Advanced Quantum Information 1: Quantum States Measurements and Evolution

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Systems and States

Classical State Space

Physical System - A collection of physical degrees of freedom that allow a closed, consistent description

State of a System - An equivalence class of experimental procedures, where each experimental procedure in the class gives the same empirical results for all possible measurements on S

A simple classical system would be an object that can either be **red**, **green** or **blue**. The state of the object would then be the probability distribution **r** over the three colours:

$$\mathbf{r} = (P(\text{red}), P(\text{green}), P(\text{blue}))$$

We then have the following ***pure states***:

$$\begin{aligned} \text{e}_0 &= (1, 0, 0) \\ \text{e}_1 &= (0, 1, 0) \\ \text{e}_2 &= (0, 0, 1) \end{aligned}$$

However we also may have some uncertainty about the state, resulting in a ***mixed state***, for example we may have the ***maximally mixed state***:

$$\mathbf{r}_{mm} = (1/3, 1/3, 1/3) = (1/3)\mathbf{e}_0 + (1/3)\mathbf{e}_1 + (1/3)\mathbf{e}_2$$

Since any state can be expressed as a combination of pure states we can understand the ***state space*** as the set of linear combinations of the pure states. For a probability distribution we constrain this set to those with unit norm, otherwise known as a ***convex combination***.

A convex combination of two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{z} = p\mathbf{x} + (1 - p)\mathbf{y}$$

where $0 \leq p \leq 1$. We can visualise $\mathbf{z}$ as lying on the line segment joining $\mathbf{x}$ and $\mathbf{y}$. The convex combination can be extended by introducing a new vector $\mathbf{w}$, such that

$$\mathbf{z}' = q\mathbf{w} + (1 - q)\mathbf{z} = q\mathbf{w} + (1 - q)p\mathbf{x} + (1 - q)(1 - p)\mathbf{y}$$

where now $\mathbf{z}'$ is visualised as lying inside the triangle formed by the three points $\{\mathbf{w}, \mathbf{x}, \mathbf{y}\}$. With this picture we can also understand the maximally mixed point $\mathbf{r}_{mm}$ as lying at the centre of the triangle.

As we move to quantum systems we will keep the notion of a convex set, which can be formally defined as a set $\mathcal{C}$ such that for any two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{C}$ we also have that

$px + (1-p)\mathbf{y}$ is also in $\mathcal{C}$ for any $0 \leq p \leq 1$.

It is useful to note that a ***simplex*** is a simple convex shapes previously discussed, for example in 1D it is the line segment and in 2D it becomes a triangle.

The notion of classical states $\mathbf{r} = (r_0, r_1, ..., r_{n-1})$ can be formally summarised by requiring the following two properties

- Non-Negativity $\implies r_i \geq 0 \quad \forall i$
- Normalisation $\implies \sum_i r_i = 1$

Evolution and Measurement in Classical Systems

There are only two operations performed on physical systems:

- Transformations
  - Required to be linear to satisfy probability theory
    $L(p\mathbf{x} + (1-p)\mathbf{y}) = pL(\mathbf{x}) + (1-p)L(\mathbf{y})$
- Measurements
  - Project the state onto a given state, ie $\mathrm{P(k)} = \mathbf{m}_k \cdot \mathbf{r}$
    $\mathbf{m}_k$ is from the collection of vectors $\{\mathbf{m}_0, \mathbf{m}_1, ..., \mathbf{m}_{d-1}\}$ which correspond to the d outcomes of the measurement and satisfies the following properties
    - Non-Negativity $\implies \text{components of } \mathbf{m}_k \geq 0 \quad \forall k$
    - Normalisation $\implies \sum_k \mathbf{m}_k = (1, 1, ..., 1)$

Quantum Theory

Moving from classical to quantum theory requires the following substitutions:

1. Classical vectors $\rightarrow$ Hermitian matrices
2. Non-negativity now pertains to the eigenvalues of these matrices
3. Normalisation now applies to the trace of the matrices
4. Classical vector dot products now become traces, $\mathbf{x} \cdot \mathbf{y} \rightarrow \mathrm{tr}(XY)$

We can formalise our description of quantum theory with the following definitions

- States
  - Non-negativity $\implies \text{eigs}(\rho) \geq 0$
  - Normalisation $\implies \mathrm{tr}(\rho) = 1$

