Turbulence and Transport in Fusion Plasmas
Part V

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Quiz

10-minute self-evaluated quiz:

1. How many eigenmodes (per wavenumber) exist in a fusion plasma?
2. How can we make more/fewer modes unstable?
3. What does a positive (negative) mode frequency mean?
4. Under what conditions is the gyrokinetic framework valid?

Quiz – Answers

1. Technically, there exists an infinite number of modes. In simulations, we capture a number equal to the product of all numerical resolutions.
2. By adjusting the drive: ∇T, ∇n, β, ν_{ei}; *or (bonus trick) by lowering the shear $\hat{s}$, see McKinney JPP 2019.*
3. It means the mode drifts in the ion (electron) direction along y (*recall: some use the opposite nomenclature!*)
4. Nearly closed Larmor orbits $\Rightarrow$ strong magnetization and $L_B \gg \rho$. Alternatively, no fast waves, $\omega \ll \Omega_{\mathrm{c}}$.

Gyrokinetics Derivation Plan

Now, we'll spend some time to sketch how to derive gyrokinetics

This may be the hardest part of the course :-)

1. write out Lagrangian and *one-form* γ
   $\Rightarrow$ contains all info about the dynamics of the system
2. define the gyrocenter coordinate system
3. derive the gyrocenter one-form Γ,
   perturb electric/magnetic fields
4. use Lie transform to average over gyromotion using consistent ordering (Lie: near-identity transformation)
5. plug gyroaveraged one-form into Euler-Lagrange equation
6. obtain gyrokinetic (full-f) Vlasov equation

Gyrokinetics review: Brizard RMP 2007

Gyrokinetics Derivation I

Common approach to get dynamics in many physics areas:

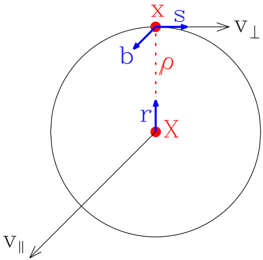
Euler-Lagrange equations $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{X}} = 0 \qquad (1)$$

with the Lagrangian for charged particles in electric & magnetic fields

$$\mathcal{L}(\mathbf{x}, \mathbf{v}) = \left(m\mathbf{v} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right)\cdot\dot{\mathbf{x}} - \left(\frac{1}{2}mv^2 + e\Phi(\mathbf{x})\right) \qquad (2)$$

Define one-form γ as

$$\gamma \equiv \mathcal{L}(\mathbf{x}, \mathbf{v})\mathrm{d}t = \overbrace{\left(m\mathbf{v} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right)}^{\gamma_{\mathbf{x}}}\cdot\mathrm{d}\mathbf{x} - \overbrace{\left(\frac{1}{2}mv^2 + e\Phi(\mathbf{x})\right)}^{-\gamma_t}\mathrm{d}t \qquad (3)$$



Coordinate transformation

$$\begin{aligned} \mathbf{x} &= \mathbf{X} + \rho(\mathbf{X})\mathbf{r}(\theta) && (4)\\ \mathbf{v} &= v_{\parallel}\mathbf{b}(\mathbf{x}) + v_{\perp}\mathbf{s}(\theta) && (5)\\ v_{\perp} &= \left(\frac{2\mu B(\mathbf{x})}{m}\right)^{1/2} && (6) \end{aligned}$$

For now, only background field: assume $L_B \gg \rho \Rightarrow \mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{X})$

Gyrokinetics Derivation II

Apply transformation to one-form (Einstein sum convention); *recall*: use transformed one-form Γ later in Euler-Lagrange

$$\Gamma_a = \gamma_b \frac{\mathrm{d}x^b}{\mathrm{d}X^a}\,, \quad a, b \text{ cover space, } v\text{-space, time coordinates} \tag{7}$$

Red terms $\propto \mathbf{r}, \mathbf{s}$ cancel under gyroaverage $\int \mathrm{d}\theta$ (i.e., $\Gamma \to \bar{\Gamma}$)

- time is not transformed, $\Gamma_t = \gamma_t = -\frac{1}{2}mv_\parallel^2 - \mu B(\mathbf{X}) - e\Phi(\mathbf{X})$
- $\Gamma_{v\parallel} = 0$ (because $\gamma_{\mathbf{v}} = 0$)
- $\Gamma_\mu = \left(mv_\parallel \mathbf{b} + mv_\perp \mathbf{s} + \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{r}\frac{B}{mv_\perp \Omega} = \mathbf{A}(\mathbf{X}) \cdot \mathbf{r}/v_\perp(\mathbf{X})$
- $\Gamma_\theta = \left(mv_\parallel \mathbf{b} + mv_\perp \mathbf{s} + \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{s}\frac{v_\perp}{\Omega} =$ $mv_\perp^2/\Omega(\mathbf{X}) + \mathbf{A}(\mathbf{X}) \cdot \mathbf{s}ev_\perp/(c\Omega(\mathbf{X}))$
- $\Gamma_\mathbf{X} = \left(mv_\parallel b_j + mv_\perp s_j + \frac{e}{c}A_j\right)\frac{\mathrm{d}x^j}{\mathrm{d}\mathbf{X}} =$ $\quad j \in \{x, y, z\}$ $mv_\parallel \mathbf{b} + mv_\perp \mathbf{s} + (e/c)\mathbf{A}(\mathbf{X}) - [\mu \mathbf{r} \cdot \mathbf{A}(\mathbf{X})/(v_\perp(\mathbf{X})B(\mathbf{X}))]\, \mathrm{d}B(\mathbf{X})/\mathrm{d}\mathbf{X}$

