Turbulence and Transport in Fusion Plasmas
Part III

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Tuesday Recap

Yesterday, we covered

- what linear and toroidal reactor designs exist, and why some are perhaps better than others
- what classical, neoclassical, and anomalous transport are
- the meaning of instability and turbulence
- how a nonlinear term transforms to Fourier space and makes the nature of three-wave coupling more transparent

Next: solve the Horton-Holland fluid model analytically

Linear Horton-Holland

Reminder: this is the **linearized Horton-Holland** model:

$$\begin{aligned}
\frac{\partial \Phi(k_\perp)}{\partial t} &= \frac{1}{1+k_\perp^2}\left(-ik_y\Phi(k_\perp) + ik_y\epsilon p(k_\perp) - \nu k_\perp^2\Phi(k_\perp)\right) \\
\frac{\partial p(k_\perp)}{\partial t} &= -ik_y(1+\eta)\Phi(k_\perp) - \chi k_\perp^4 p(k_\perp)
\end{aligned}$$

Group Work: Fluid Code

1.5 hours group work:

Preparatory calculations

1. derive ITG/ETG dispersion relation from fluid model
2. solve dispersion relation analytically to get ω_c

3.
4.
5.
6.

Analytical Solution

Dispersion relation – relating ω_c, $k_\perp$:

$$-\omega_\mathrm{c}^2(1+k_\perp^2)+i\omega_\mathrm{c}(1+k_\perp^2)k_\perp^4\chi = \omega_\mathrm{c}k_y - ik_yk_\perp^4\chi + k_y^2\epsilon(1+\eta) - i\omega_\mathrm{c}k_\perp^2\nu - k_\perp^6\nu\chi$$

Exactly solvable; but simpler structure with some additional simplifications: $k_\perp \approx k_y$; weak collisionalities ν, χ; typical $0.1 < k_\perp < 1$ (units of inverse gyroradius); thus:

$$\omega_{\mathrm{c}1,2} = \underbrace{\frac{k_y}{2+2k_\perp^2}}_{\text{drift}} \pm \underbrace{ik_y\left(\frac{(1+\eta)\epsilon}{1+k_\perp^2}\right)^{1/2}}_{\text{gradient drive}} - \underbrace{\frac{i\nu k_\perp^2}{2+2k_\perp^2} - \frac{i\chi k_\perp^4}{2}}_{\text{collisional damping}}$$

Very few models solvable by hand …
what to do for more complex physics?

Group Work: Fluid Code

1.5 hours group work:

Preparatory calculations

1. derive ITG/ETG dispersion relation from fluid model
2. solve dispersion relation analytically to get ω_{c}

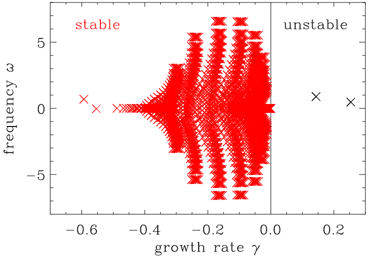
Numerical treatment: use plot routine (from prep work) to

3. implement the full dispersion relation (zeroes of $\mathrm{lhs} - \mathrm{rhs}$)
4. test numerical convergence, compare to analytical result
5. obtain, interpret instability spectrum $\gamma(k_y)$
6. scan some input parameter – e.g., η, ϵ, ν – and discuss physical meaning of results

Rule for all group work sessions: if you're stuck, just ask!

Subdominant and Stable Modes

Commonly, focus on fastest-growing mode, but recall:
at each $k_\perp$, have many linear eigenmodes



- **dominant**: $\gamma_{\max}$
- **subdominant**: $0 < \gamma < \gamma_{\max}$
- **stable**: $\gamma < 0$

Nonlinear saturation: either

- “actual” dissipation: collisions, or
- **stable modes returning energy** to $\nabla T, n$

Coupling between k, k', $k - k'$ sends energy from unstable to stable modes!

Questions & Discussion

Anything unclear that we talked about here?

