Written Exam, Tuesday May 20th 2025

Course name:	Quantum Information Technology.
Course number:	10385.
Aids:	All aids are allowed (no internet).
Duration:	4 hours.
Weighting:	All sub-problems have equal weight in the evaluation.

Problem 1: Qubit thermometry

We consider the problem of estimating the temperature T of a large object (which can be considered a thermal bath) using a qubit as a probe.

We take the qubit to have energy eigenstates $|0\rangle$ and $|1\rangle$, corresponding to energies 0 and Δ, respectively. The probe Hamiltonian is thus $\hat{H} = \Delta|1\rangle\langle 1|$.

The qubit probe is first brought into thermal equilibrium with the bath. It is then in the mixed thermal state

$$\hat{\rho}_T = \frac{1}{Z_T}\exp(-\hat{H}/T) = \frac{1}{Z_T}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\Delta/T} \end{pmatrix}, \tag{1}$$

given here in the energy eigenbasis (in units where Boltzmann's constant $k_B = 1$).

1.1 Find the normalisation constant Z_T.

The probe is now measured in the energy eigenbasis. We denote the outcomes corresponding to $|0\rangle$ and $|1\rangle$ by 0 and 1, respectively.

1.2 Show that the probabilities for the two outcomes are

$$p_0 = \frac{1}{1+e^{-\Delta/T}}, \qquad \text{and} \qquad p_1 = \frac{1}{1+e^{\Delta/T}}, \tag{2}$$

and find the (classical) Fisher information with respect to the temperature T for this measurement.

1.3 Show that (in the energy eigenbasis)

$$\dot{\hat{\rho}}_T = \frac{\dot{Z}_T}{Z_T^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

where the dot denotes derivative with respect to T.

1.4 Show that the symmetric logarithmic derivative $\hat{L}_T(\hat{\rho}_T)$ is given by

$$\hat{L}_T(\hat{\rho}_T) = \frac{\dot{Z}_T}{Z_T}\begin{pmatrix} -1 & 0 \\ 0 & e^{\Delta/T} \end{pmatrix}. \tag{4}$$

Is the energy measurement optimal for estimating T?

Problem 2: Beam splitting and randomness

Consider a setup for quantum random-number generation as shown in Figure 1. One input mode, prepared in a state $|\psi_{in}\rangle$, is mixed with a second mode in a vacuum (zero-photon) state on a beam splitter with transmittivity η. The photon number in the transmitted output mode A is measured. We denote the measurement outcome by x.

The goal is to extract randomnes from the measurement at A. The input states and the beam-splitter transmittivity are trusted (and publicly known). However, an eavesdropper (Eve) is assumed to have access to the second output of the beam splitter.

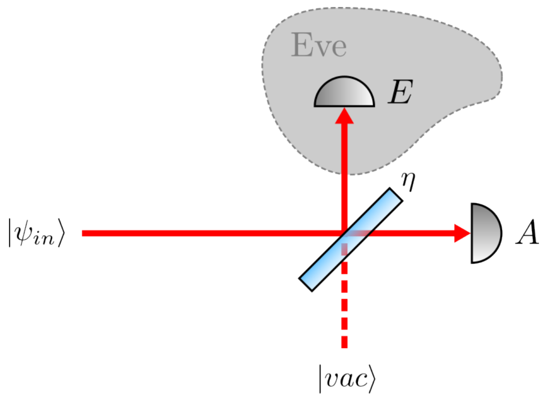

Figure 1: Beam-splitter setup for random-number generation.

Consider first a single-photon input state, $|\psi_{in}\rangle = |1\rangle$.

2.1 Find the conditional states of mode E when either $x = 0$ or $x = 1$ photons are detected in mode A. Argue that no randomness can be extracted in this case.

Next, assume that the input is a superposition $|\psi_{in}\rangle = \sqrt{\lambda}|0\rangle + \sqrt{1-\lambda}|1\rangle$, with $\lambda \in [0, 1]$ real, and that Eve performs a projective measurement of the photon number in E, with outcome y.

2.2 Find the joint state of modes A and E at the output of the beam splitter and the joint probability distribution of the outcomes $P_{AE}(x, y)$.

2.3 Show that Eve's average guessing probability (for guessing x) is given by

$$p_g = (1-\lambda)(1-\eta) + \max\{\lambda, (1-\lambda)\eta)\}. \tag{5}$$

2.4 Find the optimal choice of λ (as function of η) which minimises p_g and the corresponding expression for p_g.

Problem 3: Dual-rail photonic qubits

In dual-rail encoding, a photonic qubit is encoded using a single photon in two orthogonal modes, such that

$$|0_L\rangle = |1,0\rangle, \quad |1_L\rangle = |0,1\rangle \tag{6}$$

3.1 Describe how arbitrary single-qubit unitary operations can be implemented on such dual-rail qubits using only linear optical components. Draw schematics of setups implementing these single-qubit gates.

3.2 Describe possible physical realizations of the dual-rail modes and discuss their experimental advantages and disadvantages.

The Knill–Laflamme–Milburn (KLM) scheme enables probabilistic implementation of nonlinear gates using linear optics, ancilla photons, and measurements. Consider three dual-rail photonic qubits labeled 1, 2, and 3. Suppose you apply two consecutive controlled-phase (CZ) gates. First between qubits 1 and 2, then between qubits 2 and 3 (as indicated in Figure 2).

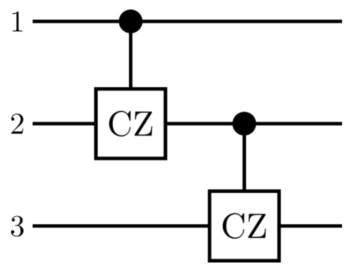

Figure 2: Circuit with three qubits and two CZ-gates.

3.3 Draw a schematic diagram of an optical circuit implementing the circuit in Figure 2 using the KLM approach.

3.4 How many ancilla photons are needed for the two-CZ circuit in Figure 2 and what is the overall success probability of the circuit?

End of set