Turbulence and Transport in Fusion Plasmas
Part I

DIFFER

M.J. Pueschel

RUHR UNIVERSITÄT BOCHUM RUB

Ruhr-Universität Bochum, February 27 – March 10, 2023

Topics

In *Turbulence and Transport in Fusion Plasmas*,
you will receive an introduction on how

- fusion energy can help us reduce carbon emissions
- instabilities cause plasma microturbulence and anomalous heat/particle losses
- to use fluid and kinetic equations to describe plasmas and evaluate instabilities
- to derive and use the gyrokinetic framework
- to write and deploy plasma simulation code
- instabilities saturate, setting transport levels
- to use reduced quasilinear models for fast flux prediction
- transport depends on plasma parameters
- to solve transport equations to predict plasma profiles

Slides, Projects, Grades

Quizzes

Occasionally:
short quiz about recent subjects, self-evaluated

Not part of final grade

Research Projects

Pick topic by March 10th

Third week:
research in groups of 2–3

Present results on Friday, *March 17th*

Unless requested otherwise, **grades will be based on the research project** (67% joint) **and presentation** (33% individual)

Lecturer

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Dutch Institute for Fundamental Energy Research
leader of the *Plasma Microturbulence* research group

Priv. Doz. at the Ruhr-Universität Bochum
local collaborations with Tjus & Grauer groups

working on plasmas since 2004 and on fusion since 2005
formerly at University of Texas, University of Wisconsin,
& Max Planck Institute for Plasma Physics

Literature

- Individual papers covering specific topics will be cited throughout the class
- Textbook about plasma physics:
  Francis F. Chen, *Introduction to Plasma Physics*, Springer Science & Business Media
- Review paper on gyrokinetic turbulence simulations:
  X. Garbet, Y. Idomura, L. Villard, and T.H. Watanabe, Nucl. Fusion **50**, 043002 (2010)
- Background on quasilinear transport modeling:
  C. Bourdelle *et al.*,
  Plasma Phys. Control. Fusion **58**, 014036 (2016)
- Review paper on integrated modeling:
  F.M. Poli, Phys. Plasmas **25**, 055602 (2018)

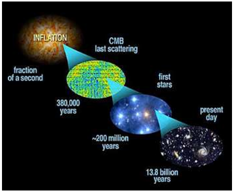
Questions

Any questions about the setup?

Cosmic Energy Sources

Given that energy in the Universe is conserved,
where does it come from and where can we find it?

- Big Bang



Unknown injection mechanism at $t = 0$

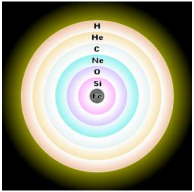
primordial nucleosynthesis

- Gravitation

Large scales:

Potential energy in planets, stars, galaxies . . .

- Fusion



Stars (+ Big Bang):

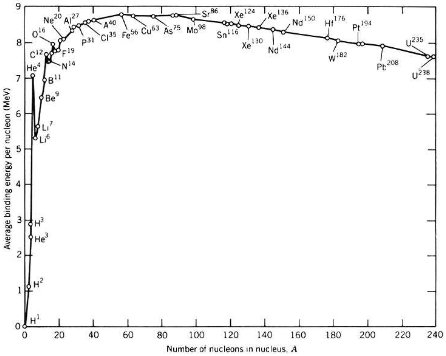
fusion of lighter into heavier elements

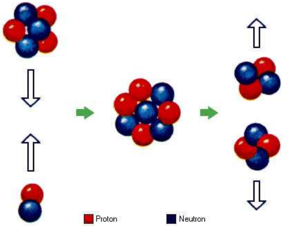
By comparison: the Sun radiates as much energy as 10^{17} *large power plants would produce*

Nuclear Fusion

Atomic nuclei: differences between elements in binding energy (due to quantum-mechanical energy states between nucleons)

$\Rightarrow$ *can convert less stable into more stable nuclei*



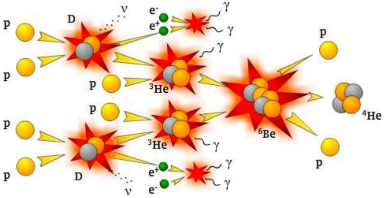


However, need to overcome Coulomb repulsion between positively charged nuclei via large collision speeds/temperature

Stellar Fusion Cycles

$\leftarrow$ M K G F A B O $\rightarrow$

lighter stars, Sun:



- **p-p cycle**:
  protons to helium (primary process in the sun)

heavier, hotter stars:

- **triple-alpha cycle**:
  key to the creation of heavier elements
- **CNO cycle**:
  carbon-catalyzed fusion of protons, highly efficient at high temperatures

Exotic Fusion Events

Big Bang

- initial state: only protons and neutrons exist
- fuse to D, T, He, Li
- heavier elements: only after the first generation of stars

Supernovae

- collapse due to missing core fuel
- fusion of heavy elements
- high energies: creation of radioactive elements

Fusion is the only process in the cosmos that can create life and sustain it in the long term

Terrestrial Considerations

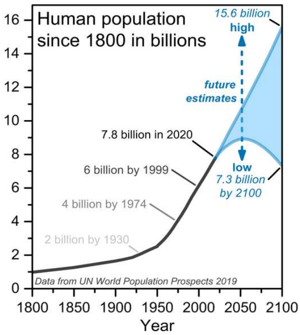
On Earth: most available energy stems (directly or indirectly) from the Sun (other sources: radioactivity, Earth's core), **exceeded by civilization's needs**

Historically: improved energy efficiency

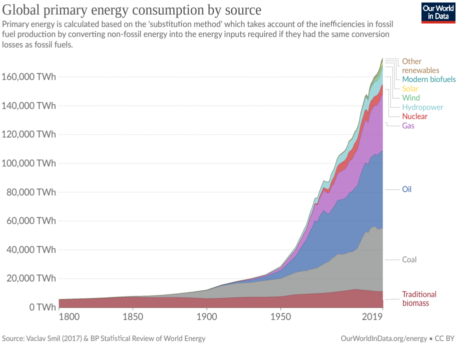
$\leftrightarrow$

accelerated growth of demand

Population growth:

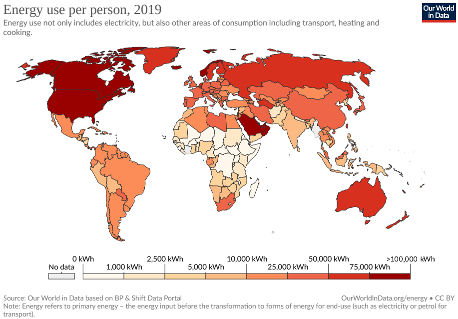


Energy use:



Distributive Justice

Per-capita energy use:



- enormous consumption in North America, Europe
- industrialization China and India
- future growth in Africa

$\leftrightarrow$ limited supply
$\leftrightarrow$ limited ability of nature to compensate the consequences

Moral and practical question:
Who should be allowed to use how much of what type of energy?

Climate Change

Causes:

- emission of greenhouse gases
- deforestation (esp. primordial)
- impact on marine life

$\Rightarrow$ global warming

Consequences:

- Reduction in biodiversity
- massive changes in precipitation
- food and water insecurity

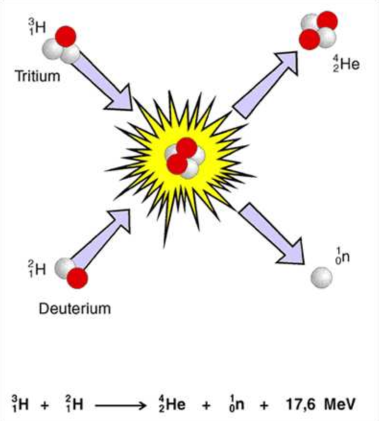
$\Rightarrow$ climate wars, refugee flows

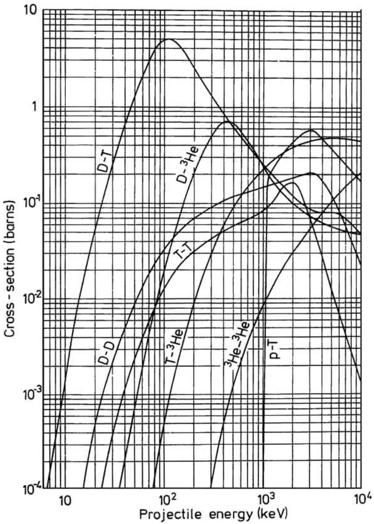
Without decisive and emcompassing action, hundreds of millions of people could die!

Presuming humans accept only minimal reductions in per-capita energy,
how do we sustain society without ruining the planet?

D-T Fusion

As we will see: **fusion** provides **safe long-term solution**
How about the most efficient fusion reaction?





- high reaction rate
- large released E
- safe ash: He

Magnetic Confinement

Sun's core ($T \approx 1.5 \times 10^7$ K): particles **confined gravitationally**

How to confine particles in a fusion reactor ($T \approx 10^8$ K)?

- at these temperatures, matter in **plasma state**
- Lorentz force: particles spiral about magnetic field lines

$\Rightarrow$ **magnetic confinement**

Similar to solar corona:

$\Rightarrow$ fusion reactors: vacuum chambers with strong magnetic fields containing hydrogen (isotope) plasmas

(*but*: density not too high, otherwise eruption/solar flares)

Lawson Criterion

If sufficiently many fusion reactions: energy released in alpha particles can compensate heat losses $\Rightarrow$ **"ignition"** (*note*: heat/particle losses = what this class is about!)

Requires high temperatures/densities, good confinement, or:

fusion triple product ($\sim$ Lawson criterion): $n T \tau_{\mathrm{E}} \gtrsim 3 \times 10^{28} \frac{\mathrm{K\,s}}{\mathrm{m}^3}$

Stars: sufficient density to get to ignition (p-p cycle)
Planets: did not get there; note brown dwarfs (D-D)

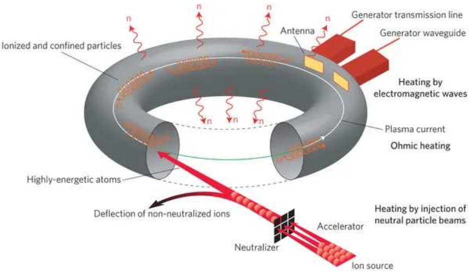
Different approaches to achieving fusion on Earth

- magnetic confinement (our focus): small-ish n, high T, τ_E
- inertial confinement: high n, high T, tiny τ_E

Note: ignition not necessary for commercial fusion!
$Q = P_{\mathrm{fusion}}/P_{\mathrm{heating}} \gtrsim 5$ suffices (ignition is $Q \to \infty$)

Plasma Heating

How can we heat a plasma to 10^8 *degrees?*



- **Ohmic heating**: induce current $\Rightarrow$ resistive heating
- **wave heating**: accelerate particles on Larmor orbits via radio waves
- **neutral-particle heating**: accelerate negative ions, strip excess electrons, inject into plasma $\Rightarrow$ collisional heating

Fueling: freeze hydrogen into pellets, accelerate in centrifuge, shoot into plasma

Timeline

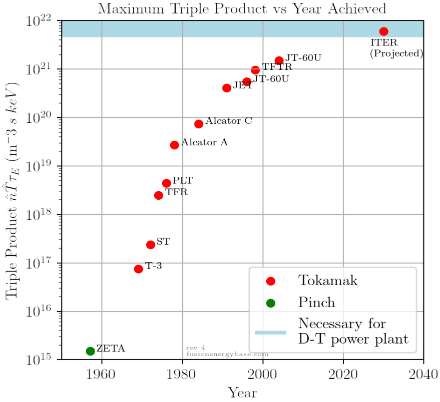
1950: Proyecto Huemul,
first fusion scam

1950/60s: classified research
in Western/Eastern Block

1970/80s: East-West
collaboration

1990s: JET (Landshut LH181)

2000s: theory catches up



2010s:
development of breakthrough concepts: ITER, (SP)ARC, . . .

Also see:

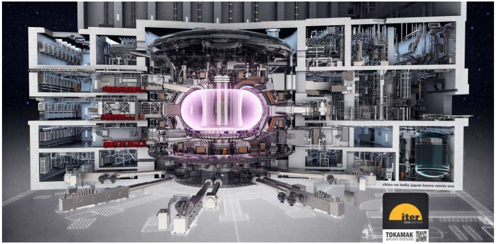
R. Herman, *Fusion – The Search for Endless Energy* (Cambridge University Press, 1990)

`https://www.fusionenergybase.com/article/measuring-progress-in-fusion-energy-the-triple-products/`

ITER – the Next Step

Key difficulty in fusion: insulation of hot plasma core

One approach: **build bigger reactors** (*but cost* $\propto$ *volume!*)



ITER:
International
Thermonuclear
Experimental
Reactor

International collaboration, biggest science experiment after ISS

Targets: $Q = 10$ for 30 minutes, $500\,\mathrm{MW}$ fusion power

Participants:

- China
- European Union
- India
- Japan
- Russia
- South Korea
- USA

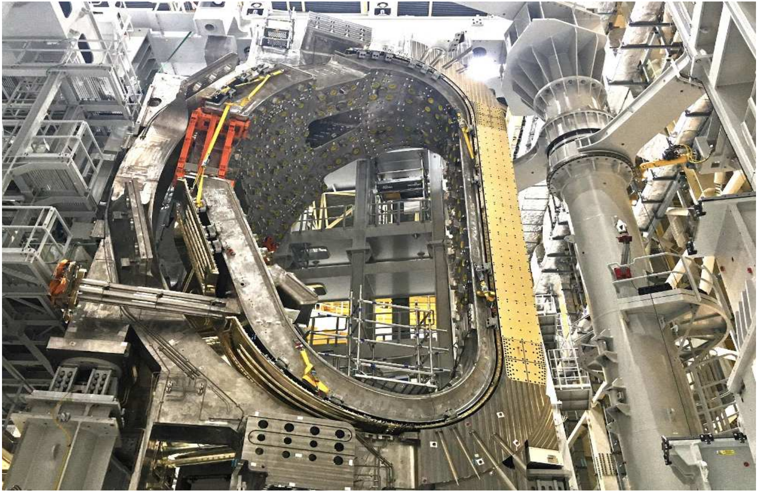
The ITER Site I

Cadarache, Southern France: ITER reactor hall completed

First experiments planned for December 2025

The ITER Site II

2021: First magnetic field coils and vacuum vessel segment installed



First experiments planned for December 2025

Power-Plant Operation

Fuel

Deuterium:
extract from water

Tritium: decays with half-life of 12 years
$\Rightarrow$ no natural sources
Create via Lithium breeding:
$\mathrm{Li} + \mathrm{n} \rightarrow \mathrm{T} + \mathrm{He}$

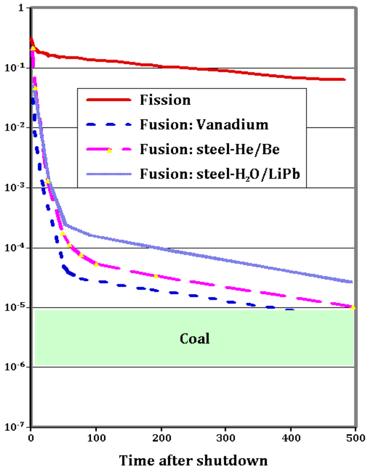
Blanket: Lithium layer absorbs fusion neutrons

- fuel available for millions of years:
  D: oceans,
  Li: Earth's crust, oceans
- operation: continuous or pulsed (ca. 12 hours, then ca. 15 minutes shut-down)
- operations inside reactor chamber: robots (e.g., repairing wall tiles)

Waste and Radioactivity

- **Fusion ash**: Helium (no waste)
- **Neutron irradiation**: transmutation of wall materials $\Rightarrow$ low-level radioactivity (compare radiotoxicity on right)
- **Radioactive fuel**: tritium burnt up in fusion reactions

Radiotoxicity of reactor components:



Group Work: Waste

45 minutes group work:

1. each group download one of
    a. K. Brodén *et al.*, Fusion Eng. Des. **42**, 1 (1998)
    b. M. Zucchetti *et al.*, Fusion Eng. Des. **136**, 1529 (2018)
2. digest the content & prepare a short presentations

After 45 minutes, each group will present one of the papers (10 minutes plus 5 minutes discussion)

Rule for all group work sessions: if you're stuck, just ask!

Risk

H. Bartels *et al.*,
Fusion Eng. Des. (1998):
accident scenarios

worst-case scenario:
airplane crash, pulverizing tritium in blanket,
airborne HTO or $\mathrm{T_2O}$

$\Rightarrow$ need to evacuate ca. $3\,\mathrm{km}$ radius for multiple days

F. Najmabadi *et al.*,
Fusion Eng. Des. (2006):
smart design can avoid even small-radius evacuation

Possible **cost problem**:

Solar-like eruptions:
"edge-localized modes", ELMs can damage wall tiles
$\Rightarrow$ *replacing very expensive*

Newer research:

ELMs can be split into many mini-ELMs (no wall damage) via

- controlled perturbation of magnetic field
- timed pellet injection

Solving the Energy Crisis?

Fusion holds enormous promise:

- hardly any CO_2
- little problematic waste
- globally available
- continually available
- compatible with current power grids
- near-inexhaustible fuel reserves

However: **no silver bullet against climate change**
requires all-of-the-above approach, with fusion playing key role

Research ongoing on

High-temperature and **high-field superconductors**
$\Rightarrow$ substantially smaller and more efficient reactors

Optimization using high-performance computing
$\Rightarrow$ reduce heat losses, improve efficiency via magnet shaping

Questions & Discussion

Who has unanswered questions about fusion energy?

Who has seen interesting fusion stuff on the news recently and wants to discuss?

TU/e Fusion M.Sc.

For those of you who are not yet spoken for M.Sc.-wise:

The Eindhoven University of Technology (TU/e) offers a **Science and Technology of Nuclear Fusion** M.Sc. program

`www.tue.nl/en/education/graduate-school/master-science-and-technology-of-nuclear-fusion`

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Ionized Gas

Want to confine & heat plasmas . . . **so what is a plasma?**

Common (*too simple*) definition: **plasma = ionized gas**

Saha equation: ion density in weakly ionized gases
Hydrogen gas:

$$\frac{n_{\mathrm{ionized}}^2}{n_{\mathrm{neutral}}} = \lambda_{\mathrm{th}}^{-3}\,\mathrm{e}^{-E_{\mathrm{ionization}}/T} \approx \frac{6\times 10^{21}}{\mathrm{cm}^3} T^{3/2}\,\mathrm{e}^{-E_{\mathrm{ionization}}/T}$$

(λ_{th}: electron de Broglie wavelength; $T[\mathrm{eV}]$)

Quick exercise: calculate the ionization level $n_{\mathrm{ionized}}/n_{\mathrm{neutral}}$ for

- room temperature
- the Sun's core
- a fusion reactor plasma

What T to expect for the *Cosmic Microwave Background*?
Use $n \sim 300\,\mathrm{cm}^{-3}$; how does T compare to the actual $\sim 1\,\mathrm{eV}$?

Plasma Definition

Plasma = **fourth state of matter**, what is the **phase transition**?

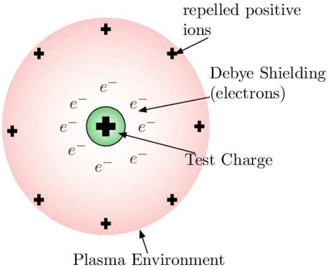
- **ionization**: very continuous, behavior of ionized gas does not change suddenly at very low/high ionization
- **weakly collisional**: when collisions completely dominate, dynamics follow physics of gases
- **collective effects**: long-range interactions of charged particles due to electromagnetic forces
  $\Rightarrow$ fundamentally new behavior, true phase transition

When and how do collective effects arise?

Debye Shielding

Consider single charge q:
repulsion of like q,
attraction of opposite q

$\Rightarrow$ "Debye shielding"
reduces effective charge



More quantitatively (1D):

$$f_{\mathrm{i,e}} = \mathrm{e}^{-m_{\mathrm{i,e}} v_{\mathrm{i,e}}^2/(2T_{\mathrm{i,e}}) - q_{\mathrm{i,e}}\Phi/T_{\mathrm{i,e}}} \quad \rightarrow \quad n_{\mathrm{i,e}} = n_0 \mathrm{e}^{-q_{\mathrm{i,e}}\Phi/T_{\mathrm{i,e}}}$$

Into Poisson equation, expand for small Φ, use $m_{\mathrm{i}} \gg m_{\mathrm{e}}$:

$$\frac{1}{4\pi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = -q_{\mathrm{i}} n_{\mathrm{i}} - q_{\mathrm{e}} n_{\mathrm{e}} \approx e n_0 \left(\frac{e\Phi}{T_{\mathrm{e}}} - \frac{Z_{\mathrm{i}} e\Phi}{T_{\mathrm{i}}}\right) \approx n_0 \frac{e^2\Phi}{T_{\mathrm{e}}}$$

Solved by $\Phi \propto \exp(-|x|/\lambda_{\mathrm{D}})$ with Debye length $\lambda_{\mathrm{D}}^2 = T_{\mathrm{e}}/(4\pi e^2 n_{\mathrm{e}})$

Debye shielding reduces Φ_{eff}, inhibits collective effects!

Plasma Waves I

So what exactly are collective effects?
Plasmas can produce any number of waves and instabilities, we will discuss many of them later

Simplest plasma wave:

- move all electrons (density n_0) a little to one side ($m_\mathrm{i} \gg m_\mathrm{e} \Rightarrow$ ions static; $T_\mathrm{e} = T_\mathrm{i} = 0$)
- perturbed electron density δn creates electric field $E = E_x$
- perturbation travels at speed $\delta v = \delta v_x$

Continuity equation:	$\partial_t \delta n + n_0 \partial_x \delta v = 0$
electrostatic force:	$\partial_t m_\mathrm{e} \delta v = -eE$
Poisson equation:	$\partial_x E = -4\pi e \delta n$

Three equations, three unknowns $\Rightarrow$ *straightforward to solve*

Group Work: Plasma Frequency

15 minutes group work:

1. solve this set of equations with a wave ansatz $\mathrm{e}^{ikx-i\omega t}$
2.

