Homework 3

Problem 1

a

From the joint pmf, we have

$$\begin{aligned} c + 0.1 + 0.1 + 0.2 + 0.1 + 0.2 + 0.1 + 0.1 &= 1 \\ c &= 0.1 \end{aligned}$$

b

From definition of joint cdf, $F_{XY}(2,1)$ will be

$$\begin{aligned} F_{XY}(2,1) &= \sum_{x\le 2,\ y\le 1} p_{XY}(j,k) \\ &= 0.1 + 0.1 + 0.2 + 0.1 \\ &= 0.5 \end{aligned}$$

c

From the table of pmf, by summing up the $p_{XY}(j,k)$ in each row / column, we have

j	1	2	3	4	otherwise
$P_X(j)$	0.3	0.3	0.2	0.2	0

k	0	1	2	otherwise
$P_Y(k)$	0.3	0.5	0.2	0

d

No, a counterexample would be $(4,1)$, where $P_{XY}(4,1) \neq P_X(4) \times P_Y(1)$

e

Yes, by calculating the covarience of X, Y, we have

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= 1.8 - 2.07 \\ &= -0.27 \neq 0, \end{aligned}$$

which means X, Y are correlated.

f

First, calculate standard variation of X, Y

$$\begin{aligned}\sigma_X &= (E(X^2) - (E(X))^2)^{1/2} \\ &= (6.2 - 2.3^2)^{1/2} \\ &= foo\end{aligned}$$

$$\begin{aligned}\sigma_Y &= (E(X^2) - (E(X))^2)^{1/2} \\ &= (1.3 - 0.9^2)^{1/2} \\ &= bar\end{aligned}$$

$$\begin{aligned}r &= Cov(X, Y)/\sigma_X\sigma_Y \\ &= foobar\end{aligned}$$

g

From the marginal pmf of Y, we have $P(Y < 1) = 0.3$, then the conditional pmf of X can be calculated using $P_{X|Y<1}(j) = P(j, Y < 1)/P(Y < 1)$, We have

j	1	2	3	4	otherwise
$P_{X\|Y<1}(j)$	1/3	1/3	0	1/3	0

h

From the conditional pmf in g, the conditional expected value of X is

$$\begin{aligned}E(X|Y < 1) &= \sum_j j * P_{X|Y<1}(j) \\ &= 7/3\end{aligned}$$

Problem 2

a

From the joint pmf of X, Z, we have

$$\begin{aligned}
P_X(x) &= \sum_{z=x}^{+\infty} \frac{3^z}{x!(z-x)!} 0.4^x 0.6^{z-x} e^{-3} \\
&= \frac{1.2^x}{x!} e^{-3} \sum_{t=0}^{+\infty} \frac{1.8^t}{t!} \quad (let\ t = z - x) \\
&= \frac{1.2^x}{x!} e^{-1.2}
\end{aligned}$$

It belongs to Poisson distribution with mean 1.2

b

From the joint pmf of X, Z, we have

$$\begin{aligned}
P_Z(z) &= \sum_{x=0}^{z} \frac{3^z}{x!(z-x)!} 0.4^x 0.6^{z-x} e^{-3} \\
&= \frac{3^z}{z!} e^{-3} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} 0.4^x 0.6^{z-x} \\
&= \frac{3^z}{z!} e^{-3}
\end{aligned}$$

It belongs to Poisson distribution with mean 3

c

For $Z = X + Y$, we have

$$\begin{aligned}
P_Y(y) &= \sum_{x=0}^{+\infty} \frac{3^{x+y}}{x!y!} 0.4^x 0.6^y e^{-3} \\
&= \frac{1.8^y}{y!} e^{-3} \sum_{x=0}^{+\infty} \frac{3^x}{x!} 0.4^x \\
&= \frac{1.8^y}{y!} e^{-1.8}
\end{aligned}$$

It belongs to Poisson distribution with mean 1.8

d

No, a counterexample will be $(x, z) = (1, 0)$, for neither of $P_X(1), P_Z(0)$ equals zero, but by definition $P_{X,Z}(1, 0) = 0$

e

Yes, for

$$\begin{aligned}
P_{X,Y}(x, y) &= P_{X,Z}(x, x + y) \quad (by\ definition) \\
&= \frac{3^{x+y}}{x!((x + y) - x)!}0.4^x 0.6^{(x+y)-x}e^{-3} \\
&= \frac{3^{x+y}}{x!y!}0.4^x 0.6^y e^{-3} \\
&= \frac{1.2^x}{x!}e^{-1.2} \times \frac{1.8^y}{y!}e^{-1.8} \\
&= P_X(x) \times P_Y(y)
\end{aligned}$$

f

From the pmf of X, Z and marginal pmf of Z, we have

$$\begin{aligned}
P_{X|Z}(x) &= (\frac{3^z}{x!(z - x)!}0.4^x 0.6^{z-x}e^{-3})/(\frac{3^z}{z!}e^{-3}) \\
&= \frac{z!}{x!(z - x)!}0.4^x 0.6^{z-x},
\end{aligned}$$

which belongs to binomial distribution with paremeter $(n, p) = (z, 0.4)$.

