Problem 1

a: From the pmf given, we have

$$\begin{aligned} 0.2 + 0.4 + a + 0.1 + 0.1 &= 1 \\ a &= 0.2 \end{aligned}$$

b: The expectation of X is

$$\begin{aligned} E[X] &= \sum xp(x) \\ &= -2 * 0.2 + (-1) * 0.4 + 0 * a + 1 * 0.1 + 2 * 0.1 \\ &= -0.5 \end{aligned}$$

, while the varience of X is

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= 4 * (0.2 + 0.1) + 1 * (0.4 + 0.1) + a * 0 - (0.4)^2 \\ &= 1.45 \end{aligned}$$

c: The sample space of X given A is $\{-2, -1\}$(omit the event "otherwise" with probability 0), and the probability of these events are

$$\begin{aligned} p_{X|A}(X = -2) &= \frac{0.2}{0.2 + 0.4} &= \frac{1}{3} \\ p_{X|A}(X = -1) &= \frac{0.4}{0.2 + 0.4} &= \frac{2}{3} \end{aligned}$$

d

$$\begin{aligned} E[X|A] &= \sum xp_{X|A}(x) \\ &= (-2) * \frac{1}{3} + (-1) * \frac{2}{3} \\ &= -\frac{4}{3} \end{aligned}$$

$$\begin{aligned} Var[X|A] &= E[X^2] - (E[X])^2 \\ &= \sum x^2 p_{X|A}(x) - (E[X])^2 \\ &= 4 * \frac{1}{3} + 1 * \frac{2}{3} - (-\frac{4}{3})^2 &= \frac{2}{9} \end{aligned}$$

e The sample space of Y is $\{-1, 1, 3, 5, 9\}$(omit the event "otherwise" with probability being 0), and the probability of these events are $0.2, 0.4, 0.2, 0.1, 0.1$ respectively.

Problem 2

a: The pmfs of Transmitter 1 and 2 are

$$P_1(x) = \frac{1}{3} * (\frac{2}{3})^{x-1}$$
$$P_2(x) = \frac{1}{2} * (\frac{1}{2})^{x-1},$$

where $x \in N^+$.
From the problem, we have $P(x) = 0.7 * P_1(x) + 0.3 * P_2(x)$, thus the pmf of X is

$$P(X) = 0.7 * \frac{1}{3} * (\frac{2}{3})^{x-1} + 0.3 * \frac{1}{2} * (\frac{1}{2})^{x-1}, \quad X \in N^+$$

b:

$$\begin{aligned} E[X] &= \sum_{1}^{+\infty} xP(x) \\ &= \sum_{1}^{+\infty} x(0.7 * P_1(x) + 0.3 * P_2(x)) \\ &= 0.7 * \sum_{1}^{+\infty} xP_1(x) + 0.3 * \sum_{1}^{+\infty} xP_2(x) \\ &= 0.7 * 3 + 0.3 * 2 \\ &= 2.7 \end{aligned}$$

c: *Theorem 1*: The expectation of X^2 is $\frac{2-p}{p^2}$, where X is a geometric random variable ranging from 1 to infinity with mean $\frac{1}{p}$.
Proof:

$$\begin{aligned} E[X^2] &= Var[X] + (E[X])^2 \\ &= \frac{1-p}{p^2} + (\frac{1}{p})^2 \\ &= \frac{2-p}{p^2} \end{aligned}$$

So, the varience of X is

$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \sum_{1}^{+\infty} x^2 P(x) - (E[X])^2 \\
&= \sum_{1}^{+\infty} x^2 (0.7 * P_1(x) + 0.3 * P_2(x)) - (E[X])^2 \\
&= 0.7 * \sum_{1}^{+\infty} x^2 P_1(x) + 0.3 \sum_{1}^{+\infty} x^2 P_2(x) - (E[X])^2 \\
&= 0.7 * 15 + 0.3 * 6 - 2.7^2 \\
&= \text{calculate later}
\end{aligned}$$