Group Work: Plasma Frequency

15 minutes group work:

1. solve this set of equations with a wave ansatz $\mathrm{e}^{ikx-i\omega t}$

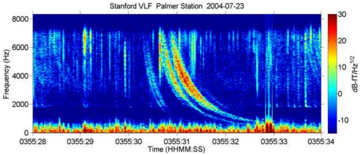
Solution: **plasma frequency** $\omega_\mathrm{p} = \sqrt{4\pi e^2 n_0/m_\mathrm{e}}$

2. evaluate ω_p for densities in a typical fusion plasma

Plasma Waves II

Before getting to instabilities, here is a sample of plasma waves (all in homogeneous magnetic fields, no pressure gradients)

Plasma oscillation	$\omega^2 = \omega_\mathrm{p}^2 + 3k^2 v_\mathrm{th}^2/2$	$\mathbf{k} \parallel \mathbf{B}$
Hybrid wave	$\omega^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2$	$\mathbf{k} \perp \mathbf{B}$
Light wave	$\omega^2 = \omega_\mathrm{p}^2 + k^2 c^2$	$\mathbf{k} \perp \mathbf{B}$
Alfvén wave	$\omega^2 = k^2 v_\mathrm{A}^2 = k^2 \frac{B^2}{4\pi n_\mathrm{i} m_\mathrm{i}}$	$\mathbf{k} \parallel \mathbf{B}$
Whistler wave	$\omega^2 = k^2 c^2 + \frac{\omega_\mathrm{p}^2}{1-\omega_\mathrm{c}/\omega}$	$\mathbf{k} \parallel \mathbf{B}$



Whister wave: created by lightning, “whistling” detectable by radio

$\Rightarrow$ *interpreting frequency spectra in plasmas can be difficult!*

Quasineutrality

When dealing with ion-electron or pair plasmas, we often assume $n_\mathrm{i} = n_\mathrm{e}$ (**neutrality**), but still allow $\Phi \neq 0$... ?!?

Underlying concept: only $\ll 1$ departures from $n_\mathrm{i} = n_\mathrm{e}$ allowed, otherwise truly extreme E fields

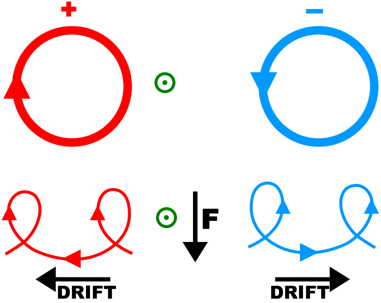
Correct vs. approximate derivation of ion sound waves (∇n):

$\delta n_\mathrm{i} = \delta n_\mathrm{e}$ **approximation**	**Correct treatment**
$m_\mathrm{i} n_0 \partial_t v_\mathrm{i} = -e n_0 \partial_x \Phi - T_\mathrm{i} \partial_x \delta n_\mathrm{i}$	$\longleftarrow$ same
adiabatic ("Boltzmann") electrons ($m_\mathrm{e} \to 0$):	Poisson equation $\partial_x E = 4\pi e(\delta n_\mathrm{i} - \delta n_\mathrm{e})$
$\delta n_\mathrm{e} = n_0 e\Phi/T = \delta n_\mathrm{i}$	$\delta n_\mathrm{e} = n_0 e\Phi/T \neq \delta n_\mathrm{i}$
continuity: $\partial_t \delta n_\mathrm{i} + n_0 \partial_x v_\mathrm{i} = 0$	$\longleftarrow$ same
$\Rightarrow \omega^2/k^2 = T_\mathrm{e}/m_\mathrm{i} + T_\mathrm{i}/m_\mathrm{i}$	$\omega^2/k^2 = \frac{1}{1+k^2\lambda_\mathrm{D}^2} T_\mathrm{e}/m_\mathrm{i} + T_\mathrm{i}/m_\mathrm{i}$

Valid for large scales — most fusion theory relies on approximate neutrality (**quasineutrality**) but uses Poisson

Particle Drifts

Fusion plasmas: **strong magnetic guide fields**
$\Rightarrow$ thermal motion along $\mathbf{B}$, slow **perpendicular drifting**



Some force $\mathbf{F} = F\hat{\mathbf{e}}_y$
(assuming $\partial_t F = 0$):

$$\begin{aligned} v_x &= v_\perp \exp(i\omega_c t) \\ v_y &= \pm i v_\perp \exp(i\omega_c t) + F/(qB) \\ \mathbf{v}_{\mathrm{drift}} &= \mathbf{F}\times\mathbf{B}/(qB^2) \end{aligned}$$

Possible F:

- gravitation
- polarization ($\partial_t E$)
  (*but*: ∂_t modifies v_x)
- inhomogeneous E field
- inhomogeneous B field

We will look at inhomogeneous fields in more detail later

Group Work: Drifts

45 minutes group work:

1. search online (e.g., in papers) for typical quantities in fusion plasmas: B, E (or Φ), turbulent frequencies ω
2. evaluate the gravitation drift for those parameters
3. derive the polarization drift for $E = E_x \propto \exp(i\omega t)$ (*need to add* E *to* $\partial_t v_x$ *force, get* $\partial_t^2 v_x = -\omega_\mathrm{c}^2 v_x + i\omega\omega_\mathrm{c} E_x/B$; use this for E-modified v_x ansatz and assume $\omega \ll \omega_\mathrm{c}$)
4. evaluate the polarization drift for those parameters

After 45 minutes, report your findings to the class. Which of these drifts, if any, is likely to be important?

Later, we will study **turbulence**, which is produced by **drift-wave instabilities**

Solution: Polarization Drift

To obtain **polarization drift**, consider $E = E_x$ acting on ions (electrons will have opposite drift sign)

No E: have $v_x = v_\perp \exp(i\omega_c t)$ and $\partial_t v_x = i\omega_c v_\perp \exp(i\omega_c t) = \omega_c v_y$
E enters via force equation $\partial_t v_x = \omega_c v_y + \omega_c E_x / B \leftarrow$ no drift in x if $\partial_t E = 0$

$$\text{Thus,} \quad \partial_t^2 v_x = -\omega_c^2 \left(v_x - \frac{i\omega}{\omega_c} \frac{E_x}{B} \right)$$

Finite E: **ansatz** $v_x = v_\perp \exp(i\omega_c t) + v_{\text{test}}$ with $v_{\text{test}} = (i\omega/\omega_c)(E_x/B)$

$$\Rightarrow \quad \partial_t^2 v_x = -\omega_c^2 v_x + (\omega_c^2 - \omega^2) v_{\text{test}} \approx -\omega_c^2 v_x + \omega_c^2 v_{\text{test}} \qquad (1)$$

when $\omega \ll \omega_c \Rightarrow$ **solves equation**, can therefore define

$$v_{\text{test}} \equiv v_{\text{pol}} = \frac{1}{\omega_c} \frac{\partial_t E}{B} \qquad (2)$$

Note: E_x also causes drift in v_y; but E oscillates, so no net motion

Questions & Discussion

What is unclear/wrong/poorly explained,
and what else would you like to know?

Turbulence and Transport in Fusion Plasmas
Part II

M.J. Pueschel

Ruhr-Universität Bochum, February 27 – March 10, 2023

Monday Recap

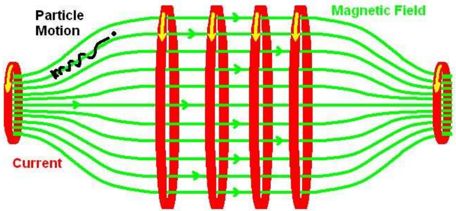
Yesterday, we covered

- how fusion energy will provide clean and safe energy
- the nature of Debye shielding and how to define plasmas
- what plasma waves and plasma drifts are

Next: how to design a confining magnetic field?

Magnetic Mirror

Early magnetic confinement: the **mirror machine**
(particles trapped between regions of large B)



Conserved: magnetic moment $\mu = \dfrac{mv_\perp^2}{2B}$, energy $E = \dfrac{1}{2}mv_\parallel^2 + \dfrac{1}{2}mv_\perp^2$

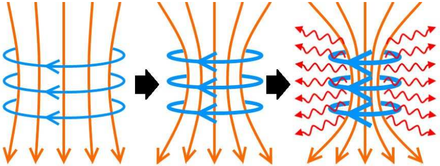
$\Rightarrow$ particles moving from small B to larger B convert $v_\parallel \to v_\perp$
at $v_\parallel = 0$, particles are **reflected** ($\leftrightarrow$ **magnetic mirror**)

If $v_\perp / v < \sqrt{B_{\min}/B_{\max}}$, **no reflection, particle slips through**
Non-zero $\nu_{\rm ei}$: trapped particles can enter this *loss cone*

$\Rightarrow$ *mirror machines have poor confinement due to end losses*

Z-Pinch

Another linear device (of Ocean's Eleven fame): the **Z-pinch**



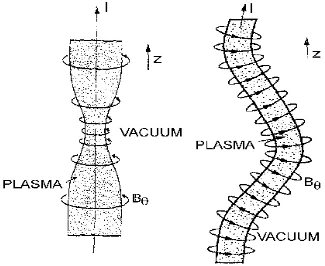
current filaments attract, compress field $\Rightarrow$ increased n, but only for short τ_E

Similar for lightning in weak rod; can use to shape metals, see

`sciencedemonstrations.fas.harvard.edu/presentations/can-crusher-magnetic-implosion`

However, no stable equilibrium! **Kink/sausage instabilities** (different from transport-relevant instabilities in fusion reactors)

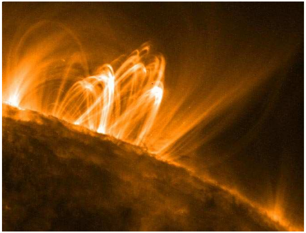
- **Sausage**: pinch effect
- **Kink**: prevents pinch from working

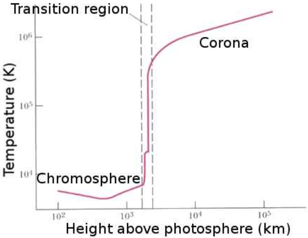


Minor Detour

A short excursion into a pinch-like phenomenon in space plasmas . . .

The Coronal Heating Problem





Decade-old question: *how can the solar corona be so hot*?

Heating mechanism:

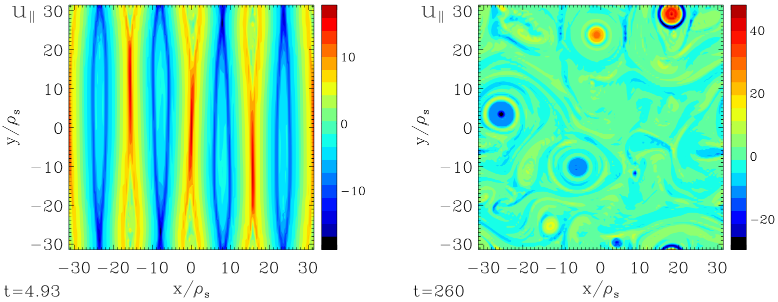
- MHD waves?
- loop footpoint motion?
- kinetic Alfvén waves?
- turbulent reconnection?
- nanoflares?

Driven Reconnection Turbulence

Using a fusion turbulence simulation code, study **magnetic reconnection** in a coronal loop (Pueschel ApJS 2014)

Driving turbulence through x-y-**periodic current sheets**,

$$\left.\frac{\partial g_j}{\partial t}\right|_{\mathrm{drive}} = -\omega_{\mathrm{dr}}\Big(g_j(k_y=0,t) - g_j(k_y=0,t=0)\Big)$$

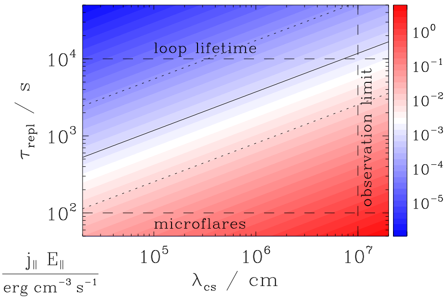


- **isotropization** despite driven current sheet
- formation of **plasmoids**, consistent with $S \gtrsim 10^4$ criterion

Application: Coronal Heating

Many turbulence simulations scanning over parameters:
extract **volumetric heating rate**

$$\frac{j_{\|}E_{\|}}{\mathrm{erg\,cm^{-3}\,s^{-1}}} = 1.5\times10^{-3}\left(\frac{n_{e0}}{10^9\,\mathrm{cm^{-3}}}\right)^{0.375}\left(\frac{T_{e0}}{10^6\,\mathrm{K}}\right)^{-0.1}\left(\frac{L_{\mathrm{ref}}}{10^9\,\mathrm{cm}}\right)^{0.2}\left(\frac{\beta}{3.5\times10^{-4}}\right)^{0.125}$$
$$\times\left(\frac{\nu_{ei}}{87\,\mathrm{s^{-1}}}\right)^{-0.1}\left(\frac{m_e/m_i}{1/1836}\right)^{-0.25}\left(\frac{T_{i0}/T_{e0}}{1.0}\right)^{0.0}\left(\frac{B_{\mathrm{rec}}}{5\,\mathrm{G}}\right)^{1.8}\left(\frac{\tau_{\mathrm{repl}}}{1100\,\mathrm{s}}\right)^{-1.5}\left(\frac{\lambda^{cs}}{1.5\times10^5\,\mathrm{cm}}\right)^{0.75}$$



Compare **observations**

Withbroe 77: $10^{-3}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

(or $L_{\mathrm{ref}}^{-1}\times10^6\,\frac{\mathrm{erg}}{\mathrm{cm^2\,s}}$)

Ofman 98: $10^{-4}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

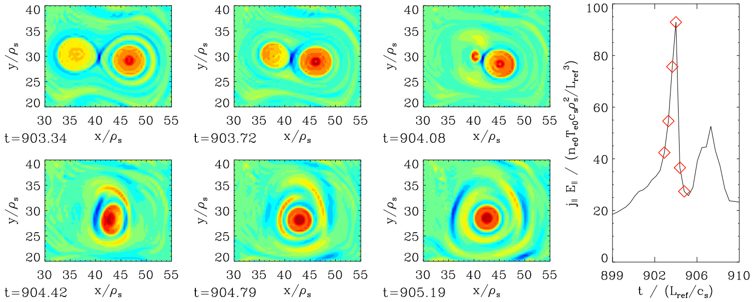
Guarrasi 14/Rosner 78:
$2\times10^{-3}\,\frac{\mathrm{erg}}{\mathrm{cm^3\,s}}$

$\Rightarrow$ observations **well within predicted range** of heating!

Nanoflares

Nanoflares: short ($t \gtrsim 20\,\mathrm{s}$) spikes in observed heating

Look at details of turbulence: **merging current filaments**



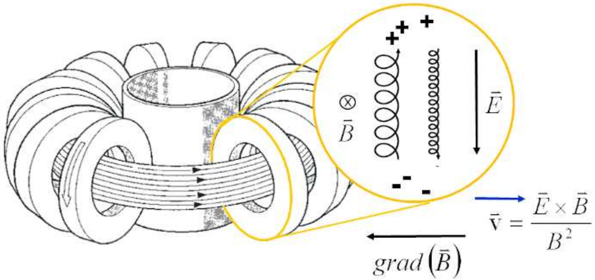
$\Rightarrow$ **mergers** are the **cause** of heating rate **spikes**

Apply to Active Regions: $7\,\mathrm{s} \lesssim t_{\mathrm{merge}} \lesssim 50\,\mathrm{s} \Rightarrow$ **match**!

Similar, but kinetic, process to the compression in a Z-pinch

Toroidal Confinement I

Bending a linear B-field into a torus… *what could go wrong?*



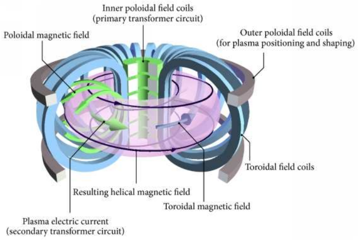
$\Rightarrow$ **near-instantaneous loss of all particles to reactor wall**

- two new plasma drifts (more later):
  $v_{\nabla B} \propto q_{\mathrm{i,e}}^{-1}$ and $v_{E \times B} \propto q_{\mathrm{i,e}}^{0}$
- $v_{\nabla B}$ separates charges, $v_{E \times B}$ lets them escape

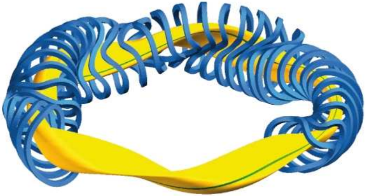
Since 1950s, many different concepts to deal with this problem

Toroidal Confinement II

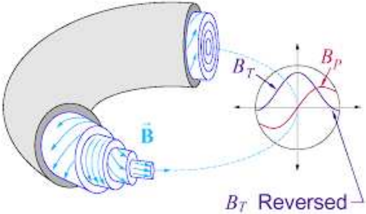
Tokamak: Soviet invention, transformer induces current



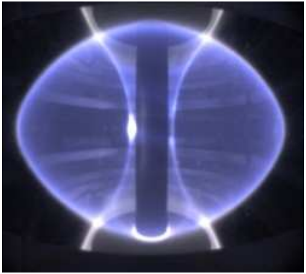
Stellarator: U.S. invention, complex coils, optimizable



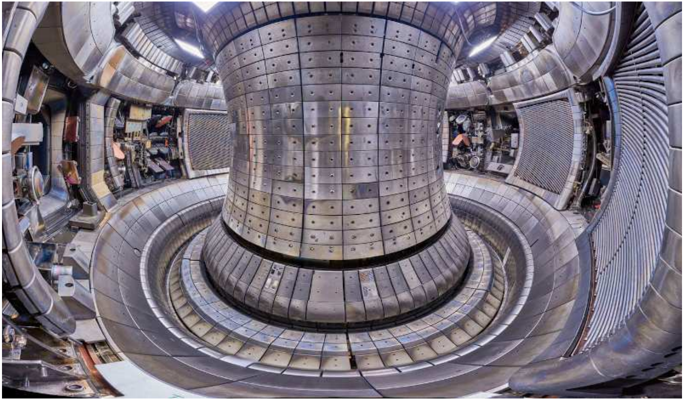
Reversed-field pinch (RFP): cheap tokamak, poor stability



Spherical Tokamak (ST): high efficiency, poor stability



Tokamak: ASDEX Upgrade



- fits 2–8 workers
- wall tiles
- divertor
- diagnostics
- ICRH antennae
- NBI shaft
- ECRH system
- pellet injector

Questions & Discussion

What's unclear? Who needs help? Who needs fresh tea?

Quiz

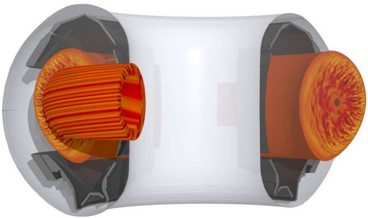
10-minute self-evaluated quiz:

1. What elements are to be used as fuel in fusion power plants, and why?
2. What are the safety considerations for fusion reactors, and how restrictive are they?
3. Why are modern fusion experiments toroidal and not linear?
4. What is the Debye length, and how does it affect whether an ionized gas is a plasma?

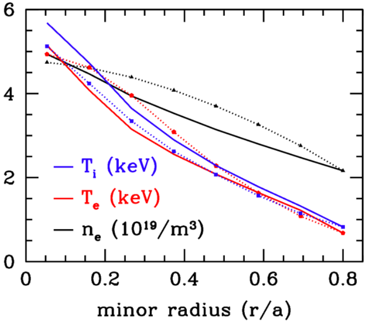
Quiz: Answers

1. Highest reaction cross-section: deuterium-tritium (*note*: alternate concepts rely on less efficient aneutronic reactions)
2. No runaway reactions, only true concern is tritium inventory; no substantial worry (but need smart choices for materials for reactor walls, steel)
3. Linear machines are subject to large end losses, poor confinement
4. Inside the Debye length, charge is shielded, no collective behavior, but gas-like

Transport



Fusion plasmas: radial distribution of n, T_i, T_e, ...



(Barnes PoP 2010)

Transport: heat/particle/momentum losses on time scale of heat/particle/momentum confinement time τ ($\longleftrightarrow$ *triple product* $nT\tau_E$)

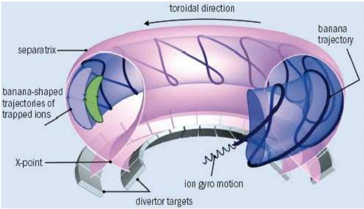
- What are the causes for these losses?
- How can we describe/predict profile evolution?

(Neo-)classical vs. Anomalous Flux

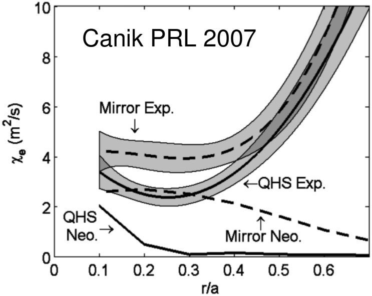
Historically, main heat loss mechanism thought to be
classical $\rightarrow$ *neoclassical* $\rightarrow$ *anomalous*

Classical: ignorably small (in 1950s, vision of tabletop reactors)

Neoclassical = collisions on banana orbits: small-ish



Anomalous: usually dominates



Historically, *anomalous* = *no idea where it comes from*,
but now used as synonym for **turbulence**

Plasma Instability

Fluctuation measurements: plasma “bubbles” like boiling water

$\Rightarrow$ *How does a physicist describe a bubbling system?*

Plasma Instability

Fluctuation measurements: plasma “bubbles” like boiling water

$\Rightarrow$ *How does a physicist describe a bubbling system?*

...we look at a **small perturbation from equilibrium**

E.g., density

$$n \rightarrow n_0 + n_1 \qquad \epsilon n_0 \sim n_1$$

In equilibrium (fusion: MHD),

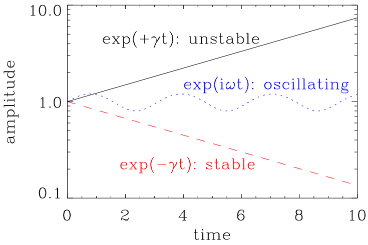
$$\begin{aligned} \frac{\partial n_0}{\partial t} &= 0 \\ \frac{\partial n_1}{\partial t} &= A n_1 + B n_1^2 + \ldots \end{aligned}$$

Linearization: drop terms $\propto \epsilon^2$ and higher

$\Rightarrow$ can solve with ansatz

$$n_1 = n_1(t=0)\exp(i\omega_c t)$$

(ω_c: complex frequency)



Describes eigenmode growing exponentially at rate γ forever

$\rightarrow$ *what stops this growth?*

Saturation and Turbulence

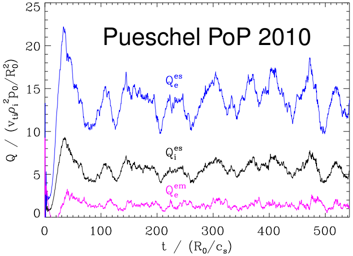
As n_1 grows, eventually $n_1^2 \sim n_1$, and **nonlinearity** becomes important

In plasmas, typically

$$\frac{\partial n_1}{\partial t} = A n_1 + \frac{\partial n_1}{\partial x}\frac{\partial n_1}{\partial y}$$

- here, quadratic nonlinearity
- saturation: **balance** of linear, nonlinear terms
- generally, **cannot solve analytically**

Solve equations numerically



quasi-stationary state after linear growth, saturation
$\Rightarrow$ **statistical analysis**

Turbulence: very difficult, decades of research in many fields

Group Work: Turbulence Basics

45 minutes group work:

Find online sources that explain

- chaos
- random walk
- the Kolmogorov spectrum

and be prepared to give a short summary.

