

Matrices and Vector Spaces

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June 23, 2021

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1 Vector Spaces

1.1 Conditions for a Vector Space

A set of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ form a linear vector space if:

- i. Set is closed under commutative and associative addition
- ii. Set is closed under distributive and associative multiplication by a scalar
- iii. Set contains a null vector
- iv. Set contains a unity vector
- v. A corresponding negative exists for all vectors

1.2 Vector Space Definitions

Span of a Set of Vectors

The set of all vectors that can be expressed as a linear sum of the original set

Linear Dependence

A set of vectors are linearly dependent if there exists a linear combination equal to the null vector

Basis Vectors

In an N-dimensional vector space V any set of N linearly independent vectors form a set of basis vectors for V

2 Inner Product

2.1 Properties of the Inner Product

Using an orthonormal basis $\hat{\mathbf{e}}$ we can find the inner product as:

$$\langle \mathbf{a} | \mathbf{b} \rangle = \langle a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + \dots + a_N \hat{\mathbf{e}}_N | b_1 \hat{\mathbf{e}}_1 + b_2 \hat{\mathbf{e}}_2 + \dots + b_N \hat{\mathbf{e}}_N \rangle \quad (1)$$

$$= \sum_{i=1}^N a_i^* b_i \langle \hat{\mathbf{e}}_i | \hat{\mathbf{e}}_j \rangle + \sum_{i=1}^N \sum_{j \neq i}^N a_i^* b_j \langle \hat{\mathbf{e}}_i | \hat{\mathbf{e}}_j \rangle \quad (2)$$

$$= \sum_{i=1}^N a_i^* b_i \quad (3)$$

2.2 Inner Product Definitions

Orthogonality

Vectors are orthogonal if $\langle \mathbf{a} | \mathbf{b} \rangle = 0$

Vector Norm

The norm of a vector is given by $|\mathbf{a}| = \langle \mathbf{a} | \mathbf{a} \rangle^{\frac{1}{2}}$

Orthonormality

A basis is orthonormal is $\langle \hat{\mathbf{e}}_i | \hat{\mathbf{e}}_j \rangle = \delta_{i,j}$

3 Linear Operators

In a vector space V with basis vectors $\hat{\mathbf{e}}_i$ we can define the action of the operator A as:

$$A \hat{\mathbf{e}}_j = \sum_{i=1}^N A_{ij} \hat{\mathbf{e}}_i \quad (4)$$

where A_{ij} is the ith component of the vector $A \hat{\mathbf{e}}_j$ in this basis and collectively the numbers A_{ij} are known as the components of the linear operator in the $\hat{\mathbf{e}}_i$ basis.

3.1 Properties of the Linear Operators

Distributivity

$$(A + B) \mathbf{x} = A \mathbf{x} + B \mathbf{x}$$

Associativity with a scalar

$$(\lambda A) \mathbf{x} = \lambda (A \mathbf{x})$$

Associativity with another linear operator

$$(AB) \mathbf{x} = A(B \mathbf{x})$$

If a linear operator is singular it does not have an inverse

4 Matrices

If a linear operator A transforms vectors from an N dimensional vector space into a vector in an M dimensional vector space, then the operator A can be represented by the matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{pmatrix} \quad (5)$$

4.1 Basic Matrix Algebra

Using the properties of linear operators we can show that matrices obey:

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad (6)$$

$$(\lambda A)_{ij} = \lambda A_{ij} \quad (7)$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \quad (8)$$

The transpose of a matrix product is given by:

$$(AB)^T = B^T A^T \quad (9)$$

as proven by:

$$(AB)_{ij}^T = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij} \quad (10)$$

4.2 Trace of a Matrix

The trace of a matrix is denoted by:

$$Tr(A) = \sum_{i=1}^N A_{ii} \quad (11)$$

The trace of a product of two matrices is independent of the order, as seen by:

$$Tr(AB) = \sum_{i=1}^N (AB)_{ii} = \sum_{i=1}^N \sum_{j=1}^N A_{ij} B_{ji} = \sum_{i=1}^N \sum_{j=1}^N B_{ji} A_{ij} = \sum_{j=1}^N (BA)_{jj} = Tr(BA) \quad (12)$$