Overall: $$\bar{\Gamma} = \left(mv_\parallel \mathbf{b} + \frac{e}{c}\mathbf{A}\right) \cdot \mathrm{d}\mathbf{X} + \frac{\mu B}{\Omega}\mathrm{d}\theta - \left(\frac{1}{2}mv_\parallel^2 + \mu B + e\Phi\right)\mathrm{d}t \tag{8}$$

Gyrokinetics Derivation III

Unfortunately, for **perturbed fields**, $L_{\delta B,\delta\Phi} \sim \rho$ (*note*: $\Phi_0 = 0$)
$\Rightarrow$ use above one-form as equilibrium Γ_0, but must transform

perturbed one-form $$\gamma_1 = \frac{e}{c}\mathbf{A}_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} - e\Phi_1(\mathbf{x})\mathrm{d}t \qquad (9)$$

Without executing gyroaverage, can write perturbed gyrocenter one-form as

$$\begin{aligned} \Gamma_1 \quad = \quad & \frac{e}{c}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathrm{d}\mathbf{X} + \frac{1}{v_\perp}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathbf{r}\mathrm{d}\mu \\ & + \frac{mv_\perp}{B_0}\mathbf{A}_1(\mathbf{X}+\rho\mathbf{r}) \cdot \mathbf{s}\mathrm{d}\theta - e\Phi_1(\mathbf{X}+\rho\mathbf{r})\mathrm{d}t \qquad (10) \end{aligned}$$

$\Gamma_1 \to \bar{\Gamma}_1$ is tricky; keep ordering consistent via **Lie transform**

Marius Sophus Lie (1842–1899)

linearized transforms: Lie groups; generators obey Lie algebra

GK derivation possible without Lie, but lots of pitfalls . . .
Alain Brizard et al.: industry for different types of GK equations

Group Work: Canonical Transformations

45 minutes group work

Find sources that explain

- canonical transformations in Hamiltonian mechanics
- phase-space conservation

and (roughly) familiarize yourself with those.
Be prepared to present your findings.

Possible source with perhaps too much info:
`ocw.mit.edu/courses/physics/8-09-classical-mechanics-iii-fall-2014/lecture-notes/MIT8_09F14_Chapter_4.pdf`

Student Lecture

Who feels comfortable giving quick explanations of . . .

What a **Canonical Transformation** is?

What **phase-space conservation** means?

Whether a Lie transform is canonical?

What canonical transforms do in terms of phase-space conservation?

Gyrokinetics Derivation IV

First-order, near-identity (so we can drop higher-order terms) **Lie transform** (Littlejohn JMP 1982)

$$\bar{\Gamma}_{1a} = \Gamma_{1a} - G_1^b \left(\frac{\partial \Gamma_{0a}}{\partial x^b} - \frac{\partial \Gamma_{0b}}{\partial x^a} \right) + \frac{\partial S_1}{\partial x^a} \tag{11}$$

Generating function G_1^b, gauge function S:

choose such that θ dependencies in one-form vanish

Elegant choice of S: so that $\bar{\Gamma}_{1v_\parallel} = \bar{\Gamma}_{1\mu} = \bar{\Gamma}_{1\theta} = 0 = G_1^t$

This implies

$$\begin{aligned} mG_1^{\mathbf{X}} \cdot \mathbf{b}_0 &= -\frac{\partial S_1}{\partial v_\parallel} && (12) \\ \frac{mc}{e} G_1^\mu &= \frac{mv_\perp}{B_0} \mathbf{A}_1 \cdot \mathbf{s} + \frac{\partial S_1}{\partial \theta} && (13) \\ \frac{mc}{e} G_1^\theta &= -\frac{1}{v_\perp} \mathbf{A}_1 \cdot \mathbf{r} - \frac{\partial S_1}{\partial \mu} && (14) \end{aligned}$$

Here and hereafter: $\mathbf{A}_1 = \mathbf{A}_1(\mathbf{X} + \rho \mathbf{r})$ (same for Φ_1)

$$\bar{\Gamma}_{1\mathbf{X}} = \frac{e}{c} \left(\mathbf{A}_1 + G_1^{\mathbf{X}} \cdot \mathbf{B}_0^* \right) - mG_1^{v\parallel} \mathbf{b}_0 + \nabla S_1 \tag{15}$$

where $$\mathbf{B}_0^* = \nabla \times \left(\mathbf{A}_0 + \frac{mc}{e} v_\parallel \mathbf{b}_0 \right) \tag{16}$$

Gyrokinetics Derivation V

Performing gyroaverage $\langle \ldots \rangle$ leaves a "fluctuating" field: $(\mathbf{A}_1, \Phi) = \langle (\mathbf{A}_1, \Phi) \rangle + (\tilde{\mathbf{A}}_1, \tilde{\Phi})$
$\Rightarrow$ do not throw away, *instead make cancel through Lie choices!*