Did your sources explain what the Lyapunov exponent is?

Student Lecture

Volunteers . . .   Who wants to tell us:

What does **chaos** mean to a physicist?

What is and when do we observe **random walk**?

What is the **Lyapunov exponent**?

What is the **Kolmogorov** spectrum?

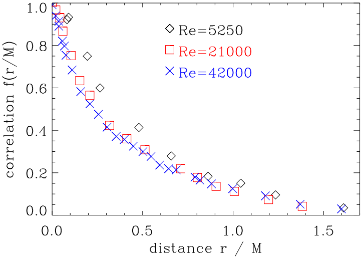
Kolmogorov Turbulence

Kolmogorov 1941

turbulence theory based on **self-similarity**

Stewart 1951

wind tunnel experiments:
no viscosity dependence
of low-ν (high-Re) turbulence



dissipation rate $\epsilon = 2\nu \int k^2 E \mathrm{d}k$, with $[E] = \frac{\mathrm{m}^3}{\mathrm{s}^2}$, $[\epsilon] = \frac{\mathrm{m}^2}{\mathrm{s}^3}$, $[\nu] = \frac{\mathrm{m}^2}{\mathrm{s}}$

dimensional analysis: $E(k) \propto u(k)^2 \lambda(k) [k\lambda]^n$ (self-similarity/scale-free)

to get rid of ν, use $[\lambda] = \mathrm{m} = [\frac{\nu^{3/4}}{\epsilon^{1/4}}]$ and $[u] = \frac{\mathrm{m}}{\mathrm{s}} = [\nu^{1/4}\epsilon^{1/4}]$

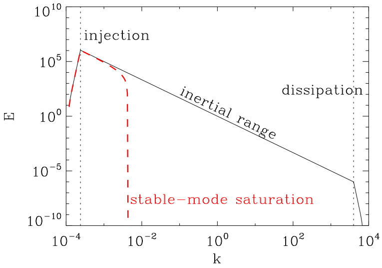
$\Rightarrow$ to make energy ν-independent, choose $n = -5/3$:

$$E(k) \propto k^n \nu^{\frac{3n+5}{4}} \epsilon^{\frac{1-n}{4}} = k^{-5/3}\epsilon^{2/3}$$

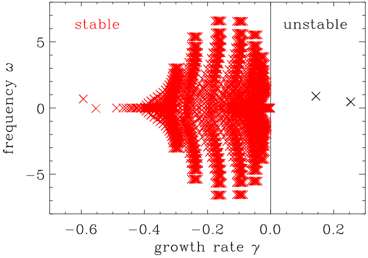
(**Kolmogorov inertial range**, confirmed in many experiments)

Beyond Kolmogorov

Kolmogorov 1941: transfer energy conservatively across scales



Full solution to linear operator A



However, instability-driven plasma systems can have energy **injection/dissipation/removal at the same scales**!

- large-scale unstable modes inject energy ($\nabla T_{\mathrm{e,i}}$, ∇n)
- 10^4+ large-scale **stable modes** remove energy

$\Rightarrow$ **no** true **inertial range**, no $5/3$ law

Fluid Model for Plasmas I

For very simple plasma description (Horton PoF 1981, 1988), can use (normalized) nonlinear **two-field fluid model** with

- electric potential $\Phi(x, y)$
- (ion or electron) pressure $p(x, y)$

$$\begin{aligned}
\frac{\partial \Phi}{\partial t} &= (\delta - \nabla_\perp^2)^{-1}\left(\overbrace{-\frac{\partial \Phi}{\partial y} + \epsilon \frac{\partial p}{\partial y}}^{\text{drifts}} + \nu \nabla_\perp^2 \Phi - \hat{z} \cdot (\nabla_\perp \Phi \times \nabla_\perp^3 \Phi)\right) \\
\frac{\partial p}{\partial t} &= \underbrace{(-1 - \eta)\frac{\partial \Phi}{\partial y}}_{\text{drive}} - \chi \nabla_\perp^4 p - \hat{z} \cdot (\nabla_\perp \Phi \times \nabla_\perp p)
\end{aligned}$$

With: $\epsilon \sim 1/\nabla n$, "collisionalities" ν, χ, coordinate $\hat{z} \parallel \mathbf{B}$, radial coordinate x, toroidal coordinate y

Physics: toroidal ion-/electron-temperature-gradient modes, accessing free energy in background ∇T via $\eta \equiv \mathrm{d} \ln T / \mathrm{d} \ln n$

Going to Fourier space easy for linear terms, $\partial/\partial x, y \to ik_{x,y}$, *but what happens to the nonlinearities?*

Group Work: Fourier Convolution

30 minutes group work:

What does the expression

$$\{f, \Phi\} \equiv \frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial x}$$

look like when in Fourier space,
$f(x,y) \to f(k_x,k_y)$ and $\Phi(x,y) \to \Phi(k_x,k_y)$, with e.g.

$$\mathcal{F}(\partial_y \Phi) = \int \mathrm{d}\mathbf{x}\, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \sum_{\mathbf{k}'} \mathrm{e}^{i\mathbf{k}'\mathbf{x}} i k'_y \Phi(\mathbf{k}')$$

Hint 1: use separate transforms (with k', k'') for $\partial_x f$, $\partial_y \Phi$ etc.
Hint 2: $\int \exp iax \to \delta_a$
Hint 3: the final form should have a single sum over $k'_{x,y}$

Solution: Fourier Convolution

$$
\begin{aligned}
\mathcal{F}(\{f,\Phi\}) &= \int \mathrm{d}\mathbf{x}\,\mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left[\frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial x}\right] \\
\mathcal{F}(\{f,\Phi\}) &= \int \mathrm{d}\mathbf{x}\,\mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left[\left(\sum_{\mathbf{k}'} \mathrm{e}^{i\mathbf{k}'\mathbf{x}} i k'_y \Phi(\mathbf{k}')\right)\left(\sum_{\mathbf{k}''} \mathrm{e}^{i\mathbf{k}''\mathbf{x}} i k''_x f(\mathbf{k}'')\right) - \ldots\right] \\
\mathcal{F}(\{f,\Phi\}) &= -\int \mathrm{d}\mathbf{x} \sum_{\mathbf{k}'}\sum_{\mathbf{k}''} \mathrm{e}^{-i(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\mathbf{x}} \left[k'_y \Phi(\mathbf{k}') k''_x f(\mathbf{k}'') - \ldots\right] \\
\mathcal{F}(\{f,\Phi\}) &= -\sum_{\mathbf{k}'}\sum_{\mathbf{k}''} \delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} \left(k'_y k''_x - k'_x k''_y\right) \Phi(\mathbf{k}') f(\mathbf{k}'') \\
\mathcal{F}(\{f,\Phi\}) &= \sum_{\mathbf{k}'} \left(k'_x k_y - k_x k'_y\right) \Phi(\mathbf{k}') f(\mathbf{k}-\mathbf{k}')
\end{aligned}
$$

Why do we care, or: what are the advantages of Fourier space?

Fluid Model for Plasmas II

In Fourier space, the model becomes

$$\begin{aligned}
\frac{\partial \Phi(k_\perp)}{\partial t} &= [\delta(k_y) + k_\perp^2]^{-1} \Bigg(-ik_y \Phi(k_\perp) + ik_y \epsilon p(k_\perp) - \nu k_\perp^2 \Phi(k_\perp) \\
&\quad - \frac{1}{2} \sum_{k'_\perp} (k'_\perp \times \hat{z} \cdot k_\perp) [(k_\perp - k'_\perp)^2 - k'^2_\perp] \Phi(k'_\perp) \Phi(k_\perp - k'_\perp) \Bigg) \\
\frac{\partial p(k_\perp)}{\partial t} &= -ik_y(1+\eta)\Phi(k_\perp) - \chi k_\perp^4 p(k_\perp) \\
&\quad - \sum_{k'_\perp} (k'_\perp \times \hat{z} \cdot k_\perp) \Phi(k'_\perp) p(k_\perp - k'_\perp)
\end{aligned}$$

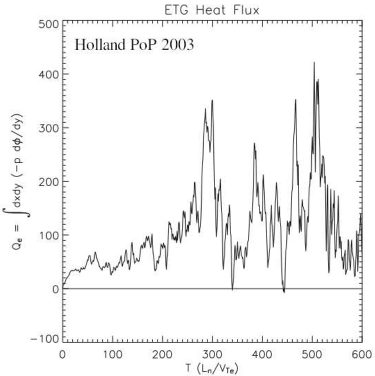
δ: adiabatic response; $\delta = {1 \atop 0}$ at ${k_y > 0 \atop k_y = 0}$ for ITG, $\delta = 1$ for ETG
(*all just a trick to get more or less correct nonlinear behavior*)

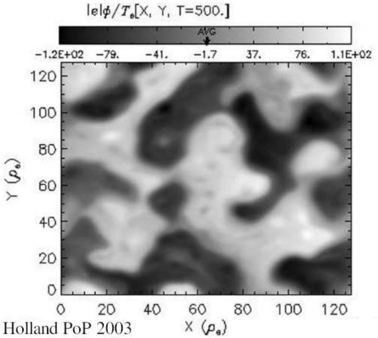
Later, more physics background, proper derivation; for now,
how do we use a model like this and what can we learn?

Instability and Saturation

Commonly, pick arbitrary initial condition (Φ, p), evolve in time

Below, ETG (we'll talk more about drift-wave physics later)





- short linear phase
- linear: mostly $k_x = 0$ to access ∇T

- nonlinearity rises early
- nonlinear coupling creates $k_x > 0$ eddies

Questions & Discussion

Anything unclear so far?

Group Work: Code Prep

30 minutes group work:

- In a programming language of your choice, write a code that plots contours of the function

  $$f(z) = (z + 0.2i)(z - 0.2i)(z + 0.3 - 0.5i)$$

  in the complex plane ($z \in \mathbb{C}$)

- Be prepared to read data from an ASCII text file and create line and contour plots based on that data; example data:

  ```
  1.33 -1.8337E+01
  1.34 -1.8591E+01
  1.35 811.429
  1.36 40238.2
  ```

Turbulence and Transport in Fusion Plasmas
Part III

M.J. Pueschel

Ruhr-Universität Bochum, February 27 – March 10, 2023

Tuesday Recap

Yesterday, we covered

- what linear and toroidal reactor designs exist, and why some are perhaps better than others
- what classical, neoclassical, and anomalous transport are
- the meaning of instability and turbulence
- how a nonlinear term transforms to Fourier space and makes the nature of three-wave coupling more transparent

Next: solve the Horton-Holland fluid model analytically

Linear Horton-Holland

Reminder: this is the **linearized Horton-Holland** model:

$$\begin{aligned}
\frac{\partial\Phi(k_\perp)}{\partial t} &= \frac{1}{1+k_\perp^2}\left(-ik_y\Phi(k_\perp) + ik_y\epsilon p(k_\perp) - \nu k_\perp^2\Phi(k_\perp)\right) \\
\frac{\partial p(k_\perp)}{\partial t} &= -ik_y(1+\eta)\Phi(k_\perp) - \chi k_\perp^4 p(k_\perp)
\end{aligned}$$

Group Work: Fluid Code

1.5 hours group work:

Preparatory calculations

1. derive ITG/ETG dispersion relation from fluid model
2. solve dispersion relation analytically to get ω_c
3.
4.
5.
6.

Analytical Solution

Dispersion relation – relating ω_c, $k_\perp$:

$$-\omega_\mathrm{c}^2(1+k_\perp^2)+i\omega_\mathrm{c}(1+k_\perp^2)k_\perp^4\chi = \omega_\mathrm{c}k_y - ik_yk_\perp^4\chi + k_y^2\epsilon(1+\eta) - i\omega_\mathrm{c}k_\perp^2\nu - k_\perp^6\nu\chi$$

Exactly solvable; but simpler structure with some additional simplifications: $k_\perp \approx k_y$; weak collisionalities ν, χ; typical $0.1 < k_\perp < 1$ (units of inverse gyroradius); thus:

$$\omega_{\mathrm{c}1,2} = \underbrace{\frac{k_y}{2+2k_\perp^2}}_{\text{drift}} \pm \underbrace{ik_y\left(\frac{(1+\eta)\epsilon}{1+k_\perp^2}\right)^{1/2}}_{\text{gradient drive}} \underbrace{- \frac{i\nu k_\perp^2}{2+2k_\perp^2} - \frac{i\chi k_\perp^4}{2}}_{\text{collisional damping}}$$

Very few models solvable by hand ...
what to do for more complex physics?

Group Work: Fluid Code

1.5 hours group work:

Preparatory calculations

1. derive ITG/ETG dispersion relation from fluid model
2. solve dispersion relation analytically to get ω_c

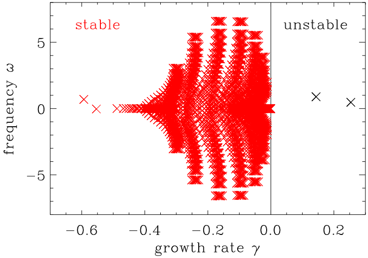
Numerical treatment: use plot routine (from prep work) to

3. implement the full dispersion relation (zeroes of $\mathrm{lhs} - \mathrm{rhs}$)
4. test numerical convergence, compare to analytical result
5. obtain, interpret instability spectrum $\gamma(k_y)$
6. scan some input parameter – e.g., η, ϵ, ν – and discuss physical meaning of results

Rule for all group work sessions: if you're stuck, just ask!

Subdominant and Stable Modes

Commonly, focus on fastest-growing mode, but recall:
at each $k_\perp$, have many linear eigenmodes



- **dominant**: $\gamma_{\max}$
- **subdominant**: $0 < \gamma < \gamma_{\max}$
- **stable**: $\gamma < 0$

Nonlinear saturation: either

- “actual” dissipation: collisions, or
- **stable modes returning energy** to $\nabla T, n$

Coupling between k, k', $k - k'$ sends energy from unstable to stable modes!

Questions & Discussion

Anything unclear that we talked about here?

Theory Frameworks

In physics, **hierarchy of models**: can describe light as *rays*, with *nonlinear optics*, or as *quantum wave functions*

$\Rightarrow$ **accuracy vs. cost/complexity**

Plasma physics: historically very simple models, but now **supercomputers allow use of high-fidelity models**

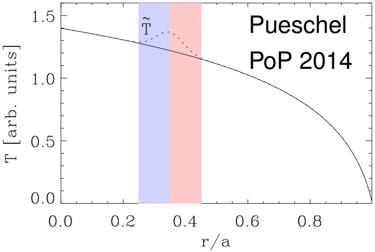
$$\begin{array}{ccccc} & & 6D & & 5D \\ & & \textit{kinetics} & \rightarrow & \textit{gyrokinetics} \\ & & \downarrow & & \downarrow \\ \textit{transport coefficients} & & \textit{fluid/MHD} & & \textit{gyrofluid} \\ 0D & & 3D & & 3D \end{array}$$

In the following, introduction to each model, (dis-)advantages

Transport Coefficients

TRANSP, B2.5, . . . : simple mock-up of turbulent flux, assume **constant heat & particle diffusivities** $\chi \sim Q/\nabla T$, $D \sim \Gamma/\nabla n$

$$\underbrace{\frac{\partial T(r)}{\partial t}}_{=0 \text{ in equil.}} = \underbrace{\nabla \cdot \chi(r) \nabla T(r)}_{\text{diffusion}} + \underbrace{S}_{\text{source}}$$



- constant χ, D: fast, easy to implement
- produces well-behaved fluxes
- **misses key physics**, e.g., ∇T_{crit}
- some folks tune χ, D to get whatever result they want

More on transport later in the course, but key research area: **study turbulence, understand & get expressions for χ, D, hand off to transport modelers**

Kinetics

Gold standard of theories: kinetics—particles at $\mathbf{x}$ moving with $\mathbf{v}$

6D, two approaches:

PIC

Particle-in-cell:

equations of motion for individual particles

problem: noise build-up

Vlasov/continuum

evolve particle distribution $f_{\mathrm{i,e}} = f_{\mathrm{i,e}}(\mathbf{x}, \mathbf{v}, t)$

harder to implement, but "cleaner" results

Solve **Vlasov** equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

but:

- **collisions** require insane resolutions $\rightarrow$ instead use *collision operator*, RHS = $C[f]$
- contains lots of unnecessary physics (light waves etc.)
- coordinates not helpful in fusion geometries

Note: get force $\mathbf{F} = \mathbf{F}(\Phi, \mathbf{A})$ from **Maxwell's equations**

Fluid Models

Earlier, we used a much simpler "fluid" model

Moments

Integrate Vlasov equation to get **moments** (6D $\rightarrow$ 3D):

$$\text{density} \quad n = \int \mathrm{d}\mathbf{v} f$$

$$\text{flow} \quad \mathbf{u} = \int \mathrm{d}\mathbf{v}\, \mathbf{v} f$$

$$\text{temperature} \quad T = \int \mathrm{d}\mathbf{v}\, \mathbf{v}^2 f$$

$$\vdots \quad \vdots$$

with potentials/fields Φ, $\mathbf{A}$

Problem: system of equations **needs closure**:

$$\frac{\partial n}{\partial t} = \ldots n + \ldots \overbrace{\int \mathrm{d}\mathbf{v}\, \mathbf{v} f}^{\mathbf{u}}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \ldots \mathbf{u} + \ldots \overbrace{\int \mathrm{d}\mathbf{v}\, \mathbf{v}^2 f}^{T}$$

- **never-ending series**
- can make assumptions, e.g., $\int \mathrm{d}\mathbf{v}\, \mathbf{v}^4 f = 0$
- tends to **miss key nonlinear physics**

Will take moments to get specific fluid model later

MHD

Special category of fluid models: **Magnetohydrodynamics**

Ideal MHD:

continuity $\frac{\partial n}{\partial t} - \nabla \cdot (n\mathbf{u}) = 0$

momentum balance

$$n\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u} = \frac{\mathbf{J} \times \mathbf{B}}{mc} - \frac{\nabla p}{m}$$

adiabaticity $\frac{\mathrm{d}}{\mathrm{d}t} \frac{p}{(mn)^{5/3}} = 0$

Ohm's law $c\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$

Ampère's law $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$

Faraday's law $\frac{\partial B}{\partial t} = -c\nabla \times \mathbf{E}$

no divergence $\nabla \cdot \mathbf{B} = 0$

Other types: RMHD, HMHD, reduced MHD, . . . , e.g., for astrophysics

Orderings

MHD validity requires

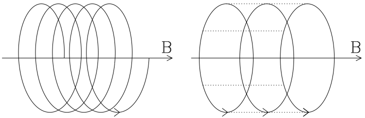
- $\lambda \gg \lambda_{\mathrm{D}}, \rho_{\mathrm{i,e}}$
- $\gamma, \omega \ll \omega_{\mathrm{p}}, \omega_{\mathrm{ci,e}}$
- $|\mathbf{u}| \ll c$
- collisional, $\nu_{\mathrm{ei}} > \gamma, \omega$
  $\Rightarrow$ no v-space dynamics
- full ionization

Fusion: MHD is used to compute magnetic equilibria, but **does not capture microinstability, turbulence**

Gyrokinetics

Littlejohn JMP 1979, 1982; Frieman PoF 1982; Dubin PoF 1983

Gyroaverage to reduce phase space $\mathbf{v}_{x,y} \rightarrow v_\perp$ from 6D to 5D



- average Larmor motion
- charged rings flow along z
- slow drifts in x, y

Why is this a big deal? *Review*: Brizard RMP 2007

- 6D to 5D means order-of-magnitude speed-up
- **gyroagerage eliminates irrelevant fast time scales** (Larmor motion, fast magnetosonic waves)
  $\Rightarrow$ **factor** 10^3 **speed-up!**

Gyrokinetics enabled turbulence studies in fusion plasmas

Later in this course: sketch of gyrokinetic framework derivation

Note: gyrofluid can be obtained from GK but not from fluid

The Gyrokinetic Equations

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\mathrm{ref}}}{JB_0}v_\|\Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\mathrm{ref}}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial\chi}{\partial x}\Gamma_{jy} - \frac{\partial\chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\mathrm{coll}}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00} - \mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01} - \mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0 \int J_0 f_j \,\mathrm{d}v_\|\,\mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2} J_1 f_j\,\mathrm{d}v_\|\,\mathrm{d}\mu$$

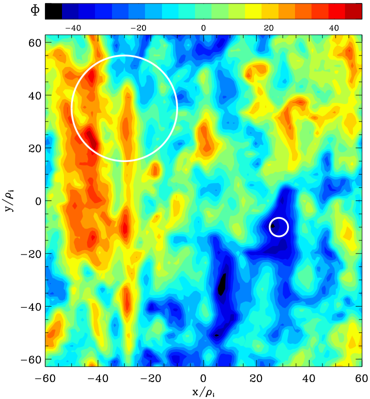
$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0) \quad \mathcal{C}_2 = -\sum_j \frac{q_j n_{j0}}{B_0}(\Gamma_0 - \Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0}\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|}F_{j0}\partial_k\chi_j q_j/(m_j v_\|) + \bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_j c)$$

Properties of Gyrokinetics

Gyroaverage I: field feels charged circle, not particle
Gyroaverage II: particle feels reduced small-scale field



$\rho > \lambda_c$ vs. $\rho < \lambda_c$

GK removes fast time scales, not spatial (but need $\rho \ll L_B$)

Drift-kinetics

When $\rho \ll \lambda_c \leftrightarrow k_\perp \rho \ll 1$: no gyroaverage, *drift-kinetics*

- slightly faster, slightly less memory needed
- easier analytics
- easier closures

Included in GK (*but not in gyrofluid*): **velocity space, wave-particle interactions, full zonal-flow physics**

Gyrokinetic Orderings

Assumptions in **general gyrokinetics**:

- no very fast time scales, $\omega \ll \Omega_{cj}$ for all species j
- Larmor orbits nearly closed, $\rho_j \ll L_B = B/|\partial B/\partial \mathbf{x}| \quad \forall\, j$

Additional common assumptions (**strong magnetization**):

- fast parallel motion $v_\| \sim v_{\mathrm{T}j} \gg v_{x,y}^{\mathrm{drifts}} \quad \forall j$
- anisotropic turbulence $k_\| \ll k_\perp \leftrightarrow \lambda_\| \gg \lambda_\perp$
- low pressure $\beta_j \equiv 8\pi n_{j0} T_{j0}/B_0^2 \ll 1 \quad \forall j$

Additional assumptions for δf (c.f. earlier $n \rightarrow n_0 + n_1$):

- small perturbations $f_1/f_0 \sim \mathbf{B}_1/\mathbf{B}_0 \ll 1$
  (*not* Φ_1/Φ_0 *since* Φ_0 *can be* $= 0$)
- fluctuation localization $\rho_j \ll L_{(n,T)} = (n,T)/|\partial(n,T)/\partial \mathbf{x}| \quad \forall\, j$