Theory Frameworks

In physics, **hierarchy of models**: can describe light as *rays*, with *nonlinear optics*, or as *quantum wave functions*

$\Rightarrow$ **accuracy vs. cost/complexity**

Plasma physics: historically very simple models, but now **supercomputers allow use of high-fidelity models**

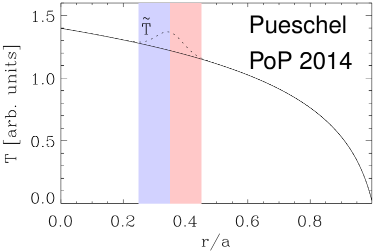
$$
\begin{array}{ccccc}
 & 6D & & 5D \\
 & kinetics & \rightarrow & gyrokinetics \\
 & \downarrow & & \downarrow \\
transport\ coefficients & fluid/MHD & & gyrofluid \\
0D & 3D & & 3D
\end{array}
$$

In the following, introduction to each model, (dis-)advantages

Transport Coefficients

TRANSP, B2.5, . . . : simple mock-up of turbulent flux, assume **constant heat & particle diffusivities** $\chi \sim Q/\nabla T$, $D \sim \Gamma/\nabla n$

$$\underbrace{\frac{\partial T(r)}{\partial t}}_{=0 \text{ in equil.}} = \underbrace{\nabla \cdot \chi(r) \nabla T(r)}_{\text{diffusion}} + \underbrace{S}_{\text{source}}$$



- constant χ, D: fast, easy to implement
- produces well-behaved fluxes
- **misses key physics**, e.g., ∇T_{crit}
- some folks tune χ, D to get whatever result they want

More on transport later in the course, but key research area: **study turbulence, understand & get expressions for χ, D, hand off to transport modelers**

Kinetics

Gold standard of theories: kinetics—particles at $\mathbf{x}$ moving with $\mathbf{v}$

6D, two approaches:

PIC

Particle-in-cell:

equations of motion for individual particles

problem: noise build-up

Vlasov/continuum

evolve particle distribution $f_{\mathrm{i,e}} = f_{\mathrm{i,e}}(\mathbf{x}, \mathbf{v}, t)$

harder to implement, but "cleaner" results

Solve **Vlasov** equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

but:

- **collisions** require insane resolutions $\rightarrow$ instead use *collision operator*, RHS = $C[f]$
- contains lots of unnecessary physics (light waves etc.)
- coordinates not helpful in fusion geometries

Note: get force $\mathbf{F} = \mathbf{F}(\Phi, \mathbf{A})$ from **Maxwell's equations**

Fluid Models

Earlier, we used a much simpler "fluid" model

Moments

Integrate Vlasov equation to get **moments** (6D → 3D):

$$\text{density} \quad n = \int d\mathbf{v} f$$

$$\text{flow} \quad \mathbf{u} = \int d\mathbf{v}\, \mathbf{v} f$$

$$\text{temperature} \quad T = \int d\mathbf{v}\, \mathbf{v}^2 f$$

$$\vdots \quad \vdots$$

with potentials/fields Φ, $\mathbf{A}$

Problem: system of equations **needs closure**:

$$\frac{\partial n}{\partial t} = \ldots n + \ldots \overbrace{\int d\mathbf{v}\, \mathbf{v} f}^{\mathbf{u}}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \ldots \mathbf{u} + \ldots \overbrace{\int d\mathbf{v}\, \mathbf{v}^2 f}^{T}$$

- **never-ending series**
- can make assumptions, e.g., $\int d\mathbf{v}\, \mathbf{v}^4 f = 0$
- tends to **miss key nonlinear physics**

Will take moments to get specific fluid model later

MHD

Special category of fluid models: **Magnetohydrodynamics**

Ideal MHD:

continuity $\frac{\partial n}{\partial t} - \nabla \cdot (n\mathbf{u}) = 0$

momentum balance

$$n\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u} = \frac{\mathbf{J} \times \mathbf{B}}{mc} - \frac{\nabla p}{m}$$

adiabaticity $\frac{\mathrm{d}}{\mathrm{d}t} \frac{p}{(mn)^{5/3}} = 0$

Ohm's law $c\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$

Ampère's law $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$

Faraday's law $\frac{\partial B}{\partial t} = -c\nabla \times \mathbf{E}$

no divergence $\nabla \cdot \mathbf{B} = 0$

Other types: RMHD, HMHD, reduced MHD, . . . , e.g., for astrophysics

Orderings

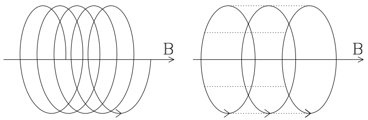
MHD validity requires

- $\lambda \gg \lambda_{\mathrm{D}}, \rho_{\mathrm{i,e}}$
- $\gamma, \omega \ll \omega_{\mathrm{p}}, \omega_{\mathrm{ci,e}}$
- $|\mathbf{u}| \ll c$
- collisional, $\nu_{\mathrm{ei}} > \gamma, \omega$
  $\Rightarrow$ no v-space dynamics
- full ionization

Fusion: MHD is used to compute magnetic equilibria, but **does not capture microinstability, turbulence**

Gyrokinetics

Littlejohn JMP 1979, 1982; Frieman PoF 1982; Dubin PoF 1983
Gyroaverage to reduce phase space $\mathbf{v}_{x,y} \to v_\perp$ from 6D to 5D



- average Larmor motion
- charged rings flow along z
- slow drifts in x, y

Why is this a big deal?

