# Example solution - reexam August 2024
# 10385 Quantum Information Technology

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## Problem 1: A squeezed state of light

### 1.1

Heisenbergs uncertainty relation for the vacuum state:

$$\Delta^2 X_v \Delta^2 P_v = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

This relation also holds for the sqq. state:

$$\Delta^2 X_{sqz} \Delta^2 P_{sqz} = \frac{1}{4} \Rightarrow \Delta^2 P_{sqz} = \frac{1/4}{1/20} = 5$$

### 1.2

The state undergoes 50% loss:

$$X_{sqz} \to \sqrt{\frac{1}{2}} X_{sqz} + \sqrt{\frac{1}{2}} X_v$$
$$P_{sqz} \to \sqrt{\frac{1}{2}} P_{sqz} + \sqrt{\frac{1}{2}} P_v$$

The measured quadrature is

$$\begin{aligned} X_\theta &= X_{\text{sqz}} \cos\theta + P_{\text{sqz}} \sin\theta \\ &\overset{\text{loss}}{\Rightarrow} \left( \sqrt{\frac{1}{2}} X_{sqz} + \sqrt{\frac{1}{2}} X_v \right) \cos\theta + \left( \sqrt{\frac{1}{2}} P_{\text{sqz}} + \sqrt{\frac{1}{2}} P_v \right) \sin\theta \end{aligned}$$

The variance is

$$\begin{aligned} \Delta^2 X_\theta &= \frac{1}{2} \left( \Delta^2 X_{sqz} + \Delta^2 X_v \right) \cos^2\theta + \frac{1}{2} \left( \Delta^2 P_{sqz} + \Delta^2 P_v \right) \sin^2\theta \\ &= \frac{1}{2} \left( \frac{1}{20} + \frac{1}{2} \right) \cos^2\theta + \frac{1}{2} \left( 5 + \frac{1}{2} \right) \sin^2\theta \\ &= \frac{11}{40} \cos^2\theta + \frac{11}{4} \sin^2\theta \end{aligned}$$

### 1.3

The output of the beam splitter

$$\begin{aligned} X_{sqz1} &= \sqrt{\tfrac{1}{2}} X_{sqz} + \sqrt{\tfrac{1}{2}} X_v, \quad X_{sqz2} = \sqrt{\tfrac{1}{2}} X_{sqz} - \sqrt{\tfrac{1}{2}} X_v \\ P_{sqz1} &= \sqrt{\tfrac{1}{2}} P_{sqz} + \sqrt{\tfrac{1}{2}} P_v, \quad P_{sqz2} = \tfrac{1}{\sqrt{2}} P_{sqz} - \sqrt{\tfrac{1}{2}} P_v \end{aligned}$$

Two-mode squeezing is obtained when the sum of the amplitude quadrature measurements and the difference of the phase quadrature measurements yield squeezing:

$$\Delta^2 \left(X_{sqz1} + X_{sqz2}\right) = \Delta^2 \left(2\sqrt{\frac{1}{2}} X_{sqz}\right) = 2\Delta^2 X_{sqz}$$

$$\Delta^2 \left(P_{sqz1} - P_{sqz2}\right) = \Delta^2 \left(2\sqrt{\frac{1}{2}} P_v\right) = 2\Delta^2 P_v$$

We thus observe two-mode squeezing in the amplitude, but not in the phase quadrature.

### 1.4

To enable continuous variable teleportation, we require two-mode squeezing in X and P to faithfully teleport the amplitude and phase quadrature of the input state. Since the above resource exhibits two-mode squeezing only in the amplitude quadrature, solely the amplitude quadrature will be teleported with noise lower than the classical teleporter.

## Problem 2: Phase estimation with quadrature measurements

### 2.1

Given that the input is a coherent state, the state after the phase shift is also a coherent state $|\alpha e^{i\theta}\rangle$ (or $|\alpha e^{-i\theta}\rangle$, the sign of $\theta$ does not matter here).

Since we have a coherent state, the expectation value is $\langle \hat{X} \rangle = \sqrt{2}\,\mathrm{Re}(\alpha e^{i\theta}) = \sqrt{2}\alpha\cos(\theta)$.

The quadrature variance in a coherent state is independent of the amplitude. It always equals the variance in the vacuum state $\langle \Delta^2 \hat{X} \rangle = \frac{1}{2}$.

A plot of $\langle \hat{X} \rangle$ is shown in Fig. 1.

### 2.2

When estimating $\theta$ from an $\hat{X}$-measurement, we can think of $\langle \hat{X} \rangle$ as a function of $\theta$. Simple error propagation then stipulates that the uncertainties are related by

$$\Delta \hat{X} = |\frac{\partial \langle \hat{X} \rangle}{\partial \theta}| \Delta\theta. \tag{1}$$

Hence, the variance obeys

$$\Delta^2 \theta = |\frac{\partial \langle \hat{X} \rangle}{\partial \theta}|^{-2} \langle \Delta^2 \hat{X} \rangle. \tag{2}$$

Inserting $\langle \hat{X} \rangle = \sqrt{2}\alpha\cos(\theta)$ and $\Delta^2 \hat{X} = \frac{1}{2}$ gives

$$\Delta^2 \theta = \frac{1}{4\alpha^2 \sin^2(\theta)}, \tag{3}$$

as desired.