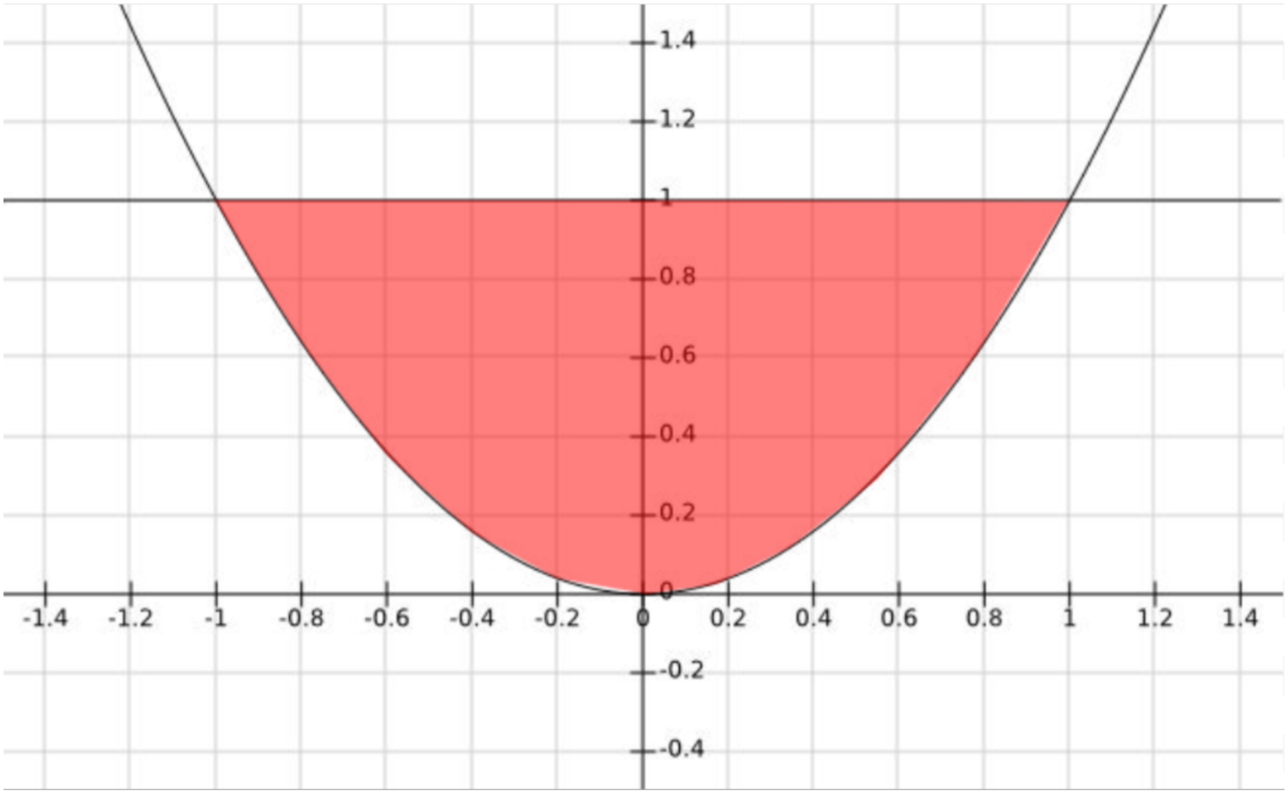
g

From the result in (f), the conditional expectation of X is $E[X|Z] = 0.4Z$.

Problem 3

a

The red region is where pdf of X, Y is non-zero, which is the intersetion of two functions: $y = x^2$ and $y = 1$.



b

From the property that pdf intergates to one, we have

$$
\begin{aligned}
\iint f_{X,Y}(x,y)\ dxdy &= 1 \\
\iint_S cdxdy &= 1 \\
c\int_{-1}^{1} dx \int_{x^2}^{1} dy &= 1 \\
c\int_{-1}^{1} dx(1-x^2) &= 1 \\
c(x-\frac{1}{3}x^3)|_{-1}^{1} &= 1 \\
c &= \frac{3}{4}
\end{aligned}
$$

, notice that I treat integral $\int dx$ as an operator, so that we can write dx right after $\int$, which seems more clear than nested $dxdydz$.

c

For any $x \notin [-1, 1]$, the pdf of X is zero. For $x \in [-1, 1]$, we have

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x,y)dy \\ &= \int_{x^2}^{1} \frac{3}{4} dy \\ &= \frac{3}{4}(1 - x^2) \end{aligned}$$

