MTH 9831

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(1).

(a) Solve the SDE $dU_t = \mu_1 U_t dt + \sigma_1 U_t dW_t \quad U_0 = 1$.

Let $y_t = \log U_t$.

$$*dy_t = \frac{1}{U_t} dU_t + \left(-\frac{1}{U_t^2}\right) d[U]_t$$

$$= \frac{1}{U_t}[\mu_1 U_t \cancel{*}dt + \sigma_1 U_1 dW_t] - \frac{1}{U_t^2}\sigma_1^2 U_t^2 dt.$$

$$= (\mu_1 - \sigma_1^2) dt + \sigma_1 dW_t.$$

$$y_t - y_0 = \int_0^t \mu_1 - \sigma_1^2 dt + \int_0^t \sigma_1 dW_s.$$

$$\log U_t - 0 = (\mu_1 - \sigma_1^2)t + \sigma_1 W_t.$$

$$U_t = e^{(\mu_1 - \sigma_1^2)t + \sigma_1 W_t}$$

(b) Solve the SDE $dV_t = a(t)dt + b(t)dW_t$, $V_0 = x$.

$$V_t - V_0 = \int_0^t a(t)dt + \int_0^t b(t) dW_t.$$

$$V_t = x + \int_0^t a(t)dt + \int_0^t b(t) dW_t.$$

(c) $\cancel{d}\ X_t = U_t V_t$

$$dX_t = \cancel{dV_t}\, U_t dV_t + V_t dU_t + d[U_t, V_t]$$

$$= U_t \cancel{(x + \int_0^t} (a(t)dt + b(t)dW_t) + V_t(\mu_1 U_t dt + \sigma_1 U_t dW_t)$$

$$+ \sigma_1 b(t) U_t dt.$$

$$= (U_t a(t) + \mu_1 U_t V_t + \sigma_1 b(t) U_t) dt + (b(t) U_t + \sigma_1 U_t V_t) dW_t$$

$$= \cancel{U_t}(a(t) U_t + \sigma_1 b(t) U_t + \mu_1 X_t) dt + (b(t) U_t + \sigma_1 X_t) dW_t$$

$$\begin{cases} (a(t) + \sigma_1 b(t)) U_t = \mu_0 \\ b(t) U_t = \sigma_0 \end{cases}$$

$$b(t) = \frac{\sigma_0}{U_t} \quad a(t) = \frac{\mu_0}{U_t} - \frac{\sigma_1 \sigma_0}{U_t} = \frac{1}{U_t}(\mu_0 - \sigma_0 \sigma_1)$$

(2). Let $z_t = \rho W_t + \sqrt{1-\rho^2} W_t'$, where W_t and W_t' are independent brownian motion

(ii) ~~Let $B_t = W_t$ $Z_t = \rho B_t + \sqrt{1-\rho^2} \cdot B_t$~~

SDEs.

$$dB_t = r B_t dt, \quad B_0 = 1$$

$$\frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dW_t, \quad S_0^1 = x^{①}$$

$$\frac{dS_t^2}{S_t^2} = \mu_2 dt + \rho \cdot \sigma_2 dW_t + \sqrt{1-\rho^2} \cdot \sigma_2 \cdot dW_t' \quad S_0^2 = x^{②}$$

$$d\tilde{B}_t = d\frac{B_t}{S_t^1} = \frac{1}{S_t^1} dB_t + \left(-\frac{B_t}{(S_t^1)^2}\right) \cdot dS_t^1 + \frac{1}{2} \cdot \left(-\frac{0 - 2S_t^1 B_t}{(S_t^1)^4}\right) \cdot d[S_t^1]_t + 0$$

$$= \frac{1}{S_t^1} r B_t \cdot dt - \frac{B_t}{(S_t^1)^2} \cdot (\mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t)$$

