MTH 9831

HW10

Weiyi Chen

Zhenfeng Liang

Mo Shen

HW 10.

(1)

(a) $X \sim \Gamma(\alpha, \lambda)$ $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}_{[0,\infty)}(x)$

C.H.F.

$$\phi_X(u) = E(e^{iux}) = \int_{-\infty}^{+\infty} e^{iux} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}_{[0,\infty)}(x) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda - iu)x} \cdot x^{\alpha-1} dx$$

Let $v = (\lambda - iu)$. $\cancel{x = \frac{1}{\lambda - iu} v}$

$$\cancel{= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-v \cdot x} \left(\frac{1}{\lambda - iu}\right)^{\alpha-1} \cdot v^{\alpha-1} d\left(\frac{1}{\lambda - iu} v\right)}$$

$$\cancel{= \left(\frac{\lambda}{\lambda - iu}\right)^{\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{-v \cdot x} \cdot v^{\alpha-1} dv}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-v \cdot x} x^{\alpha-1} dx$$

$$= \frac{\lambda^\alpha}{v^\alpha} \int_0^\infty \frac{v^\alpha}{\Gamma(\alpha)} \cdot e^{-vx} \cdot x^{\alpha-1} dx$$

$$= \left(\frac{\lambda}{v}\right)^\alpha = \left(\frac{\lambda}{\lambda - iu}\right)^\alpha.$$

$\blacksquare$

For ~~I~~ infinitely divisible.

Suppose $X \sim \Gamma(n,\lambda)$ is a sum of n independent $\Gamma(1,\lambda) \equiv EXP(\lambda)$

So, there exists a sequence of i.i.d random variables $\{X_i^{(n)}\}_{i=1}^{n}$.

$\cancel{\phi}$ C.H.F $= \left(\frac{\lambda}{\lambda - iu}\right)^n = \left(\varphi_{X_1}(u)\right)^n \quad X_i \sim EXP(\lambda)$

(b) Frullani integral

$$\left(\frac{\lambda}{\lambda - z}\right)^{\alpha} = e^{\int_0^\infty (e^{zx}-1)\frac{\alpha}{x}e^{-\lambda x}dx}.$$

~~pl~~ plug $z = iu$ in the integral.

$$\phi_X(u) = \left(\frac{\lambda}{\lambda - iu}\right)^{\alpha} = e^{\int_0^\infty \cancel{(e^{zx}-1)}(e^{iux}-1)\frac{\alpha}{x}e^{-\lambda x}dx}$$

From Lévy-Khintchine representation:

$$\phi_X(u) = E[e^{iuX}] = e^{iu\mu - \frac{\sigma^2}{2}u^2 + \int (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}(x))\,dv(x)}$$

Compare the equations.

we get $\left(\mu_0 = 0,\ \sigma = 0 \quad v(dx) = \frac{\alpha}{x}e^{-\lambda x}\mathbb{1}_{(0,\infty)}(x)dx.\right)$

To show this.

~~(##)~~ $\int x \mathbb{1}_{\{0<|x|\le 1\}} \nu(dx)$

$= \int_0^1 x \frac{a}{x} \cdot e^{-\lambda x} dx = \frac{a}{\lambda}(1-e^{-\lambda})$

Let $\mu_0 = \mu - \int x \mathbb{1}_{\{|x|\le 1\}} \nu \cdot d(x) = 0$.

Characteristic exponent:

$\Phi(u) = iu\mu_0 - \frac{\sigma^2}{2}u^2 + \int (e^{iux}-1)\, d\nu(x)$

$\nu(dx) = \frac{a}{x} e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x)$

(c) Levy triplet. $\mu_0 = 0$, $\sigma = 0$, $d\nu = \frac{a}{x} e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x)\, dx$

(d) $\varphi_{X_1 - X_2}(u) = \varphi_{X_1}(u) \cdot \varphi_{-X_2}(u)$

$= \left(\frac{\lambda_1}{\lambda_1 - iu}\right)^{a_1} \cdot \left(\frac{\lambda_2}{\lambda_2 + iu}\right)^{a_2}$

$= \exp\left[\int_0^\infty \left[(e^{iux}-1)\frac{a_1}{x} e^{-\lambda_1 x} + (e^{-iux}-1)\frac{a_2}{x} e^{-\lambda_2 x}\right] dx\right]$

Exponent $\Phi_{X_1 - X_2}(u) = \int_0^\infty (e^{iux}-1)\left(\frac{a_1}{x} e^{-\lambda_1 x} - e^{-iux}\frac{a_2}{x} e^{-\lambda_2 x}\right) dx$

$\nu(dx) = \left(\frac{a_1}{x} e^{-\lambda_1 x} - e^{iux}\frac{a_2}{x} e^{-\lambda_2 x}\right) \cdot dx$.

(2) . $Y = X_T$ $X_t \sim Levy(\mu, \sigma^2, \nu)$

$T \sim EXP(\lambda)$. $f_T(t) = \lambda e^{-\lambda t} I_{(0,\infty)}(t)$

Y is infinite divisible, there exist some i.i.d. ~~r~~ Y_j, $j = 1, 2 \cdots n$. s.t

$\varphi_n(u) = (\varphi_{Y_i}(u))^n$

For Fixed t.

$E(e^{iuX_t}) = \varphi_{X_t}(u) = (\varphi_{X_1}(u))^t = E(e^{iuX_1})^t$.

