TORICS IN *MACAULAY2*
SMOOTH PROJECTIVE TORIC VARIETIES

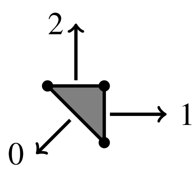
What. A ***toric variety*** over the field $\mathbb{C}$ is an irreducible variety X such that the algebraic torus $(\mathbb{C}^{\times})^d$ is a Zariski open subset of X, and the action of $(\mathbb{C}^{\times})^d$ on itself extends to an action on X.

Why. Toric varieties form part of a dictionary between algebraic geometry, commutative algebra, and combinatorial geometry. Because of their accessibility and computability, they arise essentially everywhere.

Basic Problems.
- What are smooth projective toric varieties and how do we work with them in *Macaulay2*?
- What are coherent sheaves and how do we represent them in *Macaulay2*?

Prototype. The projective plane $\mathbb{P}^2$ is the set of line through the origin in $\mathbb{C}^3$. More precisely, the multiplicative group $\mathbb{C}^{\times}$ acts diagonally on $\mathbb{C}^3$ and lines through the origin correspond to the orbits. Since the origin $\mathbf{0}$ the unique non-maximal orbit, we set $\mathbb{P}^2 := \big(\mathbb{C}^2 \setminus \{\mathbf{0}\}\big)/\mathbb{C}^{\times}$. A point $[z_0 \colon z_1 \colon z_2] \in \mathbb{P}^2$ corresponds to the line $(\lambda z_0, \lambda z_1, \lambda z_2) \in \mathbb{C}^3$ where $\lambda \in \mathbb{C}^{\times}$.

The combinatorial source of $\mathbb{P}^2$ is the following triangle in $\mathbb{R}^2$.



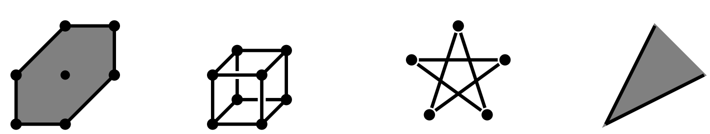
$$\Delta_2 := \operatorname{conv}\big((0,0),(-1,0),(0,-1)\big)$$

The outer normal vectors induce an exact sequence $0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \mathbb{Z}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}\right]} \mathbb{Z} \longrightarrow 0$ and applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{C}^{\times})$ yields the inclusion of tori $1 \longrightarrow \mathbb{C}^{\times} \xrightarrow{\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]} (\mathbb{C}^{\times})^3$. Combined with the entrywise action of $(\mathbb{C}^{\times})^3$ on $\mathbb{C}^3$, we obtain the diagonal action on $\mathbb{C}^3$.

Polytopes. For $d > 0$, fix the lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$. A ***lattice polytope*** $P \subset \mathbb{R}^d$ is the convex hull of a finite subset of lattice points. We assume that the smallest affine subspace containing P is $\mathbb{R}^d$. The ***vertices*** are the points that do not lie in any open line segment joining two points of P. Minkowski first established that P is the convex hull of its vertices and this is the minimal such representation.

Equivalently, P is a bounded intersection of finitely many closed half-spaces with integral normal vectors. A ***supporting hyperplane*** contains P in one of its associated closed half-spaces and contains a point in P. The unique minimal collection of supporting hyperplanes defining P correspond to the ***facets*** of P. Let n denote the number of facets.

Examples and Counterexamples.



A polytope P is ***smooth*** if, for each vertex, the facet normal vectors passing through the vertex form a subset of a basis for $\mathbb{Z}^d$.

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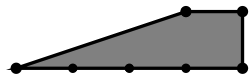
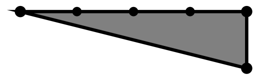
An Example and a Counterexample.  

Lemma. *The (outer) normal vectors of a smooth lattice polytope induce a short exact sequence*

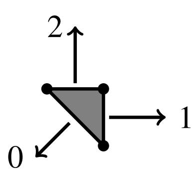
$$0 \longrightarrow \mathbb{Z}^d \xrightarrow{\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-1} \end{bmatrix}} \mathbb{Z}^n \xrightarrow{\begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} \end{bmatrix}} \mathbb{Z}^{n-d} \longrightarrow 0\,.$$

Idea of Proof. Since P is smooth, a change of basis on $\mathbb{Z}^d$ implies that $\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-1} \end{bmatrix} = \left[\begin{smallmatrix} C \\ I_d \end{smallmatrix}\right]$, so for an appropriate choice of basis on $\mathbb{Z}^{n-d}$ we have $A := \left[\mathbf{a}_0 \; \mathbf{a}_1 \; \cdots \; \mathbf{a}_{n-1}\right] = \left[I_{n-d} \;\; -C\right]$ □

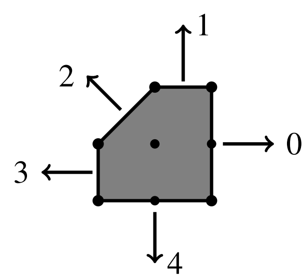
Cox ring. Since $\mathbb{C}^\times$ is an injective $\mathbb{Z}$-module, applying $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{C}^\times)$ gives the exact sequence $1 \longrightarrow (\mathbb{C}^\times)^{n-d} \xrightarrow{A^\mathsf{T}} (\mathbb{C}^\times)^n \longrightarrow (\mathbb{C}^\times)^d \longrightarrow 1$. Combined with the entrywise action of $(\mathbb{C}^\times)^n$ on $\mathbb{C}^n$, we obtain a $(\mathbb{C}^{n-d})$-action on $\mathbb{C}^n$. Algebraically, this action is encoded by the polynomial ring $S := \mathbb{C}[x_0, x_1, \dots, x_{n-1}]$ with the $\mathbb{Z}^{n-d}$-grading defined by $\deg(x_j) := \mathbf{a}_j$.

Irrelevant Ideal. A ***face*** of a polytope is the intersection with a supporting hyperplane. Each face corresponds to a unique subset of the facet hyperplanes. Consider the monomial ideal in S defined by $B_X := \bigcap_{\sigma \subseteq [n]} \langle x_j : j \in \sigma \rangle$ where the intersection is over all subsets σ corresponding to minimal non-faces.

Examples.



$B_{\Delta_2} = \langle x_0, x_1, x_2 \rangle$



$B_Q = \langle x_0, x_2 \rangle \cap \langle x_0, x_3 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_2, x_4 \rangle$

Theorem. *The quotient* $X_P := \big(\mathbb{C}^n \setminus Z(B)\big)/(\mathbb{C}^\times)^{n-d}$ *is a projective (normal) toric variety and a smooth compactification of the algebraic torus* $(\mathbb{C}^\times)^d$.

References

[M2] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at www.math.uiuc.edu/Macaulay2/.

[CLS] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

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