$$+ \frac{B_t}{(S_t^1)^3} \cdot \sigma_1^2 \cdot (S_t^1)^2 \cdot dt$$

$$= \tilde{B}_t (r - \mu_1 + \sigma_1^2) dt - \sigma_1 \tilde{B}_t dW_t.$$

$$d\tilde{S}_t^2 = d\frac{S_t^2}{S_t^1} = \frac{1}{S_t^1} dS_t^2 + \left(-\frac{S_t^2}{(S_t^1)^2}\right) \cdot dS_t^1 + \frac{1}{2}\left(-\frac{0 - 2S_t^1 \cdot S_t^2}{(S_t^1)^4}\right) \cdot d[S_t^1]_t$$

$$+ \frac{1}{2} \cdot 2 \cdot \left(-\frac{1}{(S_t^1)^2}\right) \cdot d[S_t^1, S_t^2]$$

$$= \frac{1}{S_t^1}(\mu_2 S_t^2 \cdot dt + \rho\sigma_2 \cdot S_t^2 dW_t + \sqrt{1-\rho^2}\, \sigma_2 \cdot S_2 dW_t') - \frac{S_t^2}{(S_t^1)^2} \cdot (\mu S_t^1 dt + \sigma_1 S_t^1 dW_t)$$

$$+ \frac{S_t^2}{(S_t^1)^3} \cdot (S_t^1)^2 \sigma_1^2 dt - \frac{1}{(S_t^1)^2} \cdot \sigma_1 S_t^1 \cdot \sigma_2 \cdot S_t^2 \cdot \rho \cdot dt$$

$$= \tilde{S}_t^2(\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2) dt + \tilde{S}_t^2(\rho\sigma_2 - \sigma_1) dW_t + \tilde{S}_t^2 \sqrt{1-\rho^2}\, \sigma_2 dW_t'$$

Let $dW_t + \theta_{B_t} dt = d\tilde{W}_t$

$dW_t' + \theta_{S_t^2} dt = d\tilde{W}_t'$

We need to solve: $\vec{\theta} = \begin{pmatrix} \theta_{B_t} \\ \theta_{S_t^2} \end{pmatrix}$

$$\begin{pmatrix} -\sigma_1 & 0 \\ \rho\sigma_2 - \sigma_1 & \sqrt{1-\rho^2}\,\sigma_2 \end{pmatrix} \begin{pmatrix} -\theta_{B_t} \\ -\theta_{S_t^2} \end{pmatrix} = \begin{pmatrix} \mu_1 - r - \sigma_1^2 \\ \mu_1 - \mu_2 + \rho\sigma_1\sigma_2 - \sigma_1^2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\sigma_1 & 0 \\ \rho\sigma_2 - \sigma_1 & \sqrt{1-\rho^2}\,\sigma_2 \end{pmatrix} \begin{pmatrix} \theta_{B_t} \\ \theta_{S_t^2} \end{pmatrix} = \begin{pmatrix} r + \sigma_1^2 - \mu_1 \\ \mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2 \end{pmatrix}$$

$$\theta_{B_t} = \frac{\mu_1 - r - \sigma_1^2}{\sigma_1} \qquad \theta_{S_t^2} = \frac{\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2 - (\rho\sigma_2 - \sigma_1)\theta_{B_t}}{\sqrt{1-\rho^2}\,\sigma_2}.$$

Since matrix $\begin{pmatrix} -\sigma_1 & 0 \\ \rho\sigma_2 - \sigma_1 & \sqrt{1-\rho^2}\,\sigma_2 \end{pmatrix}$ is full rank,

the $\vec{\theta}$ solution is unique, market is complete.

Suppose at T, the attainable claims pay-off

$f(B_T, S_T^1, S_T^2)$.

$$P_t = S_t \cdot \mathbb{E}^{Q}\left[\frac{f(B_T, S_T^1, S_T^2)}{S_T} \,\middle|\, \mathcal{F}_t \right]$$

Q is the equivalent martingale measure.

(b). Since dB_t doesn't have diffusion term.

