Written Exam (Reexam), Monday August 19th 2024

Course name:	Quantum Information Technology.
Course number:	10385.
Aids:	All aids are allowed (no internet).
Duration:	4 hours.
Weighting:	All sub-problems have equal weight in the evaluation.

Problem 1: A squeezed state of light

Consider a pure squeezed state of light with a maximum squeezing in its amplitude quadrature, $\hat{X}$, of 90%. The variance of the amplitude quadrature is therefore $\langle \Delta^2 \hat{X} \rangle = 1/20$ where the variance of the vacuum state is set to $\langle \Delta^2 \hat{X}_v \rangle = 1/2$.
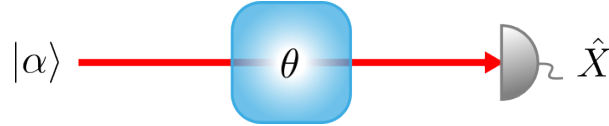
1.1 Find the Heisenberg uncertainty relation for the vacuum state. Find also the largest quadrature variance of the squeezed state.

1.2 The squeezed state undergoes a loss of 50% (simulated by a balanced beam splitter), and is subsequently measured with a homodyne detection system comprising a local oscillator with phase θ. Calculate the measured quadrature variance as a function of θ.

1.3 Now assume that the 50% lost part of the squeezed state can be measured with another homodyne detector. Investigate whether it is possible to measure two-mode squeezing in such a configuration.

Problem 2: Phase estimation with quadrature measurements

Consider a setup for estimating an optical phase shift by probing with a coherent state and subsequently performing a quadrature measurement, as shown here.

Assume that the coherent-state amplitude α is real and that the amplitude quadrature $\hat{X}$ is measured. We aim to find the precision in estimating the phase θ attainable by this measurement.

We first calculate the precision by error propagation.

2.1 Find the expectation value $\langle\hat{X}\rangle$ and variance $\langle\Delta^2\hat{X}\rangle$ of the measurement outcome. Make a sketch of $\langle\hat{X}\rangle$ as a function of θ.

2.2 Argue that, when estimating θ from the $\hat{X}$-measurement, the estimate variance is given by

$$\Delta^2\theta = \frac{\langle\Delta^2\hat{X}\rangle}{|\frac{\partial\langle\hat{X}\rangle}{\partial\theta}|^2}, \tag{1}$$

and show that $\Delta^2\theta = \frac{1}{4\alpha^2\sin^2(\theta)}$. At which phase angle θ is the best precision attained and why? (you may refer to your sketch from 2.1).

Next, we calculate the precision by using the Fisher information.

2.3 Show that the probability for obtaining the measurement outcome x is

$$p_\theta(x) = \frac{1}{\sqrt{\pi}}e^{-(x-\sqrt{2}\alpha\cos(\theta))^2}. \tag{2}$$

Hint: use the coherent-state wavefunction.

2.4 Show that the Fisher information of the distribution (2) with respect to the parameter θ is $\mathcal{F}_\theta = 4\alpha^2\sin^2(\theta)$ and argue that the precision is again $\Delta^2\theta = \frac{1}{4\alpha^2\sin^2(\theta)}$.

2.5 Find the quantum Fisher information $\mathcal{Q}_\theta$ of the output state just before the measurement. Is the quadrature measurement optimal?

Problem 3: Qutrit QKD

We will consider an extension of the BB84 quantum key-distribution protocol to qutrits (3-dimensional systems) instead of qubits.

Let $R = \{|0\rangle, |1\rangle, |2\rangle\}$ denote an orthornormal basis for a qutrit space, and let

$$|0'\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \tag{3}$$

$$|1'\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle), \tag{4}$$

$$|2'\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + \omega|2\rangle), \tag{5}$$

with $\omega = e^{i\frac{2\pi}{3}}$.

3.1 Show that the states in $R' = \{|0'\rangle, |1'\rangle, |2'\rangle\}$ are orthonormal and that $|\langle m'|n\rangle|^2 = \frac{1}{3}$ for all $m, n \in \{0, 1, 2\}$.

The protocol proceeds analogously to BB84. A first party, Alice, uses a random bit $r_a \in \{0, 1\}$ to choose between the two bases R, R' and encodes a random trit $t_a \in \{0, 1, 2\}$, i.e. she prepares the state

$$|\psi_{r_a,t_a}\rangle = \begin{cases} |t_a\rangle & \text{if } r_a = 0 \\ |t'_a\rangle & \text{if } r_a = 1 \end{cases}. \tag{6}$$

The state is transmitted to a second party, Bob, who also uses a random bit $r_b \in \{0, 1\}$ to choose the basis R or R' and performs a projective measurement, obtaining outcome t_b. The parties subsequently use an authenticated classical channel to discard rounds with $r_b \neq r_a$ and to perform error correction and privacy amplification on the remaining data.

3.2 Consider first an ideal case without any noise or eavesdropping. Argue that, in rounds that are not discarded, Alice and Bob share perfectly correlated, random trits. After running the protocol for a large number of steps N, how much randomness does Alice and Bob share, quantified in bits? How does this compare to standard BB84 with qubits?

3.3 Now imagine that an eavesdropper, Eve, performs an intercept-resend attack. Eve chooses basis R or R' according to a random bit $r_e \in \{0, 1\}$ and measures. Upon obtaining outcome t_e, she prepares the state $|\psi_{r_e,t_e}\rangle$ and sends it to Bob. Find the error rate introduced by the attack of Eve. Do you think the qutrit protocol is more or less robust to noise in general than standard BB84 with qubits? Why?

End of set