Hereafter, only consider δf*, using notation* $f_0 \rightarrow F_0, f_1 \rightarrow f$*, etc.*
$\Rightarrow$ **for us, we will require all of the above conditions**

Short Gyrokinetics History

1980s: basic gyrokinetic theory

- analytical extensions: δB, relativistic, higher orders, ...
- simple NL analytics: Similon PoF 1984, Smith PoF 1985
- NL PIC (adiabatic electrons): Lee JCP 1987

1990s: simple simulations, then gyrokinetics really takes off

- comparisons to experiment: Parker PRL 1993
- ions & electrons: Kotschenreuther PoP 1995
- importance of zonal flows: Lin PRL 1999

2000s: high-fidelity turbulence simulations, zonal flows

- ion-scale turbulence: Dimits PoP 2000, Dannert PoP 2005
- electron turbulence: Jenko PoP 2000, Dorland PRL 2000
- NL $\beta > 0$: Candy PoP 2005, Pueschel PoP 2008 & 2010

2010s: multi-scale simulations, magnetic fluctuations

- microtearing: Doerk PRL 2011, Guttenfelder PRL 2011
- i-e multi-scale: Candy PPCF 2007, Goerler PRL 2008

2020s: SOL/full-volume simulations, stellarators

Landau Damping I

One key process captured by (gyro)kinetics: **Landau damping**

Consider simple 1D Vlasov equation, perturbed distribution f:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial F_0}{\partial v} = 0 \qquad\qquad \frac{\partial E}{\partial x} = -4\pi e\int f\,\mathrm{d}v$$

Take wave ansatz for f and insert Vlasov into Poisson:

$$ikE = -4\pi e\int iE\frac{e}{m}\frac{\partial F_0/\partial v}{\omega_\mathrm{c} - kv}\mathrm{d}v \quad \text{or} \quad \frac{\omega_\mathrm{p}^2}{k^2}\int\limits_{-\infty}^{\infty}\frac{\partial \hat{F}_0/\partial v}{v - \omega_\mathrm{c}/k}\mathrm{d}v = 1$$

Lev Landau, Zh. Eksp. Teor. Fiz. **16**, 574 (1946): *how to integrate this expression properly*; for Maxwellian $F_0 \sim \exp(-v^2)$,

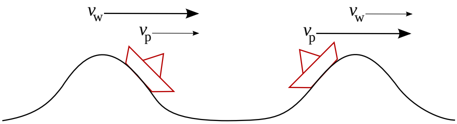
real frequency $\omega^2 = \omega_\mathrm{p}^2\left(1 + 3\frac{k^2 v_\mathrm{th}^2}{\omega^2}\right) \quad T \to 0:$ same as fluid

damping rate $\mathrm{Im}(\omega_\mathrm{c}) = -\sqrt{\pi}\frac{\omega_\mathrm{p}^4}{k^3 v_\mathrm{th}^3}\mathrm{e}^{-3/2-\omega_\mathrm{p}^2/(k^2 v_\mathrm{th}^2)}$ fluid: no damping

Landau Damping II

Thus, **waves are damped without any collisions!**
Same process in galaxy formation, with ions $\rightarrow$ stars, $qE \rightarrow F_g$

Damping comes from integrating **resonance** $v_{\rm res} - \omega_{\rm c}/k = 0$:
particles with $v \approx v_{\rm res}$ travel with wave, see little $\partial_t E$
$\Rightarrow$ **continuous acceleration in one direction**



- $v \gtrsim v_{\rm res}$: surfer behind wave *pushes wave*
- $v \lesssim v_{\rm res}$: surfer ahead of wave *pushed by wave*

Energy conservation: **particles and wave exchange energy**

Maxwellian: more particles have $v \lesssim v_{\rm res}$, **net wave damping**
However, if $\partial F_0/\partial v > 0$, can get **two-stream instability**

Group Work: Orderings

45 minutes group work

1. look up machine parameters, profiles, and fluctuation characteristics $k_\perp$, $\omega \sim \gamma$ (e.g., from papers, image search, Wikipedia) for
   a. ASDEX Upgrade
   b. JET or Wendelstein 7-X
2. for each, evaluate **MHD** validity at $r/a \approx 0.5$ and 0.9: fully ionized; $\lambda \gg \lambda_\mathrm{D}, \rho_\mathrm{i,e}$; $\omega \ll \omega_\mathrm{p}, \omega_\mathrm{ci,e}$
3. for each, evaluate **gyrokinetics** validity at same radii: *truly mandatory*: $\rho_\mathrm{i,e} \ll L_B$; *quasi-mandatory*: $\beta \ll 1$; $k_\| \ll k_\perp$; *for* δf *validity*: $B_1 \ll B_0$

If only one of T_i or T_e is available, assume $T_\mathrm{i} = T_\mathrm{e}$

Suggested resources:
NRL Plasma Formulary; Freethy RSI 2016

Questions & Discussion

Anything unclear at this time?

Turbulence and Transport in Fusion Plasmas
Part IV

DIFFER

M.J. Pueschel

RUHR UNIVERSITÄT BOCHUM RUB

Ruhr-Universität Bochum, February 27 – March 10, 2023

Wednesday Recap

Yesterday, we covered

- numerical treatment of the Horton-Holland dispersion relation
- different theory frameworks and their use
- Landau damping in kinetic theory
- what conditions have to be fulfilled so MHD and/or gyrokinetics can be used

Next: what coordinates and simulation domain should we use?

Group Work: Coordinates

45 minutes group work:

Find sources that explain

1. toroidal coordinates
2. the safety factor q as a measure of field-line pitch

and (roughly) familiarize yourself with those. Have a look at

3. all of
   `www-fusion.ciemat.es/wiki/Toroidal_coordinates`
4. as much as you feel like of
   `www-fusion.ciemat.es/wiki/Flux_coordinates`
5. bonus reading for those with a high pain threshold: pages 1–3 of P. Xanthopoulos *et al.*, Phys. Plasmas **13**, 092301 (2006)

Be prepared to present your findings.

Can you explain when/why/how field lines are (not) periodic?

Who can tell the group . . .

How are **toroidal coordinates** defined?

When are field lines periodic? Why would we care?

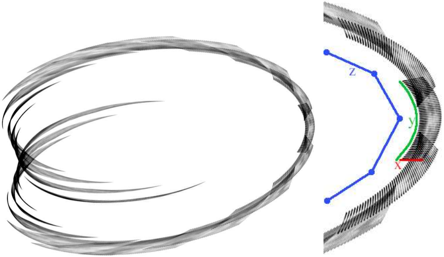
Is turbulence localized radially, toroidally, along the field?

What does all this mean for the stellarator?

Flux Tubes

Typical experiments/reactors: $k_\perp \rho_i \sim 0.1 - 1 \leftrightarrow n \sim 30 - 300$
Strong radial/toroidal localization $\Rightarrow$ **flux tube** (Beer PoP 1995)

With radial domain $L_x \gtrsim \rho_{i,e} \ll R, a$, can do Taylor expansion of n, T, q, etc. profiles, e.g., $T(r \approx r_0) \approx T_0 + (r - r_0)\mathrm{d}T/\mathrm{d}r$
A little confusing: in flux tube, both T and $\mathrm{d}T/\mathrm{d}r$ are constant!



Advantages

- cheaper (lower N_x)
- cleaner (Fourier)
- flexible (no fixed ρ^*)

Toroidal coordinates r, θ, ϕ transform to local x, y, z:

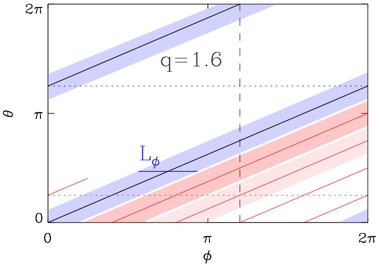
$$r = r_0 + x \qquad \theta = z - \pi \qquad \text{(circular flux surf.)}$$

$$\phi = -\frac{q_0}{r_0} y + q_0 \left(1 + \frac{\hat{s}}{r_0} x\right) \theta \qquad \hat{s} \equiv \frac{r_0}{q_0} \frac{\mathrm{d}q}{\mathrm{d}r}$$

Flux Tube vs. Flux Surface

Tokamak: a **single flux tube represents** entire **flux surface**

$\rho^* = \rho_\mathrm{i}/R_0 \ll 1$ free to pick,
$\mathcal{M} = 2\pi/L_\phi \in \mathbb{N}$ arbitrary
$\Rightarrow$ at $x = 0$, always periodic,
but not at $x \neq 0$ if $\hat{s} \neq 0$



Circular surface: parallel BC
$f(x, y, \pi) = f(x, y - 2\pi\hat{s}x, -\pi)$
(also called *twist-and-shift*)

Full system: entire flux surface (*but*: $n = 1$ means $k_y \sim k_\parallel$!)
$\Rightarrow$ need to test convergence only for $k_y^{\min} \propto L_y^{-1}$ in $-\pi \leq z < \pi$

Stellarator: **flux tubes** starting at different ϕ **differ**
$\Rightarrow$ for complete physics, need full-surface (or full-volume) code!

Ballooning Representation

Exercise: derive **parallel BC in $k_{x,y}$ Fourier space**

$$f(k_x, k_y, \pi) = (-1)^{\mathcal{N}} f(k_x + \mathcal{N} k_x^{\text{shift}}, k_y, -\pi)$$

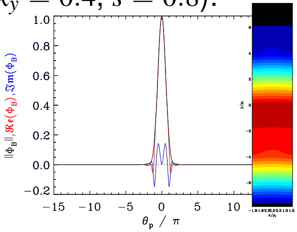
$$\mathcal{N} = 2\pi \hat{s} k_y / k_x^{\text{shift}} \stackrel{\text{commonly}}{=\!=\!=} \pm 1$$

- real space: y shift
- k-space: k_x shift
- can be used to stitch together k_x

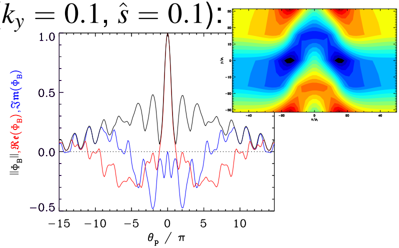
Ballooning space

Extended parallel coordinate: **ballooning angle** θ_p, usually $= 0$ at $k_x = 0$ (Candy PoP 2004)

"**ballooned**" mode ($k_y = 0.4$, $\hat{s} = 0.8$):



"**slab-like**" mode ($k_y = 0.1$, $\hat{s} = 0.1$):



Group Work: Locality

20 minutes group work

1. Determine whether the flux tube is likely to be valid for the same machines/radii as in the Orderings group work:
   Is the Taylor expansion a good approximation throughout $L_x \sim 100\rho_\mathrm{i}$?

Note: this is a simple estimate! For real applications, more thorough studies (e.g., comparing local, global) may be needed.

Questions & Discussion

Anything unclear so far?

Group Work: Equilibria

MHD: **magnetic equilibria** — no MHD instability,
fluxes from neoclassical collisions or microturbulence

1.5 hours group work:

1. Download J.W. Haverkort's write-up on equilibria:
   `http://homepage.tudelft.nl/20x40/documents/Equilibria.pdf`
2. Work through
   a. Sec. 1.1
   b. Sec. 1.2
   c. Appendix A

   for good understanding of the **Grad-Shafranov** equation
3. get R.L. Miller *et al.*, Phys. Plasmas **5**, 973 (1998),
   read Secs. 1–3, make notes about what is unclear,
   distill key findings

Then reconvene in the plenum to discuss everyone's findings

The Zoo of Instabilities

Microinstabilities: drift waves driven by pressure gradients
Note: all of them have **critical gradients**

	ITG	ETG	TEM	KBM	MT
drive	∇T_i	∇T_e	$\nabla T_\mathrm{e}, \nabla n$	$\nabla T_\mathrm{i,e} + \nabla n$	∇T_e
ρ_j scale	i	e	i	i	i
ω sign[1]	$+$	$-$	$(-)$	$+$	$-$
$\beta \nearrow$	$\gamma \searrow$	$\gamma \rightarrow$	$\gamma \rightarrow$	$\gamma \nearrow$	$\gamma \nearrow$
Φ vs. $A_\parallel$	$\gg$	$\gg$	$\gg$	$\gg$	$\lesssim$
parity[2]	$+(-)$	$+(-)$	$+(-)$	$+(-)$	$-$
slab branch[3]	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$
zonal flows[4]	$\checkmark$	$(\checkmark)$	$\times$, $\checkmark$	$\times$	$(\checkmark)$

Cause of turbulence & transport in fusion experiments

Each of the above is relevant to all of tokamak, stellarator, RFP

[1] $+(-)$ drifts in ion(electron)-direction; some use opposite nomenclature!
[2] $+(-)$ means even (odd) $\Phi(z)$ and odd (even) $A_\parallel$
[3] slab mode: parallel motion important, $|\theta_p| \gg \pi$
[4] nonlinear saturation mechanism, discussed later

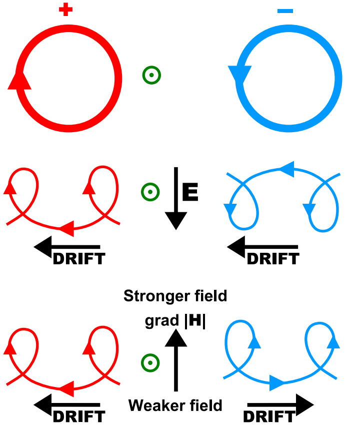
Plasma Drifts

To understand **drift-wave instabilities**, recall drifts: gradients cause perpendicular **drifts at constant** v

- electric field: "E cross B", $\mathbf{v}_E = c(\mathbf{E} \times \mathbf{B})/B^2$
- inhomogeneous guide field: "grad B" & curvature, $\mathbf{v}_{\nabla B} = v_\perp^2 (\mathbf{b} \times \nabla B)/(2B\Omega_j)$ $\mathbf{v}_\mathrm{c} = v_\parallel^2 (\nabla \times \mathbf{b})_\perp / \Omega_j$

Key properties

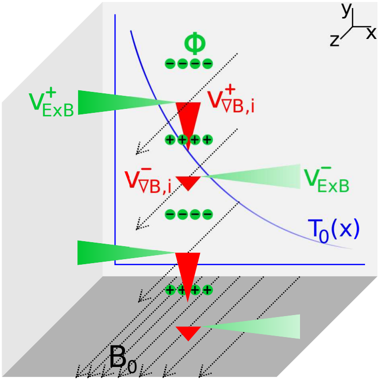
$\mathbf{v}_E$ in same direction for i, e
$\mathbf{v}_{\nabla B,\mathrm{c}}$ opposite for i, e



(adapted from: Wikipedia)

ITG & ETG Modes

ITG: Coppi PoF 1967 (linear), Dimits PoP 2000 (nonlinear)
ETG: Liu PRL 1971 (linear), Jenko PoP 2000 (nonlinear)



Toroidal ITG mode:

$\delta\Phi \to v_E \to \delta T_{\rm i} \to v_{\nabla B} \to \delta\Phi$

- slab: $k_\parallel$ instead of ∇B_0
- ETG linearly isomorphic $\Rightarrow \gamma_{\rm ITG}/v_{\rm ti} = \gamma_{\rm ETG}/v_{\rm te}$
- also called $\eta_{\rm i}$ ($\eta_{\rm e}$) mode: ∇n can stabilize
- nonlinear toroidal ITG: zonal flows (ETG, slab ITG: somewhat less)

Characteristic scales

$k_y\rho_{\rm s,e} \sim 0.1 - 1$, $k_x \sim 0 - k_y$, $\gamma \sim 0.1 - 1 v_{\rm ti,te}/L_{T{\rm i,e}}$, $\omega \sim \pm\gamma$

Trapped-Electron Modes

Linear: Coppi PRL 1974, nonlinear: Ernst PoP 2004
∇T vs. ∇n drive: Ernst PoP 2009

∇n-driven TEM:
$\delta\Phi \rightarrow v_E \rightarrow \delta n \rightarrow v_{\nabla B}^{\mathrm{e,i}} \rightarrow \delta\Phi$

- no slab equivalent
- $\nu_{\mathrm{ei}} \gtrsim \gamma$: "dissipative" DTEM
- $\nu_{\mathrm{ei}} < \gamma$: "collisionless" CTEM
- "ion" iTEM: Plunk JPP 2017
- "ubiquitous" UTEM ($\omega > 0$): Coppi PoFB 1990

Electrons trapped on outboard ($\nabla B_0 \parallel \nabla n, T$)

∇T_{e}-driven TEM works correspondingly

- ∇T TEM and ETG driven by ∇T_{e}, can be indistinguishable
- trapped-ion TIM of limited relevance

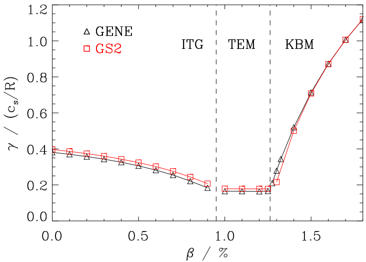
Characteristic scales

$k_y\rho_{\mathrm{s}} \sim 0.2 - 2$, $k_x \sim 0 - k_y$, $\gamma \sim 0.1 - 1 c_{\mathrm{s}}/L_{n,T\mathrm{e}}$, $\omega \sim -\gamma$

Kinetic Ballooning Modes

Linear: Tsai PoFB 1993, Hirose PRL 1994
Nonlinear: Pueschel PoP 2008 & 2010, Ishizawa NF 2013
KBM (also: “Alfvénic” AITG):
kinetic version of MHD ideal ballooning at high $n > 10$

Destabilized at high β



Driven by total gradient
$\nabla p \sim \nabla n + \nabla T_\mathrm{i} + \nabla T_\mathrm{e}$

$\beta_\mathrm{crit}^\mathrm{KBM}(k_y \to 0) \to \beta_\mathrm{crit}^\mathrm{MHD}$ at
$\alpha_\mathrm{MHD} = \beta q_0^2 R_0 (\mathrm{d}p/\mathrm{d}r)/p \approx 0.6\hat{s}$

McKinney JPP 2021: saturation requires $\beta < \beta_\mathrm{crit}^\mathrm{KBM}(k_y^\mathrm{min})$

Note: $\beta/\beta_\mathrm{crit}$ common figure of merit for electromagnetic effects

Characteristic scales

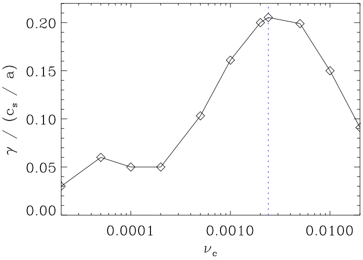
$k_y \rho_\mathrm{s} \sim 0 - 0.5$, $k_x \sim 0$, $\gamma \sim 0.1 - 1 c_\mathrm{s}/L_p$, $\omega \gg \gamma$ near β_crit

Microtearing Modes

Linear: Hazeltine PoF 1975, Drake PoF 1977
Nonlinear: Doerk PRL 2011, Guttenfelder PRL 2011

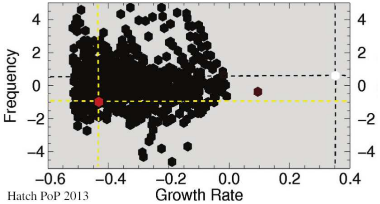
Global tearing: driven by ∇j, while **MT**: driven by β, ∇T_{e}

Energy access via ν_{c} or curvature (collisionless)



Nonlinearly, pure $Q_{\mathrm{e}}^{\mathrm{em}}$ (no $Q_{\mathrm{e,i}}^{\mathrm{es}}$ or particle flux)

Hatch PRL 2012 & PoP 2013: Subdominant MT responsible for $Q_{\mathrm{e}}^{\mathrm{em}}$ in ITG turbulence



$\Rightarrow$ first-ever example of important stable mode

Characteristic scales

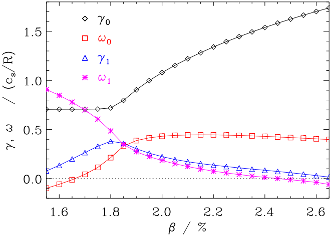
$k_y\rho_{\mathrm{s}} \sim 0.01 - 0.5$ (slab), $0.1 - 1$ (tor'l), $\gamma \sim 0.1 - 1 c_{\mathrm{s}}/L_{T\mathrm{e}}$, $\omega < 0$

Hybrid Modes

Already mentioned: ∇T TEM & ETG can join forces

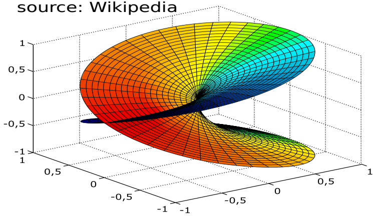
Kammerer PoP 2008, Pueschel PoP 2008:
hybrid modes combining properties of **two instabilities**

E.g., can continuously
transform KBM into TEM



Mathematically, related to
exceptional points

Riemann surface
source: Wikipedia



Walking circle in parameter
space can give different mode

Consequences:

- turbulence regime boundaries can have odd behavior
- **subcritical** linear excitation (e.g., KBM below β_{crit})

Group Work: Characteristic Scales

1 hour group work:

1. for same machines as earlier, calculate characteristic
   a. diamagnetic frequency
   b. parallel transit frequency
   c. wavelengths corresponding to $k_y\rho_\mathrm{s} = 0.3$, $k_y\rho_\mathrm{e} = 0.3$
   d. β and ballooning threshold $\beta_\mathrm{crit}^\mathrm{MHD}$

   in SI or cgs units (can look up diamagnetic frequency in A.J. Brizard, Rev. Mod. Phys. **79**, 421 (2007))

   Who can say what their physical relevance is?

2. where feasible, estimate importance of instabilities for the above machines/radii: ITG, ETG, TEM, KBM

Questions & Discussion

Anything unclear that we talked about?

Any feedback for the instructor?

Anyone still awake?

Turbulence and Transport in Fusion Plasmas
Part V

M.J. Pueschel

Ruhr-Universität Bochum, February 27 – March 10, 2023

Quiz

10-minute self-evaluated quiz:

1. How many eigenmodes (per wavenumber) exist in a fusion plasma?
2. How can we make more/fewer modes unstable?
3. What does a positive (negative) mode frequency mean?
4. Under what conditions is the gyrokinetic framework valid?

Quiz – Answers

1. Technically, there exists an infinite number of modes. In simulations, we capture a number equal to the product of all numerical resolutions.
2. By adjusting the drive: ∇T, ∇n, β, ν_{ei}; *or (bonus trick) by lowering the shear $\hat{s}$, see McKinney JPP 2019.*
3. It means the mode drifts in the ion (electron) direction along y (*recall: some use the opposite nomenclature!*)
4. Nearly closed Larmor orbits $\Rightarrow$ strong magnetization and $L_B \gg \rho$. Alternatively, no fast waves, $\omega \ll \Omega_{\mathrm{c}}$.