B_t dynamic: $dB_t = rB_t dt \quad B_0 = 1$.

$$\frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 (d\hat{W} - \theta_{B_t} dt)$$
$$= (\mu_1 - \sigma_1 \theta_{B_t}) dt + \sigma_1 d\hat{W}$$
$$\cancel{= \sigma_1 d\hat{W} \qquad S_0^1 = x^1}$$
where $\cancel{\theta_{B_t} = \frac{\mu_1 - r}{\sigma_1}}$
$$= (\mu_1 - \mu_1 + r + \sigma_1^2) dt + \sigma_1 d\hat{W}$$
$$= (r + \sigma_1^2) dt + \sigma_1 d\hat{W} \qquad S_0^1 = x^1$$

$$\frac{dS_t^2}{S_t^2} = \mu_2 dt + \rho\sigma_2 (d\hat{W} - \theta_{B_t} dt) + \sqrt{1-\rho^2}\sigma_2 (d\hat{W}' - \theta_{S_t^2} dt)$$
$$= (\mu_2 - \rho\sigma_2\theta_{B_t} - \sigma_2\sqrt{1-\rho^2} \cdot \theta_{S_t^2}) dt + \rho\sigma_2 d\hat{W} + \sqrt{1-\rho^2}\sigma_2 d\hat{W}_2$$

where $\theta_{S_t^2}$, θ_{B_t} as stated above.

$$\cancel{= (\mu_2 - \rho\sigma_2 \frac{\mu_1 - r - \sigma_1^2}{\sigma_1} - \sigma_2\sqrt{1-\rho^2}}$$

$$\theta_{S_t^2} = \frac{\mu_2 - r - \frac{\rho\sigma_2}{\sigma_1}(\mu_1 - r)}{\sqrt{1-\rho^2}\,\sigma_2}$$

$$\frac{dS_t^2}{S_t^2} = (r + \rho\sigma_1\sigma_2) dt + \rho\sigma_2 d\hat{W} + \sqrt{1-\rho^2}\sigma_2 d\hat{W}'$$

(c).

$y = \log S_t^1$

$dy = \frac{1}{S_t^1} dS_t^1 - \frac{1}{2}\frac{1}{(S_t^1)^2} d[S_t^1]_t$

$= \frac{1}{S_t^1}\left(S_t^1(r+\sigma_1^2)dt + S_t^1\sigma_1 d\hat{W}\right) - \frac{1}{2}\frac{1}{(S_t^1)^2}(S_t^1)^2 \cdot \sigma_1^2 dt$

$y_t - y_0 = (r+\cancel{*}\sigma_1^2)t + \sigma_1 \hat{W}_t - \frac{1}{2}\sigma_1^2 dt$

$S_t^1 \cancel{*} \cancel{y_t} = x^1 e^{\cancel{*}\cancel{(r+\frac{1}{2}\sigma_1^2)t+\sigma_1\hat{W}_t}(r+\frac{1}{2}\sigma_1^2)t + \sigma_1\hat{W}_t}$

$\cancel{dS_t^2}\ S_t^2 = x^2 e^{(r+\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_2^2 \cancel{dt})t + \rho\sigma_2\hat{W}_t + \sqrt{1-\rho^2}\sigma_2\hat{W}'}$

$\mathbb{1}\{S_t^1 > S_t^2\} = \mathbb{1}\{\ln x^1 + (r+\frac{1}{2}\sigma_1^2)t\cancel{\}} + \sigma_1\hat{W}_t > \ln x_2^2 + (r+\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_2^2)t + \rho\sigma_2\hat{W}_t + \sqrt{1-\rho^2}\sigma_2\hat{W}'\}$

$= \mathbb{1}\{(\sigma_1 - \rho\sigma_2)\hat{W}_t > \ln\frac{x^2}{x^1} + (\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2)t + \sqrt{1-\rho^2}\sigma_2\hat{W}_t'\}$.

$\Rightarrow = \mathbb{1}\{\cancel{\hat{W}} \cancel{>} \overset{a}{z_1} > \frac{\ln\frac{x^2}{x^1} + (\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2)t}{\sqrt{t}(\sigma_1 - \rho\sigma_2)} + \frac{\sqrt{1-\rho^2}\sigma_2}{\sigma_1 - \rho\sigma_2} z_2\}$.

where z_1, z_2 is a standard normal random variable.

