Turbulence and Transport in Fusion Plasmas
Part II

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Monday Recap

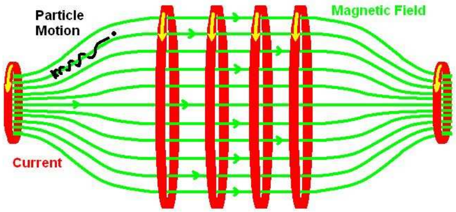
Yesterday, we covered

- how fusion energy will provide clean and safe energy
- the nature of Debye shielding and how to define plasmas
- what plasma waves and plasma drifts are

Next: how to design a confining magnetic field?

Magnetic Mirror

Early magnetic confinement: the **mirror machine**
(particles trapped between regions of large B)



Conserved: magnetic moment $\mu = \dfrac{mv_\perp^2}{2B}$, energy $E = \dfrac{1}{2}mv_\parallel^2 + \dfrac{1}{2}mv_\perp^2$

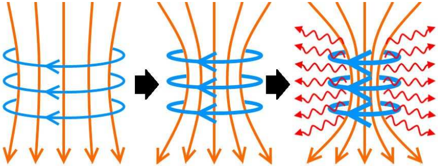
$\Rightarrow$ particles moving from small B to larger B convert $v_\parallel \to v_\perp$
at $v_\parallel = 0$, particles are **reflected** ($\leftrightarrow$ **magnetic mirror**)

If $v_\perp / v < \sqrt{B_{\mathrm{min}}/B_{\mathrm{max}}}$, **no reflection, particle slips through**
Non-zero ν_{ei}: trapped particles can enter this *loss cone*

$\Rightarrow$ *mirror machines have poor confinement due to end losses*

Z-Pinch

Another linear device (of Ocean's Eleven fame): the **Z-pinch**



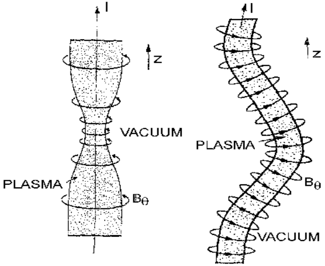
current filaments attract, compress field $\Rightarrow$ increased n, but only for short τ_E

Similar for lightning in weak rod; can use to shape metals, see

`sciencedemonstrations.fas.harvard.edu/presentations/can-crusher-magnetic-implosion`

However, no stable equilibrium!
Kink/sausage instabilities
(different from transport-relevant instabilities in fusion reactors)

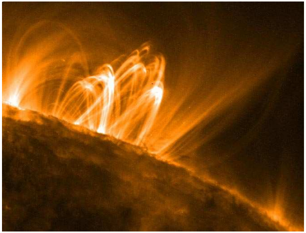
- **Sausage**: pinch effect
- **Kink**: prevents pinch from working

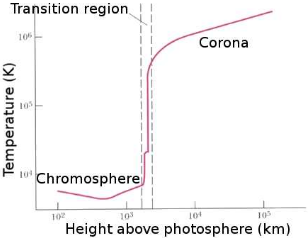


Minor Detour

A short excursion into a pinch-like phenomenon in space plasmas . . .

The Coronal Heating Problem





Decade-old question: *how can the solar corona be so hot*?

Heating mechanism:

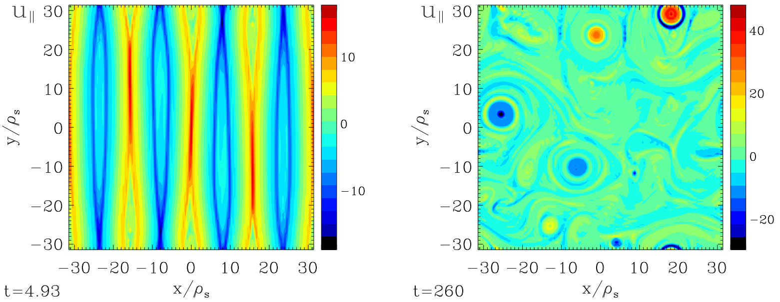
- MHD waves?
- loop footpoint motion?
- kinetic Alfvén waves?
- turbulent reconnection?
- nanoflares?

