# Problem 1

## a

From the joint pmf, we have

$$c + 0.1 + 0.1 + 0.2 + 0.1 + 0.2 + 0.1 + 0.1 = 1$$
$$c = 0.1$$

## b

From definition of joint cdf, $F_{XY}(2, 1)$ will be

$$\begin{aligned} F_{XY}(2,1) &= \sum_{x\leq 2,\ y\leq 1} p_{XY}(j,k) \\ &= 0.1 + 0.1 + 0.2 + 0.1 \\ &= 0.5 \end{aligned}$$

## c

From the table of pmf, by summing up the $p_{XY}(j, k)$ in each row / column, we have

| $j$ | 1 | 2 | 3 | 4 | otherwise |
|---|---|---|---|---|---|
| $P_X(j)$ | 0.3 | 0.3 | 0.2 | 0.2 | 0 |

| $k$ | 0 | 1 | 2 | otherwise |
|---|---|---|---|---|
| $P_Y(k)$ | 0.3 | 0.5 | 0.2 | 0 |

## d

No, a counterexample would be $(4, 1)$, where $P_{XY}(4, 1) \neq P_X(4) \times P_Y(1)$

## e

Yes, by calculating the covarience of $X, Y$, we have

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= 2.2 - 2.07 \\ &= 0.13 \neq 0, \end{aligned}$$

which means $X, Y$ are correlated.

## f

First, calculate standard variation of $X, Y$

$$\begin{aligned}\sigma_X &= (E(X^2) - (E(X))^2)^{1/2}\\ &= (6.5 - 2.3^2)^{1/2}\\ &= 1.1\end{aligned}$$

$$\begin{aligned}\sigma_Y &= (E(X^2) - (E(X))^2)^{1/2}\\ &= (1.3 - 0.9^2)^{1/2}\\ &= 0.7\end{aligned}$$

$$\begin{aligned}r &= Cov(X, Y)/\sigma_X \sigma_Y\\ &= 0.1688\end{aligned}$$

## g

From the marginal pmf of Y, we have $P(Y < 1) = 0.3$, then the conditional pmf of $X$ can be calculated using $P_{X|Y<1}(j) = P(j, Y < 1)/P(Y < 1)$, We have

| $j$ | 1 | 2 | 3 | 4 | otherwise |
|---|---|---|---|---|---|
| $P_{X\|Y<1}(j)$ | 1/3 | 1/3 | 0 | 1/3 | 0 |

## h

From the conditional pmf in $g$, the conditional expected value of $X$ is

$$\begin{aligned}E(X|Y < 1) &= \sum_j j * P_{X|Y<1}(j)\\ &= 7/3\end{aligned}$$

# Problem 2

## a

From the joint pmf of $X, Z$, we have

$$\begin{aligned} P_X(x) &= \sum_{z=x}^{+\infty} \frac{3^z}{x!(z-x)!} 0.4^x 0.6^{z-x} e^{-3} \\ &= \frac{1.2^x}{x!} e^{-3} \sum_{t=0}^{+\infty} \frac{1.8^t}{t!} \quad (let\ t = z - x) \\ &= \frac{1.2^x}{x!} e^{-1.2} \end{aligned}$$

It belongs to Poisson distribution with mean 1.2

## b

From the joint pmf of $X, Z$, we have

$$\begin{aligned} P_Z(z) &= \sum_{x=0}^{z} \frac{3^z}{x!(z-x)!} 0.4^x 0.6^{z-x} e^{-3} \\ &= \frac{3^z}{z!} e^{-3} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} 0.4^x 0.6^{z-x} \\ &= \frac{3^z}{z!} e^{-3} \end{aligned}$$

It belongs to Poisson distribution with mean 3

## c

For $Z = X + Y$, we have

$$\begin{aligned} P_Y(y) &= \sum_{x=0}^{+\infty} \frac{3^{x+y}}{x!y!} 0.4^x 0.6^y e^{-3} \\ &= \frac{1.8^y}{y!} e^{-3} \sum_{x=0}^{+\infty} \frac{3^x}{x!} 0.4^x \\ &= \frac{1.8^y}{y!} e^{-1.8} \end{aligned}$$

