Group Theory

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1 Fundamentals of a Group

1.1 What is a Group?

Groups are number systems with a set of numbers/symbols and defined operations, for example the set of integers:

$$\mathbb{Z} = \{0, +1, -1, +2, -2, ...\}$$

can be defined with the operation of addition (an example of a composition law) and remain closed.

Another group could be:

$$G = \{\phi, \Phi, D\}$$

with the a general composition law of

$$x \circ y$$

sometimes written as $x + y$, $x \times y$, xy

1.2 What is a Composition Law?

Composition laws can be defined with a composition table:

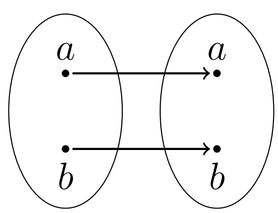
$\circ$	ϕ	Φ	D
ϕ	$\phi \circ \phi$	$\phi \circ \Phi$	$\phi \circ \mathrm{D}$
Φ	$\Phi \circ \phi$	$\Phi \circ \Phi$	$\Phi \circ \mathrm{D}$
D	$\mathrm{D} \circ \phi$	$\mathrm{D} \circ \Phi$	$\mathrm{D} \circ \mathrm{D}$

The composition table must obey the following axioms:

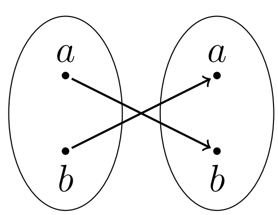
1. Closure - All composition products exist within the group
2. Identity - $e \circ g = g \circ e = g$
3. Associativity - $(x \circ y) \circ z = x \circ (y \circ z)$
4. Inverses - $\forall g \in G \, \exists \, g^{-1} \; s.t. \; g \circ g^{-1} = g^{-1} \circ g = e$

1.3 Associativity

Building a group that obeys associativity can be tricky, fortunately there is a method for this problem. Start by defining the set of permutations / bijective mappings for the set $\{a, b\}$ such that:



i

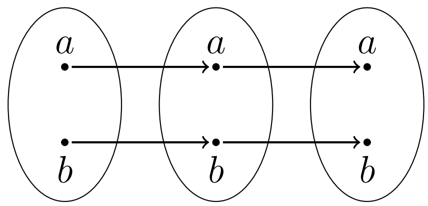


τ

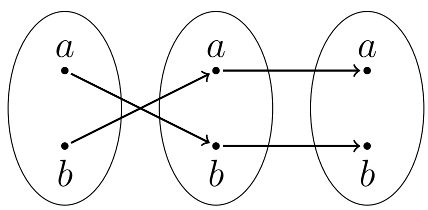
With both of these mappings defined we can then define a group with the group $G = \{i, \tau\}$ with the following composition table:

$\circ$	i	τ
i	i	τ
τ	τ	i

The compositions can be seen in the following diagrams:



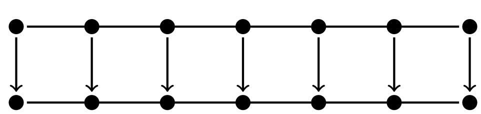
$i \circ i$



$i \circ \tau$

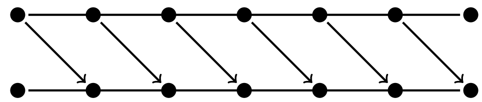
This approach represents compositions as compositions of set permutations. Making sure that the original set's compositions obey the following composition laws of the set permutations will result in associativity holding true.