Driven Reconnection Turbulence

Using a fusion turbulence simulation code, study **magnetic reconnection** in a coronal loop (Pueschel ApJS 2014)

Driving turbulence through x-y-**periodic current sheets**,

$$\left.\frac{\partial g_j}{\partial t}\right|_{\rm drive} = -\omega_{\rm dr}\Big(g_j(k_y=0,t) - g_j(k_y=0,t=0)\Big)$$



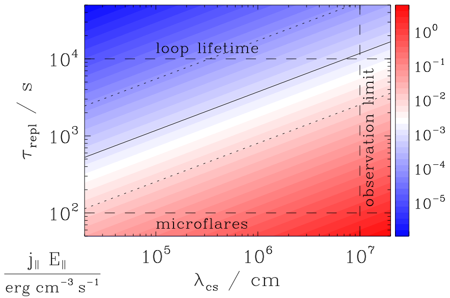
- **isotropization** despite driven current sheet
- formation of **plasmoids**, consistent with $S \gtrsim 10^4$ criterion

Application: Coronal Heating

Many turbulence simulations scanning over parameters: extract **volumetric heating rate**

$$\frac{j_\| E_\|}{\mathrm{erg\,cm^{-3}\,s^{-1}}} = 1.5\times10^{-3} \left(\frac{n_{e0}}{10^9\,\mathrm{cm^{-3}}}\right)^{0.375} \left(\frac{T_{e0}}{10^6\,\mathrm{K}}\right)^{-0.1} \left(\frac{L_{\mathrm{ref}}}{10^9\,\mathrm{cm}}\right)^{0.2} \left(\frac{\beta}{3.5\times10^{-4}}\right)^{0.125}$$

$$\times\left(\frac{\nu_{ei}}{87\,\mathrm{s^{-1}}}\right)^{-0.1} \left(\frac{m_e/m_i}{1/1836}\right)^{-0.25} \left(\frac{T_{i0}/T_{e0}}{1.0}\right)^{0.0} \left(\frac{B_{\mathrm{rec}}}{5\,\mathrm{G}}\right)^{1.8} \left(\frac{\tau_{\mathrm{repl}}}{1100\,\mathrm{s}}\right)^{-1.5} \left(\frac{\lambda^{cs}}{1.5\times10^5\,\mathrm{cm}}\right)^{0.75}$$



Compare **observations**

Withbroe 77: $10^{-3}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

(or $L_{\mathrm{ref}}^{-1}\times10^6\,\frac{\mathrm{erg}}{\mathrm{cm^2\,s}}$)

Ofman 98: $10^{-4}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

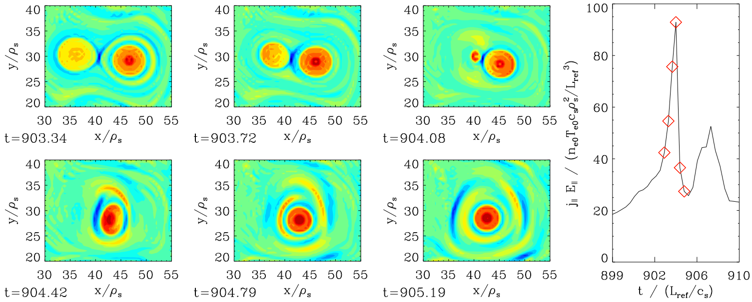
Guarrasi 14/Rosner 78: $2\times10^{-3}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

$\Rightarrow$ observations **well within predicted range** of heating!