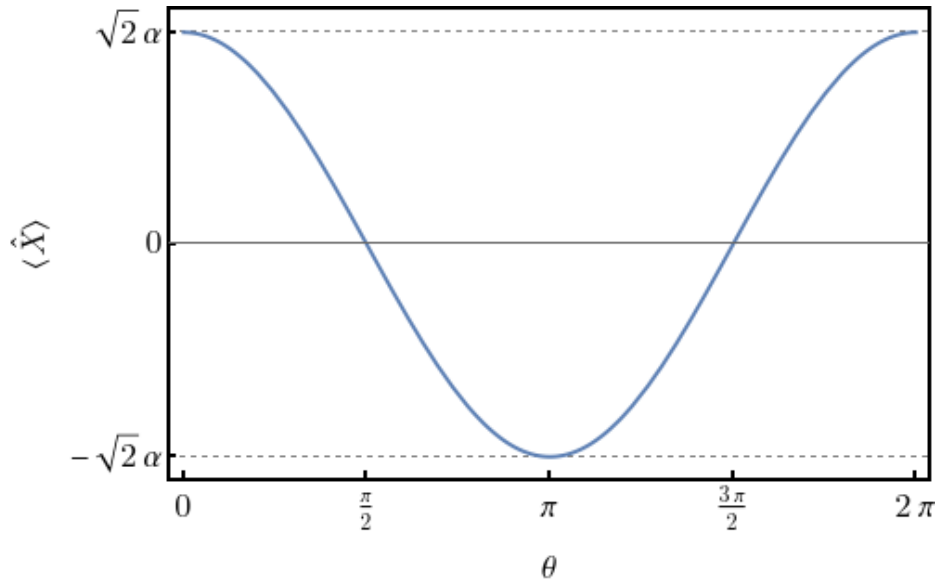


**Figure 1:** Plot of $\langle \hat{X} \rangle = \sqrt{2}\alpha \cos(\theta)$ (solid blue curve). The value is most sensitive to small changes in $\theta$ around odd multiples of $\frac{\pi}{2}$ and least sensitive around even multiples.

Clearly, $\Delta^2\theta \leq \frac{1}{4\alpha^2}$. The best precision (minimum variance) is attained when $\sin^2(\theta) = 1$ i.e. for odd multiples of $\frac{\pi}{2}$, while the variance diverges for $\sin^2(\theta) = 0$ at even multiples of $\frac{\pi}{2}$. This can also be seen from Fig. 1 as the signal is the slope of the curve while the noise (the variance $\Delta^2\hat{X}$) is constant. The signal-to-noise ratio is maximal for maximum slope and vanishes for vanishing slope.

### 2.3

The probability of outcome $x$ is given by

$$p_\theta(x) = |\langle x|\alpha e^{i\theta}\rangle|^2 = |\psi_{|\alpha e^{i\theta}\rangle}(x)|^2 = \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{2}\alpha\cos(\theta))^2}, \tag{4}$$

where we have used that $\alpha$ is real and that the position-space wave function of a coherent state $|\beta\rangle$ is given by

$$\langle x|\beta\rangle = \psi_{|\beta\rangle}(x) = \frac{1}{\pi^{\frac{1}{4}}} e^{-i\frac{p_\beta x_\beta}{2}} e^{ip_\beta x} e^{-\frac{1}{2}(x-x_\beta)^2}, \tag{5}$$

with $x_\beta = \frac{1}{\sqrt{2}}(\beta + \beta^*) = \sqrt{2}\,\mathrm{Re}(\beta)$ and $p_\beta = \frac{1}{i\sqrt{2}}(\beta - \beta^*) = \sqrt{2}\,\mathrm{Im}(\beta)$.

### 2.4

The Fisher information is given by

$$\mathcal{F}_\theta = \int_{-\infty}^{\infty} [\frac{\partial}{\partial\theta} \log(p_\theta(x))]^2 p_\theta(x) dx = \int_{-\infty}^{\infty} \frac{\dot{p}_\theta^2(x)}{p_\theta(x)} dx, \tag{6}$$

where the dot denotes $\frac{\partial}{\partial\theta}$. We first compute

$$\dot{p}_\theta(x) = \frac{\partial}{\partial\theta} \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{2}\alpha\cos(\theta))^2} = -\frac{2\sqrt{2}\alpha}{\sqrt{\pi}} e^{-(x-\sqrt{2}\alpha\cos(\theta))^2} (x - \sqrt{2}\alpha\cos(\theta))\sin(\theta). \tag{7}$$

