1 Affine and convex sets

1.1 lines and line segments

line is points of the form $y = \theta x_1 + (1-\theta)x_2$ where $\theta \in R$, and $x_1 \neq x_2$.

line segement corresponds to points where parameter θ is between 0 and 1.

1.2 affine sets

a set is *affine* iff for any $x_1, x_2 \in C$ and $\theta \in R$, we have $\theta x_1 + (1-\theta)x_2 \in C$. This idea can be generalized to more than two points. That is to say, a affine set contains every affine combination of its points: C is an affine set, $x_1, x_2, ..., x_k \in C$, and $\theta_1 + ... + \theta_k = 1$, then the point $\theta_1 x_1 + ... + \theta_k x_k$ also belongs to C.

if C is an affine set and $x_0 \in C$, then the set $V = C - x_0 = \{x \ - \ x_0 \mid x \ \in C\}$ is a subspace. thus any affine set could be expressed as a subspace plus an offset, and the subspace doesnot depend on the choice of x_0.

the set of all affine combination of points in some set C is called the *affine hull* of C:

$$\mathbf{affC} = \{\theta_1 x_1 + ... + \theta_k x_k \mid x_1, x_2, ..., x_k \in C, \ \theta_1 + ... + \theta_k = 1\}. \tag{1}$$

1.3 affine dimension and relative interior

we define the *affine dimension* of a set C as the dimension of its affine hull. if the affine dimension of C is less than n, then the set lies in a affine set $\mathbf{affC} \neq \mathbf{R}^n$. we define *relative interior* if the set C, denoted **relintC**, as its interior relative to **affC**:

$$\mathbf{relintC} = \{x \in C \mid B(x,r) \cap \mathit{affC} \subset C \ for \ some \ r > 0\} \tag{2}$$

where $B(x,r) = \{y \mid \|y - x\| \leq r\}$. we can then define the *ralative boundary* of a set C as $\mathbf{cl}C$ $\mathbf{relint}C$, where $\mathbf{cl}C$ is the closure of C.

1.4 convex sets

a seet C is *convex* if the line segment between any two points in C lies in C. roughly speaking, a set is convex if every point in the set can be seen by every other point.

we call a point of the form $\theta_1 x_1 + ... + \theta_k x_k$, where $\theta_1 + ... + \theta_k = 1$, and $\theta_i \geq 0$, a *convex combination* of the points $x_1, ..., x_k$. a set is convex iff it contains every convex combination of its points. the *convex hull* of a set C, denoted $\mathbf{conv}C$, is the set of all convex combinations of the points in C. the idea of a convex combination can be generalized to include infinite sums, integrals, and probability distributions. suppose θ_i satisfy

$$\theta_i \geq 0, i = 1, 2, ..., \quad \sum_{i=1}^{\infty} \theta_i = 1, \tag{3}$$

and $x_1, x_2, ... \in C$, where $C \subset \mathbf{R}^n$ is convex, then

$$\sum_{i=1}^{\infty} \theta_i x_i \in C, \tag{4}$$

if the series converges. more generally, suppose $C \subset \mathbf{R}^n$ is convex and x is a random vector with $x \in C$ with probability one, then $\mathbf{E}x \in C$.

1.5 cones

a set is called a *cone*, or *nonnegative homogeneous*, if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. a point of the form $\theta_1 x_1 + ... + \theta_k x_k$ with $\theta_i \geq 0$ is called *conic combination* or a *nonnegative linear combination* of x_i. a set C is a convex cone iff it contains all conic combinations of its elements. this idea could also be generalized to infinite sums and integrals. the *conic hull* of a set C is the set of all conic combinations of points in C.

2 some important examples

2.1 hyperplanes and halfspaces

a *hyperplane* is a set of the form

$$\{x \mid a^T x = b\}, \tag{5}$$

where $a \in \mathbf{R}^n,\ a \neq 0$, and $b \in \mathbf{R}$. this repersentation can inturn be expressed as

$$\{x \mid a^T(x - x_0) = 0\} = x_0 + a^\perp, \tag{6}$$

where $a^\perp$ denotes the orthogonal complement of a, *i.e.*, the set of all vectors orthogonal to it.

a hyperplane divides $\mathbf{R}^n$ into two *halfspaces*, which a set of the form

$$\{x \mid a^T x \leq b\}, \tag{7}$$

where $a \neq 0$

2.2 euclidean balls and ellipsoids

a *(euclidean) ball* in $\mathbf{R}^n$ has the form

$$B(x_c, r) = \{x \mid \|x - x_c\| \leq r\} = \{x \mid (x - x_c)^T(x - x_c) \leq r^2\}, \tag{8}$$

where $r \geq 0$, and the vector x_c is the *center* of the ball and the saclar r is its *radius*. another common repersentation for the euclidean ball is

$$B(x_c, r) = \{x_c + ru \mid \|u\| \leq 1\}, \tag{9}$$

a euclidean ball is a convex set (use the homogeneity and triangle inequality for $\|\cdot\|_2$)

a related family of convex sets is the *ellipsoids*, which have the form

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}, \tag{10}$$

where P is symmetric and positive definite. the vector $x_c \in \mathbf{R}^n$ is the *center* of the ellipsoid. the matrix P determines how far the ellipsoid extends in every direction from x_c by the square root of its eigenvalues. another common repersentation of an ellipsoid is

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}, \tag{11}$$

where A is the square and nonsingular, in fact, $A = P^{1/2}$. when the matrix A is symmetric positive semidefinite but not singular, the set is called a *degenerate ellipsoid*, its affine dimension is equal to the rank of A, and it is also convex.

2.3 norm balls and norm cones

a *norm ball* of radius r and center x_c given by $\{x \mid \|x - x_c\| \leq r\}$, is convex. the *norm cone* associated with the norm $\|\cdot\|$ is the set

$$C = \{(x, t) \mid \|x\| \leq t\} \subset \mathbf{R}^{n+1}. \tag{12}$$

it is also a convex set, as the name suggests.

2.4 polyhera

a *polyheron* is defined as the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j,\ j = 1, ..., m,\ c_j^T x = d_j,\ j = 1, ..., p\}. \tag{13}$$

it will be convenient to use the compact notation

$$\mathcal{P} = \{x \mid Ax \preceq b,\ Cx = d\}, \tag{14}$$

where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix}, \tag{15}$$

and the symbol $\preceq$ denotes *vector inequality* or *componentwise inequality* in $\mathbf{R}^m$.

2.4.1 simplexes

t.b.c.

2.4.2 convex hull description of polyhedra

t.b.c