But for random T

$E(e^{iuX_T}) = \varphi_{X_T}(u) \neq (\varphi_{X_1}(u))^T$

Because $\varphi_{X_T}(u)$ is a deterministic function in u, and $(\varphi_{X_1}(u))^T$ is a random ~~variable~~ function

~~B~~. However, we can say,

$$E(e^{iuX_T} | T = t) = E(e^{iuX_t})$$

Then, . $\varphi_{X_T}(u) = E(e^{iuX_T}) = \int_0^\infty E(e^{iuX_T} | T = t) f_T(t) dt$

$$= \lambda \int_0^\infty (\varphi_{X_1}(u))^t e^{-\lambda t} dt. \quad (\text{let } \varphi_{X_1}(u) = e^{\Phi(u)})$$

$$= \lambda \int_0^\infty e^{-(\lambda - \Phi(u))t} dt$$

$$= \cancel{\frac{\lambda}{\lambda - \Phi(u)}} \frac{\lambda}{\lambda - \Phi(u)}$$

(3)

(a) Time: $Z_t \sim$ Possion(λ) process

So, $Z_t \sim Levy(0, 0, \lambda\delta(x-1)dx)$

Space: $W_t \sim BM$

$W_t \sim Levy(0,1,0)$ process

~~Sol~~ So, subordinated process

$Y_t := W_{Z_t}$ (Time changed B.M)

We can see Y_t is flat until Z_t jumps.

$$Z_t = \begin{cases} 0 & \text{until } 1^{st} \text{ jump} \\ 1 & \text{until } 2^{nd} \text{ jump} \\ 2 & \text{until } 3^{rd} \text{ jump} \\ \vdots \end{cases}$$

$$W_{Z_t} = \begin{cases} 0 & \text{until } Z_t = 1 \\ W_1 & \text{until } Z_t = 2 \\ W_2 & \text{until } Z_t = 3 \\ \vdots \end{cases}$$

It's jump size on the k^{th} jump is

$X_k - X_{k-1} \sim N(0,1)$

Therefore, W_{Z_t} is a compound possion(λ)

with ~~indepden~~ independent $N(0,1)$ jumps

(b) Let $T_i \sim Exp(\lambda)$ is Z's jump times.

$V_j \sim Exp(\alpha)$ $j = 1, 1.$

$$Z_t = \begin{cases} 0 & \text{until } T_1 \\ V_1 & \text{until } T_1 + T_2 \\ V_2 & \text{until } T_1 + T_2 + T_3 \\ \vdots \end{cases}$$

$$W_{Z_t} = \begin{cases} 0 & \text{until } Z_t = V_1 \\ W_{V_1} & \text{until } Z_t = V_1 + V_2 \\ W_{V_1+V_2} & \text{until } Z_t = V_1 + V_2 + V_3 \\ W_{V_1+V_2+V_3} & \text{until } Z_t = V_1 + V_2 + V_3 + V_4 \end{cases} \quad \begin{array}{|l} T_1 \\ T_1 + T_2 \\ T_1 + T_2 + T_3 \\ T_1 + T_2 + T_3 + T_4 \end{array}$$

W_{Z_t} is a pure jump process. its jump size on the k^{th} jump at time $\cancel{\sum_{j=1}^{k} T_j}$ $\sum_{j=1}^{k} T_j$ is $N(0, V_k)$

We need to find $\int_0^\infty dF_{X_s}(\alpha)\,\rho(ds)$

$dF_{X_s}(x) = f_{X_s}(x)\,dx$

$d\rho(s) = \rho(ds) = \rho(s)\,ds$ as long as $d\rho << ds$

We need to find $f_{X_s}(x)$, pdf of $N(0, exp(\alpha))$ in this case.

If $V_k \sim exp(\lambda)$ is given, we can do it.

Say $V_k = v$, then the pdf is

$$f(x,v) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}}$$

Therefore,

$$E(f(x,v)) = E\left(E(f(x,v) \mid V = v)\right)$$

$$= \int_{\infty}^{+\infty} f(x,v)\, g(v)\, dv \quad (g(v) \text{ is pdf of } exp(\lambda))$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}} \cdot \lambda \cdot e^{-\lambda v} \cdot dv$$

$$= \frac{\lambda}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{v}} e^{-\lambda v - \frac{x^2}{2v}} dv$$

(4)

(a) Let $f = \ln S_t$.

$$df = f_t dt + f_s dS^c + \frac{1}{2} f_{ss} d[S^c]_2 + f(t, S_t) - f(t, S_{t-})$$

$$dS^c_t = S_{t-} \cdot dX^c_t \qquad X^c_t = \mu t + \sigma W_t.$$

$$\ln S_t - \ln S_0 = \frac{1}{S_t} dS^c_t \cancel{*} - \frac{1}{2}\frac{1}{S_t^2} d[S^c]_2 + \sum_{0<s\le t}[\ln S_s - \ln S_{s-}]$$

$$\ln S_t = \ln S_0 + \frac{S_{t-}}{S_t} dX^c_t \cancel{*} - \frac{S_{t-}^2}{2S_t^2} \cdot d[X^c_t]_2 + \sum_{0<s\le t}[\ln S_s - \ln S_{s-}]$$

$$= \ln S_0 + \frac{S_{t-}}{S_t} (\mu dt + \sigma dW_t) - \frac{S_{t-}^2}{2S_t^2} \cdot \sigma^2 dt + \sum_{0<s\le t}[\ln S_s - \ln S_{s-}]$$

$$= \ln S_0 + \left(\frac{S_{t-} \mu}{S_{t-} + \Delta S} \cancel{(\mu dt +} - \frac{S_{t-}^2}{2(S_{t-} + \Delta S)^2} \right) \cdot dt$$

$$+ \frac{S_{t-}}{S_{t-} + \Delta S} \cdot \sigma \, dW_t + \sum_{0<s\le t}[\ln S_s - \ln S_{s-}]$$