- Measurements (with m outcomes)
  - Non-negativity $\implies \text{eigs}(M_i) \geq 0 \quad \forall i \in 0...m-1$
  - Normalisation $\implies \sum_i M_i = \mathbf{I}$
- Evolutions
  - Linear transformation $\rho \to \mathcal{E}(\rho)$ such that the transformed state is another valid quantum state

Classical Theory Inside of Quantum Theory

Classical states exist within quantum state space, as seen by:

$$\mathbf{r} \to \rho = \text{diag}(r_0, r_1, ..., r_d) = \begin{pmatrix} r_0 & 0 & 0 & ... & 0 \\ 0 & r_1 & 0 & ... & 0 \\ 0 & 0 & r_2 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & r_d \end{pmatrix}$$

The classical states are only fully diagonal when we work in the computational basis, quantum theory introduces non-diagonal states due to there being superpositions of classical states.

State Space of a Qubit

A qubit can be represented by a 2×2 Hermitian matrix, typically denoted by

$$\rho = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$$

where a and b are real and c is complex, meaning that we have 4 real parameters. From the previous discussion we also know that the matrix must satisfy $\text{eigs}(\rho) \geq 0$ and $\text{tr}(\rho) = 1$.

It is convenient to decompose the matrix in the Pauli basis, defined by

$$\text{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

such that we have

$$\rho = \frac{1}{2}\left(\mathrm{I} + xX + yY + zZ\right) = \frac{1}{2}\left(\mathrm{I} + \mathbf{r}\cdot\sigma\right)$$

Since we can show that $\mathrm{eigs}(\rho) = \frac{1}{2}\left(1 \pm \sqrt{x^2+y^2+z^2}\right)$ we find that in order to satisfy the requirement that $\mathrm{eigs}(\rho) \geq 0$ we require that the norm of the ***Bloch vector*** $\mathbf{r}$ is less than or equal to unity.

Analogeous to pure classical states, pure quantum states lie on the surface of the Bloch sphere (the quantum version of classical simplex). This means that they can be written in the form

$$\rho = \left|\psi\right\rangle\left\langle\psi\right|$$

where $\left|\psi\right\rangle$ is a normalised vector.

Ensembles and Purifications of Quantum States

With any two quantum states σ_0 and σ_1 we can express any arbitrary state ρ as

$$\rho = p\sigma_0 + (1-p)\sigma_1$$

where $0 \leq p \leq 1$. We can also express it in ***ensemble decomposition*** form

$$\rho = p_0\sigma_0 + p_1\sigma_1 + \ldots$$

which can be understood as the state σ_i being randomly prepared with probability p_i. If we express this sum in terms of the Bloch vector form

$$\begin{aligned}\sigma_0 &= \frac{1}{2}\left(\mathrm{I} + \mathbf{x}\cdot\sigma\right)\\ \sigma_1 &= \frac{1}{2}\left(\mathrm{I} + \mathbf{y}\cdot\sigma\right)\end{aligned}$$

we can see that the arbitrary state becomes a convex combination

$$\rho = \frac{1}{2}\left(\mathrm{I} + \left(p\mathbf{x} + (1-p)\mathbf{y}\right)\cdot\sigma\right)$$

If we choose opposite Bloch vectors as our basis ($\mathbf{y} = -\mathbf{x}$) and we take equal amounts of both we have

$$\rho = \frac{1}{2}\sigma_0 + \frac{1}{2}\sigma_1 = \frac{1}{2}\mathrm{I}$$

which corresponds to the maximally mixed state.

Ensemble Ambiguity Paradox

Classically every probability distribution has a unique decomposition into the pure states which define the simplex. However, in quantum theory a mixed states can be decomposed multiple ways into different bases. For example, if we consider the maximally mixed state we can see it can be decomposed into both the z-basis and x-basis

$$\rho_z = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\mathrm{I}$$
$$\rho_x = \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right) = \frac{1}{2}\mathrm{I}$$

This means that we can reach the maximally mixed state by two seemingly different phenomenological routes.

