Advanced Quantum Information: Quantum States Measurements and Evolution

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Systems and States

Classical State Space

Physical System - A collection of physical degrees of freedom that allow a closed, consistent description

State of a System - An equivalence class of experimental procedures, where each experimental procedure in the class gives the same empirical results for all possible measurements on S

A simple classical system would be an object that can either be **red**, **green** or **blue**. The state of the object would then be the probability distribution **r** over the three colours:

$$\mathbf{r} = (P(\text{red}), P(\text{green}), P(\text{blue}))$$

We then have the following ***pure states***:

$$\begin{aligned} \mathrm{e}_0 &= (1,0,0) \\ \mathrm{e}_1 &= (0,1,0) \\ \mathrm{e}_2 &= (0,0,1) \end{aligned}$$

However we also may have some uncertainty about the state, resulting in a ***mixed state***, for example we may have the ***maximally mixed state***:

$$\mathbf{r}_{mm} = (1/3, 1/3, 1/3) = (1/3)\mathbf{e}_0 + (1/3)\mathbf{e}_1 + (1/3)\mathbf{e}_2$$

Since any state can be expressed as a combination of pure states we can understand the ***state space*** as the set of linear combinations of the pure states. For a probability distribution we constrain this set to those with unit norm, otherwise known as a ***convex combination***.

A convex combination of two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{z} = p\mathbf{x} + (1-p)\mathbf{y}$$

where $0 \leq p \leq 1$. We can visualise $\mathbf{z}$ as lying on the line segment joining $\mathbf{x}$ and $\mathbf{y}$. The convex combination can be extended by introducing a new vector $\mathbf{w}$, such that

$$\mathbf{z}' = q\mathbf{w} + (1-q)\mathbf{z} = q\mathbf{w} + (1-q)p\mathbf{x} + (1-q)(1-p)\mathbf{y}$$

where now $\mathbf{z}'$ is visualised as lying inside the triangle formed by the three points $\{\mathbf{w}, \mathbf{x}, \mathbf{y}\}$. With this picture we can also understand the maximally mixed point $\mathbf{r}_{mm}$ as lying at the centre of the triangle.

As we move to quantum systems we will keep the notion of a convex set, which can be formally defined as a set $\mathcal{C}$ such that for any two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{C}$ we also have that $px + (1-p)\mathbf{y}$ is also in $\mathcal{C}$ for any $0 \leq p \leq 1$.

It is useful to note that a ***simplex*** is a simple convex shapes previously discussed, for example in 1D it is the line segment and in 2D it becomes a triangle.

The notion of classical states $\mathbf{r} = (r_0, r_1, ..., r_{n-1})$ can be formally summarised by requiring the following two properties

- Non-Negativity $\implies r_i \geq 0 \quad \forall i$
- Normalisation $\implies \sum_i r_i = 1$

Evolution and Measurement in Classical Systems

There are only two operations performed on physical systems:

- Transformations

- Required to be linear to satisfy probability theory
  $L(p\mathbf{x} + (1-p)\mathbf{y}) = pL(\mathbf{x}) + (1-p)L(\mathbf{y})$

- Measurements
  - Project the state onto a given state, ie $\mathrm{P(k)} = \mathbf{m}_k \cdot \mathbf{r}$
    $\mathbf{m}_k$ is from the collection of vectors $\{\mathbf{m}_0, \mathbf{m}_1, ..., \mathbf{m}_{d-1}\}$ which correspond to the d outcomes of the measurement and satisfies the following properties
    - Non-Negativity $\implies$ $\text{components of } \mathbf{m}_k \geq 0 \quad \forall k$
    - Normalisation $\implies$ $\sum_k \mathbf{m}_k = (1, 1, ..., 1)$

Quantum Theory

Moving from classical to quantum theory requires the following substitutions:

1. Classical vectors $\rightarrow$ Hermitian matrices
2. Non-negativity now pertains to the eigenvalues of these matrices
3. Normalisation now applies to the trace of the matrices
4. Classical vector dot products now become traces, $\mathbf{x} \cdot \mathbf{y} \rightarrow \mathrm{tr}(XY)$

We can formalise our description of quantum theory with the following definitions

- States
  - Non-negativity $\implies$ $\mathrm{eigs}(\rho) \geq 0$
  - Normalisation $\implies$ $\mathrm{tr}(\rho) = 1$
- Measurements (with m outcomes)
  - Non-negativity $\implies$ $\mathrm{eigs}(M_i) \geq 0 \quad \forall i \in 0...m-1$
  - Normalisation $\implies$ $\sum_i M_i = \mathbf{I}$
- Evolutions
  - Linear transformation $\rho \rightarrow \mathcal{E}(\rho)$ such that the transformed state is another valid quantum state

Classical Theory Inside of Quantum Theory

Classical states exist within quantum state space, as seen by:

$$\mathbf{r} \rightarrow \rho = \mathrm{diag}(r_0, r_1, ..., r_d) = \begin{pmatrix} r_0 & 0 & 0 & ... & 0 \\ 0 & r_1 & 0 & ... & 0 \\ 0 & 0 & r_2 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & r_d \end{pmatrix}$$

The classical states are only fully diagonal when we work in the computational basis, quantum theory introduces non-diagonal states due to there being superpositions of classical states.

State Space of a Qubit

A qubit can be represented by a 2×2 Hermitian matrix, typically denoted by

$$\rho = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$$

where a and b are real and c is complex, meaning that we have 4 real parameters. From the previous discussion we also know that the matrix must satisfy $\mathrm{eigs}(\rho) \geq 0$ and $\mathrm{tr}(\rho) = 1$.

It is convenient to decompose the matrix in the Pauli basis, defined by

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

such that we have

$$\rho = \frac{1}{2}\left(\mathrm{I} + xX + yY + zZ\right) = \frac{1}{2}\left(\mathrm{I} + \mathbf{r} \cdot \sigma\right)$$

Since we can show that $\mathrm{eigs}(\rho) = \frac{1}{2}\left(1 \pm \sqrt{x^2 + y^2 + z^2}\right)$ we find that in order to satisfy the requirement that $\mathrm{eigs}(\rho) \geq 0$ we require that the norm of the ***Bloch vector*** $\mathbf{r}$ is less than or equal to unity.

Analogeous to pure classical states, pure quantum states lie on the surface of the Bloch sphere (the quantum version of classical simplex). This means that they can be written in the form

$$\rho = |\psi\rangle\langle\psi|$$

where $|\psi\rangle$ is a normalised vector.