Nanoflares

Nanoflares: short ($t \gtrsim 20\,\mathrm{s}$) spikes in observed heating

Look at details of turbulence: **merging current filaments**



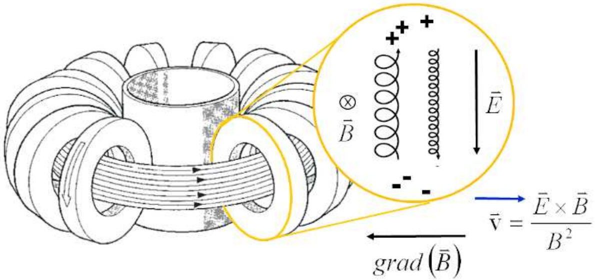
⇒ **mergers** are the **cause** of heating rate **spikes**

Apply to Active Regions: $7\,\mathrm{s} \lesssim t_{\mathrm{merge}} \lesssim 50\,\mathrm{s}$ ⇒ **match**!

Similar, but kinetic, process to the compression in a Z-pinch

Toroidal Confinement I

Bending a linear B-field into a torus… *what could go wrong?*



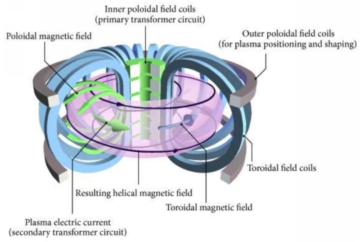
$\Rightarrow$ **near-instantaneous loss of all particles to reactor wall**

- two new plasma drifts (more later):
  $v_{\nabla B} \propto q_{\mathrm{i,e}}^{-1}$ and $v_{E\times B} \propto q_{\mathrm{i,e}}^{0}$
- $v_{\nabla B}$ separates charges, $v_{E\times B}$ lets them escape

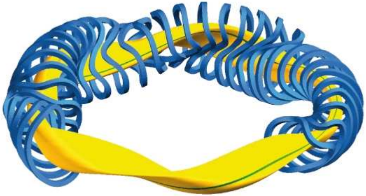
Since 1950s, many different concepts to deal with this problem

Toroidal Confinement II

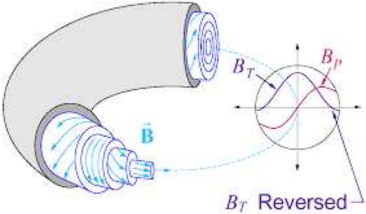
Tokamak: Soviet invention, transformer induces current



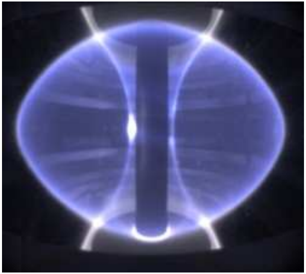
Stellarator: U.S. invention, complex coils, optimizable



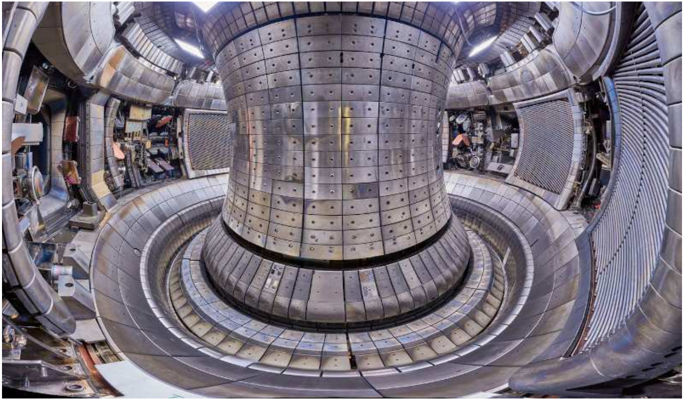
Reversed-field pinch (RFP): cheap tokamak, poor stability



Spherical Tokamak (ST): high efficiency, poor stability



Tokamak: ASDEX Upgrade



- fits 2–8 workers
- wall tiles
- divertor
- diagnostics
- ICRH antennae
- NBI shaft
- ECRH system
- pellet injector

Questions & Discussion

What's unclear? Who needs help? Who needs fresh tea?

