Homework 2

Problem 1

a: From the pmf given, we have

$$\begin{aligned} 0.2+0.4+a+0.1+0.1 &= 1 \\ a &= 0.2 \end{aligned}$$

b: The expectation of X is

$$\begin{aligned} E[X] &= \sum xp(x) \\ &= -2*0.2+(-1)*0.4+0*a+1*0.1+2*0.1 \\ &= -0.5 \end{aligned}$$

, while the varience of X is

$$\begin{aligned} Var[X] &= E[X^2]-(E[X])^2 \\ &= 4*(0.2+0.1)+1*(0.4+0.1)+a*0-(0.4)^2 \\ &= 1.45 \end{aligned}$$

c: The sample space of X given A is $\{-2,-1\}$(omit the event "otherwise" with probability 0), and the probability of these events are

$$\begin{aligned} p_{X|A}(X=-2) &= \frac{0.2}{0.2+0.4} &= \frac{1}{3} \\ p_{X|A}(X=-1) &= \frac{0.4}{0.2+0.4} &= \frac{2}{3} \end{aligned}$$

d

$$\begin{aligned} E[X|A] &= \sum xp_{X|A}(x) \\ &= (-2)*\frac{1}{3}+(-1)*\frac{2}{3} \\ &= -\frac{4}{3} \\ Var[X|A] &= E[X^2]-(E[X])^2 \\ &= \sum x^2p_{X|A}(x)-(E[X])^2 \\ &= 4*\frac{1}{3}+1*\frac{2}{3}-(-\frac{4}{3})^2 \qquad = \frac{2}{9} \end{aligned}$$

e The sample space of Y is $\{-1,1,3,5,7\}$(omit the event "otherwise" with probability being 0), and the probability of these events are $0.2, 0.4, 0.2, 0.1, 0.1$ respectively.

Problem 2

a: The pmfs of Transmitter 1 and 2 are

$$P_1(x) = \frac{1}{3} * (\frac{2}{3})^{x-1}$$
$$P_2(x) = \frac{1}{2} * (\frac{1}{2})^{x-1},$$

where $x \in N^+$.
From the problem, we have $P(x) = 0.7 * P_1(x) + 0.3 * P_2(x)$, thus the pmf of X is

$$P(X) = 0.7 * \frac{1}{3} * (\frac{2}{3})^{x-1} + 0.3 * \frac{1}{2} * (\frac{1}{2})^{x-1}, \quad X \in N^+$$

b:

$$\begin{aligned} E[X] &= \sum_{1}^{+\infty} xP(x) \\ &= \sum_{1}^{+\infty} x(0.7 * P_1(x) + 0.3 * P_2(x)) \\ &= 0.7 * \sum_{1}^{+\infty} xP_1(x) + 0.3 * \sum_{1}^{+\infty} xP_2(x) \\ &= 0.7 * 3 + 0.3 * 2 \\ &= 2.7 \end{aligned}$$

c: *Theorem 1*: The expectation of X^2 is $\frac{2-p}{p^2}$, where X is a geometric random variable ranging from 1 to infinity with mean $\frac{1}{p}$.
Proof:

$$\begin{aligned} E[X^2] &= Var[X] + (E[X])^2 \\ &= \frac{1-p}{p^2} + (\frac{1}{p})^2 \\ &= \frac{2-p}{p^2} \end{aligned}$$

So, the varience of X is

$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \sum_{1}^{+\infty} x^2 P(x) - (E[X])^2 \\
&= \sum_{1}^{+\infty} x^2 (0.7 * P_1(x) + 0.3 * P_2(x)) - (E[X])^2 \\
&= 0.7 * \sum_{1}^{+\infty} x^2 P_1(x) + 0.3 \sum_{1}^{+\infty} x^2 P_2(x) - (E[X])^2 \\
&= 0.7 * 15 + 0.3 * 6 - 2.7^2 \\
&= \text{calculate later}
\end{aligned}$$