Demanding $\bar{\Gamma}_{1\mathbf{X}} = (e/c)\langle \mathbf{A}_1 \rangle$ and taking $\mathbf{B}_0^* \cdot \bar{\Gamma}_{1\mathbf{X}}$ and $\mathbf{b}_0 \times \bar{\Gamma}_{1\mathbf{X}}$:

$$G_1^{v\|} = \frac{1}{mB_{0\|}^*}\left(\mathbf{B}_0^* \cdot \tilde{\mathbf{A}}_1 + \mathbf{B}_0^* \cdot \nabla S_1\right) \tag{17}$$

$$G_1^{\mathbf{X}} = -\frac{1}{B_{0\|}^*}\left(\mathbf{b}_0 \times \tilde{\mathbf{A}}_1 + \frac{\mathbf{B}_0^*}{m}\frac{\partial S_1}{\partial v_\|} + \frac{c}{e}\mathbf{b}_0 \times \nabla S_1\right) \tag{18}$$

Almost there, but $\bar{\Gamma}_{1t}$ looks messy; however, can drop higher-order terms and absorb fluctuating terms into $\partial S_1/\partial\theta$:

$$\bar{\Gamma}_{1t} = -e\langle \Phi_1 \rangle - e\tilde{\Phi}_1 - \frac{1}{B_{0\|}^*}\left(\frac{\mathbf{B}_0^*}{m}\frac{\partial S_1}{\partial v_\|} + \mathbf{b}_0 \times \left(\tilde{\mathbf{A}}_1 + \frac{c}{e}\nabla S_1\right)\right) \cdot \nabla(\mu B_0)$$
$$+ \frac{\mathbf{B}_0}{B_{0\|}^*}\left(\frac{e}{c}\tilde{\mathbf{A}}_1 + \nabla S_1\right) v_\| + \frac{e}{c}\left(\frac{v_\perp}{B_0}(\langle \mathbf{A}_1 \cdot \mathbf{s} \rangle + \widetilde{\mathbf{A}_1 \cdot \mathbf{s}}) + \frac{1}{m}\frac{\partial S_1}{\partial \theta}\right)\mathbf{B}_0 + \partial_t S_1 \tag{19}$$

Now, *how to evaluate the gyroaverages here and in Eq. (15)?*

Group Work: Gyroaverage

45 minutes group work

Consider a field A and at the particle position: $A(\mathbf{X} + \rho\mathbf{r})$.

1. write down the gyroaveraged $\bar{A}(\mathbf{X}) \leftrightarrow \bar{A}(k_\perp)$ using $\int \mathrm{d}\theta$ (*note*: $\mathbf{k}_\perp \cdot \mathbf{r} = k_\perp \cos(\theta - \theta_0)$)
2. reduce this expression for $\bar{A}$ as much as possible, (*hint: you can split the average into two half-orbits*) using the definition

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi \mathrm{e}^{iz\cos\tau} \cos(n\tau)\,\mathrm{d}\tau$$

   for the Bessel function of order $n \in \mathbb{N}_0$ and argument $z \in \mathbb{C}$
3. take limit $k_\perp \rho \to 0$: what does the gyroaverage do there?

Gyrokinetics Derivation VI

So now we have $\langle[\mathbf{A}_1,\Phi_1](\mathbf{X}+\rho\mathbf{r})\rangle = J_0(k_\perp\rho)[\mathbf{A}_1,\Phi_1](\mathbf{X})$

More complicated, but can write $-\frac{e}{c}\langle\mathbf{A}_1\cdot\mathbf{s}\rangle = \underbrace{\frac{J_1(k_\perp\rho)}{k_\perp\rho}}_{\bar{J}_1}\mu B_{1\|}(\mathbf{X})$

Thus, the full gyroaveraged one-form reads

$$\begin{aligned}\bar{\Gamma} = \bar{\Gamma}_0 + \bar{\Gamma}_1 \; = \; & \left(mv_\|\mathbf{b}_0 + \frac{e}{c}\mathbf{A}_0 + \frac{e}{c}J_0 A_{1\|}\mathbf{b}_0\right)\cdot \mathrm{d}\mathbf{X} + \frac{\mu B_0}{\Omega}\mathrm{d}\theta \\ & - \left(\frac{1}{2}mv_\|^2 + \mu B_0 + eJ_0\Phi_1 + \mu\bar{J}_1 B_{1\|}\right)\mathrm{d}t \end{aligned} \tag{20}$$

Plug into Euler-Lagrange $\rightarrow$ **gyrokinetic equations of motion**

$$\begin{aligned}\dot{\mu} = 0 \qquad \dot{\mathbf{X}} = & \; v_\|\mathbf{b}_0 + \tfrac{B_0}{B_{0\|}^*}\Big(v_\|\tfrac{J_0\mathbf{B}_{1\perp}}{B_0} + c\tfrac{J_0\mathbf{E}_1\times\mathbf{B}_0}{B_0^2} \\ & + \tfrac{\mu}{m\Omega}\mathbf{b}_0\times\nabla(B_0+\bar{J}_1B_{1\|}) + \tfrac{v_\|^2}{\Omega}(\nabla\times\mathbf{b}_0)_\perp\Big)\end{aligned} \tag{21}$$

$$v_\| = \mathbf{b}_0\cdot\dot{\mathbf{X}} \qquad \dot{v}_\| = \tfrac{1}{mv_\|}\dot{\mathbf{X}}\cdot\left(eJ_0\mathbf{E}_1 - \mu\nabla(B_0+\bar{J}_1B_{1\|})\right) \tag{22}$$

$$\dot{\theta} = \Omega - \tfrac{e}{mc}\tfrac{\partial}{\partial\mu}\left(\tfrac{e}{c}v_\|J_0A_{1\|} - eJ_0\Phi_1 - \mu\bar{J}_1B_{1\|}\right) \tag{23}$$

