Modelling the Timbre of a Piano String

Daniel Sun and Alice Ku*

June 18, 2023

1 Inharmonic Waves

We all learn in introductory physics that a taut, massive string has sinusoidal normal modes, governed by the 1D wave equation $\Box u = 0$. This is a good approximation to reality, and indeed, we take advantage of our knowledge of these waves to create musical instruments. One of the most ubiquitous instrument families are the keyboards, which use percussive driving of excitations on strings under tension to create pitches. The pitch is determined by the fundamental frequency of the string, and the timbre (tone quality) is determined by the higher harmonics of the string.

Oscillations following $\Box u = 0$ have a harmonic series $f_n = nf_0$, where the higher harmonics have frequencies equal to integer multiples of the fundamental frequency. This leads to a very consonant sound, since the higher harmonics sit at consonant intervals in relation to the fundamental. However, in reality, pianos are notoriously difficult to tune, because the strings of a piano have stiffness. By Euler-Bernoulli beam theory, they satisfy the following inharmonic wave equation [3]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \frac{ESK^2}{\rho} \frac{\partial^4 u}{\partial x^4} \tag{1}$$

where c is wave speed, E is the Young's modulus of the string material, S is its cross-sectional area, and K is its radius of gyration. Upon separation of variables, this PDE turns into two linear homogeneous ODEs with constant coefficients, which are easy to solve analytically. Define the following parameters:

$$B \equiv \pi^2 \frac{ESK^2}{TL^2} = \frac{\pi^3}{4} \frac{ER^4}{TL^2} \tag{2}$$

$$\omega_0 \equiv \frac{\pi c}{L} \sqrt{1+B} \tag{3}$$

where L is the length of the string. In essence, B is a measure of inharmonicity, and ω_0 is the fundamental angular frequency of the string. The spectrum and general form of the solution for the pinned oscillator (u and $\frac{\partial^2 u}{\partial x^2}$ vanish on the boundaries) are:

$$\omega_n = n\omega_0 \sqrt{\frac{1+Bn^2}{1+B}} \tag{4}$$

*Stanford University Physics 113

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \sin\left(\frac{n\pi}{L} x\right) \tag{5}$$

A piano string takes a typical value of $B \sim 10^{-3}$ and $L = 1\,\mathrm{m}$.

2 The Sound of Inharmonicity

Through Fourier analysis, we can find solutions $u(x,t)$ to equation 1 satisfying certain initial conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = f(x)$. We choose $f(x)$ to be either a delta function or a rectangular function, centered on some initial position $x_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}L$, depending on whether we wish to model a infinitely narrow strike (knife) or a reasonably thick strike (hammer). We also model what a bowed string may sound like by modeling bowing as a series of impulses separated by time δt, and taking the limit $\delta t \to 0$. This is then proportional to the time integral of $u(x,t)$. In our notebook, we study the entire parameter space described above (inharmonicity, thickness of the excitation, bowing, and choice of x_0). We generate animations for $u(x,t)$ as well as sound files of the oscillations in time, following the example of Bhatia [1]. We add an exponential damping to $u(x,t)$; details of the implementation can be found in the repository.

We finish by using our audio files to play MIDIs of two piano works by Claude Debussy: *Arabesque No. 1* and "En Bateau" from the *Petite Suite* [2]. To see the full effect of the fourth-order term, we generate a side-by-side comparison of two cases $B = 0$ and $B = 10^{-3}$. The dissonance created by the nonzero B is apparent. In reality, piano tuners don't tune piano strings by naïvely tuning the fundamental frequencies. Rather, they pay more attention to the intervals between notes on the piano, to avoid the dissonance heard here.

All source code and audio files can be found at this GitHub repository.

References

[1] Aatish Bhatia. *Making Music With Math.* Aatish Bhatia. Dec. 7, 2019. URL: `https://aatishb.com/stringmath/` (visited on 06/18/2023).

[2] Claude Debussy and Jack Sirulnikoff. *Classical MIDI Files - Download for free :: MIDI-WORLD.COM.* URL: `https://www.midiworld.com/classic.htm` (visited on 06/18/2023).

[3] Xavier Gràcia and Tomás Sanz-Perela. "The wave equation for stiff strings and piano tuning". In: *Reports@SCM* 3 (2017), pp. 1–16. ISSN: 2385-4227. DOI: `10.2436/20.2002.02.11`. arXiv: `1603.05516[physics]`. URL: `http://arxiv.org/abs/1603.05516` (visited on 06/18/2023).