Then

$$\frac{\dot{p}_\theta^2(x)}{p_\theta(x)} = 8\alpha^2 \sin^2(\theta) \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{2}\alpha\cos(\theta))^2} (x - \sqrt{2}\alpha\cos(\theta))^2 \tag{8}$$

$$= 8\alpha^2 \sin^2(\theta) |\psi_{|\beta\rangle}(x)|^2 (x - x_\beta)^2, \tag{9}$$

for $\beta = \alpha e^{i\theta}$. The integral over the last part of this expression is just the variance of $\hat{X}$ in the coherent state $|\alpha e^{i\theta}\rangle$ which is $\frac{1}{2}$, i.e.

$$\int |\psi_{|\beta\rangle}(x)|^2 (x - x_\beta)^2 = (\Delta^2 \hat{X})_{|\beta\rangle} = \frac{1}{2}. \tag{10}$$

Therefore, the Fisher information is

$$\mathcal{F}_\theta = 4\alpha^2 \sin^2(\theta), \tag{11}$$

as desired. The Crámer-Rao bound then implies that

$$\Delta^2\theta \geq \frac{1}{\mathcal{F}_\theta} = \frac{1}{4\alpha^2 \sin^2(\theta)} \tag{12}$$

for any unbiased estimator.

### 2.5

The output state just before the measurement is $|\beta\rangle = |\alpha e^{i\theta}\rangle$, as stated above. The phase shift is generated by a Hamiltonian equal to the photon-number operator, $\hat{H} = \hat{n}$. Since the output state is pure, the quantum Fisher information is given by four times the variance of $\hat{H}$ in the input state

$$\mathcal{Q}_\theta = 4(\langle \hat{n}^2 \rangle_{|\alpha\rangle} - \langle \hat{n} \rangle_{|\alpha\rangle}^2) = 4|\alpha|^2 = 4\alpha^2. \tag{13}$$

Comparing $\mathcal{F}_\theta = 4\alpha^2 \sin^2(\theta)$ for the $\hat{X}$-measurement to this, we see that the measurement is optimal only around odd multiples of $\frac{\pi}{2}$. An optimal measurement at a general angle $\theta$ is obtained by measuring a quadrature rotated from $\hat{X}$ by $\theta - \frac{\pi}{2}$.

## Problem 3: Qutrit QKD

### 3.1

We start by noting that $\omega = e^{i\frac{2\pi}{3}}$ is a root of unity with $|\omega|^2 = 1$ and $1 + \omega + \omega^2 = 0$.

The states are normalised since

$$\langle k'|k'\rangle = \frac{1}{3}(1 + |\omega^k|^2 + |\omega^{2k}|^2) = \frac{1}{3}(1 + 1 + 1) = 1. \tag{14}$$

They are also orthogonal since, for $k \neq l$

$$\langle l'|k'\rangle = \frac{1}{3}(1 + \omega + \omega^2) = 0, \tag{15}$$

by using properties of roots of unity ($\omega^* = \omega^2$ and $\omega^3 = 1$).

### 3.2

In the absence of noise and eavesdropping, the state received by Bob equals the state transmitted by Alice, which is one of the basis state in the basis determined by $r_a$. Since only rounds with $r_b = r_a$ are kept, Bob measures in the same basis and therefore obtains $t_b = t_a$ with certainty.

On average, Bob will choose the correct basis in half of the rounds, so after basis sifting, on average $\frac{1}{2}N$ rounds are kept. In each of those rounds, a perfectly random trit is shared, corresponding to a min-entropy of $H_{\min} = -\log_2(\frac{1}{3})$, where the choice of $\log_2$ implies quantification in bits. The total number of random bits is thus on average

$$\frac{1}{2}N \log_2(3). \tag{16}$$

In standard BB84 the parties also discard half of the rounds on average but share perfectly correlated random bits (not trits) in the remaining rounds, so the entropy is $\frac{1}{2}N$, which is smaller.

### 3.3

We only need to consider rounds that are not discarded, so we can assume that Bob's basis choice matches Alice's, i.e. $r_b = r_a$. If Eve chooses the correct basis, her outcome will match the preparation of Alice with probability one, and she will repreprare the sme state. No error will be introduced in this case. If Eve chooses the wrong basis, $r_e \neq r_a$ she will obtain each of the possible outcomes with probability $\frac{1}{3}$, because $|\langle m'|n\rangle|^2 = \frac{1}{3}$ as shown in 3.1, and prepare the corresponding state. The probability for Bob to obtain $t_b = t_a$ (i.e. to see no error) for any of these states is also $\frac{1}{3}$ for the same reason. Hence, the overall error rate will be

$$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \approx 33\%. \tag{17}$$

This error rate for the intercept-resend attack is higher than for BB84 with qubits, where it is 25%. That is, intercept-resend attacks are easier to detect in the qutrit protocol. Hence, one might expect the qutrit protocol to generally be more noise robust / have better security also against more general attacks. This is indeed the case, as shown by Cerf *et al.* in *Phys. Rev. Lett.* 88, 127902 (2002).