Gyrokinetic Vlasov Equation

Final goal: **Vlasov equation** (here: full-f)

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \dot{\mathbf{X}} \cdot \nabla f + \dot{v}_\parallel \frac{\partial f}{\partial v_\parallel} + \dot{\mu} \frac{\partial f}{\partial \mu} \\
&= \frac{\partial f}{\partial t} + \left(v_\parallel \mathbf{b}_0 + \frac{B_0}{B^*_{0\parallel}} (\mathbf{v}_E + \mathbf{v}_{\nabla B} + \mathbf{v}_\mathrm{c}) \right) \\
&\quad \cdot \left(\nabla f + \frac{1}{m v_\parallel} \left(e J_0 \mathbf{E}_1 - \mu \nabla (B_0 + \bar{J}_1 B_{1\parallel}) \right) \frac{\partial f}{\partial v_\parallel} \right) = 0
\end{aligned} \tag{24}$$

- parallel streaming, drifts, gradient drive, trapping
- useful to split $f \to F_0 + f_1$ for clarity, efficiency
- can add collision term on right-hand-side
- magnetic geometry “hidden away” in $\mathbf{B}_0$, drifts
- complemented by **field equations** (Maxwell)

Recall: Gyrokinetic Equations

Pueschel PoP 2011: δf equations for GENE code (`www.genecode.org`; normalization!)

$$\frac{\partial g_j}{\partial t}+\frac{1}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left(\frac{v_\|^2}{\Omega_j}+\frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x}-\kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy}-\left(\frac{\partial B_0}{\partial y}+\kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right]+\frac{B_{\rm ref}}{JB_0}v_\|\Gamma_{jz}$$

$$-\frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left[L_n^{-1}+\left(\frac{m_jv_\|^2}{2T_{j0}}+\frac{\mu B_0}{T_{j0}}-\frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial\chi}{\partial y}+\left(\frac{v_\|^2}{\Omega_j}+\frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y}+\kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+\frac{1}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy}-\frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|}+\frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left(\frac{\partial\chi}{\partial x}\Gamma_{jy}-\frac{\partial\chi}{\partial y}\Gamma_{jx}\right)=\left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j=f_j-\frac{q_j}{m_jc}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|}\quad \chi=\bar{\Phi}-\frac{v_\|}{c}\bar{A}_\|+\frac{\mu}{q_j}\bar{B}_\|\quad \Phi=\frac{\mathcal{C}_3\mathcal{M}_{00}-\mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3-\mathcal{C}_2^2}\quad B_\|=\frac{\mathcal{C}_1\mathcal{M}_{01}-\mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3-\mathcal{C}_2^2}$$

$$\mathcal{M}_{00}=\sum_j\frac{2q_j}{m_j}\pi B_0\int J_0f_j\,\mathrm{d}v_\|\,\mathrm{d}\mu\qquad \mathcal{M}_{01}=\sum_j\frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int\mu^{1/2}J_1f_j\,\mathrm{d}v_\|\,\mathrm{d}\mu$$

$$\mathcal{C}_1=\frac{k_\perp^2}{4\pi}+\sum_j\frac{q_j^2n_{j0}}{T_{j0}}(1-\Gamma_0)\quad \mathcal{C}_2=-\sum_j\frac{q_jn_{j0}}{B_0}(\Gamma_0-\Gamma_1)\quad \mathcal{C}_3=-\frac{1}{4\pi}-\sum_j\frac{m_jn_{j0}v_{Tj}}{B_0^2}(\Gamma_0-\Gamma_1)$$

$$A_\|=\left(\sum_j\frac{8\pi^2q_jB_0}{m_jc}\int v_\|J_0g_j\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)\left(k_\perp^2+\sum_j\frac{8\pi^2q_j^2B_0}{m_jc^2T_{j0}}\int v_\|^2J_0^2F_{j0}\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk}=\partial_kg_j+\partial_{v_\|}F_{j0}\partial_k\chi_jq_j/(m_jv_\|)+\bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_jc)$$

Questions & Discussion

Anything unclear so far?

Anything clear so far?

Moments Preparation

Coming up: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2.
3.
4.

Reducing Gyrokinetics I

Terms vanishing due to $\beta = 0$, drop nonlinearity

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left[\left(\frac{\partial B_0}{\partial x} - \kappa_2 \frac{\partial B_0}{\partial z} \right) \Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \Gamma_{jx} \right] + \frac{B_{\mathrm{ref}}}{J B_0} v_\| \Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2} \right) L_{Tj}^{-1} \right] \left[c \frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j} \right) \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z} \right) \right]$$

$$+ \frac{1}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \frac{4\pi v_\|^2}{B_0 \Omega_j} \frac{\partial p_{j0}}{\partial x} \Gamma_{jy} - \frac{B_{\mathrm{ref}}}{J B_0} \frac{\mu}{m_j} \frac{\partial B_0}{\partial z} \frac{\partial f_j}{\partial v_\|} + \frac{c F_{j0}}{B_{\mathrm{ref}}} \frac{B_0}{B^*_{0\|}} \left(\frac{\partial \chi}{\partial x} \Gamma_{jy} - \frac{\partial \chi}{\partial y} \Gamma_{jx} \right) = \left. \frac{\partial f_j}{\partial t} \right|_{\mathrm{coll}}$$

