MTH9831

HW5

Weiyi Chen

Zhenfeng Liang

Mo Shen

(1). $dW_t = \frac{1}{\sqrt{R_t}} \sum_{k=1}^{n} B_k(t) dB_k(t)$

(a) $[dW_t]_t = \left(\frac{1}{\sqrt{R_t}} \sum_{k=1}^{n} B_k(t) dB_k(t)\right)\left(\frac{1}{\sqrt{R_t}} \sum_{k=1}^{n} B_k(t) dB_k(t)\right)$

Since we know $dB_i(t) \cdot dB_j(t) = 0$ for $i \neq j$.

$dB_i(t) \cdot dB_i(t) = t$

$$[dW_t]_t = \frac{1}{R_t}\left(B_1^2(t)[dB_1(t)]_t + B_2^2(t)[dB_2(t)]_t + \cdots B_n^2(t)[dB_n(t)]_t \right]$$

$$= \frac{1}{R_t}\left[B_1^2(t) + B_2^2(t) + \cdots B_n^2(t)\right] t$$

$$= t$$

(b) $dR_t = d\left(\sum_{k=1}^{n} B_k^2(t)\right)$

Let $f(t, B_1, B_2 \cdots B_n) = \sum_{k=1}^{n} B_k^2(t)$

$f_t \cancel{ft} = 0$ $\quad f_{B_1} \cancel{x2} = 2B_1(t)$ $\quad f_{B_2} = 2B_2(t)$

$\cdots f_{B_n} \cancel{x} = 2B_n(t)$ $\qquad \nabla f = \begin{bmatrix} 2B_1 \\ \vdots \\ 2B_n \end{bmatrix}$

$\Delta^2 f = 2n$

$$dR_t = \cancel{\nabla f} \begin{bmatrix} 2B_1 \\ \vdots \\ 2B_n \end{bmatrix} \cdot \begin{bmatrix} dB_1 \\ \vdots \\ dB_n \end{bmatrix} + \frac{1}{2} \times 2n \cdot dt = 2\sum_{k=1}^{n} B_k dB_k + n dt$$

$$dW_t = \frac{1}{\sqrt{R_t}} \sum_{k=1}^{n} B_k(t)\, dB_k(t).$$

$$2\sqrt{R_t}\, dW_t = 2\sum_{k=1}^{n} B_k(t)\, dB_k(t)$$

So: $dR_t = 2\sum_{k=1}^{n} B_k(t)\, dB_k(t) + n\,dt$.

(c). $X_t = \sqrt{R_t} = \sqrt{\sum_{k=1}^{n} B_k^2(t)}$

Let $f = \sqrt{\sum_{k=1}^{n} B_k^2(t)}$

$f_t = 0$. $f_{B_1} = \frac{1}{2}\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} \cdot 2B_1 = B_1 \cdot \left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}}$.

$$\nabla f = \begin{bmatrix} B_1\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} \\ \vdots \\ B_n\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} B_1/X_t \\ \vdots \\ B_n/X_t \end{bmatrix}$$

$\cancel{\Delta^2 f} \Rightarrow f_{B_1B_1} = \left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} + B_1 \cdot \left(-\frac{1}{2}\right)\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{3}{2}} \cdot 2B_1$

$$= \left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} \cdot \left(1 - B_1^2 \left(\sum_{k=1}^{n} B_k^2(t)\right)^{-1}\right).$$

$$\Delta^2 f = \left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} \left(n - \frac{B_1^2 + B_2^2 + \cdots B_n^2}{\left(\sum_{k=1}^{n} B_k^2(t)\right)}\right)$$

$$= (n-1)\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}}.$$

$$dX_t = \frac{1}{2}(n-1)\left(\sum_{k=1}^{n} B_k^2(t)\right)^{-\frac{1}{2}} dt + \sum_{k=1}^{n} \frac{B_k}{X_t} dB_k$$

$$= \frac{n-1}{2X_t}\, dt + \sum_{k=1}^{n} \frac{B_k}{X_t}\, dB_k$$

$$dW_t = \frac{1}{X_t} \sum_{k=1}^{n} B_k(t) dB_k$$

So: $dW_t + \frac{n-1}{2X_t}dt = \frac{1}{X_t}\sum_{k=1}^{n} B_k(t)dB_k(t) + \frac{n-1}{2X_t}dt$

$= dX_t$.

(2) ~~$du = u_t dt + u_s dS\, u_{S_t} dS_t + u$~~

$du = u_t dt + u_s ds + u_a da$

$+ \frac{1}{2}u_{ss}\, d[S]_t + \frac{1}{2}u_{aa}\, d[a]_t + u_{sa}\, d[S,a]$

$= u_t dt + u_s \cdot (S \cdot \sigma\, dB) + u_a \cdot S \cdot dt$.

$+ \frac{1}{2}u_{ss} \cdot S^2 \cdot \sigma^2 \cdot dt + \frac{1}{2}u_{aa} \cdot S^2 (dt)^2$

$+ u_{s \cdot a} \cdot S^2 \cdot \sigma \cdot (dt \cdot dB)$.

Since $(dt)^2 \to 0$, $|dB \cdot dt| \to 0$.