Problem 3

a: The pdf should satisfy the constraint that $\int_{-\infty}^{+\infty} f(x)dx = 1$, in this case

$$\begin{aligned}
\int_0^1 cx(2+x)dx &= 1 \\
c(1+\frac{1}{3}) &= 1 \\
c &= \frac{3}{4}
\end{aligned}$$

b: The cdf of X is

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)dt \\
&= \int_0^x \frac{3}{4} t(2+t)dt \\
&= \frac{3}{4}x^2 + \frac{1}{4}x^3
\end{aligned}$$

for $x \in [0,1]$, and for all $x < 0, F(x) = 0$, and all $x > 1, F(x) = 1$. In conclusion, the cdf is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{4}x^2 + \frac{1}{4}x^3 & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

c: The expectation of X is

$$\begin{aligned}E[X] &= \int_{-\infty}^{+\infty} xf(x)dx \\ &= \int_0^1 x * \frac{3}{4}x(2+x)dx \\ &= \frac{3}{4}(\frac{2}{3}+\frac{1}{4}) \\ &= \frac{11}{16},\end{aligned}$$

and the varience is

$$\begin{aligned}Var[X] &= E[X^2] - (E[X])^2 \\ &= \int_0^1 \frac{3}{4}x^2 * x(2+x)dx - (\frac{11}{16})^2 \\ &= \frac{3}{4} * (\frac{1}{5}+\frac{1}{2}) - \frac{11}{16}^2 \\ &= 0.0523.\end{aligned}$$

d: The probability of event A is given by

$$\begin{aligned}P(A) &= F(0.8) - F(0.4) \\ &= \frac{3}{4}(0.8^2 - 0.4^2) + \frac{1}{4}(0.8^3 - 0.4^3) \\ &= \frac{59}{125}\end{aligned}$$

e: The conditional pdf(denoted by $f_{X|A}$) will be

$$\begin{aligned}f_{X|A}(x|A) &= \frac{d}{dx}F_{X|A}(x) \\ &= \lim_{\epsilon\to 0} P(X \in [x, x+\epsilon]|A) \\ &= \lim_{\epsilon\to 0} P(X \in [x, x+\epsilon])/P(A) \\ &= f(x)/P(A) \\ &= \frac{375}{236}x(2+x),\end{aligned}$$

for $x \in (0.4, 0.8)$, otherwise $f_{X|A}(x|A) = 0$. Thus, the conditional expectation is

