1 definition

if the n-th term of a sequence can be expressed as a function of previous terms

$$x_n = F(x_{n-k}, ..., x_{n-1}) + g_n,\ n > k, \tag{1}$$

then this equation is called a k-th order *recurrence relation*, the values $x_1, ..., x_k$ are called *initial conditions*, if $g_n \equiv 0$, the recurrence equation is called *homogeneous*, otherwise it's called *non-homogeneous*

2 solving first order recurrences

this class of recurrences can be solved by *iteration*, which is, i think, taught in high school, thus i omit it.

3 solving second order recurrences

this class of recurrences can be solved using *a characterstic equation*, when we have constant coefficients.

example:Fibonacci numbers the Fibonacci sequence is defined by:

$$a_n = a_{n-1} + a_{n-2}, a_0 = 0, a_1 = 1 \tag{2}$$

we obtain a polynomial equation, which is called *a characterstic equation*

$$\lambda^2 = \lambda + 1 \tag{3}$$

which has two roots, $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$, therefore, λ_1^n and λ_2^n and their linear combination $c_1\lambda_1^n + c_2\lambda_2^n$ are solutions to this recurrence, then, from initia conditions, we find c_1 and c_2.

4 multiple roots

what if some of the roots of the characterstic equation are the same?

example

$$a_n = 2a_{n-1} - a_{n-2} \tag{4}$$

the characterstic equation has two identical roots $\lambda_1 = \lambda_2 = 1$, so the first solution is 1^n, to get the second solution, we consider a new equation

$$b_n = (2+\epsilon)b_{n-1} - (1+\epsilon)b_{n-2} \tag{5}$$

, if $\epsilon \to 0$, then b_n approaches a_n, the characterstic equation for b_n has two roots $1, 1+\epsilon$, and $c_1 = 1 - \frac{1}{\epsilon}, c_2 = \frac{1}{\epsilon}$, thus $b_n = (1 - \frac{1}{\epsilon} + \frac{1}{\epsilon} * (1+\epsilon)^n)$, consider when $\epsilon \to 0$, we have $a_n = 1 + n$.

Theorom let λ be a root of a multiplicity p of the characterstic equation, then $\lambda^n, n\lambda^n, ..., n^{p-1}\lambda^n$ are all solutions to the recurrence.

5 non-homogeneous equations

Theorom a recurrence of the form

$$x_n + c_1x_{n-1} + .. + c_kx_{n-k} = b^nP(n) \tag{6}$$

where c_k and b are all constants, and $P(n)$ is a polynomial of the order d can be transformed into the characterstic equation

$$(r^k + c_1r^{k-1} + ... + c_k)(r-b)^{d+1} = 0 \tag{7}$$

6 generating functions

example we are going to derive a generating function for this sequence

$$a_n - 3a_{n-1} + 2a_{n-2} = 0, a_0 = 0, a_1 = 1 \tag{8}$$

first, we define

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{9}$$

then we have

$$f(x) - 3xf(x) + 2x^2 f(x) = a_0 + a_1 x - 3a_0 x = x \tag{10}$$

thus,

$$f(x) = \frac{1}{1-2x} - \frac{1}{1-x} \tag{11}$$

by means of a geometric series, we have

$$f(x) = (2-1)x + (2^2-1)x^2 + ... \tag{12}$$