So. $du = (u_t + u_a \cdot S + \frac{1}{2}u_{ss} \cdot S^2\sigma^2)dt + u_s \cdot S\sigma dB$

~~$= (u_t + u_{A_t} \cdot S_t + \frac{1}{2}u_{S_tS_t} S_t^2 \sigma_t^2(S_t))dt + u_s \cdot S_t \sigma_t(S_t)dB$~~

In order to let u as a martingale,

~~let $(u_t + u_{A_t} S_t + \frac{1}{2}u_{S_tS_t} S_t^2 \sigma_t(S_t)) = 0$.~~

Let $u_t + u_a S + \frac{1}{2}u_{ss} \cdot S^2\sigma^2 = 0$.

(3). ~~S_t is only invst~~

$S_t \Rightarrow$ martingale . $E(S_t) = E(S_0) = S$

$V_t \Rightarrow$ CIR process

$E(V_t) = e^{-\lambda T} V_0 + (1 - e^{-\lambda T}) m = m \quad (V_0 = m)$

$dS_t V_t = S_t \cdot dV_t + V_t dS_t + d[S_t, V_t]$

$= S_t \cdot dV_t + V_t dS_t + \rho\eta \cdot S_t \cdot V_t \cdot dt$

$S_t V_t = S_0 V_0 + \int_0^T S_t \, dV_t + \int_0^T V_t \, dS_t + \rho\eta \cdot \int_0^T S_t V_t dt$

$= S_0 V_0 + \int_0^T S_t \cdot (\lambda(m - V_t) dt + \eta \sqrt{V_t} dB_t)$

$+ \int_0^T V_t \sqrt{V_t} \cdot S_t (\rho dB_t + \sqrt{1-\rho^2} dW_t) + \rho\eta \int_0^T S_t V_t dt$

$E(S_t V_t) = S_0 V_0 + E(\int_0^T S_t \lambda (m - V_t) dt) + \rho\eta E(\int_0^T S_t V_t dt)$

$= S_0 V_0 + \lambda m \int_0^T E(S_T) dt - \lambda \int_0^T E(S_t V_t) dt$

$+ \rho\eta \int_0^T E(S_t V_t) dt$

Let $f = E(S_t V_t)$.

$f' = \lambda m \cdot S - \lambda f + \rho\eta \cdot f = \lambda S m + (\rho\eta - \lambda) f$

$f' + (\lambda - \rho\eta) f = \lambda S m$

$(e^{(\lambda - \rho\eta)t} \cdot f)' = \lambda S m \cdot e^{(\lambda - \rho\eta)t}$

$$e^{(\lambda-\rho\eta)t} f = \lambda sm \int e^{(\lambda-\rho\eta)t} dt + C$$

$$= \cancel{\lambda sm} \frac{\lambda sm}{\lambda-\rho\eta} e^{(\lambda-\rho\eta)t} + C$$

$$f = \frac{\lambda sm}{\lambda-\rho\eta} + C \cdot e^{-(\lambda-\rho\eta)t}$$

$$f_0 = E(S_0 \cdot V_0) = sm = \frac{\lambda sm}{\lambda-\rho\eta} + C = sm$$

$$C = sm\left(1 - \frac{\lambda}{\lambda-\rho\eta}\right) = -\cancel{sm} \frac{sm \cdot \rho\eta}{\lambda-\rho\eta}$$

$$\cancel{f}\ E(S_t V_t) = \frac{sm}{\lambda-\rho\eta}\left(\lambda - \rho\eta e^{-(\lambda-\rho\eta)t}\right)$$

$$Cov(S_t, V_t) = E(S_t \cdot V_t) - E(S_t)E(V_t)$$

$$= \frac{sm}{\lambda-\rho\eta}\left(\lambda - \rho\eta e^{-(\lambda-\rho\eta)t}\right) - sm$$

$$= \frac{sm\rho\eta}{\lambda-\rho\eta}\left(1 - e^{-(\lambda-\rho\eta)t}\right)$$

(b)

① When $\rho < 0$, $\lambda - \rho\eta > 0$, $e^{-(\lambda-\rho\eta)t} < 1$

So. $Cov(S_t \cdot V_t) < 0$

② When $\rho = 0$. $Cov(S_t, V_t) = 0$

③ When $\rho > 0$ if $\lambda - \rho\eta > 0$, $e^{-(\lambda-\rho\eta)t} < 1$

$Cov(S_t, V_t) > 0$

if $\lambda - \rho\eta < 0$, $e^{-(\lambda-\rho\eta)t} > 1$, $Cov(S_t, V_t) > 0$

(4)

(a) ~~$\mathbb{R}$~~

Under $\mathbb{P}$, $X \sim N(\mu, \sigma^2)$

Under $\tilde{\mathbb{P}}$, $Y \sim N(0, \sigma^2)$

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{d\tilde{\mathbb{P}}/dy}{d\mathbb{P}/dx} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-0)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

$$= e^{\frac{\mu^2 - 2\mu x}{2\sigma^2}}$$

(b) $d\tilde{\mathbb{P}}/dy = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x^t \Sigma^{-1} x)\right)$

$d\mathbb{P}/dy = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^t \Sigma^{-1} (x-\mu)\right)$

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\frac{1}{2}\left((x-\mu)^t \Sigma^{-1} (x-\mu) - x^t \Sigma^{-1} x\right)\right)$$