Multiple Quantum Systems and the Partial Trace

States can be defined on systems composed of multiple subsystems, denoted by a Hilbert space $\mathcal{H}_{AB}$ over the bipartite system AB. In general a state in $\mathcal{H}_{AB}$ is denoted as

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

Some particularly useful states of this form are the Bell states

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$
$$|\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B\right)$$
$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right)$$
$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right)$$

When working with joint states it is important to introduce the partial trace, which is defined as

$$\rho_A = \mathrm{tr}(\rho_{AB}) = \sum_k \langle k|_B \, \rho_A \otimes \rho_B \, |k\rangle_B$$

Purifications of a Mixed Quantum State

Given the mixed state ρ_A, which has the eigen-decomposition

$$\rho_A = \sum_{k=0}^{d-1} \lambda_k \, |e_k\rangle \langle e_k|$$

we can define a purification of ρ_A onto a joint system AB as

$$|\psi\rangle_{AB} = \sum_k \sqrt{\lambda_k} \, |e_k\rangle_A \otimes |k\rangle_B$$

In this form the mixed-ness of ρ_A appears to arise from its entanglement with the states $|k\rangle_B$.

Isometries

A more general technique is:

1. Embed system B into a larger system C
2. Apply a unitary in C

For example, given the maximally mixed state $\rho_A = \frac{1}{2}\mathbf{I}$, we can purify it to

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

The prufiying system B can be embedded into a three level (qutrit) system C, which has the basis $\{|0\rangle_C, |1\rangle_C, |2\rangle_C\}$, in many different ways, for example

$$\begin{aligned} W &: |0\rangle_B \to |0\rangle_C \\ W &: |1\rangle_B \to |2\rangle_C \end{aligned}$$

Or more in another (more simple) manner as

$$\begin{aligned} W &: |0\rangle_B \to |0\rangle_C \\ W &: |1\rangle_B \to |1\rangle_C \end{aligned}$$

We can then apply a 3×3 unitary V to this mapped state, such as

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

which is unitary for $\omega = e^{2\pi i/3}$. This results in the following purification

$$|\Psi\rangle_{AB} \rightarrow (\mathrm{I}_A \otimes V_C W_B) |\Psi_{AB}\rangle = |\Phi\rangle_{AC}$$

If we start with $\rho_A = \frac{1}{2}\mathrm{I}_A$ we find

$$\begin{aligned} |\Phi\rangle_{AC} &= \frac{1}{\sqrt{6}}[|0\rangle_A \otimes (|0\rangle_C + |1\rangle_C + |2\rangle_C) + |1\rangle_A \otimes (|0\rangle_C + \omega |1\rangle_C + \omega^2 |2\rangle_C)] \\ &= \sqrt{p_0} |\phi_0\rangle_A \otimes |0\rangle_C + \sqrt{p_1} |\phi_1\rangle_A \otimes |1\rangle_C + \sqrt{p_2} |\phi_2\rangle_A \otimes |2\rangle_C \end{aligned}$$

where now state ρ_A has been purified into the non-orthogonal pure states $\{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle\}$. This process of embedding a system into a larger system by the total unitary transformation $U = VW$ is known as an isometry.

General Quantum Measurements

It is useful to consider a measurement with k outcomes as corresponding to a set of k matrices $\{M_i\}$ where $\mathrm{eigs}(M_i) \geq 0$ and $\sum_i M_i = \mathrm{I}$. The probability of the i^{th} measurement result is given by

$$P_i = \mathrm{tr}[M_i \rho]$$

The most general type of measurement is the ***POVM*** measurement (Positive Operator Valued Measurement). Note that a positive operator is one for which $\mathrm{eigs}(M) \geq 0$.

The following are some examples of POVM measurements

Projective Measurements

For example

$$\mathcal{M} = \{M_0 = |+\rangle \langle +| , M_1 = |-\rangle \langle -|\}$$

which are he projectors onto the $+1$ and -1 eigenstates of σ_x.

Convex Mixtures of Projectors

For example

$$\mathcal{M} = \left\{ M_0 = \frac{1}{2} |0\rangle \langle 0| , M_1 = \frac{1}{2} |1\rangle \langle 1| , M_2 = \frac{1}{2} |+\rangle \langle +| , M_3 = \frac{1}{2} |-\rangle \langle -| \right\}$$

In this case we have a 50% chance of a projective measurement in the σ_z basis, and 50% chance of the σ_x basis. By performing this type of measurement many times we can therefore estimate both bases.