Quiz

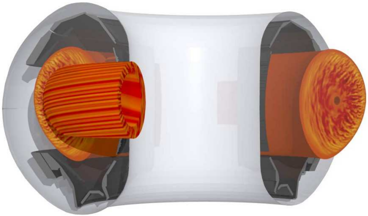
10-minute self-evaluated quiz:

1. What elements are to be used as fuel in fusion power plants, and why?
2. What are the safety considerations for fusion reactors, and how restrictive are they?
3. Why are modern fusion experiments toroidal and not linear?
4. What is the Debye length, and how does it affect whether an ionized gas is a plasma?

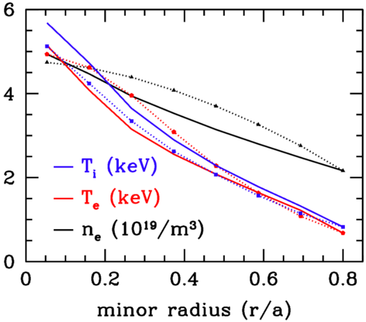
Quiz: Answers

1. Highest reaction cross-section: deuterium-tritium (*note*: alternate concepts rely on less efficient aneutronic reactions)
2. No runaway reactions, only true concern is tritium inventory; no substantial worry (but need smart choices for materials for reactor walls, steel)
3. Linear machines are subject to large end losses, poor confinement
4. Inside the Debye length, charge is shielded, no collective behavior, but gas-like

Transport



Fusion plasmas: radial distribution of n, T_i, T_e, . . .



(Barnes PoP 2010)

Transport: heat/particle/momentum losses on time scale of heat/particle/momentum confinement time τ ($\longleftrightarrow$ *triple product* $nT\tau_E$)

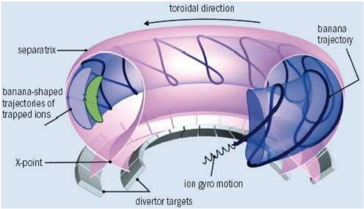
- What are the causes for these losses?
- How can we describe/predict profile evolution?

(Neo-)classical vs. Anomalous Flux

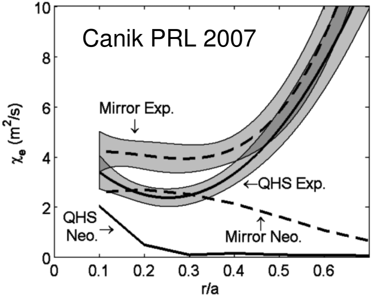
Historically, main heat loss mechanism thought to be
classical $\rightarrow$ *neoclassical* $\rightarrow$ *anomalous*

Classical: ignorably small (in 1950s, vision of tabletop reactors)

Neoclassical = collisions on banana orbits: small-ish



Anomalous: usually dominates



Historically, *anomalous* = *no idea where it comes from*,
but now used as synonym for **turbulence**

Plasma Instability

Fluctuation measurements: plasma “bubbles” like boiling water

$\Rightarrow$ *How does a physicist describe a bubbling system?*

Plasma Instability

Fluctuation measurements: plasma "bubbles" like boiling water

$\Rightarrow$ *How does a physicist describe a bubbling system?*

...we look at a **small perturbation from equilibrium**

E.g., density

$$n \to n_0 + n_1 \qquad \epsilon n_0 \sim n_1$$

In equilibrium (fusion: MHD),

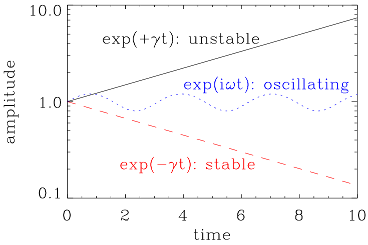
$$\begin{aligned}\frac{\partial n_0}{\partial t} &= 0 \\ \frac{\partial n_1}{\partial t} &= A n_1 + B n_1^2 + \ldots\end{aligned}$$

