# Problems for tropical geometry tutorial

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(1) Newton polytopes and Minkowski sums.

(a) Show that the vertices of the Minkowski sum of two polytopes $P_1$ and $P_2$ have the form $v_1 + v_2$ where $v_1$ and $v_2$ are vertices of $P_1$ and $P_2$ respectively.

(b) Show that any face $F$ of the Minkowski sum of two polytopes $P_1$ and $P_2$ can be written uniquely as $F = F_1 + F_2$ where $F_1$ and $F_2$ are faces of $P_1$ and $P_2$ respectively.

(c) Show that the Newton polytope of the product $fg$ of two polynomials $f$ and $g$ is the Minkowski sum of the Newton polytopes of $f$ and $g$. What can you say about multiplicities?

(d) Show the tropical hypersurface $\mathcal{T}(fg)$ of the product $fg$ is the union of the tropical hypersurfaces $\mathcal{T}(f)$ and $\mathcal{T}(g)$.

(e) Describe the Newton polytope of the polynomial

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

Do you know its name? Describe the tropical hypersurface of the polytope.

(2) Show that the following polynomial is irreducible over $\mathbb{C}$:

$$x + 2x^2 + x^3 + y + 3xy + 5y^2$$

(3) Use Puiseux's theorem to prove Eisenstein's criterion for irreducibility of polynomials over $\mathbb{Q}$ (Look it up on Wikipedia if you don't remember). For polynomials satisfying the criterion with a prime $p$, what is the Newton polygon with respect to the p-adic valuation? What does it tell you about the p-adic valuations of the roots?

(4) Draw the tropical hypersurface and the dual subdivision of the Newton polytope for the polynomial

$$f = t + x + tx^2 + y + xy + ty^2 \in \mathbb{C}\{\{t\}\}[x, y].$$

(5) Draw the intersection of tropical curves corresponding to the following polynomials:

$$x + y - y^2 + 2xy + xy^2 \text{ and } x - y + x^2y - y^2 + 2xy^2.$$

Explain why the two polynomials have finitely many common zeroes in $(\mathbb{C}^*)^2$.

(6) Let $f = w^2 + x^2 + y^2 + z^2$ and $g = w^3 + x^3 + xyz$ be polynomials in $\mathbb{C}[w, x, y, z]$.

(a) Explain why $\mathcal{T}(f) \cap \mathcal{T}(g) \neq \mathcal{T}(\langle f, g\rangle)$.

(b) Compute the tropical variety $\mathcal{T}(\langle f, g\rangle)$ using the software Gfan, with the commands `gfan_tropicalstartincone` and `gfan_tropicaltraverse`.

(7) Let $K$ be the field of Puiseux series over $\mathbb{C}$, and let $f = a_1 + a_2x + a_3y$ and $g = b_1x + b_2y + b_3xy$ be polynomials in $K[x, y]$ where $a_1, a_2, a_3, b_1$, and $b_3$ have valuation 0 and $b_2$ has valuation 1.

(a) Draw the two corresponding tropical curves. What is their intersection?

(b) Explain why $f$ and $g$ have two common zeroes (counted with multiplicities), both lying in $(K^*)^2$.

(c) Do some examples to make a conjecture about where the valuations these two points are located.

(d) Prove your conjecture.

(e) **Open problem** Generalize to two arbitrary polynomials in two variables. What can you say about the location of valuations of their common zeroes just by looking at their tropical curves? (Some necessary conditions are known. See the paper "Tropical Images of Intersection Points" by Ralph Morrison. It is an open question whether these conditions are sufficient.)