mu \cdot u_x = V \qquad 0 < t < T$$

$$u(x,T) = h(x)$$

into:

$$\frac{1}{v}\left(v_t + \frac{\sigma^2}{2}\left(v_{xx} - \frac{v_x^2}{v}\right) + \frac{\sigma^2}{2} \frac{v_x^2}{v} + \mu \cdot v_x \right) = V$$

$$v_t + \mu v_x + \frac{\sigma^2}{2} v_{xx} = V \cdot v$$

$$v(x,T) = e^{h(x)} \; \cancel{= g(x)}$$

Let $Y(X_t,t) = v(X_t,t)\, e^{-\int_0^t V(X_s,s)ds} = v(X_t,t) \cdot g(t)$

$$dY = g(t)(v_t + g'(t) \cdot V)dt + g(t) v_x \cdot dX + \frac{1}{2} g(t) v_{xx} \cdot d[X]_t$$

$$= g(t)\left(v_t + \mu v_x + V \cdot g'(t) + \frac{1}{2}\sigma^2 v_{xx}\right)dt + \sigma \cdot g(t) v_x \, dW_t$$

$$= g(t)\left(v_t + \mu v_x + \frac{1}{2}\sigma^2 v_{xx} - v \cdot V\right)dt + \sigma g(t) v_x dW_t$$

$$= \sigma g(t) v_x dW_t$$

Integrate $\quad v(X_T,T) g(T) - v(X_t,t) g(t) = \int_t^T \sigma g(s) v_x \cdot dW_s$

$$v(X_t,t) = e^{h(x)} \cdot e^{-\int_t^T V(X_s,s)ds} - \int_t^T \sigma \cdot e^{-\int_t^s V(X_\tau,\tau)d\tau} \cdot v_x \cdot dW_s$$

Take the conditional expectation

$$E_{t,x}(v(X_t,t)) = v(X_t,t) = E_{t,x}\left[e^{h(X_T)} \cdot e^{-\int_t^T V(X_s,s)ds}\right]$$

$$u(X_t,t) = \ln v(X_t,t) = \ln\left(E_{t,x}\left[e^{h(X_T)} \cdot e^{-\int_t^T V(X_s,s)ds}\right]\right).$$

16) $u_t + \frac{1}{2}u_{xx} + \mu(x)u_x = 0 \quad t < T$

$u(x,T) = h(x)$

We know the representation of $u(x,t)$

$u(x,t) = \mathbb{E}_{t,x}[h(X_T)]$

where $dX_t = \mu(X_t)dt + d\cancel{B_t}\,W_t$

$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^t \mu(X_s)dW_s - \frac{1}{2}\int_0^t \mu^2(X_t)ds}$ $\quad X_t$ is a B.M under $\hat{\mathbb{P}}$

$\cancel{u(x_0,t) = \mathbb{E}}$

$\cancel{\mathbb{E}}\, u(X_0;0) = \hat{\mathbb{E}}[h(X_T)(\frac{d\mathbb{P}}{d\hat{\mathbb{P}}})_T]$

$u(X_0,0) = \hat{\mathbb{E}}(u(x,t)\cdot(\frac{d\mathbb{P}}{d\hat{\mathbb{P}}})_t]$

contional on $X_t = x$

$u(x,t) \cancel{=} \cancel{\hat{\mathbb{E}}_{t,x}} = \hat{\mathbb{E}}_{t,x}[h(X_T)\cdot e^{\int_t^T \mu(X_s)dW_s + \frac{1}{2}\int_t^T \mu^2(X_s)ds}]$

$= \hat{\mathbb{E}}[h(X_T)\cdot e^{\int_t^T \mu(X_s)(dX_s - \mu(X_s)\cdot ds) + \frac{1}{2}\int_t^T \mu^2(X_s)ds}]$

$= \hat{\mathbb{E}}[h(X_T)\cdot e^{\int_t^T \mu(X_s)dX_s - \frac{1}{2}\int_t^T \mu^2(X_s)ds}]$

$= \hat{\mathbb{E}}[h(B_T)\cdot e^{\int_t^T \mu(B_s)dB_s - \frac{1}{2}\int_t^T \mu^2(B_s)ds}]$