Linearization: drop terms $\propto \epsilon^2$ and higher

$\Rightarrow$ can solve with ansatz

$$n_1 = n_1(t=0)\exp(i\omega_c t)$$

(ω_c: complex frequency)



Describes eigenmode growing exponentially at rate γ forever

$\rightarrow$ *what stops this growth?*

Saturation and Turbulence

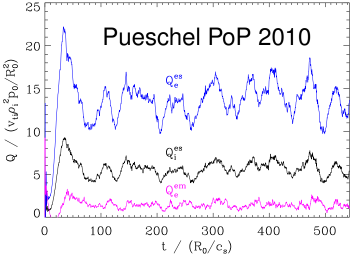
As n_1 grows, eventually $n_1^2 \sim n_1$, and **nonlinearity** becomes important

In plasmas, typically

$$\frac{\partial n_1}{\partial t} = A n_1 + {\color{red}\frac{\partial n_1}{\partial x}\frac{\partial n_1}{\partial y}}$$

- here, quadratic nonlinearity
- saturation: **balance** of linear, nonlinear terms
- generally, **cannot solve analytically**

Solve equations numerically



quasi-stationary state after linear growth, saturation
$\Rightarrow$ **statistical analysis**

Turbulence: very difficult, decades of research in many fields

Group Work: Turbulence Basics

45 minutes group work:

Find online sources that explain

- chaos
- random walk
- the Kolmogorov spectrum

and be prepared to give a short summary.

Did your sources explain what the Lyapunov exponent is?

Student Lecture

Volunteers . . .   Who wants to tell us:

What does **chaos** mean to a physicist?

What is and when do we observe **random walk**?

What is the **Lyapunov exponent**?

What is the **Kolmogorov** spectrum?

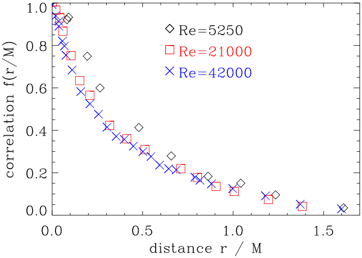
Kolmogorov Turbulence

Kolmogorov 1941

turbulence theory based on **self-similarity**

Stewart 1951

wind tunnel experiments:
no viscosity dependence
of low-ν (high-Re) turbulence



dissipation rate $\epsilon = 2\nu \int k^2 E \mathrm{d}k$, with $[E] = \frac{\mathrm{m}^3}{\mathrm{s}^2}$, $[\epsilon] = \frac{\mathrm{m}^2}{\mathrm{s}^3}$, $[\nu] = \frac{\mathrm{m}^2}{\mathrm{s}}$

dimensional analysis: $E(k) \propto u(k)^2 \lambda(k) [k\lambda]^n$ (self-similarity/scale-free)

to get rid of ν, use $[\lambda] = \mathrm{m} = [\frac{\nu^{3/4}}{\epsilon^{1/4}}]$ and $[u] = \frac{\mathrm{m}}{\mathrm{s}} = [\nu^{1/4}\epsilon^{1/4}]$

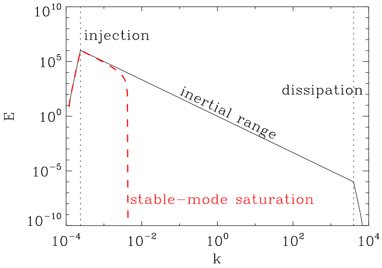
$\Rightarrow$ to make energy ν-independent, choose $n = -5/3$:

$$E(k) \propto k^n \nu^{\frac{3n+5}{4}} \epsilon^{\frac{1-n}{4}} = k^{-5/3}\epsilon^{2/3}$$

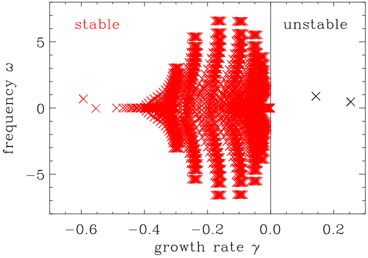
(**Kolmogorov inertial range**, confirmed in many experiments)

Beyond Kolmogorov

Kolmogorov 1941: transfer energy conservatively across scales



Full solution to linear operator A



However, instability-driven plasma systems can have energy **injection/dissipation/removal at the same scales**!