Gyrokinetics Derivation Plan

Now, we'll spend some time to sketch how to derive gyrokinetics

This may be the hardest part of the course :-)

1. write out Lagrangian and *one-form* γ
   $\Rightarrow$ contains all info about the dynamics of the system
2. define the gyrocenter coordinate system
3. derive the gyrocenter one-form Γ,
   perturb electric/magnetic fields
4. use Lie transform to average over gyromotion using consistent ordering (Lie: near-identity transformation)
5. plug gyroaveraged one-form into Euler-Lagrange equation
6. obtain gyrokinetic (full-f) Vlasov equation

Gyrokinetics review: Brizard RMP 2007

Gyrokinetics Derivation I

Common approach to get dynamics in many physics areas:

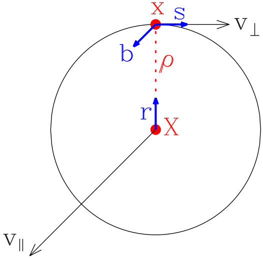
Euler-Lagrange equations $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{X}} = 0 \qquad (1)$$

with the Lagrangian for charged particles in electric & magnetic fields

$$\mathcal{L}(\mathbf{x}, \mathbf{v}) = \left(m\mathbf{v} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right) \cdot \dot{\mathbf{x}} - \left(\frac{1}{2}mv^2 + e\Phi(\mathbf{x})\right) \qquad (2)$$

Define one-form γ as

$$\gamma \equiv \mathcal{L}(\mathbf{x}, \mathbf{v})\mathrm{d}t = \overbrace{\left(m\mathbf{v} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right)}^{\gamma_{\mathbf{x}}} \cdot \mathrm{d}\mathbf{x} - \overbrace{\left(\frac{1}{2}mv^2 + e\Phi(\mathbf{x})\right)}^{-\gamma_t} \mathrm{d}t \qquad (3)$$



Coordinate transformation

$$\mathbf{x} = \mathbf{X} + \rho(\mathbf{X})\mathbf{r}(\theta) \qquad (4)$$

$$\mathbf{v} = v_{\|}\mathbf{b}(\mathbf{x}) + v_{\perp}\mathbf{s}(\theta) \qquad (5)$$

$$v_{\perp} = \left(\frac{2\mu B(\mathbf{x})}{m}\right)^{1/2} \qquad (6)$$

For now, only background field: assume $L_B \gg \rho \Rightarrow \mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{X})$

Gyrokinetics Derivation II

Apply transformation to one-form (Einstein sum convention); *recall*: use transformed one-form Γ later in Euler-Lagrange

$$\Gamma_a = \gamma_b \frac{\mathrm{d}x^b}{\mathrm{d}X^a}\,, \quad a, b \text{ cover space, } v\text{-space, time coordinates} \tag{7}$$

Red terms $\propto \mathbf{r}, \mathbf{s}$ cancel under gyroaverage $\int \mathrm{d}\theta$ (i.e., $\Gamma \to \bar{\Gamma}$)

- time is not transformed, $\Gamma_t = \gamma_t = -\frac{1}{2}mv_\|^2 - \mu B(\mathbf{X}) - e\Phi(\mathbf{X})$
- $\Gamma_{v\|} = 0$ (because $\gamma_\mathbf{v} = 0$)
- $\Gamma_\mu = \left(mv_\|\mathbf{b} + mv_\perp\mathbf{s} + \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{r}\frac{B}{mv_\perp\Omega} = \mathbf{A}(\mathbf{X}) \cdot \mathbf{r}/v_\perp(\mathbf{X})$
- $\Gamma_\theta = \left(mv_\|\mathbf{b} + mv_\perp\mathbf{s} + \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{s}\frac{v_\perp}{\Omega} =$ $mv_\perp^2/\Omega(\mathbf{X}) + \mathbf{A}(\mathbf{X}) \cdot \mathbf{s}ev_\perp/(c\Omega(\mathbf{X}))$
- $\Gamma_\mathbf{X} = \left(mv_\|b_j + mv_\perp s_j + \frac{e}{c}A_j\right)\frac{\mathrm{d}x^j}{\mathrm{d}\mathbf{X}} =$ $\quad j \in \{x, y, z\}$
  $mv_\|\mathbf{b} + mv_\perp\mathbf{s} + (e/c)\mathbf{A}(\mathbf{X}) - [\mu\mathbf{r} \cdot \mathbf{A}(\mathbf{X})/(v_\perp(\mathbf{X})B(\mathbf{X}))]\,\mathrm{d}B(\mathbf{X})/\mathrm{d}\mathbf{X}$

Overall: $$\bar{\Gamma} = \left(mv_\|\mathbf{b} + \frac{e}{c}\mathbf{A}\right) \cdot \mathrm{d}\mathbf{X} + \frac{\mu B}{\Omega}\mathrm{d}\theta - \left(\frac{1}{2}mv_\|^2 + \mu B + e\Phi\right)\mathrm{d}t \tag{8}$$

Gyrokinetics Derivation III

Unfortunately, for **perturbed fields**, $L_{\delta B,\delta\Phi} \sim \rho$ (*note*: $\Phi_0 = 0$)
$\Rightarrow$ use above one-form as equilibrium Γ_0, but must transform

perturbed one-form $$\gamma_1 = \frac{e}{c}\mathbf{A}_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} - e\Phi_1(\mathbf{x})\mathrm{d}t \qquad (9)$$

Without executing gyroaverage, can write perturbed gyrocenter one-form as

$$\begin{aligned}\Gamma_1 \quad = \quad & \frac{e}{c}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathrm{d}\mathbf{X} + \frac{1}{v_\perp}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathbf{r}\mathrm{d}\mu \\ & + \frac{mv_\perp}{B_0}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathbf{s}\mathrm{d}\theta - e\Phi_1(\mathbf{X}+\rho\mathbf{r})\mathrm{d}t \qquad (10)\end{aligned}$$

$\Gamma_1 \to \bar{\Gamma}_1$ is tricky; keep ordering consistent via **Lie transform**

Marius Sophus Lie (1842–1899)

linearized transforms: Lie groups; generators obey Lie algebra

GK derivation possible without Lie, but lots of pitfalls . . .
Alain Brizard et al.: industry for different types of GK equations

Group Work: Canonical Transformations

45 minutes group work

Find sources that explain

- canonical transformations in Hamiltonian mechanics
- phase-space conservation

and (roughly) familiarize yourself with those.
Be prepared to present your findings.

Possible source with perhaps too much info:
`ocw.mit.edu/courses/physics/8-09-classical-mechanics-iii-fall-2014/lecture-notes/MIT8_09F14_Chapter_4.pdf`

Student Lecture

Who feels comfortable giving quick explanations of . . .

What a **Canonical Transformation** is?

What **phase-space conservation** means?

Whether a Lie transform is canonical?

What canonical transforms do in terms of phase-space conservation?

Gyrokinetics Derivation IV

First-order, near-identity (so we can drop higher-order terms) **Lie transform** (Littlejohn JMP 1982)

$$\bar{\Gamma}_{1a} = \Gamma_{1a} - G_1^b \left(\frac{\partial \Gamma_{0a}}{\partial x^b} - \frac{\partial \Gamma_{0b}}{\partial x^a} \right) + \frac{\partial S_1}{\partial x^a} \qquad (11)$$

Generating function G_1^b, gauge function S:

choose such that θ dependencies in one-form vanish

Elegant choice of S: so that $\bar{\Gamma}_{1v_\|} = \bar{\Gamma}_{1\mu} = \bar{\Gamma}_{1\theta} = 0 = G_1^t$

$$\begin{aligned} \text{This implies} \quad m G_1^{\mathbf{X}} \cdot \mathbf{b}_0 &= -\frac{\partial S_1}{\partial v_\|} & (12) \\ \frac{mc}{e} G_1^\mu &= \frac{m v_\perp}{B_0} \mathbf{A}_1 \cdot \mathbf{s} + \frac{\partial S_1}{\partial \theta} & (13) \\ \frac{mc}{e} G_1^\theta &= -\frac{1}{v_\perp} \mathbf{A}_1 \cdot \mathbf{r} - \frac{\partial S_1}{\partial \mu} & (14) \end{aligned}$$

Here and hereafter: $\mathbf{A}_1 = \mathbf{A}_1(\mathbf{X} + \rho \mathbf{r})$ (same for Φ_1)

$$\bar{\Gamma}_{1\mathbf{X}} = \frac{e}{c} \left(\mathbf{A}_1 + G_1^{\mathbf{X}} \cdot \mathbf{B}_0^* \right) - m G_1^{v_\|} \mathbf{b}_0 + \nabla S_1 \qquad (15)$$

$$\text{where} \quad \mathbf{B}_0^* = \nabla \times \left(\mathbf{A}_0 + \frac{mc}{e} v_\| \mathbf{b}_0 \right) \qquad (16)$$

Gyrokinetics Derivation V

Performing gyroaverage $\langle \ldots \rangle$ leaves a "fluctuating" field: $(\mathbf{A}_1, \Phi) = \langle (\mathbf{A}_1, \Phi) \rangle + (\tilde{\mathbf{A}}_1, \tilde{\Phi})$
$\Rightarrow$ do not throw away, *instead make cancel through Lie choices!*

Demanding $\bar{\Gamma}_{1\mathbf{X}} = (e/c)\langle \mathbf{A}_1 \rangle$ and taking $\mathbf{B}_0^* \cdot \bar{\Gamma}_{1\mathbf{X}}$ and $\mathbf{b}_0 \times \bar{\Gamma}_{1\mathbf{X}}$:

$$G_1^{v\|} = \frac{1}{mB_{0\|}^*}\left(\mathbf{B}_0^* \cdot \tilde{\mathbf{A}}_1 + \mathbf{B}_0^* \cdot \nabla S_1\right) \tag{17}$$

$$G_1^{\mathbf{X}} = -\frac{1}{B_{0\|}^*}\left(\mathbf{b}_0 \times \tilde{\mathbf{A}}_1 + \frac{\mathbf{B}_0^*}{m}\frac{\partial S_1}{\partial v_\|} + \frac{c}{e}\mathbf{b}_0 \times \nabla S_1\right) \tag{18}$$

Almost there, but $\bar{\Gamma}_{1t}$ looks messy; however, can drop higher-order terms and absorb fluctuating terms into $\partial S_1/\partial\theta$:

$$\bar{\Gamma}_{1t} = -e\langle \Phi_1 \rangle - e\tilde{\Phi}_1 - \frac{1}{B_{0\|}^*}\left(\frac{\mathbf{B}_0^*}{m}\frac{\partial S_1}{\partial v_\|} + \mathbf{b}_0 \times \left(\tilde{\mathbf{A}}_1 + \frac{c}{e}\nabla S_1\right)\right) \cdot \nabla(\mu B_0)$$
$$+ \frac{\mathbf{B}_0}{B_{0\|}^*}\left(\frac{e}{c}\tilde{\mathbf{A}}_1 + \nabla S_1\right) v_\| + \frac{e}{c}\left(\frac{v_\perp}{B_0}(\langle \mathbf{A}_1 \cdot \mathbf{s} \rangle + \widetilde{\mathbf{A}_1 \cdot \mathbf{s}}) + \frac{1}{m}\frac{\partial S_1}{\partial \theta}\right)\mathbf{B}_0 + \partial_t S_1 \tag{19}$$

Now, *how to evaluate the gyroaverages here and in Eq. (15)?*

Group Work: Gyroaverage

45 minutes group work

Consider a field A and at the particle position: $A(\mathbf{X} + \rho\mathbf{r})$.

1. write down the gyroaveraged $\bar{A}(\mathbf{X}) \leftrightarrow \bar{A}(k_\perp)$ using $\int \mathrm{d}\theta$ (*note*: $\mathbf{k}_\perp \cdot \mathbf{r} = k_\perp \cos(\theta - \theta_0)$)
2. reduce this expression for $\bar{A}$ as much as possible, (*hint: you can split the average into two half-orbits*) using the definition

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi \mathrm{e}^{iz\cos\tau} \cos(n\tau)\,\mathrm{d}\tau$$

   for the Bessel function of order $n \in \mathbb{N}_0$ and argument $z \in \mathbb{C}$
3. take limit $k_\perp \rho \to 0$: what does the gyroaverage do there?

Gyrokinetics Derivation VI

So now we have $\langle[\mathbf{A}_1, \Phi_1](\mathbf{X}+\rho\mathbf{r})\rangle = J_0(k_\perp\rho)[\mathbf{A}_1, \Phi_1](\mathbf{X})$
More complicated, but can write $-\frac{e}{c}\langle\mathbf{A}_1\cdot\mathbf{s}\rangle = \underbrace{\frac{J_1(k_\perp\rho)}{k_\perp\rho}}_{\bar{J}_1}\mu B_{1\|}(\mathbf{X})$

Thus, the full gyroaveraged one-form reads

$$\begin{aligned}\bar{\Gamma} = \bar{\Gamma}_0 + \bar{\Gamma}_1 \; = \; & \left(mv_\|\mathbf{b}_0 + \frac{e}{c}\mathbf{A}_0 + \frac{e}{c}J_0A_{1\|}\mathbf{b}_0\right)\cdot \mathrm{d}\mathbf{X} + \frac{\mu B_0}{\Omega}\mathrm{d}\theta \\ & - \left(\frac{1}{2}mv_\|^2 + \mu B_0 + eJ_0\Phi_1 + \mu\bar{J}_1B_{1\|}\right)\mathrm{d}t \qquad (20)\end{aligned}$$

Plug into Euler-Lagrange $\rightarrow$ **gyrokinetic equations of motion**

$$\begin{aligned}\dot{\mu} = 0 \qquad \dot{\mathbf{X}} = & \; v_\|\mathbf{b}_0 + \frac{B_0}{B_{0\|}^*}\Bigg(v_\|\frac{J_0\mathbf{B}_{1\perp}}{B_0} + c\frac{J_0\mathbf{E}_1\times\mathbf{B}_0}{B_0^2} \\ & + \frac{\mu}{m\Omega}\mathbf{b}_0\times\nabla(B_0+\bar{J}_1B_{1\|}) + \frac{v_\|^2}{\Omega}(\nabla\times\mathbf{b}_0)_\perp\Bigg) \qquad (21)\end{aligned}$$

$$v_\| = \mathbf{b}_0\cdot\dot{\mathbf{X}} \qquad \dot{v}_\| = \frac{1}{mv_\|}\dot{\mathbf{X}}\cdot\left(eJ_0\mathbf{E}_1 - \mu\nabla(B_0+\bar{J}_1B_{1\|})\right) \quad (22)$$

$$\dot{\theta} = \Omega - \frac{e}{mc}\frac{\partial}{\partial\mu}\left(\frac{e}{c}v_\|J_0A_{1\|} - eJ_0\Phi_1 - \mu\bar{J}_1B_{1\|}\right) \qquad (23)$$

Gyrokinetic Vlasov Equation

Final goal: **Vlasov equation** (here: full-f)

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \dot{\mathbf{X}} \cdot \nabla f + \dot{v}_\parallel \frac{\partial f}{\partial v_\parallel} + \dot{\mu} \frac{\partial f}{\partial \mu} \\
&= \frac{\partial f}{\partial t} + \left(v_\parallel \mathbf{b}_0 + \frac{B_0}{B^*_{0\parallel}} \left(\mathbf{v}_E + \mathbf{v}_{\nabla B} + \mathbf{v}_\mathrm{c} \right) \right) \\
&\quad \cdot \left(\nabla f + \frac{1}{m v_\parallel} \left(e J_0 \mathbf{E}_1 - \mu \nabla \left(B_0 + \bar{J}_1 B_{1\parallel} \right) \right) \frac{\partial f}{\partial v_\parallel} \right) = 0
\end{aligned}
\tag{24}
$$

- parallel streaming, drifts, gradient drive, trapping
- useful to split $f \to F_0 + f_1$ for clarity, efficiency
- can add collision term on right-hand-side
- magnetic geometry “hidden away” in $\mathbf{B}_0$, drifts
- complemented by **field equations** (Maxwell)

Recall: Gyrokinetic Equations

Pueschel PoP 2011: δf equations for GENE code (`www.genecode.org`; normalization!)

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\rm ref}}{JB_0}v_\|\Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial\chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial\chi}{\partial x}\Gamma_{jy} - \frac{\partial\chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00} - \mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01} - \mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0\int J_0 f_j \mathrm{d}v_\|\mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2}J_1 f_j \mathrm{d}v_\|\mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j\frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0) \quad \mathcal{C}_2 = -\sum_j\frac{q_j n_{j0}}{B_0}(\Gamma_0-\Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j\frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0-\Gamma_1)$$

$$A_\| = \left(\sum_j\frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j \mathrm{d}v_\|\mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j\frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0}\mathrm{d}v_\|\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|}F_{j0}\partial_k\chi_j q_j/(m_j v_\|) + \bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_j c)$$

Questions & Discussion

Anything unclear so far?

Anything clear so far?

Moments Preparation

Coming up: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2.
3.
4.

Reducing Gyrokinetics I

Terms vanishing due to $\beta = 0$, drop nonlinearity

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left[\left(\frac{\partial B_0}{\partial x} - \kappa_2 \frac{\partial B_0}{\partial z} \right) \Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \Gamma_{jx} \right] + \frac{B_{\mathrm{ref}}}{J B_0} v_\| \Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2 T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2} \right) L_{Tj}^{-1} \right] \left[c \frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \right]$$

$$+ \frac{1}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \frac{4\pi v_\|^2}{B_0 \Omega_j} \frac{\partial p_{j0}}{\partial x} \Gamma_{jy} - \frac{B_{\mathrm{ref}}}{J B_0} \frac{\mu}{m_j} \frac{\partial B_0}{\partial z} \frac{\partial f_j}{\partial v_\|} + \frac{c F_{j0}}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left(\frac{\partial \chi}{\partial x} \Gamma_{jy} - \frac{\partial \chi}{\partial y} \Gamma_{jx} \right) = \left. \frac{\partial f_j}{\partial t} \right|_{\mathrm{coll}}$$

$$g_j = f_j - \frac{q_j}{m_j c} \bar{A}_\| \frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c} \bar{A}_\| + \frac{\mu}{q_j} \bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3 \mathcal{M}_{00} - \mathcal{C}_2 \mathcal{M}_{01}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1 \mathcal{M}_{01} - \mathcal{C}_2 \mathcal{M}_{00}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2 q_j}{m_j} \pi B_0 \int J_0 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j \pi (2 B_0 / m_j)^{3/2}}{c k_\perp} \int \mu^{1/2} J_1 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}} (1 - \Gamma_0) \quad \mathcal{C}_2 = -\sum_j \frac{q_j n_{j0}}{B_0} (\Gamma_0 - \Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2} (\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8 \pi^2 q_j B_0}{m_j c} \int v_\| J_0 g_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \right) \left(k_\perp^2 + \sum_j \frac{8 \pi^2 q_j^2 B_0}{m_j c^2 T_{j0}} \int v_\|^2 J_0^2 F_{j0} \, \mathrm{d}v_\| \, \mathrm{d}\mu \right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0} \partial_k \chi_j q_j / (m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j / (m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_\mathrm{c} = 0 \Rightarrow$ **no collisions**
3.
4.

Reducing Gyrokinetics II

Terms vanishing due to $\nu_{\rm c} = 0$

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}} \frac{B_0}{B^*_{0\|}} \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left[\left(\frac{\partial B_0}{\partial x} - \kappa_2 \frac{\partial B_0}{\partial z} \right) \Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \Gamma_{jx} \right] + \frac{B_{\rm ref}}{J B_0} v_\| \Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\rm ref}} \frac{B_0}{B^*_{0\|}} \left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2} \right) L_{Tj}^{-1} \right] \left[c \frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \right]$$

$$+ \frac{1}{B_{\rm ref}} \frac{B_0}{B^*_{0\|}} \frac{4\pi v_\|^2}{B_0 \Omega_j} \frac{\partial p_{j0}}{\partial x} \Gamma_{jy} - \frac{B_{\rm ref}}{J B_0} \frac{\mu}{m_j} \frac{\partial B_0}{\partial z} \frac{\partial f_j}{\partial v_\|} + \frac{c F_{j0}}{B_{\rm ref}} \frac{B_0}{B^*_{0\|}} \left(\frac{\partial \chi}{\partial x} \Gamma_{jy} - \frac{\partial \chi}{\partial y} \Gamma_{jx} \right) = \left. \frac{\partial f_j}{\partial t} \right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c} \bar{A}_\| \frac{\partial F_{j0}}{\partial v_\|} \qquad \chi = \bar{\Phi} - \frac{v_\|}{c} \bar{A}_\| + \frac{\mu}{q_j} \bar{B}_\| \qquad \Phi = \frac{\mathcal{C}_3 \mathcal{M}_{00} - \mathcal{C}_2 \mathcal{M}_{01}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2} \qquad B_\| = \frac{\mathcal{C}_1 \mathcal{M}_{01} - \mathcal{C}_2 \mathcal{M}_{00}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j} \pi B_0 \int J_0 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j \pi (2B_0/m_j)^{3/2}}{c k_\perp} \int \mu^{1/2} J_1 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}} (1 - \Gamma_0) \qquad \mathcal{C}_2 = - \sum_j \frac{q_j n_{j0}}{B_0} (\Gamma_0 - \Gamma_1) \qquad \mathcal{C}_3 = - \frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2} (\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c} \int v_\| J_0 g_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \right) \left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}} \int v_\|^2 J_0^2 F_{j0} \, \mathrm{d}v_\| \, \mathrm{d}\mu \right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0} \partial_k \chi_j q_j / (m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j / (m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_\mathrm{c} = 0 \Rightarrow$ **no collisions**
3. assume $\partial_k B_0 = 0 \Rightarrow$ **homogeneous magnetic field**
4.