Review: Brizard RMP 2007

- 6D to 5D means order-of-magnitude speed-up
- **gyroagerage eliminates irrelevant fast time scales**
  (Larmor motion, fast magnetosonic waves)
  $\Rightarrow$ **factor** 10^3 **speed-up!**

Gyrokinetics enabled turbulence studies in fusion plasmas

Later in this course: sketch of gyrokinetic framework derivation
Note: gyrofluid can be obtained from GK but not from fluid

The Gyrokinetic Equations

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\mathrm{ref}}}{JB_0}v_\|\Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial\chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\mathrm{ref}}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\mathrm{ref}}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial\chi}{\partial x}\Gamma_{jy} - \frac{\partial\chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\mathrm{coll}}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00} - \mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01} - \mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0\int J_0 f_j \mathrm{d}v_\|\mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2}J_1 f_j\mathrm{d}v_\|\mathrm{d}\mu$$

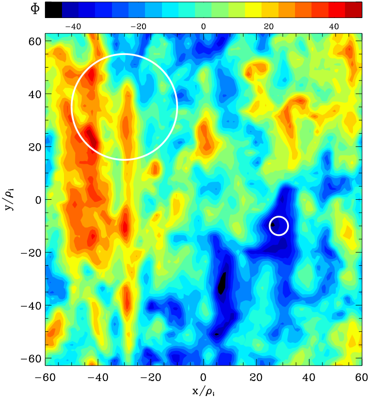
$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j\frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0) \quad \mathcal{C}_2 = -\sum_j\frac{q_j n_{j0}}{B_0}(\Gamma_0-\Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j\frac{m_j n_{j0}v_{Tj}}{B_0^2}(\Gamma_0-\Gamma_1)$$

$$A_\| = \left(\sum_j\frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j\mathrm{d}v_\|\mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j\frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0}\mathrm{d}v_\|\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|}F_{j0}\partial_k\chi_j q_j/(m_j v_\|) + \bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_j c)$$

Properties of Gyrokinetics

Gyroaverage I: field feels charged circle, not particle
Gyroaverage II: particle feels reduced small-scale field



$\rho > \lambda_c$ vs. $\rho < \lambda_c$

GK removes fast time scales, not spatial (but need $\rho \ll L_B$)

Drift-kinetics

When $\rho \ll \lambda_c \leftrightarrow k_\perp \rho \ll 1$:
no gyroaverage, *drift-kinetics*

- slightly faster, slightly less memory needed
- easier analytics
- easier closures

Included in GK (*but not in gyrofluid*): **velocity space, wave-particle interactions, full zonal-flow physics**

Gyrokinetic Orderings

Assumptions in **general gyrokinetics**:

- no very fast time scales, $\omega \ll \Omega_{cj}$ for all species j
- Larmor orbits nearly closed, $\rho_j \ll L_B = B/|\partial B/\partial \mathbf{x}| \quad \forall\, j$

Additional common assumptions (**strong magnetization**):

- fast parallel motion $v_\| \sim v_{\mathrm{T}j} \gg v_{x,y}^{\mathrm{drifts}} \quad \forall j$
- anisotropic turbulence $k_\| \ll k_\perp \leftrightarrow \lambda_\| \gg \lambda_\perp$
- low pressure $\beta_j \equiv 8\pi n_{j0} T_{j0}/B_0^2 \ll 1 \quad \forall j$

Additional assumptions for δf (c.f. earlier $n \to n_0 + n_1$):

- small perturbations $f_1/f_0 \sim \mathbf{B}_1/\mathbf{B}_0 \ll 1$
  (*not* Φ_1/Φ_0 *since* Φ_0 *can be* $= 0$)
- fluctuation localization $\rho_j \ll L_{(n,T)} = (n,T)/|\partial(n,T)/\partial \mathbf{x}| \quad \forall\, j$

Hereafter, only consider δf*, using notation* $f_0 \to F_0, f_1 \to f$*, etc.*
$\Rightarrow$ **for us, we will require all of the above conditions**

Short Gyrokinetics History

1980s: basic gyrokinetic theory

- analytical extensions: δB, relativistic, higher orders, ...
- simple NL analytics: Similon PoF 1984, Smith PoF 1985
- NL PIC (adiabatic electrons): Lee JCP 1987

