Linear Optics Quantum Computation: an Overview

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Quantum Information Processing with Linear Optics

Quantum Optics

Classical Electromagnetism

EM waves emerge from the source free Maxwell equations

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t)$$

This has the positive and negative field solutions

$$\mathbf{E}(\mathbf{r}, t) = i \sum_k \left(\frac{\hbar \omega_k}{2} \right)^{1/2} \left[a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + a_k^* \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \right]$$

The energy associated with these fields is given by

$$H = \frac{1}{2} \int_V \left(\mathbf{E}^2 + \mathbf{B} \right) \mathrm{d}\mathbf{r} = \sum_k \left(\frac{\hbar \omega_k}{2} \right) a_k a_k^*$$

Quantisation

To quantise the field $\mathbf{E}(\mathbf{r}, t)$ we convert the coefficients a_k into mode operators (aka annihilation and creation operators), which satisfy the canonical commutation relation

$$\left[\hat{a}_i, \hat{a}_j^\dagger \right] = \delta_{ij}$$

This results in the energy defined as

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + 1/2\right)$$

which resembles a harmonic oscillator.

The eigenstates of the quantised field Hamiltonian are Fock states $|n\rangle$, corresponding to a mode of the field. We can then define the number operator as

$$\hat{n}|n\rangle = \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$$

Given the canonical commutation relation we can define the action of the mode operators on the Fock states as

$$\begin{aligned} \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned}$$

We can express the annihilation operator as a linear combination of two Hermitian operators, eg position and momentum operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{p} + i\hat{p}\right)$$

where $\hat{Q}_1$ is the in-phase component and $\hat{Q}_2$ is the out of phase component. These operators then satisfy the following relations relation

$$\begin{aligned} [\hat{q}, \hat{p}] &= i \\ \Delta\hat{q}\Delta\hat{p} &\geq \frac{1}{2} \end{aligned}$$

where the second relation corresponds to the Heisenberg uncertainty principle.

For a general Fock state $|n\rangle$ we have

$$\langle Q_1|n\rangle = \Psi_n(Q_1) = \frac{H_n(\frac{Q_1}{\sqrt{2}})}{\sqrt{2^n n!}\sqrt{\pi}} \exp\left(-\frac{Q_1^2}{4}\right)$$

were $H_n(x)$ are the Hermite polynomials.

Coherent States

The uncertainty of a state with respect to an operator $\hat{A}$ is defined as

$$(\Delta\hat{A})^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

The superposition of Fock states with minimum uncertainty is known as the coherent state $|\alpha\rangle$, where $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. We can then define the coherent state in terms of the displacement operator $D(\alpha)$

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger)|0\rangle$$

💡 To show the final expression use the Campbell-Baker-Haussdorff identity

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

Just as we previously decomposed the annihilation operator we can define the following decomposition of the complex amplitude α

$$\alpha = \frac{1}{\sqrt{2}}\left(q_0 + ip_0\right)$$

We can then express the displacement operator in terms of position and momentum operators

$$\hat{D}(\alpha) = \exp(ip_0\hat{q} - iq_0\hat{p}) = \exp\left(-\frac{ip_0q_0}{2}\right)\exp\left(ip_0\hat{q}\right)\exp\left(-iq_0\hat{p}\right)$$

Given that in the position representation $\hat{p} = -i\partial/\partial q$ we can understand

$$\exp(-iq_0\hat{p})\Psi(q) = \exp\left(-q_0\frac{\partial}{\partial q}\right)\Psi(q) = \Psi(q - q_0)$$

as a translation operator.

💡 To see this try differentiating both sides

With this result we can see that a displaced vacuum state (a coherent state) is given by

$$\begin{aligned}\Psi_\alpha(q) &= \hat{D}(\alpha)\Psi(q)\\ &= \Psi_0(q-q_0)\exp\left(ip_0q - \frac{ip_0q_0}{2}\right)\\ &= \pi^{-1/4}\exp\left(-\frac{(q-q_0)^2}{2} + ip_0q - \frac{ip_0q_0}{2}\right)\end{aligned}$$

Similarly in momentum space we see

$$\Psi_\alpha(p) = \pi^{-1/4}\exp\left(-\frac{(p-p_0)^2}{2} - iq_0p + \frac{ip_0q_0}{2}\right)$$

Given these expressions we can see that the wavefunction of a coherent state is a Gaussian in phase space.

Squeezed States

We can reduce the width of this 2D Gaussian with respect to one quadrature at the expense of increased uncertainty in the other quadrature. To perform this "squeezing" we apply the Squeeze operator

$$S(\epsilon) = \exp\left(\frac{1}{2}\epsilon^* \hat{a}^2 - \frac{1}{2}\epsilon(\hat{a}^\dagger)^2\right)$$

where $\epsilon = re^{i\varphi}$ is some complex number defining the axis of the squeezing operation.

Linear Optics

An optical component is linear if its output modes are linear combinations of the input modes, eg

$$\hat{b}_j^\dagger = \sum_k M_{jk}\hat{a}_k^\dagger$$

Linear optical components are constructed from phase shifters and beam splitters, which apply the following transformations

$$\begin{aligned} U_\phi &= e^{in\phi} \\ U_{BS} &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix} = \exp\left(-i\theta\left(e^{i\phi}\hat{a}_k^\dagger \hat{a}_l + e^{-i\phi}\hat{a}_k\hat{a}_l^\dagger\right)\right) \end{aligned}$$

the transformation on an input mode l can be understood as the transformation $\hat{a}_l^\dagger|0\rangle \to \sum_m U_{ml}a_m^\dagger|0\rangle$. For example if we have the state $|mn\rangle_{12}$ incident on a beam splitter we see the following transformation:

$$\begin{aligned} |mn\rangle_{12} = \frac{(\hat{a}_1^\dagger)^m}{\sqrt{m!}}\frac{(\hat{a}_2^\dagger)^n}{\sqrt{n!}}|00\rangle &\to \frac{1}{\sqrt{m!}}\left(\sum_i U_{i1}\hat{a}_i^\dagger\right)\frac{1}{\sqrt{n!}}\left(\sum_j U_{j1}\hat{a}_j^\dagger\right)|00\rangle \\ &= \frac{1}{\sqrt{m!n!}}\left(\hat{a}_1^\dagger\cos\theta\ - \hat{a}_2^\dagger e^{-i\phi}\sin\theta\right)^m\left(\hat{a}_1^\dagger e^{-i\phi}\sin\theta\ + \hat{a}_2^\dagger\cos\theta\right)^n|00\rangle \end{aligned}$$

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LOQC and Quantum Error Correction