$$g_j = f_j - \frac{q_j}{m_j c} \bar{A}_\| \frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c} \bar{A}_\| + \frac{\mu}{q_j} \bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3 \mathcal{M}_{00} - \mathcal{C}_2 \mathcal{M}_{01}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1 \mathcal{M}_{01} - \mathcal{C}_2 \mathcal{M}_{00}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j} \pi B_0 \int J_0 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j \pi (2B_0/m_j)^{3/2}}{c k_\perp} \int \mu^{1/2} J_1 f_j \, \mathrm{d}v_\| \, \mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}} (1 - \Gamma_0) \quad \mathcal{C}_2 = - \sum_j \frac{q_j n_{j0}}{B_0} (\Gamma_0 - \Gamma_1) \quad \mathcal{C}_3 = - \frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2} (\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c} \int v_\| J_0 g_j \, \mathrm{d}v_\| \, \mathrm{d}\mu \right) \left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}} \int v_\|^2 J_0^2 F_{j0} \, \mathrm{d}v_\| \, \mathrm{d}\mu \right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0} \partial_k \chi_j q_j / (m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j / (m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_{\mathrm{c}} = 0 \Rightarrow$ **no collisions**
3.
4.

Reducing Gyrokinetics II

Terms vanishing due to $\nu_c = 0$

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j}+\frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x}-\kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy}-\left(\frac{\partial B_0}{\partial y}+\kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right]+\frac{B_{\rm ref}}{JB_0}v_\|\Gamma_{jz}$$

$$-\frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1}+\left(\frac{m_j v_\|^2}{2T_{j0}}+\frac{\mu B_0}{T_{j0}}-\frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial\chi}{\partial y}+\left(\frac{v_\|^2}{\Omega_j}+\frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y}+\kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+\frac{1}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy}-\frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|}+\frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial\chi}{\partial x}\Gamma_{jy}-\frac{\partial\chi}{\partial y}\Gamma_{jx}\right)=\left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|}\qquad \chi = \bar\Phi - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \qquad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00}-\mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3-\mathcal{C}_2^2}\qquad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01}-\mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3-\mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0\int J_0 f_j \,\mathrm{d}v_\|\,\mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2}J_1 f_j\,\mathrm{d}v_\|\,\mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi}+\sum_j\frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0)\qquad \mathcal{C}_2 = -\sum_j\frac{q_j n_{j0}}{B_0}(\Gamma_0-\Gamma_1)\qquad \mathcal{C}_3 = -\frac{1}{4\pi}-\sum_j\frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0-\Gamma_1)$$

$$A_\| = \left(\sum_j\frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)\left(k_\perp^2+\sum_j\frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0}\,\mathrm{d}v_\|\,\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|}F_{j0}\partial_k\chi_j q_j/(m_j v_\|) + \bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_\mathrm{c} = 0 \Rightarrow$ **no collisions**
3. assume $\partial_k B_0 = 0 \Rightarrow$ **homogeneous magnetic field**
4.

Reducing Gyrokinetics III

Terms vanishing due to $\partial_k B_0 = 0$

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2\frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_{\rm ref}}{JB_0}v_\|\Gamma_{jz}$$

$$- \frac{F_{j0}}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right]\left[c\frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j\Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1\frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\frac{4\pi v_\|^2}{B_0\Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_{\rm ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_{\rm ref}}\frac{B_0}{B^*_{0\|}}\left(\frac{\partial \chi}{\partial x}\Gamma_{jy} - \frac{\partial \chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_{\rm coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\|\frac{\partial F_{j0}}{\partial v_\|} \quad \chi = \bar{\Phi} - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \quad \Phi = \frac{\mathcal{C}_3\mathcal{M}_{00} - \mathcal{C}_2\mathcal{M}_{01}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2} \quad B_\| = \frac{\mathcal{C}_1\mathcal{M}_{01} - \mathcal{C}_2\mathcal{M}_{00}}{\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0\int J_0 f_j \mathrm{d}v_\|\mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j\pi(2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2}J_1 f_j\mathrm{d}v_\|\mathrm{d}\mu$$

$$\mathcal{C}_1 = \frac{k_\perp^2}{4\pi} + \sum_j\frac{q_j^2 n_{j0}}{T_{j0}}(1-\Gamma_0) \quad \mathcal{C}_2 = -\sum_j\frac{q_j n_{j0}}{B_0}(\Gamma_0 - \Gamma_1) \quad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j\frac{m_j n_{j0}v_{Tj}}{B_0^2}(\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j\frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j \mathrm{d}v_\|\mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j\frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0}\mathrm{d}v_\|\mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|}F_{j0}\partial_k\chi_j q_j/(m_j v_\|) + \bar{A}_\|\partial_k\partial_{v_\|}F_{j0}q_j/(m_j c)$$

Moments Preparation

Rest of today: what to do with those equations?
We'll simplify them and implement the result in a simulation code