Let $\alpha = \frac{\ln\frac{x^2}{x^1}\cancel{t} + (\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2)t}{\sqrt{t}(\sigma_1 - \rho\sigma_2)}$ $\quad \beta = \frac{\sqrt{1-\rho^2}\sigma_2}{\sigma_1 - \rho\sigma_2}$.

$= \mathbb{1}\{z_1 > \alpha + \beta z_2\}$

Because in this EMM $\frac{P}{S_0^1} = \tilde{E}\left(\frac{(S_T^1 - S_T^2)^+}{S_T^1}\right)$

$P = S_0^1 \int_{-\infty}^{+\infty}\int_{\alpha+\beta z_2}^{+\infty} \cancel{S_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right) dz_1\, dz_2$

~~(2)~~.

(3)

(a). ~~Let $h_t(X) = \int \sigma$~~

$\sigma(X_t) = (1-X_t^2)\cdot\beta \quad h_t(X_t) = \int \frac{1}{\sigma(X_t)}dx = \frac{1}{\beta}\int \frac{1}{1-X^2}dx$

$h_t(x) = \frac{1}{\beta}\int_0^t \frac{1}{2}\left(\frac{1}{1+X} + \frac{1}{1-X}\right)dx = \left(\frac{1}{2}\ln|1+X| - \frac{1}{2}\ln|1-X|\Big|_0^t\right)\cdot\frac{1}{\beta}$

$h_t(X) = \frac{1}{2\beta}\ln\left|\frac{1+X_t}{1-X_t}\right| - 0$.

$dh_t(X) = \frac{1}{\beta}\frac{1}{1-X_t^2}dX_t + \frac{1}{2\beta}\left(\frac{0-(-2X_t)}{(1-X_t^2)^2}\right)\cdot d[X]_t$.

$\cancel{= -(\alpha+\beta^2}$

$= \frac{1}{\beta(1-X_t^2)}\cdot\left(-(\alpha+\beta^2X_t)(1-X_t^2)dt + \beta(1-X_t^2)dW_t\right) + \frac{1}{\beta}\frac{X_t}{(1-X_t^2)^2}\cdot\beta^2(1-X_t^2)^2dt$

$\cancel{= \frac{1}{\beta}(-(\alpha+\beta^2X_t)dt}$

$= \frac{1}{\beta}\left(-(\alpha+\beta^2X_t)dt + dW_t + \beta_t^2 X_t\cdot dt\right)$.

$= \frac{1}{\beta}(-\alpha\, dt + dW_t)$ ~~$= \frac{1}{2\beta}\ln\left|\frac{1+X_t}{1-X_t}\right|$~~

$\frac{1}{2\beta}\ln\left|\frac{1+X_t}{1-X_t}\right| = -\frac{\alpha}{\beta}\cdot t + \frac{1}{\beta}W_t$.

$X_t = \frac{e^{-2\alpha t+2W_t}+1}{e^{-2\alpha t+2W_t}-1}$

Or:

$X_t = \frac{e^{-2\alpha t+2W_t}-1}{e^{-2\alpha t+2W_t}+1}$.

(b) $\sigma(x) = \beta(1-X_t^2)$.

$$h_t(x) = \int_0^t \frac{1}{\sigma(X_s)} ds = \frac{1}{2\beta} \ln\left|\frac{1+X_t}{1-X_t}\right|$$

$$dh_t(x) = \frac{1}{\beta(1-X_t^2)} dX_t + \frac{1}{\beta} \frac{X_t}{\cancel{1+X}(1-X_t^2)^2} d[X]_t$$

$$\cancel{= \frac{1}{\beta(1-X_t^2)}}$$

From part (a)

we see

$$dh_t(x) = \frac{1}{\beta} dW_t$$

$$\frac{1}{\beta} W_t = \frac{1}{2\beta} \ln\left|\frac{1+X_t}{1-X_t}\right|$$

From part (a)

we know:

$$X_t = \frac{e^{2W_t}-1}{e^{2W_t}+1} \quad \text{or} \quad X_t = \frac{e^{2W_t}+1}{e^{2W_t}-1}.$$