It belongs to Poisson distribution with mean 1.8

### d

No, a counterexample will be $(x, z) = (1, 0)$, for neither of $P_X(1), P_Z(0)$ equals zero, but by definition $P_{X,Z}(1, 0) = 0$

### e

Yes, for

$$\begin{aligned} P_{X,Y}(x, y) &= P_{X,Z}(x, x + y) \quad (by\ definition) \\ &= \frac{3^{x+y}}{x!((x + y) - x)!} 0.4^x 0.6^{(x+y)-x} e^{-3} \\ &= \frac{3^{x+y}}{x!y!} 0.4^x 0.6^y e^{-3} \\ &= \frac{1.2^x}{x!} e^{-1.2} \times \frac{1.8^y}{y!} e^{-1.8} \\ &= P_X(x) \times P_Y(y) \end{aligned}$$

### f

From the pmf of $X, Z$ and marginal pmf of $Z$, we have

$$\begin{aligned} P_{X|Z}(x) &= \left(\frac{3^z}{x!(z - x)!} 0.4^x 0.6^{z-x} e^{-3}\right) / \left(\frac{3^z}{z!} e^{-3}\right) \\ &= \frac{z!}{x!(z - x)!} 0.4^x 0.6^{z-x}, \end{aligned}$$

which belongs to binomial distribution with paremeter $(n, p) = (z, 0.4)$.

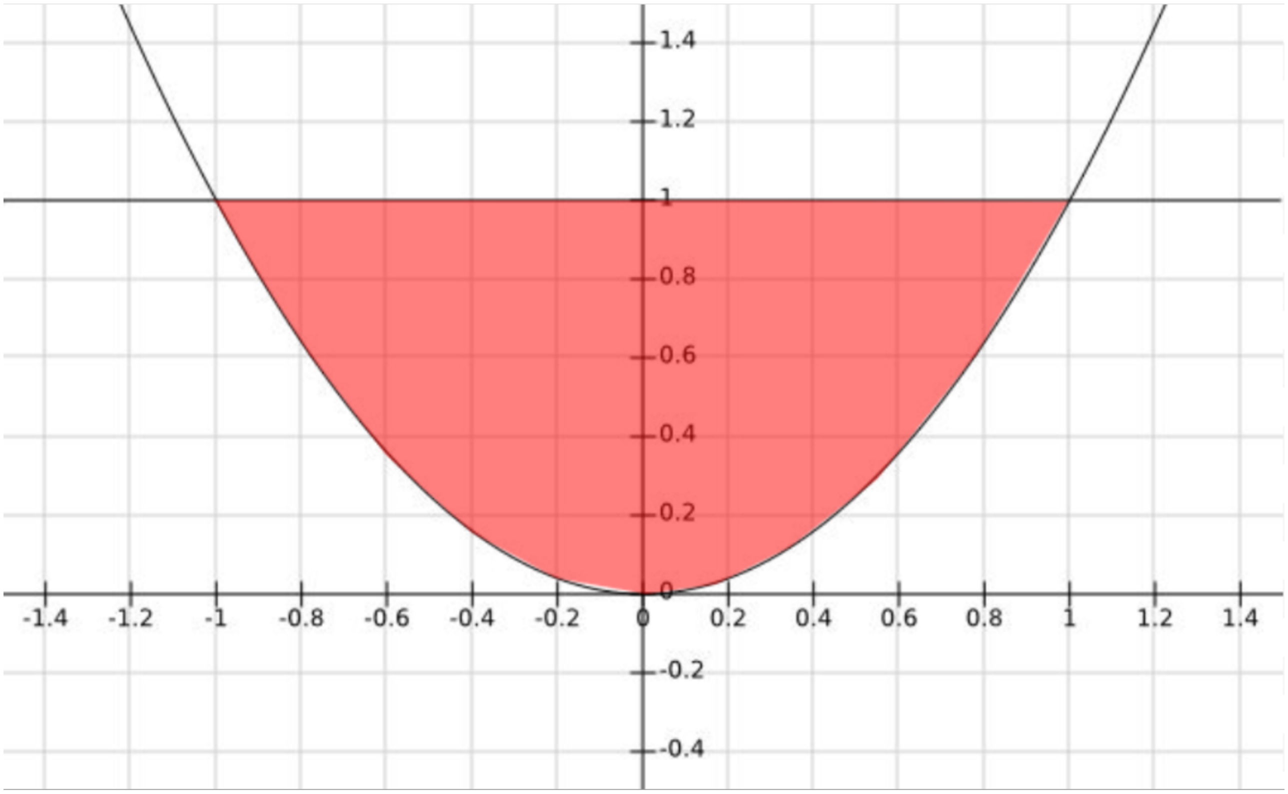
### g

From the result in $(f)$, the conditional expectation of $X$ is $E[X|Z] = 0.4Z$.