Start with **linear** gyrokinetic equations in Pueschel PoP 2011,

1. assume $\beta = 0 \Rightarrow$ **no magnetic fluctuations**
2. assume $\nu_\mathrm{c} = 0 \Rightarrow$ **no collisions**
3. assume $\partial_k B_0 = 0 \Rightarrow$ **homogeneous magnetic field**
4. assume $k_\perp \rho_j \ll 1 \Rightarrow$ **drift-kinetic limit**

Reducing Gyrokinetics IV

$J_0(k_\perp \rho_j) \to 1$ due to $k_\perp \rho_j \ll 1$, terms altered ($\lambda_\mathrm{D} \to 0$ and $\Gamma_0 \approx 1 - v_{Tj}^2 k_\perp^2/(2\Omega_j)$)

$$\frac{\partial g_j}{\partial t} + \frac{1}{B_\mathrm{ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j}\right)\left[\left(\frac{\partial B_0}{\partial x} - \kappa_2 \frac{\partial B_0}{\partial z}\right)\Gamma_{jy} - \left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z}\right)\Gamma_{jx}\right] + \frac{B_\mathrm{ref}}{JB_0} v_\| \Gamma_{jz}$$

$$- \frac{F_{j0}}{B_\mathrm{ref}}\frac{B_0}{B_{0\|}^*}\left[L_n^{-1} + \left(\frac{m_j v_\|^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right) L_{Tj}^{-1}\right]\left[c\frac{\partial \chi}{\partial y} + \left(\frac{v_\|^2}{\Omega_j} + \frac{\mu B_0}{m_j \Omega_j}\right)\left(\frac{\partial B_0}{\partial y} + \kappa_1 \frac{\partial B_0}{\partial z}\right)\right]$$

$$+ \frac{1}{B_\mathrm{ref}}\frac{B_0}{B_{0\|}^*}\frac{4\pi v_\|^2}{B_0 \Omega_j}\frac{\partial p_{j0}}{\partial x}\Gamma_{jy} - \frac{B_\mathrm{ref}}{JB_0}\frac{\mu}{m_j}\frac{\partial B_0}{\partial z}\frac{\partial f_j}{\partial v_\|} + \frac{cF_{j0}}{B_\mathrm{ref}}\frac{B_0}{B_{0\|}^*}\left(\frac{\partial \chi}{\partial x}\Gamma_{jy} - \frac{\partial \chi}{\partial y}\Gamma_{jx}\right) = \left.\frac{\partial f_j}{\partial t}\right|_\mathrm{coll}$$

$$g_j = f_j - \frac{q_j}{m_j c}\bar{A}_\| \frac{\partial F_{j0}}{\partial v_\|} \qquad \chi = J_0 \Phi - \frac{v_\|}{c}\bar{A}_\| + \frac{\mu}{q_j}\bar{B}_\| \qquad \Phi = \frac{\mathcal{C}_3 \mathcal{M}_{00} - \mathcal{C}_2 \mathcal{M}_{01}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2} \qquad B_\| = \frac{\mathcal{C}_1 \mathcal{M}_{01} - \mathcal{C}_2 \mathcal{M}_{00}}{\mathcal{C}_1 \mathcal{C}_3 - \mathcal{C}_2^2}$$

$$\mathcal{M}_{00} = \sum_j \frac{2q_j}{m_j}\pi B_0 \int J_0 f_j \mathrm{d}v_\| \mathrm{d}\mu \qquad \mathcal{M}_{01} = \sum_j \frac{q_j \pi (2B_0/m_j)^{3/2}}{ck_\perp}\int \mu^{1/2} J_1 f_j \mathrm{d}v_\| \mathrm{d}\mu$$

$$\mathcal{C}_1 = \underbrace{\frac{k_\perp^2}{4\pi}}_{\lambda_\mathrm{D}=0} + \sum_j \frac{q_j^2 n_{j0}}{T_{j0}} \underbrace{(1 - \Gamma_0)}_{v_{Tj}^2 k_\perp^2/(2\Omega_j)} \qquad \mathcal{C}_2 = -\sum_j \frac{q_j n_{j0}}{B_0}(\Gamma_0 - \Gamma_1) \qquad \mathcal{C}_3 = -\frac{1}{4\pi} - \sum_j \frac{m_j n_{j0} v_{Tj}}{B_0^2}(\Gamma_0 - \Gamma_1)$$

$$A_\| = \left(\sum_j \frac{8\pi^2 q_j B_0}{m_j c}\int v_\| J_0 g_j \mathrm{d}v_\| \mathrm{d}\mu\right)\left(k_\perp^2 + \sum_j \frac{8\pi^2 q_j^2 B_0}{m_j c^2 T_{j0}}\int v_\|^2 J_0^2 F_{j0} \mathrm{d}v_\| \mathrm{d}\mu\right)^{-1}$$

$$\Gamma_{jk} = \partial_k g_j + \partial_{v_\|} F_{j0} \partial_k \chi_j q_j/(m_j v_\|) + \bar{A}_\| \partial_k \partial_{v_\|} F_{j0} q_j/(m_j c)$$

