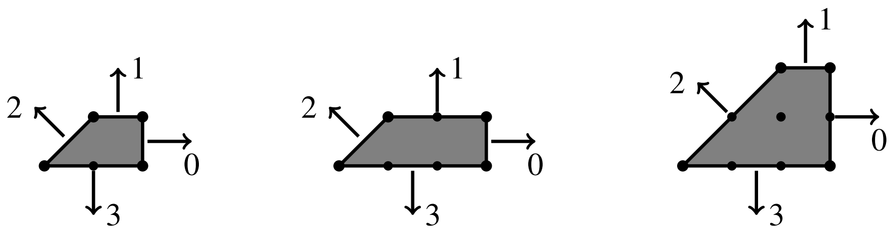
TORICS IN *MACAULAY2*
EMBEDDINGS AND COHERENT SHEAVES

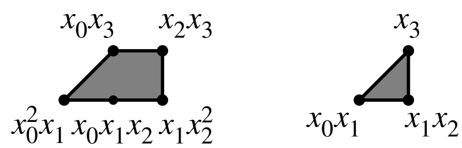
Equivalence relation. Since the quotient construction of a projective toric variety depends only on the outer normal vectors and the hyperplanes passing through the vertices, more than one polytope produces the same toric variety.

Examples. The following polytopes all produce the 1-th Hirzebruch surface Y.



By identifying monomials with lattice points, the graded components of the Cox ring correspond to polytopes.

Examples. Consider the Cox ring $S = \mathbb{C}[x_0, x_1, x_2, x_3]$ where the $\mathbb{Z}^2$-grading is induced by the columns of $\left[\begin{smallmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$. The components $S_{(1,1)}$ and $S_{(0,1)}$ correspond to the following polytopes.



When the monomials in a graded component $S_{\mathbf{u}}$ of the Cox ring have no common zeroes, they define a map into projective space. The kernel of the associated map of polynomial rings is the defining ideal of the image.

Example. The kernel of $\mathbb{C}[y_0, y_1, \dots, y_4] \to \mathbb{C}[x_0, x_1, x_2, x_3]$ defined by $y_0 \mapsto x_2x_3$, $y_1 \mapsto x_0x_3$, $y_2 \mapsto x_1^2x_2$, $y_3 \mapsto x_0x_1x_2$, $y_4 \mapsto x_0^2x_1$ is $\langle y_3^2 - y_2y_4, y_1y_3 - y_0y_4, y_1y_2 - y_0y_3 \rangle$.

Proposition. *If the polytope associated to a graded component of the Cox ring is equivalent to the defining polytope of the toric variety, then monomials defined a closed embedding.*

Idea of Proof. Show that it is enough to consider the torus-fixed points. At the toric fixed-points, show explicitly that the map separates points and tangent vectors. □

Proposition. *A degree corresponds to a closed embedding if and only if it is in the interior of the cone* $\bigcap_{\boldsymbol{x}^{\mathbf{u}} \in \mu(B)} \operatorname{pos}(\deg(x_j) : u_j \neq 0) \subset \mathbb{Z}^{n-d}$ *where the intersection is over the minimal generators of the irrelevant ideal.*

Example. On $\mathbb{P}^d$, the irrelevant ideal is $B_{\mathbb{P}^d} = \langle x_0, x_1, \dots, x_d \rangle$. Since

$$\operatorname{pos}\big(\deg(x_j)\big) = \operatorname{pos}\big([1]\big) = \mathbb{R}_{\geqslant 0},$$

the positive degrees correspond to closed embeddings.

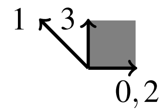
Date: 17 June 2014.

2010 *Mathematics Subject Classification.* 14M25, 14Q99.

Example. On Y, the irrelevant ideal is $B_Y = \langle x_2x_3, x_0x_1, x_0x_3, x_0x_1\rangle$. Since we have

$$\operatorname{pos}\big(\deg(x_0), \deg(x_1)\big) \cap \operatorname{pos}\big(\deg(x_2), \deg(x_3)\big) \cap \operatorname{pos}\big(\deg(x_1), \deg(x_2)\big) \cap \operatorname{pos}\big(\deg(x_2), \deg(x_3)\big)$$
$$= \operatorname{pos}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]\right) \cap \operatorname{pos}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = (\mathbb{R}_{\geqslant 0})^2,$$

the entrywise positive degrees correspond to closed embeddings.



Conjecture. The closed embedding of a smooth projective toric variety is projectively normal and defined by quadric equations.

Sheaves. On affine space, finitely generated modules corresponding to coherent $\mathscr{O}_{\mathbb{A}^n}$-modules. Moreover, graded modules correspond to $(\mathbb{C}^\times)^{n-d}$-equivariant sheaves.

Theorem (Cox)**.** *The category of coherent sheaves on* $X = \big(\mathbb{C}^n \setminus Z(B)\big)/(\mathbb{C}^\times)^{n-d}$ *is equivalent to the quotient of the category of finitely generated* $\mathbb{Z}^{n-d}$*-graded* S*-modules by the full subcategory of* B*-torsion modules.*

Example. On $\mathbb{P}^d$, the irrelevant ideal in $B = \langle x_0, x_1, \dots, x_d\rangle$, so a B-torsion module is simply a module of finite length. Hence, two modules that agree in all sufficiently large degrees correspond to the same coherent sheaf.

Example. On Y, the irrelevant ideal is $B_Y = \langle x_2x_3, x_0x_1, x_0x_3, x_0x_1\rangle$ and the $\mathbb{Z}^2$-grading is induced by the columns of $\left[\begin{smallmatrix}1 & -1 & 1 & 0\\ 0 & 1 & 0 & 1\end{smallmatrix}\right]$. Hence, two modules that agree in all sufficiently positive degrees correspond to the same coherent sheaf.

Examples. In this dictionary, twists of the Cox ring $S(\mathbf{u})$ corresponds to line bundles. In particular, we have $\operatorname{Pic}(X) = \mathbb{Z}^{n-d}$ and $H^0\big(X, \mathscr{O}_X(\mathbf{u})\big) = S_{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{Z}^{n-d}$. WARNING: The notation $\mathscr{O}_X(\mathbf{u})$ depends on the choice of a basis for $\operatorname{Pic}(X)$.

Comments on Proof. Given a $\mathbb{Z}^{n-d}$-graded S-module M, the induced coherent sheaf $\widetilde{M}$ satisfies $H^0(U_\sigma, \widetilde{M}) = \big(M[x_j : j \notin \sigma]\big)_{\mathbf{0}}$. Conversely, given a coherent $\mathscr{O}_X$-module, an associated $\mathbb{Z}^{n-d}$-graded S-module arises from a truncation of $H^0_*(\mathscr{M}) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^{n-d}} H^0\big(X, \mathscr{M} \otimes \mathscr{O}_X(\mathbf{u})\big)$. □

REFERENCES

[M2] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at www.math.uiuc.edu/Macaulay2/.

[CLS] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

GREGORY G. SMITH, DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, K7L 3N6, CANADA
E-mail address: ggsmith@mast.queensu.ca