Reducing Gyrokinetics III

Terms vanishing due to $\partial_k B_0 = 0$

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\rm ref}}{JB_0}v_\|\Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial \chi}{\partial x}\Gamma_{jy} - \frac{\partial \chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|} \qquad \chi = \bar{\Phi} - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \qquad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00} - \mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2} \qquad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01} - \mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0 \int J_0 f_j \mathrm{d}v_\| \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2} J_1 f_j \mathrm{d}v_\| \mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0) \quad \mathcal{C}_2 = -\sum_j \frac{q_j n_{j0}}{B_0}(\Gamma_0 - \Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j \mathrm{d}v_\| \mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0} \mathrm{d}v_\| \mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0}\partial_k \chi_j q_j/(m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j/(m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_\mathrm{c} = 0 \Rightarrow$ **no collisions**
3. assume $\partial_k B_0 = 0 \Rightarrow$ **homogeneous magnetic field**
4. assume $k_\perp \rho_j \ll 1 \Rightarrow$ **drift-kinetic limit**

Reducing Gyrokinetics IV

$J_0(k_\perp \rho_j) \to 1$ due to $k_\perp \rho_j \ll 1$, terms altered ($\lambda_D \to 0$ and $\Gamma_0 \approx 1 - v_{Tj}^2 k_\perp^2/(2\Omega_j)$)

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2 \frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\rm ref}}{JB_0} v_\| \Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0 \Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial \chi}{\partial x}\Gamma_{jy} - \frac{\partial \chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\| \frac{\partial F_{j0}}{\partial v_\|} \qquad \chi = J_0 \Phi - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \qquad \Phi = \frac{\mathcal{C}_3 \mathcal{M}_{00} - \mathcal{C}_2 \mathcal{M}_{01}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2} \qquad B_\| = \frac{\mathcal{C}_1 \mathcal{M}_{01} - \mathcal{C}_2 \mathcal{M}_{00}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0 \int J_0 f_j \mathrm{d}v_\| \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j \pi (2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2} J_1 f_j \mathrm{d}v_\| \mathrm{d}\mu$$

$$\mathcal{C}_1 = \underbrace{\frac{k_\perp^2}{4\pi}}_{\lambda_D = 0} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}} \underbrace{(1 - \Gamma_0)}_{v_{Tj}^2 k_\perp^2/(2\Omega_j)} \qquad \mathcal{C}_2 = -\sum_j \frac{q_j n_{j0}}{B_0}(\Gamma_0 - \Gamma_1) \qquad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j \mathrm{d}v_\| \mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0} \mathrm{d}v_\| \mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0} \partial_k \chi_j q_j/(m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j/(m_j c)$$

Normalization

Reduced δf drift-kinetic equations:

$$\frac{\partial f_j}{\partial t} = -v_\parallel \frac{\partial f_j}{\partial z} - \frac{q_j m_j}{v_\parallel}\frac{\partial F_{j0}}{\partial v_\parallel}\frac{\partial \Phi}{\partial z} - \frac{cF_{j0}}{B_\mathrm{ref}}\left[L_n^{-1} + \left(\frac{m_j v_\parallel^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right] ik_y\Phi \quad (25)$$

$$\Phi = \frac{\sum_j \frac{2q_j}{m_j}\pi B_0 \int f_j \mathrm{d}v_\parallel \mathrm{d}\mu}{\sum_j \frac{q_j^2 n_{j0}}{T_{j0}}\frac{v_{Tj}^2 k_\perp^2}{2\Omega_j}}\,;\; k_y > 0,\ \text{adiabatic } \mathrm{e}^- \rightarrow \frac{\frac{2q_\mathrm{i}}{m_\mathrm{i}}\pi B_0 \int f_\mathrm{i} \mathrm{d}v_\parallel \mathrm{d}\mu}{\frac{q_\mathrm{e}^2 n_\mathrm{e0}}{T_\mathrm{e0}} + \frac{q_\mathrm{i}^2 n_\mathrm{i0}}{T_\mathrm{i0}}\frac{v_{T\mathrm{i}}^2 k_\perp^2}{2\Omega_\mathrm{i}}} \quad (26)$$

Suitable normalization: $x, y \rightarrow \rho$, $z \rightarrow L_z \gg \rho$, $v \rightarrow v_\mathrm{th}$, $t \rightarrow L_z/v_\mathrm{th}$, $\omega_{T\mathrm{i}} = L_z/L_{T\mathrm{i}} = -L_z \mathrm{d}\ln T_\mathrm{i0}/\mathrm{d}x$, $\omega_n = L_z/L_n = -L_z \mathrm{d}\ln n_0/\mathrm{d}x$

Normalize to mass, temperature, density of *singly-charged ions*

$$\frac{\partial f}{\partial t} = -v_\parallel \frac{\partial f}{\partial z} - v_\parallel F_0 \frac{\partial \Phi}{\partial z} - \left[\omega_n + \left(v_\parallel^2 + \mu - \frac{3}{2}\right)\omega_{T\mathrm{i}}\right] F_0 ik_y\Phi\,, \quad F_0 = \frac{1}{\pi^{3/2}}\mathrm{e}^{-v_\parallel^2 - \mu} \quad (27)$$

$$\Phi = \pi \int f \mathrm{d}v_\parallel \mathrm{d}\mu \qquad [\text{normalization: } \Phi \rightarrow (T_\mathrm{i0}/e)\rho/L_z\,;\; f/F_0 \sim \rho/L_z] \quad (28)$$

Note: no coupling between μ points

Group Work: Taking Moments

45 minutes group work:

1. perform coordinate transformation $(v_x, v_y) \to \mu$ for $\int \mathrm{d}\mathbf{v}$

Start with Eqs. (27) and (28), then

2. integrate out μ, redefining $\pi \int_0^\infty f(v_\|, \mu)\mathrm{d}\mu \to f(v_\|)$

3.

4.

5.

Group Work: Taking Moments

45 minutes group work:

1. perform coordinate transformation $(v_x, v_y) \to \mu$ for $\int \mathrm{d}\mathbf{v}$

Start with Eqs. (27) and (28), then

2. integrate out μ, redefining $\pi \int_0^\infty f(v_\|, \mu)\mathrm{d}\mu \to f(v_\|)$

Answer:

$$\frac{\partial f}{\partial t} = -v_\| \frac{\partial f}{\partial z} - v_\| F_0 \frac{\partial \Phi}{\partial z} - \left[\omega_n + \left(v_\|^2 - \frac{1}{2}\right) \omega_{T\mathrm{i}}\right] F_0 i k_y \Phi$$

$$\Phi = \int_{-\infty}^{\infty} f \mathrm{d}v_\| \qquad F_0 = \frac{1}{\pi^{1/2}} \mathrm{e}^{-v_\|^2}$$

3. take moments of the reduced drift-kinetic equations, get (linear) fluid model with n, $u_\|$, Φ
4. is the model closed? — if not, how might one do that?
5. how does the model compare to the earlier two-field model?

Questions & Discussion

Anything unclear that we talked about here?

Interpreting Eigenmodes

Now: how to interpret, produce linear data

Linear equations: **no coupling between different** k_y
$\Rightarrow$ *all real-space information contained in ballooning function*

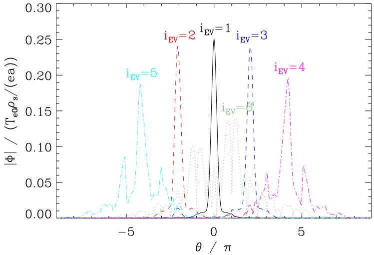
Available data

All information for a mode:

- eigenvalue (γ, ω)
- eigenvector $f(k_x, z, v_\parallel, \mu)$
  $\rightarrow \Phi(k_x, z) \rightarrow \Phi(\theta_p)$

but can have many modes i_{EV}

Pueschel PRL 2016: low-$\hat{s}$ stellarator



From $\Phi(t)$ (or $f(t)$), **dominant eigenvalue** easy to extract

- growth rate γ: fit straight line to log plot, or $\dfrac{\ln \frac{\Phi(t+\Delta t)}{\Phi(t)}}{\Delta t}$
- frequency ω: phase difference between $\Phi(t)$, $\Phi(t+\Delta t)$

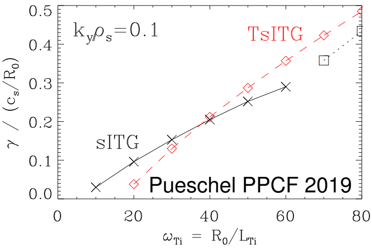
Modes' **ballooning structure**: depends on k_y, geometry, $\nabla n, T$

Excitation States I

Early theory (e.g., Coppi PoF 1967): infinite number of solutions

$$\left.\begin{array}{c}\dfrac{\gamma}{c_{\rm s}/R_0}\\ \dfrac{\omega}{c_{\rm s}/R_0}\end{array}\right\} = \frac{\ell}{2} k_y \omega_{T\rm i}^{1/2} \hat{s}^{1/2} \sqrt{\frac{T_{\rm i}}{T_{\rm e}}} \qquad \Phi_\ell(\vartheta) = {\rm e}^{-\frac{\vartheta^2}{2}} \underbrace{(-1)^\ell {\rm e}^{\vartheta^2} \frac{{\rm d}^\ell}{{\rm d}\vartheta^\ell} {\rm e}^{-\vartheta^2}}_{\text{Hermite polynomial}}$$

Implying $\gamma(\ell \to \infty) \to \infty$*! How is this possible?*



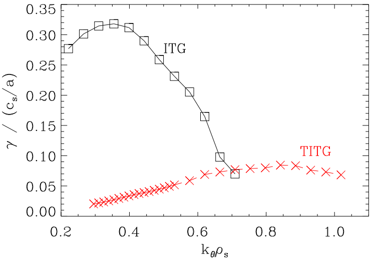
In reality:

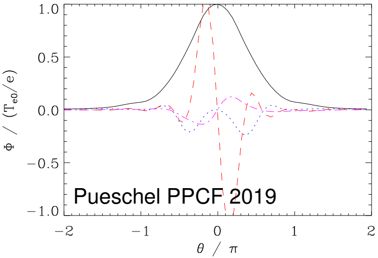
ℓ **stabilizes** at typical gradients, need **large gradients** to have higher ℓ dominate

Here: $\ell_{\rm sITG} = 0$, $\ell_{\rm TsITG} = 1$

Excitation States II

Realistic example with strong gradients, higher-ℓ states:
NSTX #129016 (lithium coating study; Guttenfelder NF 2013)





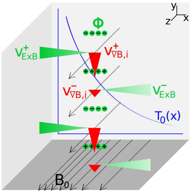
ITG: $\mathrm{Re} + \mathrm{Im}$ — TITG: $\mathrm{Re} + \mathrm{Im}$

$\Phi(t) \propto \exp(i\omega t)$, thus $\mathrm{Re}\,\Phi$ and $\mathrm{Im}\,\Phi$ change continuously

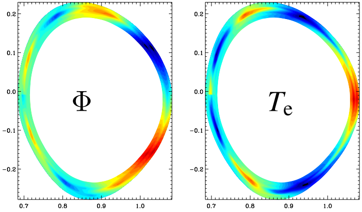
- different ℓ coexist, not exact same properties
- even (odd) Hermites: even (odd) Φ (and odd (even) $A_{||}$)
- $\Rightarrow$ even/odd **parity not sufficient for mode ID**

Phases

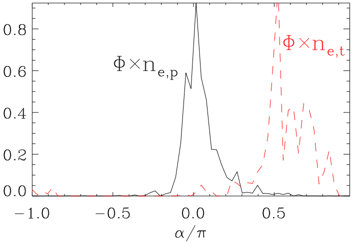
Complex phase of eigenfunction $\leftrightarrow$ position along y



Recall: $\pi/2$ shift between Φ and $\{n, T\}$ ideal for growth



Can be expressed as **complex phase angle**



Above case:

- passing e$^-$: $\alpha_{\Phi\times n} = 0$
- trapped e$^-$: $\alpha_{\Phi\times n} = \pi/2$

$\Rightarrow \nabla n$-TEM

Relation to Turbulence

Nonlinear physics discussed a little later, but for the moment:

- **fluxes often scale like** γ
- on large scales, (non-)linear **phases match** in most cases
- **eigenmode widths correlate**
  with turbulent ballooning widths
- fluxes tend to peak at lower k_y than does γ
  (explanation will be given later)

Transport researcher: “nonlinear often well-captured by linear”
Turbulence researcher: “nonlinear often deviates from linear”

Group Work: IV vs. EV

30 minutes group work:

Digest sources that explain the difference between the

- initial-value and the
- eigenvalue

approach to linear simulations. Be prepared to present your findings.

Which one can be used, and how, to obtain multiple eigenvalues/-vectors at a single wavenumber?

Student Lecture

Anyone, who can explain . . .

Why do different eigenvalues exist at each k_y?

How does **matrix inversion** differ from **iterative solving**?

What is an **eigenvalue simulation**?

Bonus: how to get **subdominant modes from IV** runs?

Demonstration

Let's have a look at all that in action!

With the GENE Diagnostics Tool, we can

1. extract eigenvalues
2. look at evolving and converged mode structures; IV vs. EV
3. evaluate phases
4. compare moments, compute quasilinear ratios (e.g., Q/n^2)
5. identify different instabilities

See `gene-diag.pdf` for a summary

Turbulence and Transport in Fusion Plasmas
Part VI

M.J. Pueschel

Ruhr-Universität Bochum, February 27 – March 10, 2023

Demonstration

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2. look at evolving and converged mode structures; IV vs. EV
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4. compare moments, compute quasilinear ratios (e.g., Q/n^2)
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See `gene-diag.pdf` for a summary

Questions & Discussion

Who has any questions and is not afraid to ask?

Group Work: Finite Differencing

1 hour group work:

Either find your own sources or look at
Pueschel CPC 2010 to learn

- how finite differencing works
- how the stencil notation works and what the stencil order is

Be prepared to present your findings.

Bonus for those who are not dissuaded by the math: learn

- *how to derive your own stencils*

¿Voluntarios?

Explain what **finite differencing** means!

What is a **stencil**?

What can be done at the **domain boundary**?

Are there **alternatives** to finite differencing?

Group Work: Hyperdiffusion

45 minutes group work

With the Fourier transform of $\Phi(x \pm \Delta x)$ being $\Phi(k)\exp(\pm ik\Delta x)$,

1. evaluate the Fourier transform of $\partial_x \Phi$ with stencil $[-1\ 0\ 1]$
2. show that it produces a modified $ik\Phi \to i\tilde{k}\Phi$
3. evaluate $\tilde{k}$ in the limits of $k \to 0$ and $k \to \pi/\Delta x$

4.

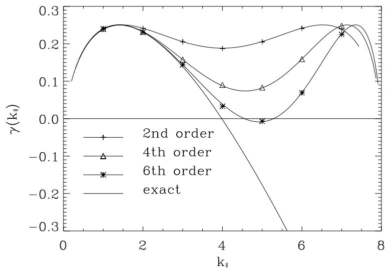
High-k Excitation

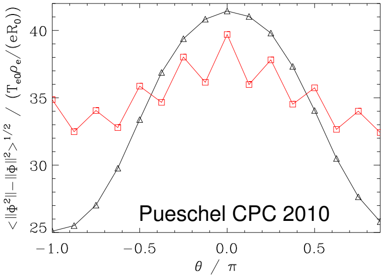
Solution:

$$k \to 0 : \quad \tilde{k} = \frac{\sin k\Delta x}{\Delta x} \to k \quad \checkmark$$

$$k \to \frac{\pi}{\Delta x} : \quad \tilde{k} = \frac{\sin k\Delta x}{\Delta x} \to 0 \quad \times$$

As $k \to \pi/\Delta x$, this stencil **reproduces the physics of $k \to 0$!**
$\Rightarrow$ **high-k modes** (physically stable) **take over** simulations





What can we do about that?

Group Work: Hyperdiffusion

45 minutes group work

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1. evaluate the Fourier transform of $\partial_x \Phi$ with stencil $[-1\ 0\ 1]$
2. show that it produces a modified $ik\Phi \to i\tilde{k}\Phi$
3. evaluate $\tilde{k}$ in the limits of $k \to 0$ and $k \to \pi/\Delta x$

4. show that adding a term $\propto k^n$ with even n to the Vlasov equation can solve this problem

Group Work: Time Stepping

45 minutes group work:

In our code, we will use an **explicit Runge-Kutta** method to evolve the distribution function in time.

Note: already in the code, but good to understand!

1. Figure out how that works (e.g., Wikipedia)
2. Find sources (lecture notes, online, ...) that show and explain the stability regions of explicit Runge-Kutta schemes in the eigenvalue plane
3. Rejoin the plenum and discuss your findings

What happens when we choose a too-small Δt?

Questions & Discussion

Any questions before we write our own simulation code?

Code Prep

Perhaps 30–60 minutes, but really as-long-as-it-takes effort:

Set up your system to be able compile and run code written in Fortran 90, e.g., by having the open-source compiler `gfortran` installed.

Familiarize yourself with basic Fortran 90 syntax, e.g. via `pages.mtu.edu/~shene/COURSES/cs201/NOTES/fortran.html` *(be sure to use the correct tilde character when pasting into browser)*.

If you are stuck, get help from one of the others in your group of from the lecturer!

Group Work: Vlasov Code

2 hours group work

1. take `dkVlasov.F90` and search for missing code marked "to be implemented"
2. implement our linear, μ-integrated drift-kinetic equations

$$\frac{\partial f}{\partial t} = -v_\parallel \frac{\partial f}{\partial z} - v_\parallel F_0 \frac{\partial \Phi}{\partial z} - \left[\omega_n + \left(v_\parallel^2 - \frac{1}{2}\right)\omega_{T\mathrm{i}}\right] F_0 i k_y \Phi + \eta \frac{\partial^2 f}{\partial z^2}$$

$$\Phi = \int f \mathrm{d}v_\parallel \quad F_0 = \frac{1}{\pi^{1/2}} \mathrm{e}^{-v_\parallel^2} \quad \frac{\partial}{\partial z} \rightarrow [-1\ 0\ 1] \quad \frac{\partial^2}{\partial z^2} \rightarrow [1\ -2\ 1]$$

3. come up with an initial condition $F(t=0)$, either for a single or a spectrum of $k_\parallel$
4. run the code and perform **convergence** checks: **local independence of numerical parameters**
5. reproduce the $k_\parallel$ dependence in Pueschel CPC 2010 for different diffusion coefficients

Questions & Discussion

Anything that we covered in need of clarification?

Turbulence and Transport in Fusion Plasmas
Part VII

DIFFER

M.J. Pueschel

RUHR UNIVERSITÄT BOCHUM RUB

Ruhr-Universität Bochum, February 27 – March 10, 2023

Monday Recap

Yesterday, we covered

- the ins and outs of numerical schemes:
  - finite differencing
  - numerical (hyper-)diffusion
  - explicit time stepping
- how to implement equations in and then deploy a simulation code

Any questions, especially about the code implementation?

Quiz

10-minute self-evaluated quiz:

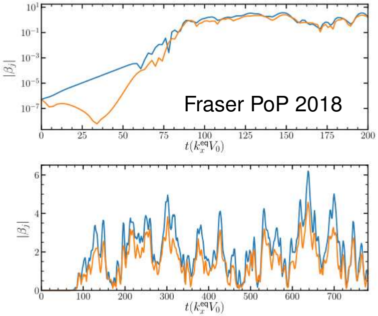
1. In the gyrokinetics derivation, what is $J_0(k_\perp \rho)$, and what does it do?
2. What is the excitation state ℓ of a mode?
3. When does a centered finite-differencing stencil fail?
4. What is the difference between diffusion and hyperdiffusion, which one do you prefer, and why?

Quiz – Answers

1. J_0 is the Bessel function, capturing how a particle with Larmor radius ρ "feels" a field (e.g., Φ) with correlation length $\sim k_\perp^{-1}$.
2. The mode's quantum state ℓ is related to how wiggly the mode is (*Hermite number*); even ℓ are ballooning parity, odd are tearing parity.
3. Centered stencils fail on scales not properly resolved by the grid spacing, $k \sim \Delta x^{-1}$.
4. Diffusion $\propto k^2$, hyperdiffusion $\propto k^{4,6,\cdots}$. Larger exponents can damp high k better but require larger stencils and more boundary points. k^4 is a common compromise.

Instability Saturation

Saturation: amplitudes stop growing



Two mechanisms:

- energy source depleted ("*quasilinear flattening*")
- nonlinearity balances linear terms ("*turbulent saturation*") $\Rightarrow$ heating, fueling maintains $\nabla T, n$

quasi-stationary: $\dfrac{\partial f}{\partial t} \approx 0 \;\Rightarrow\; \mathcal{L} = -\sum_{k'} \left(k'_x k_y - k'_y k_x\right) \Phi(k') f(k-k')$

$\Rightarrow$ **triplet interactions** shuffle energy from unstable to stable (*recall: Kolmogorov cascade vs. large-scale stable modes*)

Secondary Instability

fixed $\Phi(0, k'_y)$: $f(k_x, 0)$ and $f(k-k')$ grow exponentially $\Rightarrow \gamma_{\mathrm{sec}}$

Group Work: Zonal Flows

45 minutes group work:

Find sources that explain

- what zonal flows are and in what physical systems they matter
- whether – and if yes, when – zonal flows are important in fusion plasmas

and familiarize yourself with those.
Be prepared to present your findings.

Student Lecture: Zonal Flows

Please tell us:

What are zonal flows?

How do they affect turbulence?

Aside from fusion plasmas, where do they exist?

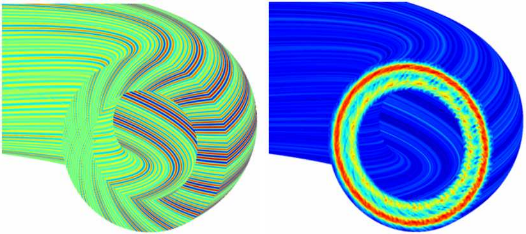
When do they and when do they not have an impact?

Perhaps some of you had run into Secondary Instability when researching zonal flows . . . who wants to tell us about that?

Higher-Order Instability

1. background gradients $\rightarrow$ linear instability
   zonal structure $\rightarrow$ radial structure
2. linear mode $\rightarrow$ secondary instability
   radial structure $\rightarrow$ zonal structure
3. secondary mode $\rightarrow$ tertiary instability
   zonal structure $\rightarrow$ radial structure

$\vdots \qquad \vdots$

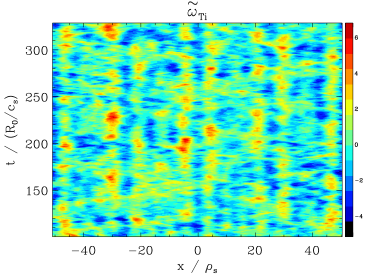


Tertiary Instability

Rogers PRL 2000, Waltz PoP 2010, Pueschel PoP 2013:
Tertiary can be **new mode** or **modified primary**

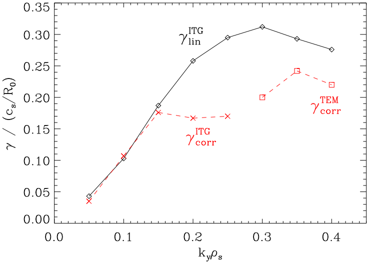
Here: mixed ITG-TEM scenario (Pueschel PoP 2013)

Profile corrugations:



$\Rightarrow$ locally/temporally enhanced gradients, may **influence linear modes**

Tertiaries (full g corrugations):



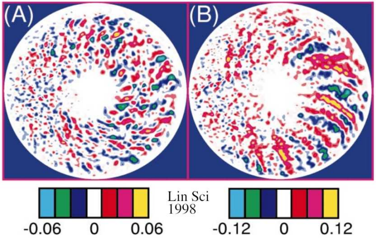
$\Rightarrow$ **little impact** near NL transport peak, zonal flows stabilize at higher k_y

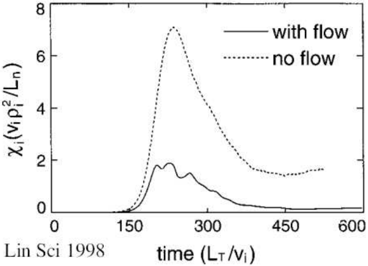
Note: zonal flows are effectively Φ corrugations

ITG Saturation

Toroidal ITG: strongly relies on **zonal flows** for saturation

Lin Sci 1998: first global nonlinear ITG simulations





- for two decades, *shearing paradigm*: ZFs shearing eddies, providing cascade to small scales
- now: **large-scale stable modes** responsible for saturation
- *note*: kinetic (vs. adiabatic) electrons affect zonal-flow dynamics $\Rightarrow \gamma_{\mathrm{adi}} \sim \gamma_{\mathrm{kin}}$ but $Q_{\mathrm{adi}} \ll Q_{\mathrm{kin}}$

Slab ITG: zonal flows less prominent but still matter

Questions & Discussion

Let us take a break and ponder all the things we learned so far:

- take 30 minutes to go over the lecture notes
- note down things that are conceptually opaque
- together, we will try to clear up anything that requires it

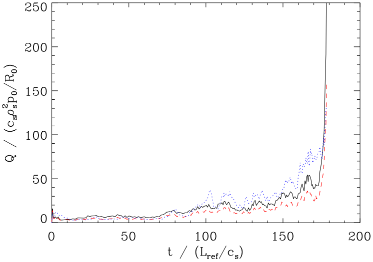
Hereafter, we will go into some current research into nonlinear aspects of fusion plasma turbulence . . .