Trine Measurement

$$\mathcal{M} = \left\{ \frac{\sqrt{2\cos\theta}}{\sqrt{1+\cos\theta}} |0\rangle \langle 0| \ , \ \frac{1}{\sqrt{1+\cos\theta}} |\psi_1\rangle \langle \psi_1| \ , \ \frac{1}{\sqrt{1+\cos\theta}} |\psi_2\rangle \langle \psi_2| \right\}$$

where $|\psi_1\rangle = \sin(\theta/2) |0\rangle - \cos(\theta/2) |1\rangle$ and $|\psi_2\rangle = \sin(\theta/2) |0\rangle + \cos(\theta/2) |1\rangle$.

Since none of these measurements are projectors this does not form a convex mixture of projectors.

Non-Rank-1 POVM Measurements

Rank-1 measurements operators have at least one non-zero eigenvalue. For a non-rank-1 example consider a two qubit (4D) system with measurements defined in terms of Bell states:

$$\mathcal{M} = \{ M_0 = |\psi^-\rangle \langle \psi^-| , M_1 = |\psi^+\rangle \langle \psi^+| + |\phi^+\rangle \langle \phi^+| + |\phi^-\rangle \langle \phi^-| \}$$

These operators are still projectors since $M_i^2 = M_i$. In this case these operators determine is the two qubits are effectively in singlet or triplet states ($J = 0$ or $J = 1$).

Near Continuous Measurements

We could define a set of measurements as

$$\begin{aligned} \mathcal{M}_1 &= \{\mathrm{I}\} \\ M_\infty &= \{dt\, |t\rangle \langle t| : 0 \leq t \leq 2\pi\} \end{aligned}$$

where $|t\rangle = \cos(t) |0\rangle + \sin(t) |1\rangle$.

The first set of measurements $\mathcal{M}_1$ is the trivial case of simply asking if the system exists. The second set is valid if we require

$$\int_0^{2\pi} dt\, |t\rangle \langle t| = \mathrm{I}$$

Both of these are POVM measurements.

Quantum Channels

Evolutions have to be linear mappings in order to obey probability theory, as well as needing to return valid quantum states. The general form of a mapping is therefore

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

where $\sum_i A_i^\dagger A_i = \mathrm{I}$. This mapping is known as a ***completely-positive trace-preserving map*** (CPTP). The set of operators $\{A_i\}$ are known as the ***Kraus operators***.

Probability Conservation

We can see the trace is conserved since

$$\begin{aligned}
\operatorname{tr}\left(\mathcal{E}(\rho)\right) &= \operatorname{tr}\left(\sum_i A_i \rho A_i^\dagger\right) \\
&= \sum_i \operatorname{tr}\left(A_i \rho A_i^\dagger\right) \\
&= \sum_i \operatorname{tr}\left(A_i^\dagger A_i \rho\right) \\
&= \operatorname{tr}\left(\sum_i A_i^\dagger A_i \rho\right) \\
&= \operatorname{tr}\left(\rho\right) = \mathrm{I}
\end{aligned}$$

Examples

Unitary and Isometry Dynamics

Isometries are valid quantum channels since we only require $U^\dagger U = \mathrm{I}$ and not $UU^\dagger = \mathrm{I}$. For example we could define

$$\begin{aligned}
U\left|0\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|2\right\rangle\right) \\
U\left|1\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|1\right\rangle + \left|3\right\rangle\right)
\end{aligned}$$

which despite being an isometry between a single qubit Hilbert space to a 4D quantum system, is still a valid quantum channel.

Since we can write any state in Bloch vector form, $\rho = \frac{1}{2}(\mathbf{I} + \mathbf{r} \cdot \sigma)$, any unitary mapping will have the form

$$\mathcal{E}(\rho) = U\rho U^\dagger = \frac{1}{2}(\mathbf{I} + U(\mathbf{r} \cdot \sigma) = \frac{1}{2}(\mathbf{I} + (\mathbf{s} \cdot \sigma))$$

We can write any unitary (ignoring a global phase) as $U = e^{i\theta \hat{\mathbf{n}} \cdot \sigma}$, ie a rotation about $\hat{\mathbf{n}}$ by an angle θ. This means that $\mathbf{s} = O\mathbf{r}$, where O is an orthogoanl rotation matrix about $\hat{\mathbf{n}}$.