1990s: simple simulations, then gyrokinetics really takes off

- comparisons to experiment: Parker PRL 1993
- ions & electrons: Kotschenreuther PoP 1995
- importance of zonal flows: Lin PRL 1999

2000s: high-fidelity turbulence simulations, zonal flows

- ion-scale turbulence: Dimits PoP 2000, Dannert PoP 2005
- electron turbulence: Jenko PoP 2000, Dorland PRL 2000
- NL $\beta > 0$: Candy PoP 2005, Pueschel PoP 2008 & 2010

2010s: multi-scale simulations, magnetic fluctuations

- microtearing: Doerk PRL 2011, Guttenfelder PRL 2011
- i-e multi-scale: Candy PPCF 2007, Goerler PRL 2008

2020s: SOL/full-volume simulations, stellarators

Landau Damping I

One key process captured by (gyro)kinetics: **Landau damping**

Consider simple 1D Vlasov equation, perturbed distribution f:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial F_0}{\partial v} = 0 \qquad \frac{\partial E}{\partial x} = -4\pi e \int f \mathrm{d}v$$

Take wave ansatz for f and insert Vlasov into Poisson:

$$ikE = -4\pi e \int iE\frac{e}{m}\frac{\partial F_0/\partial v}{\omega_\mathrm{c} - kv}\mathrm{d}v \quad \text{or} \quad \frac{\omega_\mathrm{p}^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial \hat{F}_0/\partial v}{v - \omega_\mathrm{c}/k}\mathrm{d}v = 1$$

Lev Landau, Zh. Eksp. Teor. Fiz. **16**, 574 (1946): *how to integrate this expression properly*; for Maxwellian $F_0 \sim \exp(-v^2)$,

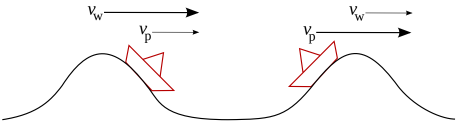
real frequency $\omega^2 = \omega_\mathrm{p}^2\left(1 + 3\frac{k^2 v_\mathrm{th}^2}{\omega^2}\right)$ $\quad T \to 0:$ same as fluid

damping rate $\mathrm{Im}(\omega_\mathrm{c}) = -\sqrt{\pi}\frac{\omega_\mathrm{p}^4}{k^3 v_\mathrm{th}^3}\mathrm{e}^{-3/2-\omega_\mathrm{p}^2/(k^2 v_\mathrm{th}^2)}$ $\quad$ fluid: no damping

Landau Damping II

Thus, **waves are damped without any collisions!**
Same process in galaxy formation, with ions $\rightarrow$ stars, $qE \rightarrow F_g$

Damping comes from integrating **resonance** $v_{\rm res} - \omega_{\rm c}/k = 0$:
particles with $v \approx v_{\rm res}$ travel with wave, see little $\partial_t E$
$\Rightarrow$ **continuous acceleration in one direction**



- $v \gtrsim v_{\rm res}$: surfer behind wave *pushes wave*
- $v \lesssim v_{\rm res}$: surfer ahead of wave *pushed by wave*

Energy conservation: **particles and wave exchange energy**

Maxwellian: more particles have $v \lesssim v_{\rm res}$, **net wave damping**
However, if $\partial F_0/\partial v > 0$, can get **two-stream instability**

Group Work: Orderings

45 minutes group work

1. look up machine parameters, profiles, and fluctuation characteristics $k_\perp$, $\omega \sim \gamma$ (e.g., from papers, image search, Wikipedia) for
   a. ASDEX Upgrade
   b. JET or Wendelstein 7-X
2. for each, evaluate **MHD** validity at $r/a \approx 0.5$ and 0.9: fully ionized; $\lambda \gg \lambda_\mathrm{D}, \rho_\mathrm{i,e}$; $\omega \ll \omega_\mathrm{p}, \omega_\mathrm{ci,e}$
3. for each, evaluate **gyrokinetics** validity at same radii: *truly mandatory*: $\rho_\mathrm{i,e} \ll L_B$; *quasi-mandatory*: $\beta \ll 1$; $k_\| \ll k_\perp$; *for δf validity*: $B_1 \ll B_0$

If only one of T_i or T_e is available, assume $T_\mathrm{i} = T_\mathrm{e}$

Suggested resources:
NRL Plasma Formulary; Freethy RSI 2016

Questions & Discussion

Anything unclear at this time?