## Problem 3

### a

The red region is where pdf of $X, Y$ is non-zero, which is the intersetion of two functions: $y = x^2$ and $y = 1$.




## b

From the property that pdf intergates to one, we have

$$\iint f_{X,Y}(x,y)\ dxdy = 1$$

$$\iint_S c dxdy = 1$$

$$c\int_{-1}^{1}\int_{x^2}^{1} dy\ dx = 1$$

$$c\int_{-1}^{1}(1-x^2)\ dx = 1$$

$$c(x-\frac{1}{3}x^3)|_{-1}^{1} = 1$$

$$c = \frac{3}{4}$$

## c

For any $x \notin [-1, 1]$, the pdf of $X$ is zero. For $x \in [-1, 1]$, we have

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x, y)dy \\ &= \int_{x^2}^{1} \frac{3}{4}dy \\ &= \frac{3}{4}(1 - x^2) \end{aligned}$$

For any $y \notin [0, 1]$, the pdf of $Y$ is zero. For $y \in [0, 1]$, we have

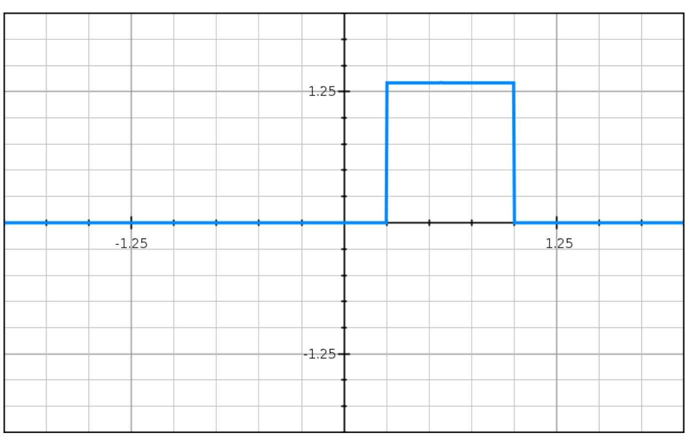
$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x, y)dx \\ &= \int_{-y^{1/2}}^{y^{1/2}} \frac{3}{4} \\ &= \frac{3}{2}y^{1/2} \end{aligned}$$

## d

By definition,

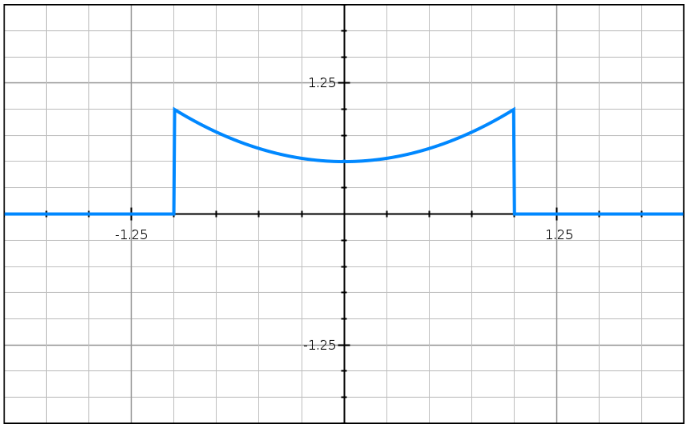
$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\ &= \frac{3/4}{3/4(1 - x^2)} \\ &= \frac{1}{1 - x^2} \end{aligned}$$

only for $y \in [x^2, 1]$, otherwise 0.



### e

Notice that $Y$ belongs to uniform distribution in $[x^2, 1]$ once $X$ is given. We have $E[Y|X] = \frac{1}{2}(1+x^2)$ for $y \in [0,1]$, otherwise 0.