The High-β Runaway

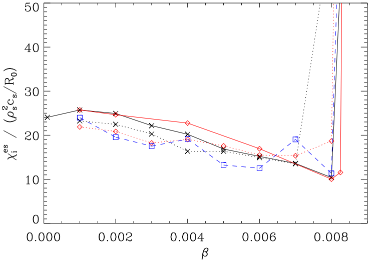
Need more proof that zonal flows matter?

For 10 years, strange, unexplained simulation behavior:

no saturation of turbulence above pressure $\beta_{\mathrm{crit}} < \beta_{\mathrm{KBM}}$



Seen in many codes: GENE, GYRO, GKW

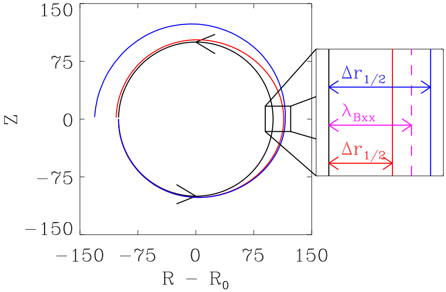


Key observation: **zonal flows absent** as fluxes take off

Field Line Decorrelation

Recall: ITG is ballooning-parity – finite β causes odd-parity B_x

Non-tearing B_x: field line leaves magnetic surface from inboard, maximum Δr at outboard, returns (almost) to B_0 surface



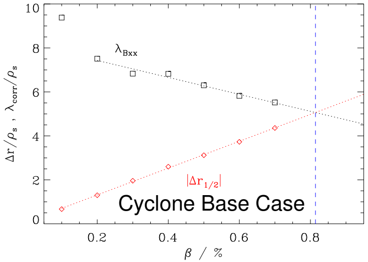
- $\Delta r_{1/2}$ is the radial displacement at $\theta = 0$
- λ_{Bxx} is the correlation length of B_x along x

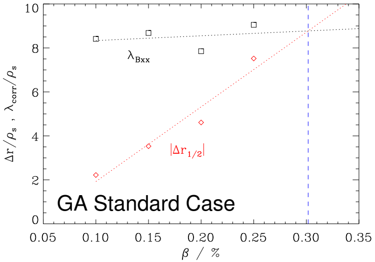
However: if field line decorrelates, **second half turn** becomes **independent of first**, no return to original position

Zonal flows shorted out $\Rightarrow$ **Non-Zonal Transition** (NZT)

The Non-Zonal Transition

Can field line decorrelation really explain the runaway?





$\Rightarrow$ **excellent prediction** of runaway β by decorrelation!

Consequences for realistic applications:

heat transport time scale in NZT-marginal state can be $\sim \gamma^{-1}$

$\Rightarrow$ **stiff profiles**, cannot increase plasma pressure anymore

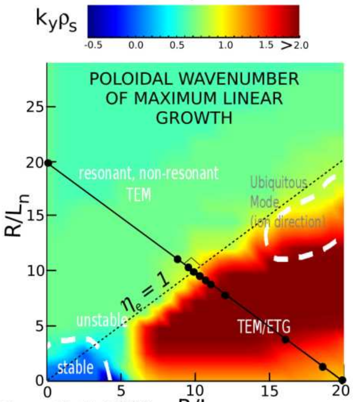
Note: indications that NZT may limit profiles in TCV, JT60-SA

TEM Saturation

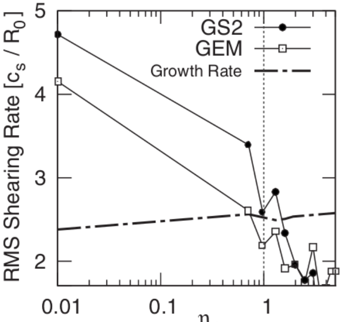
Merz PRL 2008: remove $\Phi_{\rm ZF}$ in ∇T-TEM $\Rightarrow$ no change in Q
Ernst PoP 2009: scan $\eta_{\rm e} = \omega_{T{\rm e}}/\omega_n$, see ∇n-TEM $\rightarrow$ ∇T-TEM

Shearing Rate

Rule-of-thumb: $\omega_{\rm s} = k_x^2 \Phi > 10\gamma \leftrightarrow$ ZFs matter; but: *ill-conditioned*



Ernst PoP 2009



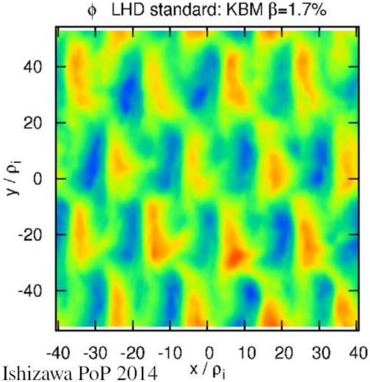
adapted from Ernst PoP 2009

- ∇T-TEM: ZFs irrelevant
- ∇n-TEM: ZFs important

Saturation without ZFs

*How do ∇T-**TEMs** saturate without zonal flows?*
Lang PoP 2008: **zonal density** = density corrugation

Ishizawa PoP 2013:
KBMs saturate via **diagonal band structures** in real and k-space



Ishizawa PoP 2014

But: **saturation without zonal** Φ mostly **poorly understood**, often through many different inefficient $(k, k', k - k')$ triplets

Why care? Nonlinear saturation very difficult to describe, being able to reduce it makes analytical approaches much easier!

Discussion: Mixing Length

Let us discuss how linear physics can describe turbulence:

1. What are the key observables in fusion plasma turbulence?
2.
3.

Discussion: Mixing Length

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   $Q, \Gamma \rightarrow \chi, D$
2. What linear information is available?
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   γ, ω, $k_\perp$, eigenmode Φ (or even f)
3. Dimensionally, how can we combine linear into nonlinear quantities?

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Let us discuss how linear physics can describe turbulence:

1. What are the key observables in fusion plasma turbulence?
   Q, $\Gamma \to \chi$, D
2. What linear information is available?
   γ, ω, $k_\perp$, eigenmode Φ (or even f)
3. Dimensionally, how can we combine linear into nonlinear quantities?
   $[\chi] = [D] = \mathrm{m}^2/\mathrm{s} \Rightarrow \gamma/k_\perp^2$

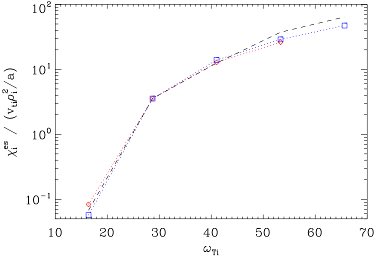
$\Rightarrow$ so-called **quasilinear**, or **mixing-length**, model:
much **faster** than nonlinear but less reliable,
whether QL is valid reveals key physics!

Quasilinear Flux Estimating

QL in Gyrokinetics

- perform scan over some parameter (e.g., ω_T or $\hat{s}$)
- use linear simulations to construct $Q_{\rm QL}$
- fix $Q_{\rm QL}$ to nonlinear $Q_{\rm NL}$ at single point in scan

Example: high-∇T turbulence (Pueschel PPCF 2019)



More on QL in next course part

Base model used in the remainder of the turbulence part:

$-(\nabla T)/T = R/L_T$

growth rate of mode j

$$Q_{\rm QL} = \omega_T \sum_{j,k_y} C(k_y) \frac{\gamma(j,k_y) w(j,k_y)}{\langle k_\perp(k_y,j)^2\rangle} \begin{array}{l} \leftarrow \text{QL weight } (\Phi \times T \text{ phase}) \\ \leftarrow \int_{-3\pi}^{3\pi} k_\perp^2 \Phi^2 \mathrm{d}\theta \,/\, \int_{-3\pi}^{3\pi} \Phi^2 \mathrm{d}\theta \end{array}$$

model constant

No stable mode modes in here, how to account for them?

Saturation Math

Terry PoP 2018: using Monday's Φ, p model for toroidal ITG, **statistical** "EDQNM" **closure** to get nonlinear properties

Conceptual derivation (playing fast and loose with the math):

- assume every k has one unstable, one stable eigenmode
- unstable (stable) mode amplitude $\beta_1 \sim |\Phi_1|$ ($\beta_2 \sim |\Phi_2|$)
- assume nonlinear interactions $\sim \beta'\beta'' = \beta(k')\beta(k-k')$ (sum over k' implied) dominated by zonal flow $\beta_\mathrm{z} \sim |\Phi_\mathrm{ZF}|$

energy $|\partial_t \beta_{1,2}|^2 \mp 2\gamma_{1,2}|\beta_{1,2}|^2 = 2\langle\beta'_\mathrm{z}\beta''\beta_{1,2}\rangle \quad |\partial_t\beta_\mathrm{z}|^2 + 2\nu|\beta_\mathrm{z}|^2 = 2\langle\beta'\beta''\beta_\mathrm{z}\rangle$

Evaluating $\langle\beta\beta'\beta''\rangle$ gives $\langle\beta\beta'\beta''\beta'''\rangle$, where one can assume only in-phase modes contribute ($\leftrightarrow$ *many metronomes*)

$$\frac{1}{2}\partial_t|\beta_1|^2 - \gamma_1|\beta_1|^2 = \frac{|\beta'_\mathrm{z}|^2(|\beta_1|^2 + \langle\beta_1\beta_2\rangle + \langle\beta''_1\beta''_2\rangle + |\beta''_2|^2)}{\underbrace{i\omega''_2 + i\omega'_\mathrm{z} - i\omega^*_1}_{\text{complex frequencies}}}$$

What does that mean and how can we use it?

Triplet Correlation Time

Triplet correlation time $\tau = \dfrac{1}{i\omega_{\text{stable}} + i\omega_{\text{ZF}} - i\omega^*_{\text{ITG}}}$

measures how resonantly ($\tau \to \infty$) modes interact nonlinearly;
large $\tau \Rightarrow$ strong transfer to stable modes $\Rightarrow$ **low flux**

Also from Terry '18:

$$|\beta_{1,2}|^2 \approx \frac{\nu}{\mathrm{Re}(\tau)}$$

and

$$Q = -\sum_k k_y \left(|\beta_1|^2 + |\beta_2|^2\right)$$
$$= \sum_k k_y \frac{\gamma_1}{k_\perp^2} \frac{\nu}{\mathrm{Re}(\tau)} \frac{1}{k_y^2 \epsilon^{1/2}}$$

Linear vs. Nonlinear τ

Essential: **linear** τ_{lin} **good proxy** for **nonlinear** τ_{NL}

$$\tau_{\text{lin}} = \frac{-i}{\omega_{\text{MITG}}(-k_x^{\text{ZF}}, k_y) - \omega^*_{\text{ITG}}(0, k_y)}$$

couples *streamer instability*, *sideband stable mode*, $\gamma_{\text{MITG}} = -\gamma_{\text{ITG}}$, and $\omega_{\text{ZF}} \approx 0$

Key modification $\mathrm{Re}(\tau)^{-1}$ can be incorporated in gyrokinetic QL

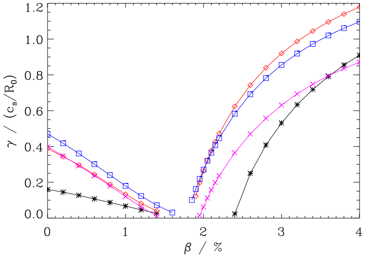
Improved quasilinear model $Q_{\text{i}}^{\text{es}} = \omega_{Ti} \sum_{k,j} \mathcal{C}_k \dfrac{\gamma_{kj} w_{kj}}{\langle k_{\perp kj}^2 \rangle \mathrm{Re}(\tau_{kj})}$

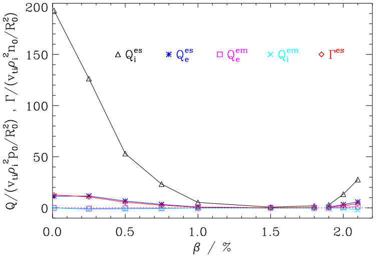
Electromagnetic Stabilization I

$\beta \equiv \frac{\text{kinetic pressure}}{\text{magnetic pressure}} > 0$ causes magnetic fluctuations

Hirose PoP 2000: ITG undergoes **linear electromagnetic stabilization** – field-line bending changes curvature

Pueschel PoP '08, '10: **nonlinearly**, even **more stabilization**





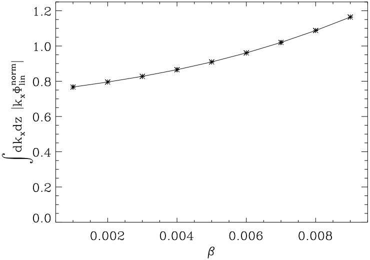
- stabilization up to $\beta_{\text{crit}}^{\text{KBM}}$; *note*: $\beta_{\text{crit}}^{\text{nonlinear}} = \beta_{\text{crit}}^{\text{linear}}$
- *here*, not considering β-induced Q_e^{em}

What causes **nonlinear electromagnetic stabilization** (NES)?

Electromagnetic Stabilization II

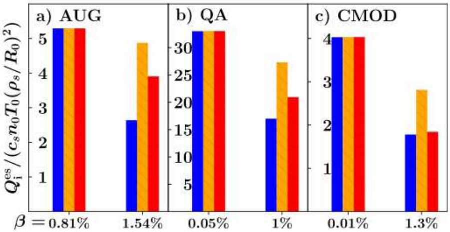
Application of τ-corrected QL model to this situation:

As β is increased,
streamer and sideband
ITG closer to resonance



Mode width increases
(Pueschel PoP 2013)

$\Rightarrow$ lower fluxes,
captured by τ correction
(Whelan PRL 2018, PoP 2019)



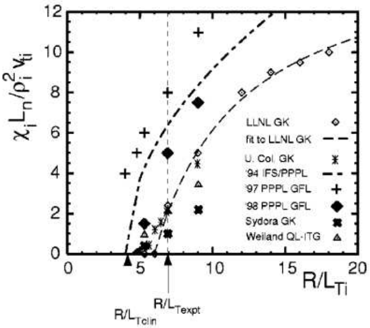
nonlinear, standard QL, corrected QL

$\Rightarrow$ can now **predict and explain** finite-β behavior

Note: further stabilization possible by fast ions
(Citrin PRL 2013, Di Siena NF 2018)

The Dimits Shift I

Dimits PoP 2000: **nonlinear upshift of critical gradient**



Dimits Regime

For $\omega_{Ti,\mathrm{crit}}^{\mathrm{lin}} < \omega_{Ti} < \omega_{Ti,\mathrm{crit}}^{\mathrm{NL}}$:

- strong zonal flows
- highly intermittent turbulence, transport
- small but finite amplitudes at $k_y > 0$

Many attempts to explain physics (e.g., secondary instability), but none predictive . . .

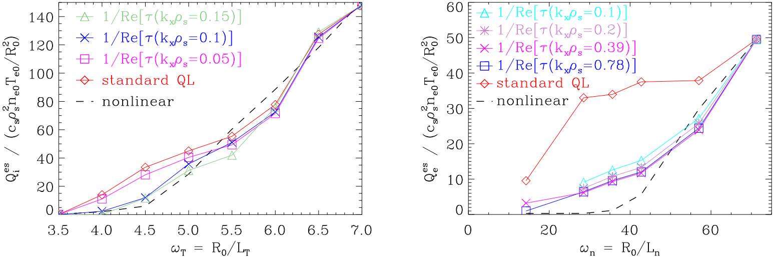
Up until now, **no QL model** that quantitatively **predicts** $\Delta\omega_{T,\mathrm{crit}}$

How about our new model with stable modes and τ?

The Dimits Shift I

Pueschel NF 2021: test of τ correction for two cases,

pure ITG scenario: reversed-field pinch case:



- new model performs much better at low gradients
- **upshift of critical gradient** is **captured**
- ITG case: also need to take **sideband stabilization** into account: perfect saturation efficiency when $\gamma_{\max}(k_x \neq 0) < 0$

Additional theory background: Terry PRL 2021, Li PoP 2021

Questions

Who has need for answers and further understanding?

Group Work: Scans & Quasilinear

2 hours group work

Using your drift-kinetic Vlasov code,

1. conduct a 2D parameter scan, over either k_y-$\omega_{T\mathrm{i}}$ or k_y-ω_n at some fixed $k_{||}$ (suggested range $0.1 \leq k_y \leq 0.8$)
2. based thereupon, construct a simple quasilinear model

$$Q_{\mathrm{i}}^{\mathrm{QL}} = \omega_{T\mathrm{i}} \sum_{k_y} \frac{\gamma(k_y)}{\langle k_y^2 \rangle} , \qquad \langle k_y^2 \rangle = \frac{\int k_y^2 |\Phi(k_y)|^2 \mathrm{d}z}{\int |\Phi(k_y)|^2 \mathrm{d}z} = k_y^2$$

   to gauge heat flux scalings of slab ITG turbulence

3. extract the $\Phi \times T_{\mathrm{i}}$ phase and include it in the model, normalized so $\alpha_{\Phi\times T} = \pi/2$ yields 100% $Q_{\mathrm{i}}^{\mathrm{QL}}$

Turbulence and Transport in Fusion Plasmas
Part VIII

DIFFER

M.J. Pueschel

RUHR UNIVERSITÄT BOCHUM RUB

Ruhr-Universität Bochum, February 27 – March 10, 2023

Tuesday Recap

Yesterday, we went over

- saturation of linear instabilities:
  - zonal flows & stable modes
  - ITG vs. ∇T-TEM vs. ∇n-TEM vs. KBM saturation
  - loss of saturation in the non-zonal transition
- quasilinear transport modeling using linear gyrokinetics
- improved modeling with the τ correction

Now, let's finish the quasilinear tasks with our Vlasov code . . .

Group Work Help: T_i

How to evaluate the ion temperature fluctuation?

Definitions for density & pressure fluctuations (normalized):

$$
\begin{aligned}
n_\mathrm{i} &= n_{\mathrm{i}0}\, \pi B_0 \int f_\mathrm{i}\, \mathrm{d}v_\|\mathrm{d}\mu \\
p_\mathrm{i} &= n_{\mathrm{i}0} T_{\mathrm{i}0}\, \pi B_0 \int v^2 f_\mathrm{i}\, \mathrm{d}v_\|\mathrm{d}\mu
\end{aligned}
$$

Further noting

$$\delta p = \delta(nT) = n_0 \delta T + T_0 \delta n \text{ and } v^2 = v_\|^2 + v_\perp^2$$

we can write a definition for T_i based on n_i and p_i

Group Work: Scans & Quasilinear

2 hours group work

Using your drift-kinetic Vlasov code,

1. conduct a 2D parameter scan, over either k_y-$\omega_{T\mathrm{i}}$ or k_y-ω_n at some fixed $k_{||}$ (suggested range $0.1 \leq k_y \leq 0.8$)
2. based thereupon, construct a simple quasilinear model

$$Q_\mathrm{i}^\mathrm{QL} = \omega_{T\mathrm{i}} \sum_{k_y} \frac{\gamma(k_y)}{\langle k_y^2 \rangle} \,, \qquad \langle k_y^2 \rangle = \frac{\int k_y^2 |\Phi(k_y)|^2 \mathrm{d}z}{\int |\Phi(k_y)|^2 \mathrm{d}z} = k_y^2$$

   to gauge heat flux scalings of slab ITG turbulence

3. extract the $\Phi \times T_\mathrm{i}$ phase and include it in the model, normalized so $\alpha_{\Phi \times T} = \pi/2$ yields 100% Q_i^QL

Group Work: Fluid Quasilinear

2 hours group work:

Returning to the (linear) Horton-Holland model solution,

$$\omega_{c1,2} = \frac{k_y}{2+2k_\perp^2} \pm ik_y\left(\frac{(1+\eta)\epsilon}{1+k_\perp^2}\right)^{1/2} - \frac{i\nu k_\perp^2}{2+2k_\perp^2}$$

1. construct a quasilinear $Q_{\mathrm{QL}} = \eta\epsilon\sum_{k_y} C(k_y)\gamma/k_\perp^2$ with $k_x = 0$
2. evaluate this model for scans over η and ν, assuming the base case $Q_{\mathrm{NL}}(k_y) \leftrightarrow C(k_y)$ to be an inverse parabola peaked at $k_y = 0.5$, with no flux below/above $k_y = 0.1/0.9$
3. add in the τ correction, assuming $\omega_{\mathrm{ZF}} = 0$ and $k_x^{\mathrm{ZF}} = 0.1$
4. repeat the scans and compare the two models

For those who are interested: P.-Y. Li *et al.*, Phys. Plasmas **28**, 102507 (2021) has a Horton-Holland comparison NL-QL(τ)

Discussion: Benchmark Point

Let us have a short discussion:

What would go into an “ideal” benchmark parameter point?
Benchmark: comparison of different simulation codes

- Why would we want to do this?
- Where in our vast parameter space should we do this?

Questions & Discussion

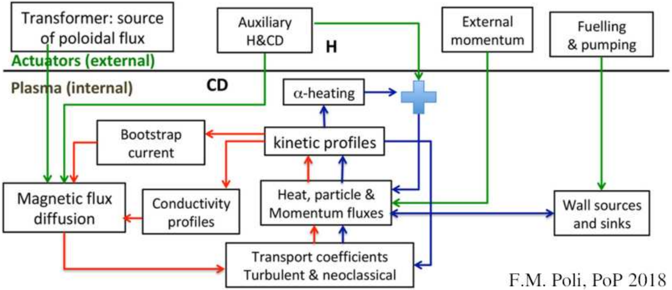
Any questions about anything nonlinear? Or linear?