For any $y \notin [0, 1]$, the pdf of Y is zero. For $y \in [0, 1]$, we have

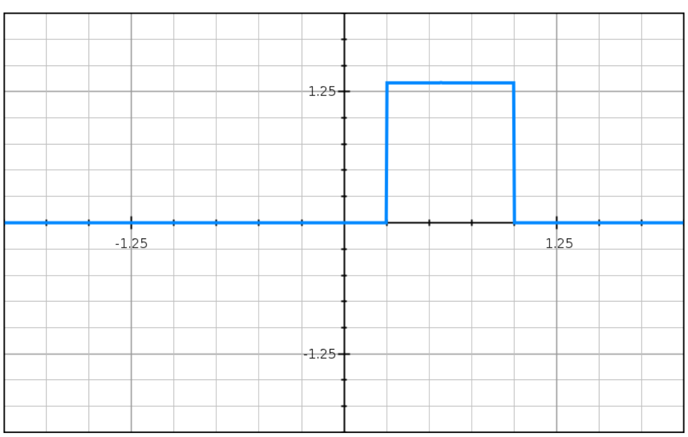
$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x,y)dx \\ &= \int_{-y^{1/2}}^{y^{1/2}} \frac{3}{4} \\ &= \frac{3}{2} y^{1/2} \end{aligned}$$

d

By definition,

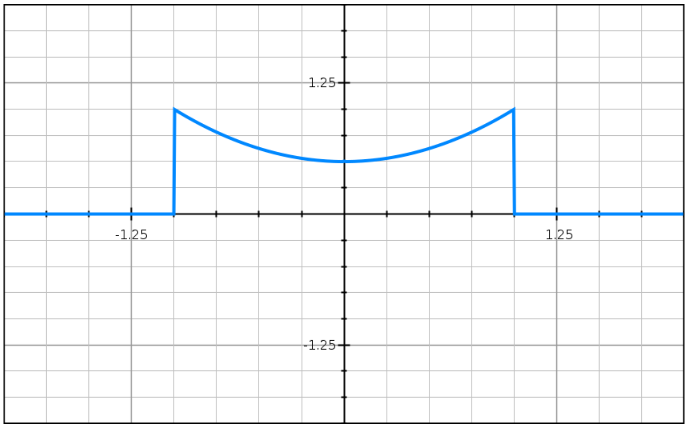
$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{3/4}{3/4(1 - x^2)} \\ &= \frac{1}{1 - x^2} \end{aligned}$$

only for $y \in [x^2, 1]$, otherwise 0.



e

Notice that Y belongs to uniform distribution in $[x^2, 1]$ once X is given. We have $E[Y|X] = \frac{1}{2}(1 + x^2)$ for $y \in [0, 1]$, otherwise 0.



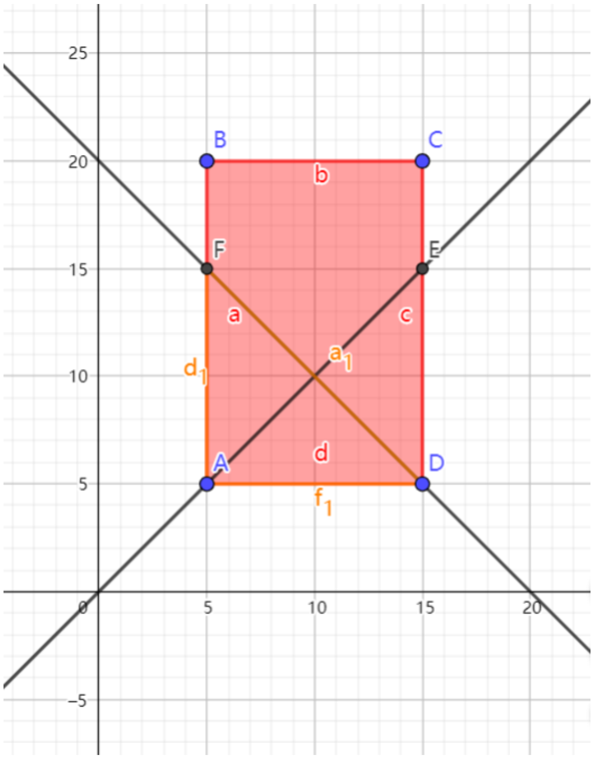
Problem 4

a

From the fact that X, Y are independent, we have

$$\begin{aligned} f_{X,Y}(x, y) &= f_X(x) f_Y(y) \\ &= \frac{1}{15 - 5} \times \frac{1}{20 - 5} \\ &= \frac{1}{150} \end{aligned}$$

for $x \in [5, 15]$ and $y \in [5, 20]$, otherwise 0. The region where PDF is non zero is the rectangle $ABCD$.



b

The probability of $Y > X$ is the space ratio of polygon $ABCE$ to rectangle $ABCD$, which is $\frac{2}{3}$

b

The probability of $X + Y < 20$ is the space ratio of triangle ADF to rectangle $ABCD$, which is $\frac{1}{3}$