Applying this perspective to the group of real integers Z we can view the number line as a sequence of points and the integers as mappings or permutations between those sequences. For example 0 can be visualised as the following mapping:



0

And +1 can be visualised as:



$+1$

1.4 Identity

An identity is required for a valid group, it is often represented with $e, i, 0, 1$. The identity is defined to compose with any other group elements to produce that group element. Ie it holds that:

$$e \circ g = g \circ e = g$$

2 Symmetric Groups

A symmetric group is denoted as S_n where $n \in \mathbb{N}$. The group is the number of set permutations within the set of natural numbers less than or equal to n. The order of a group is the number of elements in the group, which follows the law:

$$|S_n| = n!$$

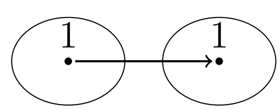
2.1 S_1

$$S_1 = \{i\}$$

with the composition table:

$\circ$	i
i	i

which corresponds to the following set permutation:



i

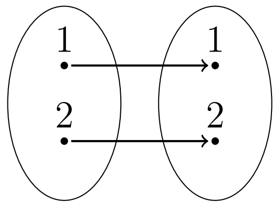
2.2 S_2

$$S_2 = \{i, \tau\}$$

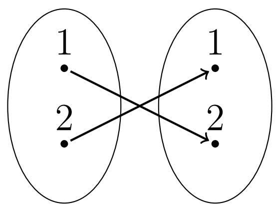
with the following composition table:

∘	i	τ
i	i	τ
τ	τ	i

which corresponds to the following set permutations:



i



τ

2.3 S_3

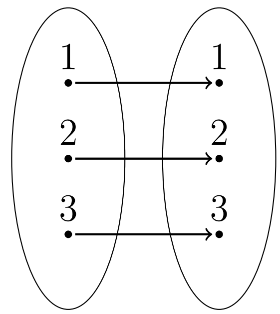
$$S_3 = \{i, \tau_{12}, \tau_{13}, \tau_{23}, \sigma, \sigma^2\}$$

with the following composition table:

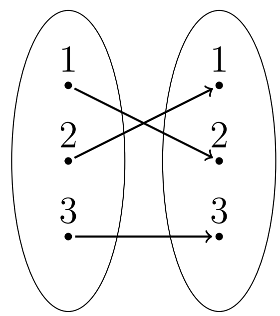
∘	i	τ_{12}	τ_{13}	τ_{23}	σ	σ^2
i	i	τ_{12}	τ_{13}	τ_{23}	σ	σ^2
τ_{12}	τ_{12}	i	σ	σ^2	τ_{13}	τ_{23}
τ_{13}	τ_{13}	σ^2	i	σ	τ_{23}	τ_{12}
τ_{23}	τ_{23}	σ	σ^2	i	τ_{12}	τ_{13}
σ	σ	τ_{23}	τ_{12}	τ_{13}	σ^2	i
σ^2	σ^2	τ_{13}	τ_{23}	τ_{12}	i	σ

Note: this composition table is non-symmetric, therefore it is non-commutative.

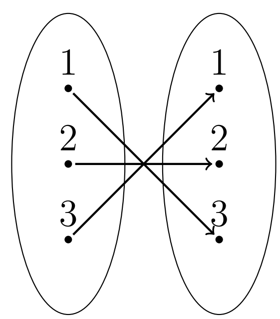
as it corresponds to the following set permutations:



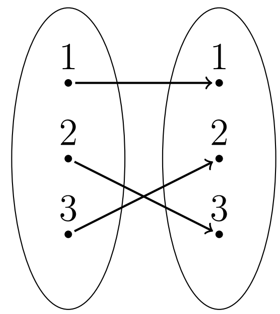
i



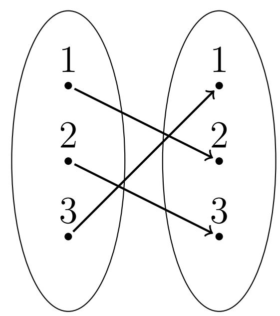
τ_{12}



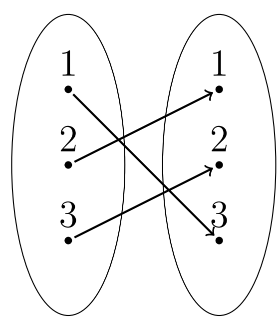
τ_{13}



τ_{23}



σ



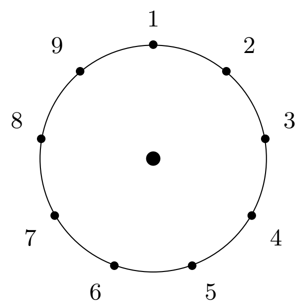
σ^2

3 Finite Cyclic Groups

Finite cyclic groups typically are denoted by C_n, where $n \in \mathbb{N}$. They contain all the cyclic permutations of the set of natural numbers less than or equal to n.

A useful way to understand the finite cyclic groups is a wheel pinned at the centre with the numbers 1 to n arranged clockwise around the edge, as seen below. The group C_n corresponds to the set of permutations corresponding to rotations of the wheel.

For example C_9 corresponds to all the possible rotational states of the following circle:



In general we have:

$$C_n = \{i, \sigma, \sigma^2, ..., \sigma^{n-1}\}$$

The first two cyclic groups for $n = 1, 2$ are equivalent to the symmetric groups, the similarities end at $n = 3$ however, with the group given as:

$$C_3 = \{i, \sigma, \sigma^2\}$$

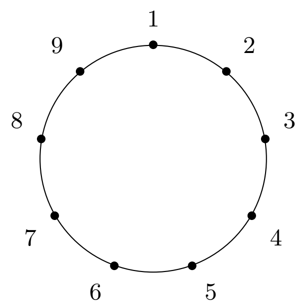
with the following composition table:

$\circ$	i	σ	σ^2
i	i	σ	σ^2
σ	σ	σ^2	i
σ^2	σ^2	i	σ

which corresponds to the set permutations defined in the symmetric group section.

4 Dihedral Groups

The dihedral groups are denoted by D_n, where $n \in \mathbb{N}$. They contain all the permutations within the finite cyclic group, with the addition of permutations achieved by flipping the elements in a second axis. This can be best understood by considering all the possible permutations of a disk when the disk is rotated and when it is lifted off the 2D plane and flipped over. For example D_9 can be represented by the following circle below, which has no pin at the center.



The order of the dihedral group is given as:

$$|D_n| = 2n$$

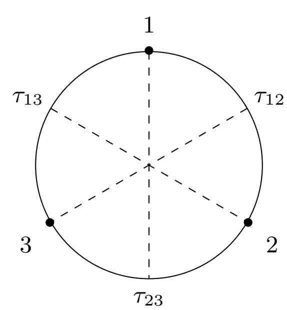
It can be shown that:

$$D_1 = S_1 = C_1$$

$$D_2 = S_2 = C_2$$

$$D_3 = S_3$$

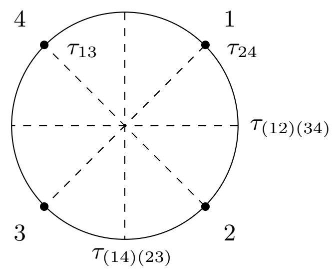
For $n = 3$ we can see the divergence from the finite cyclic group arise from imagining a flip of the disk along any of the following dashed lines. A flip on these lines results in a transpose, ie the $\tau_{12}, \tau_{13}, \tau_{23}$ elements.



At $n = 4$ we see divergence from the symmetric group as well.

$$D_4 = \{i, \sigma, \sigma^2, \sigma^3, \tau_{13}, \tau_{24}, \tau_{(12)(34)}, \tau_{(14)(23)}\}$$

with the transposed permutations (made possible by flips along the third dimension) shown below.



