Problems for tropical geometry tutorial

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June 2014

(1) Let $f = x^3+y^3+z^3+xyz$ and $I = \langle f \rangle$ be the principal ideal generated by f. Describe all initial ideals, the Gröbner fan, and the tropical variety of of I.

(2) Find the tropical variety of the curve parameterized by

$$x = 1 + x^2 + tx^3$$

$$y = t + x + 3x^2$$

(3) We wish to compute the number of common zeroes in $(\mathbb{C}^*)^2$ of the following two polynomials we saw yesterday: $x + y - y^2 + 2xy + xy^2$ and $x - y + x^2y - y^2 + 2xy^2$. We first perturb the coefficients, for example, like this:

$$tx + y - y^2 + 2txy + txy^2 \text{ and } tx - y + x^2y - ty^2 + 2txy^2.$$

Draw the two corresponding tropical curves and check that they intersect in finitely many points. Find the the mixed volume of the Newton polytopes by computing the number of intersection points, counted with multiplicities. Bernstein's Theorem says that this is equal to the number of common zeroes of the two original polynomials in $(\mathbb{C}^*)^2$.

(4) Let P_d be the triangle with vertices at $(0,0),(0,d)$, and $(d,0)$. Show that the mixed volume of P_a and P_b is ab for any positive integers a and b. Relate this to the Bézout theorem.

(5) Let P be a full-dimensional lattice polytope in $\mathbb{R}^n$ containing the origin. Let $f_1,\ldots,f_{n+1}$ be polynomials with generic coefficients in n variables whose Newton polytopes are $d_1P,\ldots,d_{n+1}P$ respectively, for some positive integers $d_1,\ldots,d_{n+1}$. They parameterize a hypersurface in $\mathbb{C}^n$. Show that the Newton polytope of the defining equation is a simplex by computing its tropical hypersurface. What are the vertices of the polytope?

Fact: Mixed volume is symmetric, multi-linear, and the mixed volume $\mathrm{MV}(P_1,\ldots,P_n)$, where $P_1 = P_2 = \cdots = P_n = P$, is equal to $n!\mathrm{Vol}(P)$.