Any feedback for the fella doin' so much talkin'?

Integrated Modeling

Now that we understand turbulence: **integrated modeling**

Fusion reactor: **multi-component**, **multi-physics** system
⇒ need to use many models, account for complex interactions



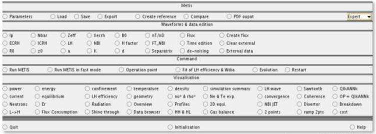
F.M. Poli, PoP 2018

Whole-Device Modeling: a few frameworks, e.g. Romanelli 2014

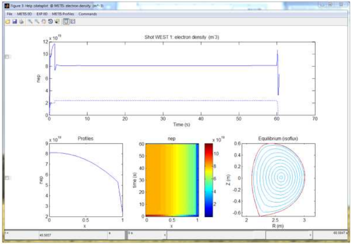
Shout-out DIFFER expert: Jonathan Citrin, head of IMT group

Flight Simulator

Aside from fundamental understanding, **goal: flight simulator**



Artaud NF 2018



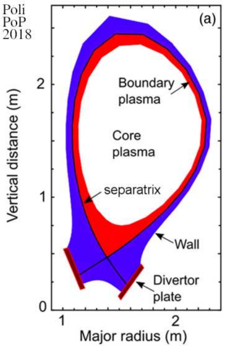
- simulate plasma evolution ahead of experiment
- ensure good performance, no disruptions
- determine good settings for heating, fueling, etc.
- *ideally modular*: many code coupling choices

See, e.g., Felici PPCF 2012, Artaud NF 2018, Janky FED 2021

Need for speed: **cannot use full-physics models**

Core vs. Edge

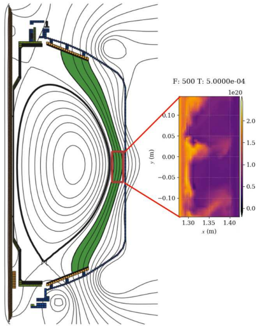
So far, this class has focused on **core plasmas**



Currently hot research topic: **plasma edge**

- pedestal
- separatrix
- scrape-off layer (SOL)

Edge gyrokinetics



(e.g., Shi JPP 2017)

For our purposes, will ignore scrape-off layer—2D transport!
(plus: open field lines, sheath boundary, full-f, neutrals, . . .)

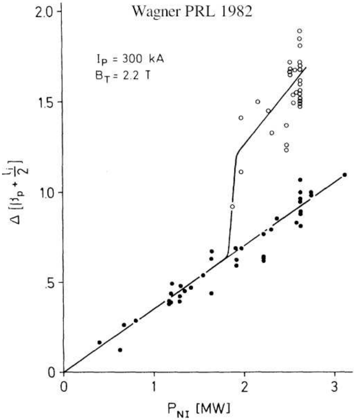
H-Mode & Pedestal

ASDEX: transition **L**ow- $\rightarrow$ **H**igh-confinement **mode**

Heating power boost

$\Rightarrow$ sudden increase in Φ_0

$\Rightarrow$ lower Q, higher τ_E



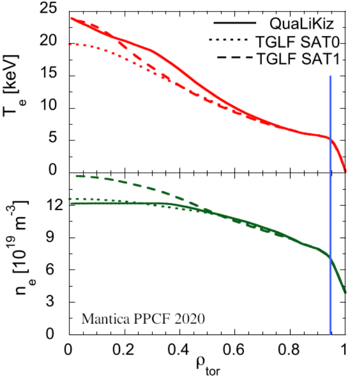
caused by ion orbit losses?

MHD pedestal models:

EPED (Snyder NF 2011)

IMEP (Luda NF 2020)

However, common to just put boundary inside pedestal:



Sources

Fusion reactors: need **heat, particle**, possibly momentum **sources** $\Rightarrow$ compensate losses, refuel D-T

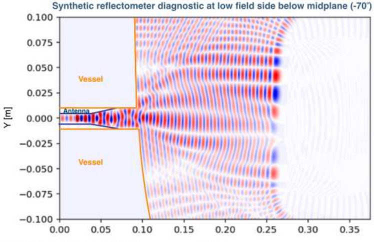
Heat sources:

- ICRH
- ECRH
- NBI

Particle sources:

- pellets
- gas puffing
- *wall erosion*

ECRH: rays; ICRH: need **full-wave modeling**



Da Silva JINST 2019

NBI: localized heat/particle/momentum source $\Rightarrow$ **current drive**

Transport Equations: Particles

How to describe transport and profile evolution mathematically?

Vlasov: $$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_j(\mathbf{x},\mathbf{v},t) = \mathcal{C}[f_j]$$

Term by term, take 0th moment:

$$\int \mathrm{d}\mathbf{v}\frac{\partial f_j}{\partial t} = \frac{\partial n_j}{\partial t} \qquad \int \mathrm{d}\mathbf{v}\,\mathcal{C}[f_j] = 0 \text{ (particle conservation)}$$

$$\int \mathrm{d}\mathbf{v}\,\mathbf{v}\cdot\frac{\partial f_j}{\partial \mathbf{x}} = \nabla\cdot(n_j\mathbf{u}_j) \equiv \nabla\cdot\mathbf{\Gamma}_j \quad (\mathbf{\Gamma}\text{: particle flux})$$

$$\int \mathrm{d}\mathbf{v}\left(E_a + \frac{v_b}{c}B_c\right)\frac{\partial f_j}{\partial v_a} = \underbrace{\left[\left(E_a + \frac{v_b}{c}B_c\right)f_j\right]_{-\infty}^{\infty}}_{=0\ [f(v\to\infty)=0]} - \int \mathrm{d}\mathbf{v}\,\underbrace{\frac{\partial}{\partial v_a}\left(E_a + \frac{v_b}{c}B_c\right)}_{=0\ (\partial_v E=0\,\&\,\partial v_a/\partial v_b=0)} f_j$$

$\Rightarrow$ particle balance $\dfrac{\partial n_j}{\partial t} + \nabla\cdot\mathbf{\Gamma}_j = S_j$ (S_j: external particle sources)

Transport Equations: Heat I

Similar to particle balance: **heat/energy balance,** $\frac{m}{2}\int v^2 \mathrm{d}\mathbf{v}$

Vlasov: $\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_j(\mathbf{x},\mathbf{v},t) = \mathcal{C}[f_j]$

$$\frac{m_j}{2}\int \mathrm{d}\mathbf{v}\frac{\partial v^2 f_j}{\partial t} = \frac{3}{2}\frac{\partial p_j}{\partial t}$$

$$\frac{m_j}{2}\int \mathrm{d}\mathbf{v}\,\mathbf{v}^3\cdot\frac{\partial f_j}{\partial \mathbf{x}} = \nabla\cdot\mathbf{Q}_j \quad (\mathbf{Q}\text{: energy/heat flux})$$

$$\frac{q_j}{2}\int \mathrm{d}\mathbf{v}\,v^2\mathbf{E}\cdot\frac{\partial f_j}{\partial \mathbf{v}} = \underbrace{\frac{q_j}{2}\left[v^2 E f_j\right]_{-\infty}^{\infty}}_{=0\ [v^2 f(v\to\infty)=0]} - q_j\mathbf{E}\cdot\int \mathrm{d}\mathbf{v}\,\mathbf{v}f_j = -q_j n_j \mathbf{E}\cdot\mathbf{u}_j$$

$$\frac{m}{2c}\int \mathrm{d}\mathbf{v}\,v^2\mathbf{v}\times\mathbf{B}\cdot\frac{\partial f_j}{\partial \mathbf{v}} = \ldots = 0 \text{ (the magnetic field does no work)}$$

However, the collision term does not vanish this time ...

Transport Equations: Heat II

$$\frac{m_j}{2}\int \mathrm{d}\mathbf{v}\, v^2 \mathcal{C}[f_j] = \frac{m_j}{2}\int \mathrm{d}\mathbf{v}\, (\mathbf{v}-\mathbf{u}_j)^2 \mathcal{C}[f_j] + m_j \mathbf{u}_j \cdot \int \mathrm{d}\mathbf{v}\, \mathbf{v}\mathcal{C}[f_j] \equiv Q_{\mathrm{ei}} + \mathbf{u}_j \cdot \mathbf{F}_j$$

with F: collisional friction force; Q_{ei}: collisional energy exchange

$\Rightarrow$ energy balance $\frac{3}{2}\frac{\partial p_j}{\partial t} + \nabla \cdot \mathbf{Q}_j = Q_{\mathrm{ei}} + Q_{j,\mathrm{fus}} + Q_{j,\mathrm{ext}} + \mathbf{u}_j \cdot (\mathbf{F} + q_j n_j \mathbf{E})$

(gray terms: negligible for purely **radial transport**)

$Q_{j,\mathrm{fus}}$:	fusion power source (j: fusion alphas)
$Q_{j,\mathrm{ext}}$:	external heating: ECRH, ICRH, NBI
$\mathbf{Q}, \boldsymbol{\Gamma}$:	**turbulent** (or neoclassical) **fluxes**

However, still in Cartesian coordinates
Need to **account for** $\mathrm{d}V = \mathrm{d}V(r) = A(r)\mathrm{d}r$ **dependence!**

Group Work: Transport Equations

30 minutes group work:

1. Perform volume integration combined with radial derivative on particle balance

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \mathbf{\Gamma}_j = S_j$$

   Hint 1: use Gauss divergence theorem for flux term
   Hint 2: can insert unity factors like $\frac{\mathrm{d}r}{\mathrm{d}r}$ *and* $\frac{\int \mathrm{d}A\mathrm{d}r}{\int \mathrm{d}A\mathrm{d}r}$

2. Express particle balance in terms of flux-surface averages

$$\langle \xi \rangle \equiv \frac{\int\limits_A \int\limits_0^r \xi \, \mathrm{d}A\mathrm{d}r}{\int\limits_A \int\limits_0^r \mathrm{d}A\mathrm{d}r} \, , \quad \text{noting that } V' \equiv \frac{\mathrm{d}V}{\mathrm{d}r}$$

   (each resultant term should have a $\langle \cdot \rangle$ *and a* V'*)*

Group Work: Solution

$$\frac{\partial n_j}{\partial t} + \nabla\cdot\mathbf{\Gamma}_j = S_j$$

$$\frac{\partial}{\partial r}\int_A\int_0^r \frac{\partial n_j}{\partial t}\mathrm{d}A\mathrm{d}r = \frac{\partial}{\partial t}\int_A n_j\mathrm{d}A = \frac{\partial}{\partial t}\int_A n_j\,\mathrm{d}A\frac{\mathrm{d}r}{\mathrm{d}r}\frac{\int_A\int_0^r \mathrm{d}A\mathrm{d}r}{\int_A\int_0^r \mathrm{d}A\mathrm{d}r} = \frac{\partial V'\langle n_j\rangle}{\partial t}$$

$$\frac{\partial}{\partial r}\int_A\int_0^r S_j\,\mathrm{d}A\mathrm{d}r = V'\langle S_j\rangle$$

$$\frac{\partial}{\partial r}\int_A\int_0^r \nabla\cdot\mathbf{\Gamma}_j\,\mathrm{d}A\mathrm{d}r = \frac{\partial}{\partial r}\int_A \mathbf{\Gamma}_j\cdot\hat{\mathbf{n}}\,\mathrm{d}A = \frac{\partial}{\partial r}\int_A \Gamma_{j,r}\,\mathrm{d}A\frac{\mathrm{d}r}{\mathrm{d}r}\frac{\int_A\int_0^r \mathrm{d}A\mathrm{d}r}{\int_A\int_0^r \mathrm{d}A\mathrm{d}r} = \frac{\partial V'\langle\Gamma_{j,r}\rangle}{\partial r}$$

Typical notation: brackets $\langle\cdot\rangle$ are dropped, $\Gamma_j \equiv \Gamma_{j,r}$

Overall : $$\frac{\partial V' n_j}{\partial t} + \frac{\partial V'\Gamma_j}{\partial r} = V' S_j$$

Transport Equations: Summary

For two-component plasma, ignoring current diffusion:

$$\frac{\partial V' n_\mathrm{i}}{\partial t} + \frac{\partial V' \Gamma_\mathrm{i}}{\partial r} = V' S_\mathrm{i}$$

$$n_\mathrm{e} = n_\mathrm{i} \qquad \Gamma_\mathrm{e} = \Gamma_\mathrm{i} \qquad P_\mathrm{ei} = -P_\mathrm{ie}$$

$$\frac{3}{2}\frac{\partial V' n_\mathrm{i} T_\mathrm{i}}{\partial t} + \frac{\partial V' Q_\mathrm{i}}{\partial r} = V'(P_\mathrm{ie} + P_\mathrm{i,fus} + P_\mathrm{i,ext})$$

$$\frac{3}{2}\frac{\partial V' n_\mathrm{e} T_\mathrm{e}}{\partial t} + \frac{\partial V' Q_\mathrm{e}}{\partial r} = V'(-P_\mathrm{ie} + P_\mathrm{e,ext})$$

Circular flux surfaces: $V = 2\pi^2 r^2 R \Rightarrow V' = 4\pi^2 r R$,
thus, can replace $V' \rightarrow r$ in above expressions

Typically, need to evaluate $10^3 - 10^5$ times
$\Rightarrow$ require fast models for Q, Γ

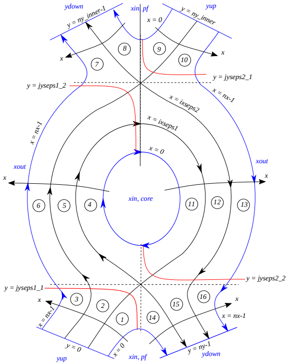
MHD & Neoclassical

Recall: Grad-Shafranov

$$R^2 \nabla \cdot \frac{\nabla \psi}{R^2} = -R^2 \frac{\partial p}{\partial \psi} - F \frac{\partial F}{\partial \psi}$$

used to get B_0 equilibrium

Transport modeling:
prescribe **shape of last closed flux surface** (LCFS) *or* (magnetic/inductive) **coil currents**



Complement Grad-Shafranov solver with **MHD stability code**

Also need code to **evaluate neoclassical fluxes**, especially when looking at stellarators (*any idea why?*)

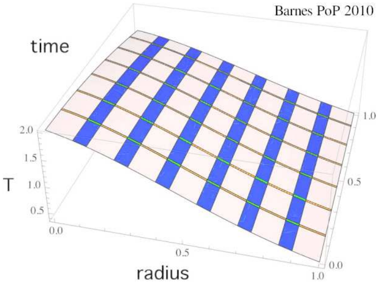
Turbulent Transport

Turbulent transport: **trade-off** between **speed** and **accuracy**

Can get turbulent Q, Γ from

- nonlinear gyrokinetics
- quasilinear gyrokinetics
- reduced QL models (QuaLiKiz, TGLF)
- simple transport coefficients and critical-gradient models

Coupling multiple nonlinear simulations at different r/a:

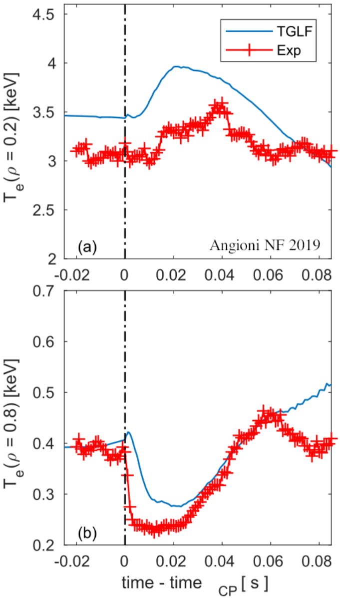


Trinity, Tango frameworks

For real-time control, flight simulators, need **machine learning/neural networks** (can be based off gyrokinetics and/or reduced QL models; e.g., van de Plassche PoP 2020)

Application I

Impurity injection: lowering edge T, expect inward propagation



Experiments (AUG/CMOD/DIII-D): edge "**cold pulse**" surprisingly **causes T increase in core**

Note: low n_e discharge, ∇n-TEM turbulence regime

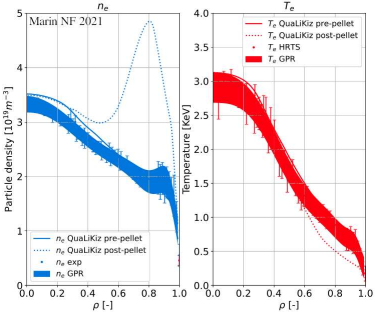
Poorly understood until recently (Angioni NF 2019):

1. impurities decrease T_{edge}, increase $n_{e,edge}$
2. impurities, electrons move inward
3. n_e boost flattens ∇n_e
4. TEM reduced $\Rightarrow$ lower Q_e $\Rightarrow$ core T_e increases

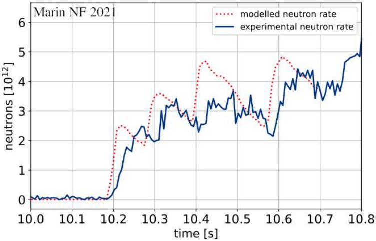
Application II

Key assumption for fusion reactors: **deuterium & tritium from pellets need to reach the plasma center**

Marin NF 2021: use JINTRAC+QuaLiKiz to analyze D pellet in H plasma $\Rightarrow$ *does* D *move inward*?



ITG causes D **pinch**, boosts core n_{D} & D-D reactions
$\Rightarrow$ **more fusion neutrons**



Post-injection: massive $\pm\nabla n_{\mathrm{i}}$
$r/a > 0.8$: TEM; $r/a < 0.8$: ITG

Caveat/future work: JINTRAC requires $T_{\mathrm{H}} = T_{\mathrm{D}}$

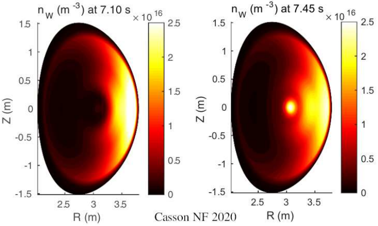
Application III

Recall: tritium retention means no C but W wall needed,
but W **in core leads to radiative collapse** of T_e

Casson NF 2020: study tungsten accumulation in JET

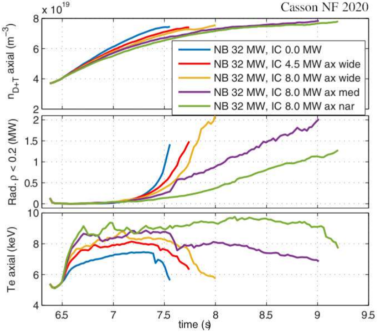
$\Gamma_\mathrm{W} < 0$ due to
neoclassical convection

$\Rightarrow$ fast core accumulation



ICRH known to help $\Rightarrow$ why?

By reducing main ion ∇n,
boost ITG $\Rightarrow$ turbulent $\Gamma_\mathrm{W} > 0$



ICRH against W not as important in ITER (AUG prefers ECRH)
Rather, the study shows we **can explain/predict the experiment**

Questions & Discussion

Any questions about integrated modeling? Other comments?

Turbulence and Transport in Fusion Plasmas
Part XI

M.J. Pueschel

Ruhr-Universität Bochum,   February 27 – March 10, 2023

Wednesday Recap

Yesterday, we

- constructed and deployed quasilinear models based on drift-kinetics and our fluid model
- learned what transport modeling and, more generally, integrated modeling means
- derived transport equations
- saw how integrated modeling can help us understand experiments better

Next step: write our own transport code

Group Work: Integrated Modeling

0.5 days group work:

1. [optional, if using IDL template] Install `gnudatalanguage`, be set up to run `Tsolve.pro`
2. Using either ITER-like parameters in SI or appropriate normalized units, implement ion temperature balance

$$\frac{3}{2}\frac{\partial r n_\mathrm{i} T_\mathrm{i}}{\partial t} - \frac{\partial}{\partial r}\left(r n_\mathrm{i} \chi \frac{\partial T_\mathrm{i}}{\partial r} \right) = r P_\mathrm{ext}$$

   (assume constant-in-time n_i and χ profiles, Gaussian power deposition)
3. Think up sensible boundary conditions
4. Test what happens when Δt chosen too large
5. Does the behavior match expectations? Qualitative & time scale?
6. How sensitive is the fusion power to $T_\mathrm{i}(r = a)$?

Group Work: Integrated Modeling

7. Get Artaud NF 2010 (good source for research project), implement T_e equation and $Q_\mathrm{ie,ei}$ energy exchange
8. Implement a more realistic diffusivity model,
   $\chi = \chi_\mathrm{gyroBohm}(\omega_T - \omega_{T,\mathrm{crit}})^2$ ponder ITG vs. TEM modifier
   (take above $\omega_{T,\mathrm{crit}}$ from Guo PoFB 1993, Eq. (22))
9. Add a small mock-up neoclassical flux, can be constant in r
10. Add current, q, $\hat{s}$ profiles according to Wesson, *Tokamaks*

$$
\begin{aligned}
\text{current density} \quad j(r) &= j_0 \left(1 - \frac{r^2}{a^2}\right)^3 \\
\text{current} \quad I(r) &= 2\pi \int_0^r j(r') r' \mathrm{d}r' \\
\text{safety factor} \quad q(r) &= \frac{2\pi r^2 B_\phi}{\mu_0 I(r) R}
\end{aligned}
$$

11. Explore the impact of all these improvements

Group Work: Journal Club

0.5 days group work:

Pick a turbulence paper, read, digest, present; suggestions:

- M. Albergante *et al.*, *Microturbulence driven transport of energetic ions in the ITER steady-state scenario*, Nucl. Fusion **50**, 084013 (2010)
- R.E. Waltz *et al.*, *Gyrokinetic simulation tests of quasilinear and tracer transport*, Phys. Plasmas **16**, 072303 (2009)
- P. Mantica *et al.*, *Progress and challenges in understanding core transport in tokamaks in support to ITER operations*, Plasma Phys. Control. Fusion **62**, 014021 (2020)
- F. Merz and F. Jenko, *Nonlinear Saturation of Trapped Electron Modes via Perpendicular Particle Diffusion*, Phys. Rev. Lett. **100**, 035005 (2008)

Research Project Suggestions I

Pick one of these or come up with your own

Turbulence topics

- Write and deploy an eigenvalue solver, test against initial-value or theory, investigate stable modes
- Perform and interpret scans over physical parameters with an upgraded version of our Vlasov code
- Write a drift-kinetic dispersion relation solver including the plasma Z function
- Analytically produce a closed fluid model for slab ITG and compare it to the results from the drift-kinetic Vlasov code

Research Project Suggestions II

Pick one of these or come up with your own

Transport topics

Study core predictions and pedestal scaling impact on scenario performance, comparing to literature; focus on one of

- scalings of magnetic field and machine size
- impact of total current and current profile
- coupling/decoupling of T_i and T_e in strongly electron-heated discharges