Normalization

Reduced δf drift-kinetic equations:

$$\frac{\partial f_j}{\partial t} = -v_\parallel \frac{\partial f_j}{\partial z} - \frac{q_j m_j}{v_\parallel}\frac{\partial F_{j0}}{\partial v_\parallel}\frac{\partial \Phi}{\partial z} - \frac{cF_{j0}}{B_{\mathrm{ref}}}\left[L_n^{-1} + \left(\frac{m_j v_\parallel^2}{2T_{j0}} + \frac{\mu B_0}{T_{j0}} - \frac{3}{2}\right)L_{Tj}^{-1}\right] ik_y\Phi \quad (25)$$

$$\Phi = \frac{\sum_j \frac{2q_j}{m_j}\pi B_0 \int f_j \mathrm{d}v_\parallel \mathrm{d}\mu}{\sum_j \frac{q_j^2 n_{j0}}{T_{j0}}\frac{v_{Tj}^2 k_\perp^2}{2\Omega_j}}\ ;\ k_y > 0, \text{ adiabatic } \mathrm{e}^- \rightarrow \frac{\frac{2q_\mathrm{i}}{m_\mathrm{i}}\pi B_0 \int f_\mathrm{i} \mathrm{d}v_\parallel \mathrm{d}\mu}{\frac{q_\mathrm{e}^2 n_{\mathrm{e}0}}{T_{\mathrm{e}0}} + \frac{q_\mathrm{i}^2 n_{\mathrm{i}0}}{T_{\mathrm{i}0}}\frac{v_{T\mathrm{i}}^2 k_\perp^2}{2\Omega_\mathrm{i}}} \quad (26)$$

Suitable normalization: $x, y \rightarrow \rho$, $z \rightarrow L_z \gg \rho$, $v \rightarrow v_{\mathrm{th}}$, $t \rightarrow L_z/v_{\mathrm{th}}$, $\omega_{T\mathrm{i}} = L_z/L_{T\mathrm{i}} = -L_z \mathrm{d}\ln T_{\mathrm{i}0}/\mathrm{d}x$, $\omega_n = L_z/L_n = -L_z \mathrm{d}\ln n_0/\mathrm{d}x$
Normalize to mass, temperature, density of *singly-charged ions*

$$\frac{\partial f}{\partial t} = -v_\parallel \frac{\partial f}{\partial z} - v_\parallel F_0 \frac{\partial \Phi}{\partial z} - \left[\omega_n + \left(v_\parallel^2 + \mu - \frac{3}{2}\right)\omega_{T\mathrm{i}}\right] F_0 ik_y\Phi\,, \quad F_0 = \frac{1}{\pi^{3/2}}\mathrm{e}^{-v_\parallel^2-\mu} \quad (27)$$

$$\Phi = \pi \int f \mathrm{d}v_\parallel \mathrm{d}\mu \qquad \text{[normalization: } \Phi \rightarrow (T_{\mathrm{i}0}/e)\rho/L_z\ ;\ f/F_0 \sim \rho/L_z] \quad (28)$$

Note: no coupling between μ points

Group Work: Taking Moments

45 minutes group work:

1. perform coordinate transformation $(v_x, v_y) \to \mu$ for $\int \mathrm{d}\mathbf{v}$

Start with Eqs. (27) and (28), then

2. integrate out μ, redefining $\pi \int_0^\infty f(v_\parallel, \mu)\mathrm{d}\mu \to f(v_\parallel)$

3.

4.

5.

Group Work: Taking Moments

45 minutes group work:

1. perform coordinate transformation $(v_x, v_y) \to \mu$ for $\int \mathrm{d}\mathbf{v}$

Start with Eqs. (27) and (28), then

2. integrate out μ, redefining $\pi \int_0^\infty f(v_\|, \mu)\mathrm{d}\mu \to f(v_\|)$

Answer:

$$\frac{\partial f}{\partial t} = -v_\| \frac{\partial f}{\partial z} - v_\| F_0 \frac{\partial \Phi}{\partial z} - \left[\omega_n + \left(v_\|^2 - \frac{1}{2}\right)\omega_{T\mathrm{i}}\right] F_0 i k_y \Phi$$

$$\Phi = \int_{-\infty}^{\infty} f \mathrm{d}v_\| \qquad F_0 = \frac{1}{\pi^{1/2}} \mathrm{e}^{-v_\|^2}$$

3. take moments of the reduced drift-kinetic equations, get (linear) fluid model with n, $u_\|$, Φ
4. is the model closed? — if not, how might one do that?
5. how does the model compare to the earlier two-field model?

Questions & Discussion

Anything unclear that we talked about here?

Interpreting Eigenmodes

Now: how to interpret, produce linear data

Linear equations: **no coupling between different** k_y
$\Rightarrow$ *all real-space information contained in ballooning function*

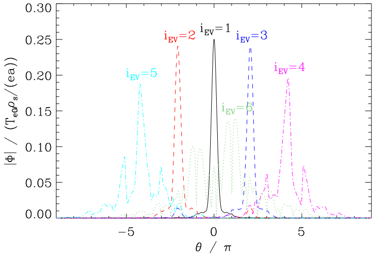
Available data

All information for a mode:

- eigenvalue (γ, ω)
- eigenvector $f(k_x, z, v_\parallel, \mu)$
  $\rightarrow \Phi(k_x, z) \rightarrow \Phi(\theta_p)$

but can have many modes i_{EV}

Pueschel PRL 2016: low-$\hat{s}$ stellarator



From $\Phi(t)$ (or $f(t)$), **dominant eigenvalue** easy to extract

- growth rate γ: fit straight line to log plot, or $\dfrac{\ln \frac{\Phi(t+\Delta t)}{\Phi(t)}}{\Delta t}$
- frequency ω: phase difference between $\Phi(t)$, $\Phi(t+\Delta t)$