(c) $\sigma(X_t) = 2\sqrt{X_t}$ $\sigma'(x) = X^{-\frac{1}{2}}$.

$$h_t(X) = \int_0^t \frac{1}{\sigma(X)} dX = \int_0^t \frac{1}{2} X^{-\frac{1}{2}} dX = X_t^{\frac{1}{2}} - X_0^{\frac{1}{2}}$$

$$dh_t(X) = \cancel{X\, 2\sqrt{}}\ \frac{1}{2} X^{-\frac{1}{2}} \cdot dX_t + \frac{1}{2}\left(-\frac{1}{4} \cdot X^{-\frac{3}{2}}\right) \cdot d[X]_t.$$

$$\cancel{= \frac{1}{2} X^{-\frac{1}{2}} (dt + 2\sqrt{X_t} dW_t) - \frac{1}{2}\frac{1}{4} X^{-\frac{3}{2}} \cdot 4 X_t dt}.$$

$$\cancel{= \left(\frac{1}{2} X^{-\frac{1}{2}} - X^{-\frac{1}{2}}\right) dt + dW_t}$$

$$= \frac{1}{2} X^{-\frac{1}{2}} (dt + 2\sqrt{X_t} dW_t) - \frac{1}{2} \times \frac{1}{4} X^{-\frac{3}{2}} \cdot 4 X_t dt.$$

$$= dW_t$$

So.

$$h_t(X) - h_0(X) = W_t = X_t^{\frac{1}{2}} - X_0^{\frac{1}{2}}.$$

$$X_t = \left(X_0^{\frac{1}{2}} + W_t\right)^2$$

(d) $\sigma_t(X) = b\,\mathrm{sech}\,X_t$.

$$h_t(X) = \int_0^t \frac{1}{b}\cosh X_s\,dX_s = \frac{1}{b}\sinh X_t - 0.$$

$$dh_t(X) = \frac{1}{b\,\mathrm{sech}\,X_t}\,dX_t + \frac{1}{2}\cdot\frac{1}{b}(\sinh X_t)\,d[X]_t.$$

$$= \frac{1}{b\,\mathrm{sech}\,X_t}\left[-\tanh X_t\left(a + \frac{b^2}{2}\mathrm{sech}^2 X\right)dt + b\,\mathrm{sech}\,X_t\,dW_t\right]$$

$$+ \frac{1}{2}\frac{1}{b}\sinh X_t\cdot b^2\,\mathrm{sech}^2 X_t\cdot dt$$

$$= -\frac{1}{b}\sinh X_t\left(a + \frac{b^2}{2}\mathrm{sech}^2 X\right)dt + \frac{1}{2}\frac{1}{b}\cdot b^2\sinh X_t\,\mathrm{sech}^2 X$$

$$+ dW_t.$$

$$= -\frac{a}{b}\sinh X_t\cdot dt + dW_t.$$

$$= -a\cdot h_t(X)\,dt + dW_t.$$

Let $Y_t = e^{\int_0^t a\,ds}\cdot h_t(X) = e^{at}\cdot h_t(X)$

$$dY_t = e^{at}\,dh_t(X) + a\cdot Y_t\cdot dt$$

$$= e^{at}(-a h_t(X)\,dt + dW_t) + a\cdot e^{at}h_t(X)\cdot dt.$$

$$= e^{at}\cdot dW_t.$$

$Y_t - Y_0 = \int_0^t e^{as}\,dW_s.$ $\quad \cancel{Y_0 = e^0 h_0(X) = \frac{1}{b}\sin X\,\frac{1}{b}\sinh}$

$Y_0 = \frac{1}{b}\sinh X_0.$

$Y_t = e^{at}h_t(X) = \frac{1}{b}\sinh X_0 + \int_0^t e^{as}\,dW_s$

(page 10) $h_t(X) = \frac{e^{-at}}{b}\sinh X_0 + e^{-at}\int_0^t e^{as}\,dW_s \quad X_t = \sinh^{-1}\left(e^{-at}\sinh X_0 + b e^{-at}\int_0^t e^{as}\,dW_s\right)$