- large-scale unstable modes inject energy ($\nabla T_{\mathrm{e,i}}$, ∇n)
- 10^4+ large-scale **stable modes** remove energy

$\Rightarrow$ **no** true **inertial range**, no $5/3$ law

Fluid Model for Plasmas I

For very simple plasma description (Horton PoF 1981, 1988), can use (normalized) nonlinear **two-field fluid model** with

- electric potential $\Phi(x, y)$
- (ion or electron) pressure $p(x, y)$

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t} &= (\delta - \nabla_\perp^2)^{-1}\left(-\overbrace{\frac{\partial \Phi}{\partial y} + \epsilon \frac{\partial p}{\partial y}}^{\text{drifts}} + \nu \nabla_\perp^2 \Phi - \hat{z} \cdot (\nabla_\perp \Phi \times \nabla_\perp^3 \Phi)\right) \\
\frac{\partial p}{\partial t} &= \underbrace{(-1 - \eta)\frac{\partial \Phi}{\partial y}}_{\text{drive}} - \chi \nabla_\perp^4 p - \hat{z} \cdot (\nabla_\perp \Phi \times \nabla_\perp p)
\end{aligned}
$$

With: $\epsilon \sim 1/\nabla n$, "collisionalities" ν, χ, coordinate $\hat{z} \parallel \mathbf{B}$, radial coordinate x, toroidal coordinate y

Physics: toroidal ion-/electron-temperature-gradient modes, accessing free energy in background ∇T via $\eta \equiv \mathrm{d}\ln T/\mathrm{d}\ln n$

Going to Fourier space easy for linear terms, $\partial/\partial x, y \to ik_{x,y}$, *but what happens to the nonlinearities?*

Group Work: Fourier Convolution

30 minutes group work:

What does the expression

$$\{f, \Phi\} \equiv \frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial x}$$

look like when in Fourier space,
$f(x, y) \to f(k_x, k_y)$ and $\Phi(x, y) \to \Phi(k_x, k_y)$, with e.g.

$$\mathcal{F}(\partial_y \Phi) = \int \mathrm{d}\mathbf{x}\, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \sum_{\mathbf{k}'} \mathrm{e}^{i\mathbf{k}'\mathbf{x}} i k'_y \Phi(\mathbf{k}')$$

Hint 1: use separate transforms (with k', k'') for $\partial_x f$, $\partial_y \Phi$ etc.
Hint 2: $\int \exp iax \to \delta_a$
Hint 3: the final form should have a single sum over $k'_{x,y}$

Solution: Fourier Convolution

$$
\begin{aligned}
\mathcal{F}(\{f,\Phi\}) &= \int \mathrm{d}\mathbf{x}\,\mathrm{e}^{-i\mathbf{k}\mathbf{x}}\left[\frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial x}\right] \\
\mathcal{F}(\{f,\Phi\}) &= \int \mathrm{d}\mathbf{x}\,\mathrm{e}^{-i\mathbf{k}\mathbf{x}}\left[\left(\sum_{\mathbf{k}'}\mathrm{e}^{i\mathbf{k}'\mathbf{x}} i k'_y \Phi(\mathbf{k}')\right)\left(\sum_{\mathbf{k}''}\mathrm{e}^{i\mathbf{k}''\mathbf{x}} i k''_x f(\mathbf{k}'')\right) - \ldots\right] \\
\mathcal{F}(\{f,\Phi\}) &= -\int \mathrm{d}\mathbf{x}\sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\mathrm{e}^{-i(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\mathbf{x}}\left[k'_y\Phi(\mathbf{k}')k''_x f(\mathbf{k}'') - \ldots\right] \\
\mathcal{F}(\{f,\Phi\}) &= -\sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}\left(k'_y k''_x - k'_x k''_y\right)\Phi(\mathbf{k}')f(\mathbf{k}'') \\
\mathcal{F}(\{f,\Phi\}) &= \sum_{\mathbf{k}'}\left(k'_x k_y - k_x k'_y\right)\Phi(\mathbf{k}')f(\mathbf{k}-\mathbf{k}')
\end{aligned}
$$