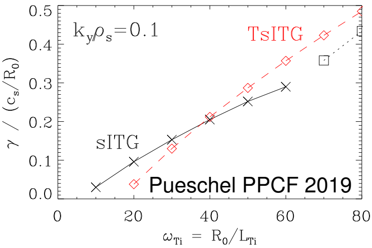
Modes' **ballooning structure**: depends on k_y, geometry, $\nabla n, T$

Excitation States I

Early theory (e.g., Coppi PoF 1967): infinite number of solutions

$$\left.\begin{matrix}\frac{\gamma}{c_s/R_0} \\ \frac{\omega}{c_s/R_0}\end{matrix}\right\} = \frac{\ell}{2} k_y \omega_{T\mathrm{i}}^{1/2} \hat{s}^{1/2} \sqrt{\frac{T_\mathrm{i}}{T_\mathrm{e}}} \qquad \Phi_\ell(\vartheta) = \mathrm{e}^{-\frac{\vartheta^2}{2}} \underbrace{(-1)^\ell \mathrm{e}^{\vartheta^2} \frac{\mathrm{d}^\ell}{\mathrm{d}\vartheta^\ell} \mathrm{e}^{-\vartheta^2}}_{\text{Hermite polynomial}}$$

Implying $\gamma(\ell \to \infty) \to \infty$*! How is this possible?*



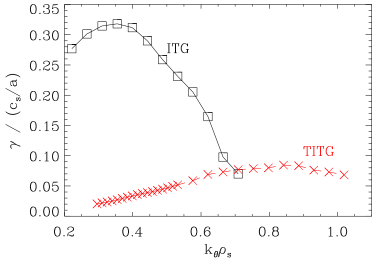
In reality:

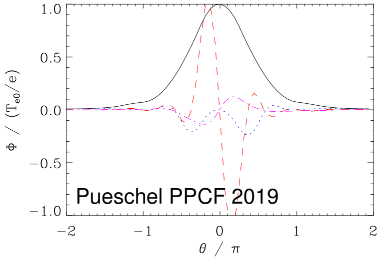
ℓ **stabilizes** at typical gradients, need **large gradients** to have higher ℓ dominate

Here: $\ell_{\mathrm{sITG}} = 0$, $\ell_{\mathrm{TsITG}} = 1$

Excitation States II

Realistic example with strong gradients, higher-ℓ states:
NSTX #129016 (lithium coating study; Guttenfelder NF 2013)





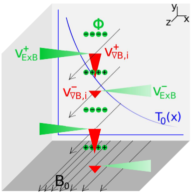
ITG: $\mathrm{Re} + \mathrm{Im}$ — TITG: $\mathrm{Re} + \mathrm{Im}$

$\Phi(t) \propto \exp(i\omega t)$, thus $\mathrm{Re}\,\Phi$ and $\mathrm{Im}\,\Phi$ change continuously

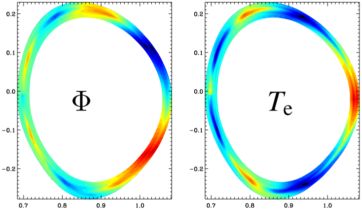
- different ℓ coexist, not exact same properties
- even (odd) Hermites: even (odd) Φ (and odd (even) $A_{\|}$)
- $\Rightarrow$ even/odd **parity not sufficient for mode ID**

Phases

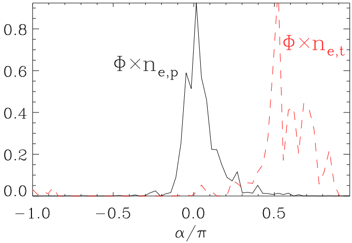
Complex phase of eigenfunction $\leftrightarrow$ position along y



Recall: $\pi/2$ shift between Φ and $\{n, T\}$ ideal for growth



Can be expressed as
complex phase angle



Above case:

- passing e^-: $\alpha_{\Phi \times n} = 0$
- trapped e^-: $\alpha_{\Phi \times n} = \pi/2$

$\Rightarrow \nabla n$-TEM

Relation to Turbulence

Nonlinear physics discussed a little later, but for the moment:

- **fluxes often scale like** γ
- on large scales, (non-)linear **phases match** in most cases
- **eigenmode widths correlate** with turbulent ballooning widths
- fluxes tend to peak at lower k_y than does γ (explanation will be given later)

Transport researcher: “nonlinear often well-captured by linear”
Turbulence researcher: “nonlinear often deviates from linear”

Group Work: IV vs. EV

30 minutes group work:

Digest sources that explain the difference between the

- initial-value and the
- eigenvalue

approach to linear simulations. Be prepared to present your findings.

Which one can be used, and how, to obtain multiple eigenvalues/-vectors at a single wavenumber?

Student Lecture

Anyone, who can explain . . .

Why do different eigenvalues exist at each k_y?

How does **matrix inversion** differ from **iterative solving**?

What is an **eigenvalue simulation**?

Bonus: how to get **subdominant modes from IV** runs?

Demonstration

Let's have a look at all that in action!

With the GENE Diagnostics Tool, we can

1. extract eigenvalues
2. look at evolving and converged mode structures; IV vs. EV
3. evaluate phases
4. compare moments, compute quasilinear ratios (e.g., Q/n^2)
5. identify different instabilities

See `gene-diag.pdf` for a summary