Matrices and Vector Spaces

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Contents

1 Vector Spaces **1**
1.1 Conditions for a Vector Space 1
1.2 Vector Space Definitions 1

2 Inner Product **2**
2.1 Properties of the Inner Product 2
2.2 Inner Product Definitions 2

3 Linear Operators **2**
3.1 Properties of the Linear Operators 2

4 Matrices **3**
4.1 Basic Matrix Algebra 3
4.2 Trace of a Matrix 3

1 Vector Spaces

1.1 Conditions for a Vector Space

A set of vectors **a**,**b**,**c**,... form a linear vector space if:

i. Set is closed under commutative and associative addition

ii. Set is closed under distributative and associative multiplication by a scalar

iii. Set contains a null vector

iv. Set contains a unity vector

v. A corresponding negative exists for all vectors

1.2 Vector Space Definitions

Span of a Set of Vectors

The set of all vectors that can be expressed as a linear sum of the original set

Linear Dependence
A set of vectors are linearly independent if there exists a linear combination equal to the null vector

Basis Vectors
In an N-dimensional vector space V any set of N linearly independent vectors form a set of basis vectors for V

2 Inner Product

2.1 Properties of the Inner Product

Using an orthonormal basis $\hat{\mathbf{e}}$ we can find the inner product as:

$$\begin{aligned} \langle \mathbf{a}|\mathbf{b}\rangle &= \langle a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + ... + a_N\hat{\mathbf{e}}_N | b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + ... + b_N\hat{\mathbf{e}}_N\rangle && (1) \\ &= \Sigma_{i=1}^N a_i^* b_i \langle \hat{\mathbf{e}}_i|\hat{\mathbf{e}}_j\rangle + \Sigma_{i=1}^N \Sigma_{j\neq i}^N a_i^* b_i \langle \hat{\mathbf{e}}_i|\hat{\mathbf{e}}_j\rangle && (2) \\ &= \Sigma_{i=1}^N a_i^* b_i && (3) \end{aligned}$$

2.2 Inner Product Definitions

Orthogonality
Vectors are orthogonal if $\langle \mathbf{a}|\mathbf{b}\rangle = 0$

Vector Norm
The norm of a vector is given by $|\mathbf{a}| = \langle \mathbf{a}|\mathbf{a}\rangle^{\frac{1}{2}}$

Orthonormality
A basis is orthonormal is $\langle \hat{\mathbf{e}}_i|\hat{\mathbf{e}}_j\rangle = \delta_{i,j}$

3 Linear Operators

In a vector space V with basis vectors $\hat{\mathbf{e}}_i$ we can define the action of the operator A as:

$$A\hat{\mathbf{e}}_j = \Sigma_{i=1}^N A_{ij}\hat{\mathbf{e}}_i \quad (4)$$

where A_{ij} is the ith component of the vector $A\hat{\mathbf{e}}_j$ in this basis and collectively the numbers A_{ij} are known as the components of the linear operator in the $\hat{\mathbf{e}}_i$ basis.

3.1 Properties of the Linear Operators

Distributivity
(A + B) $\mathbf{x}$ = A $\mathbf{x}$ + B $\mathbf{x}$

Associativity with a scalar
$(\lambda A)\mathbf{x} = \lambda(A\mathbf{x})$

Associativity with another linear operator
$(AB)\mathbf{x} = A(B\mathbf{x})$

If a linear operator is singular it does not have an inverse

4 Matrices

If a linear operator A transforms vectors from an N dimensional vector space into a vector in an M dimensional vector space, then the operator A can be represented by the matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} & ... & A_{1N} \\ A_{21} & A_{22} & ... & A_{2N} \\ ... & ... & ... & ... \\ A_{M1} & A_{M2} & ... & A_{MN} \end{pmatrix} \tag{5}$$

4.1 Basic Matrix Algebra

Using the properties of linear operators we can show that matrices obey:

$$(A+B)_{ij} = A_{ij} + B_{ij} \tag{6}$$

$$(\lambda A)_{ij} = \lambda A_{ij} \tag{7}$$

$$(AB)_{ij} = \Sigma_k A_{ik} B_{kj} \tag{8}$$

The transpose of a matrix product is given by:

$$(AB)^T = B^T A^T \tag{9}$$

as proven by:

$$(AB)^T_{ij} = (AB)_{ji} = \Sigma_k A_{jk} B_{ki} = \Sigma_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij} \tag{10}$$

4.2 Trace of a Matrix

The trace of a matrix is denoted by:

$$Tr(A) = \Sigma_{i=1}^N A_{ii} \tag{11}$$

The trace of a product of two matrices is independent of the order, as seen by:

$$Tr(AB) = \Sigma_{i=1}^N (AB)_{ii} = \Sigma_{i=1}^N \Sigma_{j=1}^N A_{ij} B_{ji} = \Sigma_{i=1}^N \Sigma_{j=1}^N B_{ji} A_{ij} = \Sigma_{j=1}^N (BA)_{jj} = Tr(BA) \tag{12}$$