Why do we care, or: what are the advantages of Fourier space?

Fluid Model for Plasmas II

In Fourier space, the model becomes

$$\frac{\partial \Phi(k_\perp)}{\partial t} = [\delta(k_y) + k_\perp^2]^{-1}\Big(-ik_y\Phi(k_\perp) + ik_y\epsilon p(k_\perp) - \nu k_\perp^2 \Phi(k_\perp)$$

$$-\frac{1}{2}\sum_{k'_\perp}(k'_\perp \times \hat{z}\cdot k_\perp)[(k_\perp - k'_\perp)^2 - k'^2_\perp]\Phi(k'_\perp)\Phi(k_\perp - k'_\perp)\Big)$$

$$\frac{\partial p(k_\perp)}{\partial t} = -ik_y(1+\eta)\Phi(k_\perp) - \chi k_\perp^4 p(k_\perp)$$

$$-\sum_{k'_\perp}(k'_\perp \times \hat{z}\cdot k_\perp)\Phi(k'_\perp)p(k_\perp - k'_\perp)$$

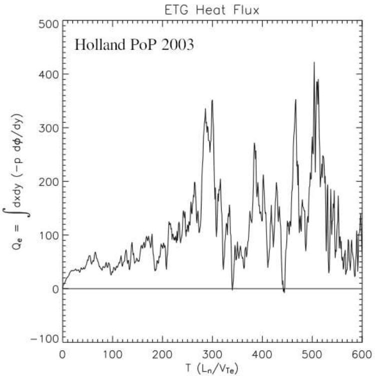
δ: adiabatic response; $\delta = {1 \atop 0}$ at ${k_y > 0 \atop k_y = 0}$ for ITG, $\delta = 1$ for ETG
(*all just a trick to get more or less correct nonlinear behavior*)

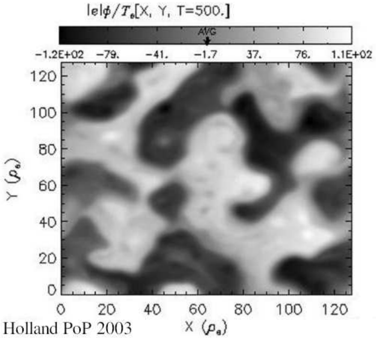
Later, more physics background, proper derivation; for now,
how do we use a model like this and what can we learn?

Instability and Saturation

Commonly, pick arbitrary initial condition (Φ, p), evolve in time

Below, ETG (we'll talk more about drift-wave physics later)





- short linear phase
- linear: mostly $k_x = 0$ to access ∇T

- nonlinearity rises early
- nonlinear coupling creates $k_x > 0$ eddies

Questions & Discussion

Anything unclear so far?

Group Work: Code Prep

30 minutes group work:

- In a programming language of your choice, write a code that plots contours of the function

  $$f(z) = (z + 0.2i)(z - 0.2i)(z + 0.3 - 0.5i)$$

  in the complex plane ($z \in \mathbb{C}$)

- Be prepared to read data from an ASCII text file and create line and contour plots based on that data; example data:

```
1.33 -1.8337E+01
1.34 -1.8591E+01
1.35 811.429
1.36 40238.2
```