TORICS IN *MACAULAY2*
TUTORIAL #1

1. SMOOTH PROJECTIVE TORIC VARIETIES

To work with toric varieties in *Macaulay2*, use the command

```
needsPackage "NormalToricVarieties";
```

1.1. **Projective Space.** How is projective d-space realized as a quotient?

(a) For a few small values of d, execute the following commands in *Macaulay2*.
```
PP = projectiveSpace d;
rays PP
max PP
```
What are the outer normal vectors? Which subsets of the facet hyperplanes correspond to the vertices? What polytope(s) defines projective d-space?

(b) How does one determine the degrees of the variables in the ring S = ring PP? What is the short exact sequence of free abelian groups used to define the action of $\mathbb{C}^\times$ on $\mathbb{C}^{d+1}$?

(c) Which points $[z_0 : z_1 : \cdots : z_d]$ in projective d-space belong to the algebraic torus $(\mathbb{C}^\times)^d$? How does the action of $(\mathbb{C}^\times)^d$ extend to an action on projective d-space?

(d) For $2 \leqslant d \leqslant 4$, describe the orbits of $(\mathbb{C}^\times)^d$ in projective d-space. How are these orbits related to the defining polytope? How can one compute these in *Macaulay2*?

1.2. **Hirzebruch Surfaces.** How are the Hirzebruch surfaces realized as quotients?

(a) For a few small positive values of a, execute the following commands in *Macaulay2*.
```
Y = hirzebruchSurface a;
rays Y
max Y
isSmooth Y
isProjective Y
```
What are the outer normal vectors? Which subsets of the facet hyperplanes correspond to the vertices? What polytope(s) define the a-th Hirzebruch surface?

(b) How is the Cox ring of Y graded in *Macaulay2*? Which monomials have degree {1,1}? What is the action of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^4$ which has Y as the quotient? Under this action, what are the orbits of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^4$? Which orbits are "irrelevant"? How do you find the irrelevant ideal in *Macaulay2*?

(c) In analogy with projective d-space, how do 4-tuples $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}^4$ correspond to points in Y? Which points belong to the algebraic torus $(\mathbb{C}^\times)^2$ in Y? How does the action of $(\mathbb{C}^\times)^d$ extend to an action on projective d-space? Describe these orbits and relate them to the defining polytope.

1.3. Irrelevant Ideals. How are irrelevant ideals of a smooth projective toric variety related to Stanley–Reisner ideals? For a few small positive values of d and various nonnegative values of i, execute the following commands in *Macaulay2*.

```
X = smoothFanoToricVariety(d,i);
max X
B = ideal X
primaryDecomposition B
dual monomialIdeal X
```

How are the minimal generators of the irrelevant ideal related to the vertices of the defining polytope? How does one prove this for a general smooth projective toric variety?

1.4. Open Affine Covering. What is the canonical open affine covering of a toric variety?

(a) For each subset σ of the facet hyperplanes corresponding to a vertex, describe the subsets in projective d-space consisting of the points $[z_0 : z_1 : \cdots : z_d]$ with $z_j = 1$ for all $j \notin \sigma$.

(b) For each subset σ of max Y, describe the subsets in a-the Hirzebruch surface consisting of the points $[z_0 : z_1 : z_2 : z_3]$ with $z_j = 1$ for all $j \notin \sigma$.

(c) For each subset σ of the facet hyperplanes corresponding to a vertex, consider the $\mathbb{C}$-algebra $S[z_j^{-1} : j \notin \sigma]_{\mathbf{0}}$ (i.e. the degree-zero component of localized Cox ring). Find generators for the $\mathbb{C}$-algebra. How are the generators related to vectors from the vertex to other lattice points in the defining polytope? What does the smoothness of the polytope imply about relations among the generators?

1.5. Kleinschmidt Varieties. What are the smooth projective toric varieties arising from polytopes with dimension d and $d + 2$ facets?

(a) For a few small positive values of d and various increasing lists L of at most d-1 nonnegative integers, execute the following commands in *Macaulay2*.
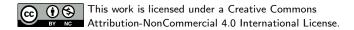
```
X = kleinschmidt(d,L);
rays X, max X
degrees ring X, ideal X
```

What polytope(s) define the associated Kleinschmidt variety?

(b) If X is a smooth projective toric variety such that the Cox ring S has a $\mathbb{Z}^2$-grading, then (up to equivalence) which matrices induce the grading on S?

REFERENCES

[M2] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at www.math.uiuc.edu/Macaulay2/.

[CLS] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

GREGORY G. SMITH, DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, K7L 3N6, CANADA
E-mail address: ggsmith@mast.queensu.ca