Dephasing

Consider the CPTP map $\mathcal{E} = \{A_0, A_1\}$ with Kraus operators

$$\begin{aligned} A_0 &= |0\rangle \langle 0| \\ A_1 &= |1\rangle \langle 1| \end{aligned}$$

which is a valid channel since $\sum_i A_i^\dagger A_i = \mathbf{I}$. Recalling that the Bloch vector is given as

$$\rho = \frac{1}{2}(\mathbf{I} + \mathbf{r} \cdot \sigma) = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

we find

$$\mathcal{E}(\rho) = \frac{1}{2}(\mathbf{I} + \mathbf{s} \cdot \sigma) = \frac{1}{2} \begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}$$

where the phase information has been lost, a process known as **dephasing**, returning a purely classical state.

The Fully Depolarising Map

For a quantum system S the **completely deplolarising** map is given by

$$\mathcal{E}(\rho) = \sigma \quad \forall \rho$$

where σ is some state within S. The corresponding d^2 Kraus operators will given by

$$A_{ij} = \sqrt{\lambda_i}\, |e_i\rangle \langle j|$$

where we have defined $\sigma = \sum_i \lambda_i |e_i\rangle\langle e_i|$. We can see that these are the correct Kraus operators since

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_{i,j} A_{ij}\rho A_{ij}^\dagger \\ &= \sum_{i,j} \lambda_i |e_i\rangle\langle j|\rho|j\rangle\langle e_i| \\ &= \sum_i \lambda_i |e_i\rangle\langle e_i| = \sigma\end{aligned}$$

The "Measure-Record-and-Update" Quantum Channel

This channel is defined as

$$\mathcal{E}(\rho) = \sum_k |k\rangle\langle k|_X \otimes B_k\rho B_k^\dagger$$

It takes system A and outputs in on a bipartite system XB. This yields a **classical-quantum state** since the first component is always diagonal in the computational basis. The process is:

1. Input state is measured with the result recorded in the classical states of the register X
2. The measured state is then outputted as system B

The corresponding Kraus operators are

$$A_k = |k\rangle_X \otimes B_k$$

which are valid provided B_k are valid Kraus operators.

We can inspect the output of the channel, $\sigma_{XB} = \mathcal{E}(\rho)$:

$$\begin{aligned}\mathrm{tr}_B(\sigma_{XB}) &= \sum_k |k\rangle\langle k|_X \,\mathrm{tr}_B\left[B_k\rho B_k^\dagger\right] \\ &= \sum_k \mathrm{tr}\left[M_k\rho\right]|k\rangle\langle k|_X \\ &= \sum_k p_k |k\rangle\langle k|_X\end{aligned}$$

where we define $M_k = B_k^\dagger B_k$ and $p_k = \mathrm{tr}[M_k\rho]$. Since $\sum_k M_k = \sum_k B_k^\dagger B_k = \mathrm{I}$ then we know that $\{M_k\}$ defines a POVM measurement.

Tracing-Out is a Quantum Channel

This corresponds to discarding part of the system, for example we could have the channel

$$\mathcal{E}(\rho_{AB}) = \mathrm{tr}_B(\rho_{AB}) = \rho_A$$

which can be decomposed into the Kraus operators

$$A_i = \mathrm{I}_A \otimes \langle i|$$

State-Preparation is a Quantum Channel

We can create a quantum state with the mapping $1 \to \rho$, foor some fixed ρ. This map has the Kraus decomposition

$$A_i = \sqrt{\lambda_i}\,|e_i\rangle$$

where

$$\rho = \sum_i \lambda_i\,|e_i\rangle\,\langle e_i|$$

General Aspects of Quantum Channels

Quantum channels are formally defined as any linear mapping

$$\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$$

such that for any $\rho \in \mathcal{B}(\mathcal{H}_\mathcal{A})$ we have

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

for some matrices $\{A_i\}$ that obey $\sum_i A_i^\dagger A_i = \mathrm{I}$.

Combinations of Quantum Channels are Quantum Channels

If $\mathcal{E} = \{A_i\}$ is a quantum channel from system A to B, and $\mathcal{F} = \{B_k\}$ is a quantum channel from system B to C, then $\mathcal{F} \circ \mathcal{E}$ is another quantum channel from A to C.

We can show this is true by verifying that C_{ik}, defined by

$$\mathcal{F} \circ \mathcal{E}(\rho) = \mathcal{F}(\mathcal{E}(\rho)) = \sum_{i,k} B_k A_i \rho A_i^\dagger B_k^\dagger = \sum_{i,k} C_{i,k} \rho C_{i,k}^\dagger$$

are valid Kraus operators.