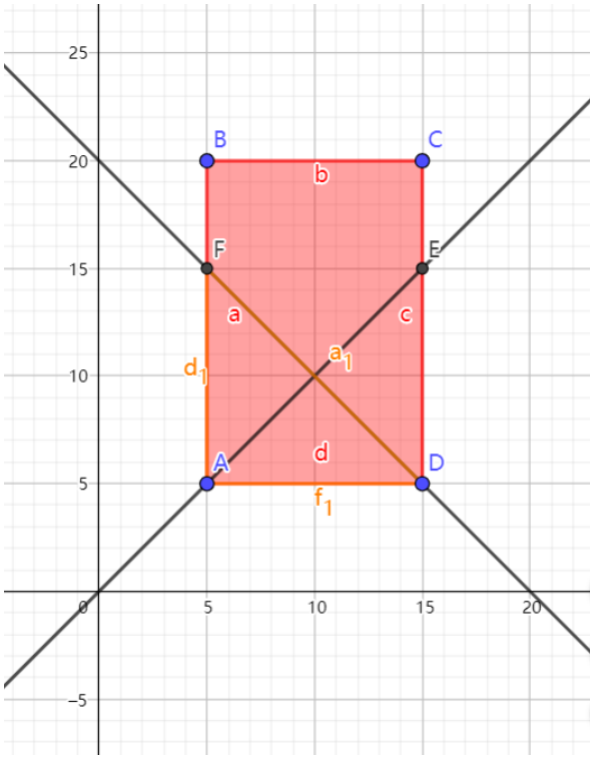

## Problem 4

### a

From the fact that $X, Y$ are independent, we have

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{15-5} \times \frac{1}{20-5} \\ &= \frac{1}{150} \end{aligned}$$

for $x \in [5, 15]$ and $y \in [5, 20]$, otherwise 0. The region where PDF is non zero is the rectangle $ABCD$.




## b

The probability of $Y > X$ is the space ratio of polygon $ABCE$ to rectangle $ABCD$, which is $\frac{2}{3}$

## b

The probability of $X + Y < 20$ is the space ratio of triangle $ADF$ to rectangle $ABCD$, which is $\frac{1}{3}$

# Problem 5

By substituting $(x, y)$ with $(z, u)$, we have

$$\begin{aligned} f_X(x) &= \frac{1}{2}e^{-\frac{1}{2}x} \\ f_Y(y) &= 2e^{-2y} \\ f_{X,Y} &= e^{-\frac{1}{2}x-2y} \\ \|\frac{\partial(x,y)}{\partial(z,u)}\| &= \frac{1}{2} \\ f_{Z,U}(z,u) &= f_{X,Y}(x(z,u),y(z,u))\|\frac{\partial(x,y)}{\partial(z,u)}\| \\ &= \frac{1}{2}e^{-\frac{5}{4}z+\frac{3}{4}u} \end{aligned}$$

for all $(z, u) \in \{(z, u)|z > |u|\}$,

## a

$$\begin{aligned} f_Z(z) &= \int_{-z}^{z} f_{Z,U}(z,u)du \\ &= \frac{1}{2}\int_{-z}^{z} e^{-\frac{5}{4}z+\frac{3}{4}u}du \\ &= \frac{1}{2}e^{-\frac{5}{4}z}(\frac{4}{3}e^{\frac{3}{4}u})|_{-z}^{z} \\ &= \frac{2}{3}(e^{-\frac{1}{2}z} - e^{-2z}) \end{aligned}$$

for $z \geq 0$

## b

$$\begin{aligned} f_U(u) &= \int_{|u|}^{+\infty} f_{Z,U}(z,u)dz \\ &= \frac{1}{2}\int_{|u|}^{+\infty} e^{-\frac{5}{4}z+\frac{3}{4}u}dz \\ &= \frac{1}{2}e^{\frac{3}{4}u}(-\frac{4}{5}e^{\frac{5}{4}z})|_{|u|}^{+\infty} \\ &= \frac{2}{5}e^{\frac{3}{4}u-\frac{5}{4}|u|} \end{aligned}$$

for $u \in R$