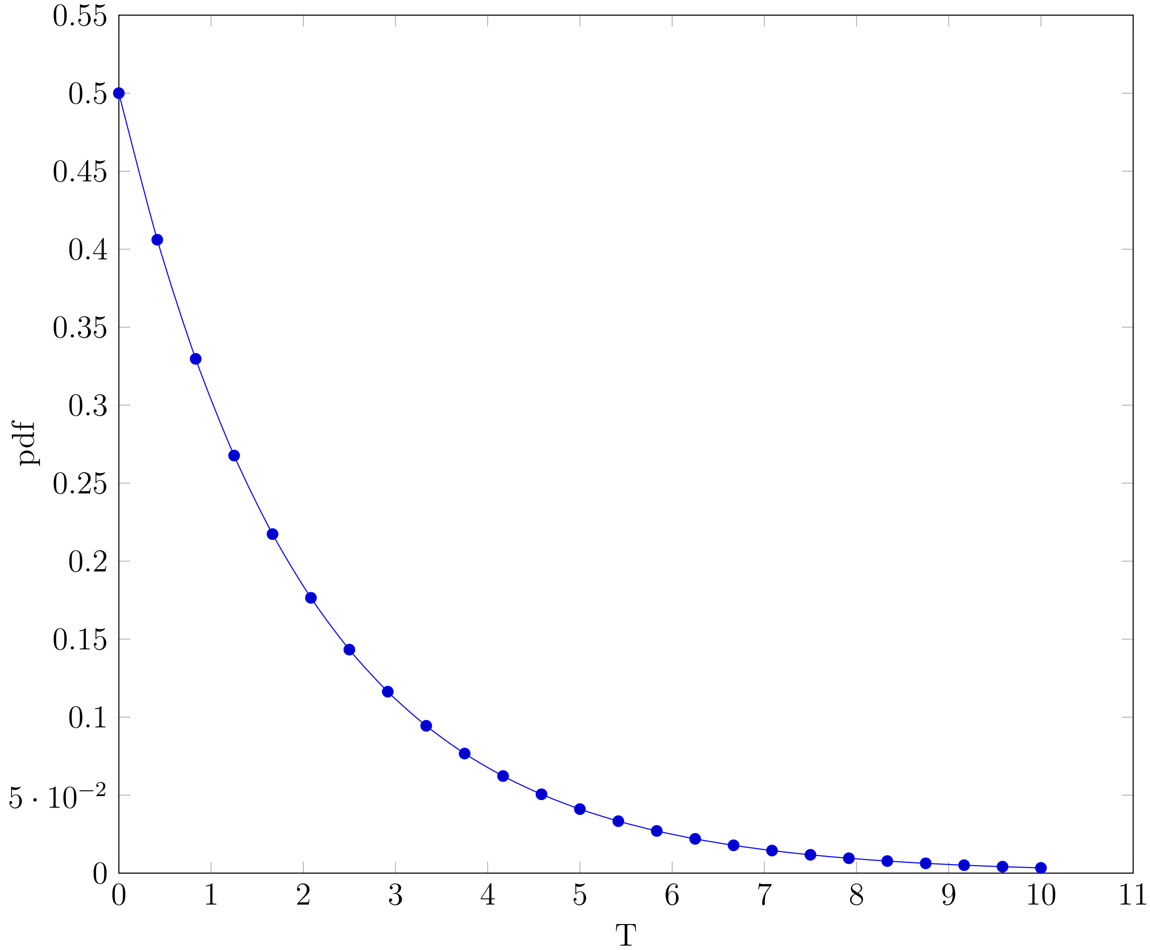
$$\begin{aligned}E[X|A] &= \int_{0.4}^{0.8} xf_{X|A}(x)dx \\ &= \frac{375}{236}\int_{0.4}^{0.8} x * x(2+x)dx \\ &= \frac{375}{236} * \frac{148}{375} \\ &= 0.6271\end{aligned}$$

Problem 4

The pdf of T is

$$f(t) = \begin{cases} 0.5e^{-0.5t} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a:



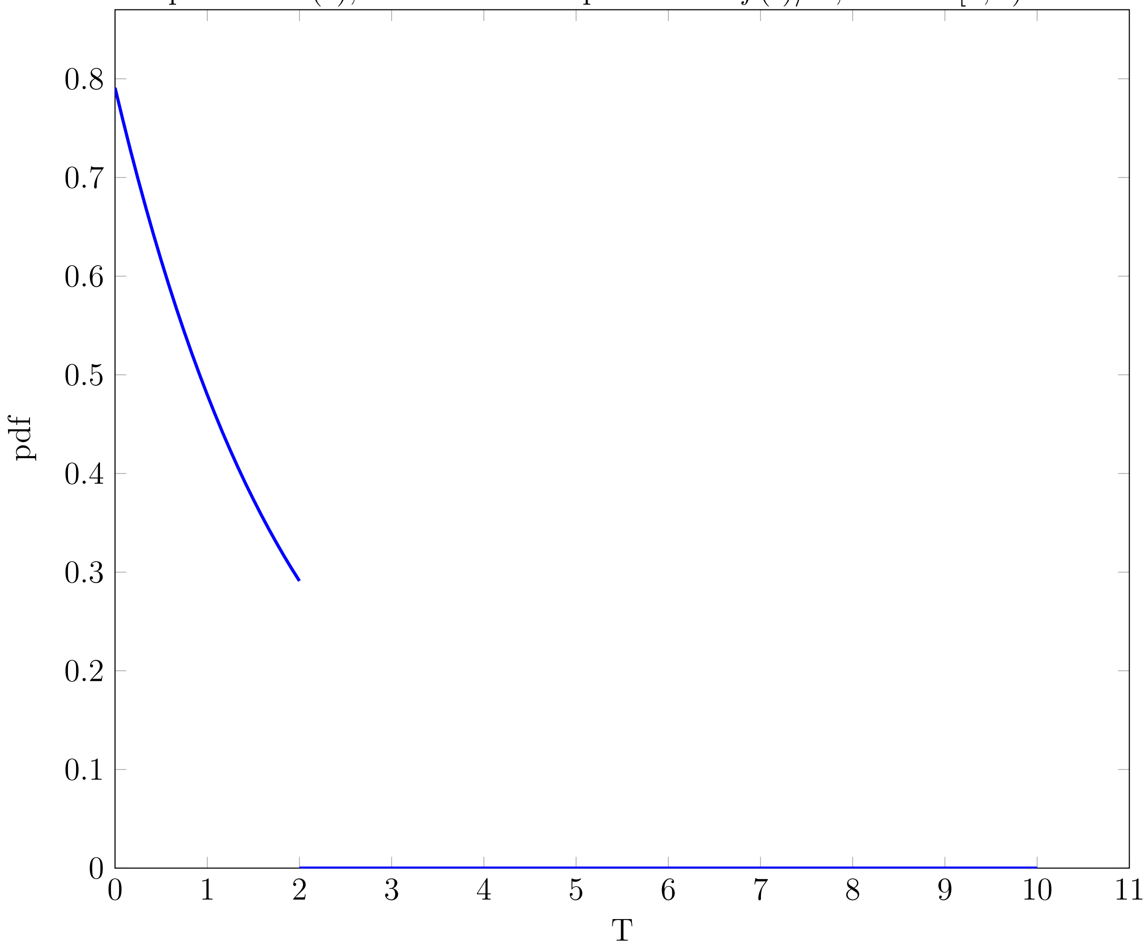
b: The expectation of T is

$$\begin{aligned}
E[T] &= \int_0^{+\infty} \frac{1}{2} t e^{-\frac{1}{2}t} dt \\
&= 2\int_0^{+\infty} u e^{-u} du, \quad (let\ u = \frac{1}{2}t) \\
&= 2\int_0^{+\infty} -u de^{-u} \\
&= 2(-ue^{-u}|_0^{+\infty} + \int_0^{+\infty} e^{-u} du) \\
&= 2
\end{aligned}$$

c: The probability of the waiting time being less than 2 mins is given by

$$\begin{aligned}
P &= \int_0^{2} f(t) dt \\
&= \int_0^{2} \frac{1}{2} e^{-\frac{1}{2}t} dt \\
&= \int_0^{1} e^{-u} du, \quad (same as 4.(b)) \\
&= 1 - e^{-1}
\end{aligned}$$

So from the proof in 3.(e), the condiitonal pdf will be $f(t)/P$, for $x \in [0, 2)$



d: The conditional probability is

$$
\begin{aligned}
E[T \mid A] &= \int_0^2 tf(t \mid A)dt \\
&= \frac{1}{2 - 2e^{-1}} \int_0^2 te^{-\frac{1}{2}t}dt \\
&= \frac{2}{1 - e^{-1}} \int_0^1 ue^{-u}du \\
&= \frac{2}{1 - e^{-1}} (-u - 1)e^{-u}|_0^1 \\
&= \frac{2e - 4}{e - 1}
\end{aligned}
$$

e: The probability that waiting time exceeds 4 minutes is

$$\begin{aligned} P(T > 4) &= \int_4^{+\infty} f(t)dt \\ &= \int_4^{+\infty} \frac{1}{2} e^{-\frac{1}{2}t} dt \\ &= \int_2^{+\infty} e^{-u} du \\ &= e^{-2} \end{aligned}$$

f: From the memoryless property of the exponential distributuion, we have

$$\begin{aligned} P(T \geq 5 \mid T \geq 4) &= P(T \geq 1) \\ &= \int_1^{+\infty} \frac{1}{2} e^{-\frac{1}{2}t} dt \\ &= \int_{\frac{1}{2}}^{+\infty} e^{-u} du \\ &= e^{-\frac{1}{2}} \end{aligned}$$