5 Group Isomorphisms

For an example of isomorphic groups consider the group S_2 which is defined as:

$$S_2 = \{i, \tau\}$$

with the following composition table:

$\circ$	i	τ
i	i	τ
τ	τ	i

We can then define another group $G = \{1, -1\}$ with the composition law of multiplication, giving us the following composition table:

$\times$	1	-1
1	1	-1
-1	-1	1

We can then say that S_2 is isomorphic to G, this can be written as:

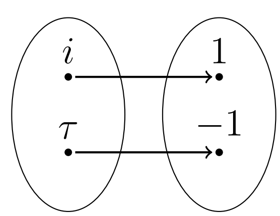
$$S_2 \approx G$$

This is because to change one group into the other group you'd just need to relabel the symbols.

To show a generic group is isomorphic to another we must show that there is an isomorphism (a relabelling / 1-to-1 mapping function) that exists between the groups. Therefore an isomorphism can be expressed as ϕ in the following:

$$\phi : S_2 \mapsto G$$

where ϕ is a bijective and surjective mapping, in this case it can be illustrated as:



$$\phi$$

For the mapping to be an isomorphism we require that the composition table is entirely isomorphic. For an aribtrary group and the isomorphism ϕ we would have the following composition tables:

$\circ$	... y ...
... x ...	x $\circ$ y

$\circ$	... $\phi(y)$ ...
... $\phi(x)$ ...	$\phi(x) \circ \phi(y)$

For the isomorphism to hold we require that the elements of the composition table, which must be elements of the group due to closure, follow the ϕ mapping. Therefore we have:

$$\phi(x) \circ \phi(y) = \phi(x \circ y) \;\; \forall x, y \in G$$

5.1 Example of a Non-Isomorphic Group

Up to order 3 all groups are isomorphic, the first example of non-isomorphic groups arises at $n = 4$.

Consider the finite cyclic group:

$$C_4 = \{i, \sigma, \sigma^2, \sigma^3\}$$

with the composition table:

$\circ$	i	σ	σ^2	σ^3
i	i	σ	σ^2	σ^3
σ	σ	σ^2	σ^3	i
σ^2	σ^2	σ^3	i	σ
σ^3	σ^3	i	σ	σ^2

Now consider the Klein 4 group, denoted by V_4. This group can be understood by considering the set of natural numbers up to and including 4. The group includes the identity permutation and a set of transpositions.

$$V_4 = \{i, \tau_{12}, \tau_{34}, \tau_{(12)(34)}\}$$

The composition table for this group is given as:

$\circ$	i	τ_{12}	τ_{34}	$\tau_{(12)(34)}$
i	i	τ_{12}	τ_{34}	$\tau_{(12)(34)}$
τ_{12}	τ_{12}	i	$\tau_{(12)(34)}$	τ_{34}
τ_{34}	τ_{34}	$\tau_{(12)(34)}$	i	τ_{12}
$\tau_{(12)(34)}$	$\tau_{(12)(34)}$	τ_{34}	τ_{12}	i

If we now compare C_4 with V_4 we can see the structure is not the same. This can be seen in the case of squaring a permutation, where in V_4 this results in the identity permutation whilst in C_4 they do not necessarily. Therefore these groups are not isomorphic.

5.2 Isomorphism Classes

An isomorphic class is the class of groups that are all isomorphic to one another.

5.3 Isomorphism between addition and multiplication

Consider the group of addition of real numbers denoted by G:

$$G = (\mathbb{R}, +)$$

and the group of positive real numbers under multiplication:

$$G' = (\mathbb{R}^+, \times)$$

To show the groups are isomorphic we need to demonstrate that a mapping function ϕ exists such that:

$$\phi : G \mapsto G'$$

This mapping function can be shown to be the exponential function:

$$\phi(x) = e^x$$

We know that e^x is a one-to-one function, therefore it is a valid bijective mapping. To demonstrate that it is a valid isomorphism we need to show that:

$$\phi(x \circ y) = \phi(x) \circ \phi(y)$$

which can be seen to be true as it is a fundamental property of e^x that:

$$e^{x+y} = e^x e^y$$

Therefore addition of the real numbers is isomorphic to multiplication of the positive real numbers.