**Solutions 5**

**1. a)** This chain is clearly ergodic. The transition matrix is

$$\begin{pmatrix} 1-p & p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & p & 1-p \end{pmatrix}$$

Assume that the detailed balance equation is satisfied. Then

$$\pi_1^*/2 = \pi_2^* p = \pi_0^* p$$

We conclude that

$$\pi_0^* = \pi_2^* = \frac{1}{2(1+p)} \quad \pi_1^* = \frac{p}{1+p}$$

It is then easy to verify that $\pi^* = \pi^* P$, and so this is indeed a stationary distribution, which obviously satisfies the detailed balance equation.

**b)** We know that $\lambda_0 = 1$, and so, to compute the eigenvalues, we must solve the equations

$$\begin{aligned} 2-2p &= 1+\lambda_1+\lambda_2 \\ -p(1-p) &= \lambda_1\lambda_2 \end{aligned}$$

Solving this, we obtain that $\lambda_1 = 1-p$ and $\lambda_2 = -p$. So $\lambda_* = \max(p, 1-p)$ and the spectral gap is given by $\gamma = 1-\lambda_* = \min(p, 1-p)$.

**c)** For $p = \frac{1}{N}$, the spectral gap is $\gamma = \frac{1}{N}$. From the theorem seen in class, we know that $\|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{\exp(-\gamma n)}{2\sqrt{\pi_i}}$, so here,

$$\max_{i\in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2}\sqrt{\frac{1+1/N}{1/N}}\exp(-n/N) \leq \sqrt{N}\exp(-n/N) = \exp\left(\frac{\log N}{2} - \frac{n}{N}\right)$$

Taking therefore $n \geq N\left(\frac{\log N}{2} + c\right)$ with $c > 0$ sufficiently large (more precisely, $c = \log(1/\varepsilon)$) ensures that the maximum total variation norm is below $\varepsilon$.

**d)** For $p = 1-\frac{1}{N}$, the spectral gap is again $\gamma = \frac{1}{N}$. So

$$\max_{i\in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2}\sqrt{2(2-1/N)}\exp(-n/N) \leq \exp(-n/N)$$

Taking therefore $n \geq cN$ with $c = \log(1/\varepsilon)$ ensures that the maximum total variation norm is below $\varepsilon$, so

$$T_\varepsilon \leq N\log(1/\varepsilon)$$

**2. a)** The transition matrix being doubly stochastic, the stationary distribution is uniform (i.e. $\pi_i = \frac{1}{2N}$ for every $i \in S$) and satisfies the detailed balance equation.

**b)** Solving the equation $P\phi^{(1)} = \lambda\phi^{(1)}$, we obtain

$$\begin{aligned} \tfrac{N-1}{N}\, a + \tfrac{1}{N}\, b &= \lambda\, a \\ \tfrac{N-1}{N}\, a - \tfrac{1}{N}\, b &= \lambda\, b \end{aligned}$$

which is saying that $\lambda$ is an eigenvalue of the $2 \times 2$ matrix

$$\begin{pmatrix} \frac{N-1}{N} & \frac{1}{N} \\ \frac{N-1}{N} & -\frac{1}{N} \end{pmatrix} = \begin{pmatrix} 1-\delta & \delta \\ 1-\delta & -\delta \end{pmatrix}$$

where we have set $\delta = \frac{1}{N}$. These eigenvalues are given by

$$\lambda_\pm = \frac{1 - 2\delta \pm \sqrt{(1-2\delta)^2 + 8\delta(1-\delta)}}{2} = \frac{1 - 2\delta \pm \sqrt{1 + 4\delta - 4\delta^2}}{2}$$

For $\delta$ small (i.e. $N$ large), the largest of these 2 eigenvalues is $\lambda_+$, which is approximately given by

$$\lambda_+ \simeq \frac{1 - 2\delta + (1 + 2\delta - 4\delta^2)}{2} = 1 - 2\delta^2 = 1 - \frac{2}{N^2}$$

so the spectral gap $\gamma \simeq \frac{2}{N^2}$.

**c)** By the theorem seen in class,

$$\max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \le \frac{\sqrt{2N}}{2} \exp(-\gamma n) \le \sqrt{2} \exp\left(\frac{\log N}{2} - \frac{2n}{N^2}\right)$$

is below $\varepsilon$ for $n \ge \frac{N^2}{2}\left(\frac{\log N}{2} + \log(\sqrt{2}/\varepsilon)\right)$, so

$$T_\varepsilon \le \frac{N^2}{2}\left(\frac{\log N}{2} + \log(\sqrt{2}/\varepsilon)\right)$$