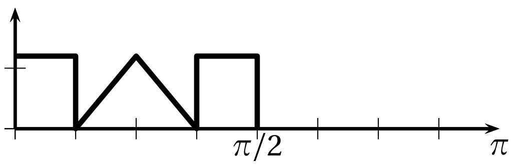
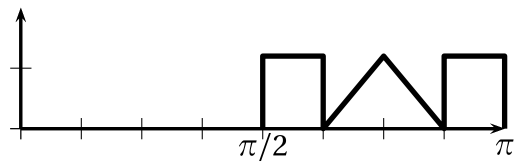
COM-303 - Signal Processing for Communications

Solutions for Homework #10

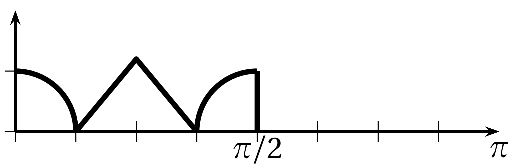
Solution 1. Multirate Signal Processing



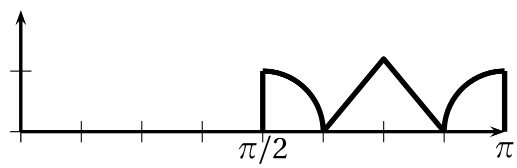
$Y_1(e^{j\omega})$



$Y_2(e^{j\omega})$



$Y_3(e^{j\omega})$



$Y_4(e^{j\omega})$

Solution 2. Multirate identities

(a) Let us denote the downsampling by 2 and upsampling by 2 operations by $D_2\{\cdot\}$ and $U_2\{\cdot\}$ respectively.

- Downsampling by 2 followed by filtering by $H(z)$ can be written as

$$\begin{aligned} Y(z) &= H(z)D_2\{X(z)\} \\ &= \frac{1}{2}H(z)\big(X(z^{1/2}) + X(-z^{1/2})\big). \end{aligned}$$

Filtering by $H(z^2)$ followed by downsampling by 2 can be written as

$$\begin{aligned}
Y(z) &= D_2\{H(z^2)X(z)\} \\
&= \frac{1}{2}\big(H(z)X(z^{1/2}) + H(z)X(-z^{1/2})\big) \\
&= \frac{1}{2}H(z)\big(X(z^{1/2}) + X(-z^{1/2}))\big).
\end{aligned}$$

The two operations are thus equivalent.

- Filtering by $H(z)$ followed by upsampling by 2 can be written as

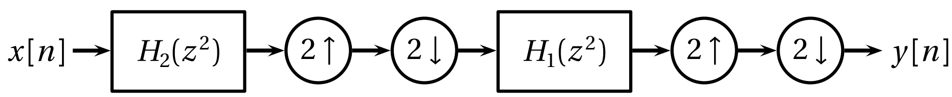
$$\begin{aligned}
Y(z) &= U_2\{H(z)X(z)\} \\
&= H(z^2)X(z^2).
\end{aligned}$$

Upsampling by 2 followed by filtering by $H(z^2)$ can be written as

$$\begin{aligned}
Y(z) &= H(z^2)U_2\{X(z)\} \\
&= H(z^2)X(z^2).
\end{aligned}$$

The two operations are thus equivalent.

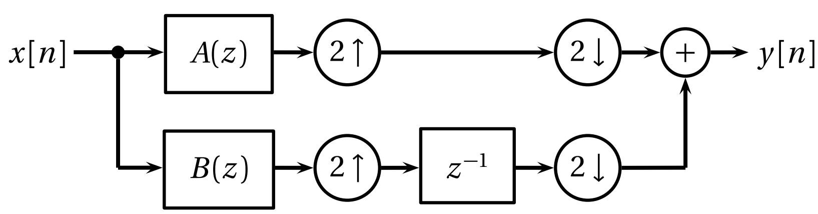
(b) Using the identities proven in (a), the system can be redrawn as



Upsampling by N immediately followed by downsampling by N leaves the signal unchanged so the transfer function of this system is given by

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z).$$

(c) Again using (a), our system is equivalent to



The lower branch contains an upsampler followed by a delay and a downsampler. The output of such a system is easily seen to be 0. Thus only the upper branch remains and the final transfer function of the system is given by

$$\frac{Y(z)}{X(z)} = A(z).$$

(d) System 1 is described by the following equation

$$\begin{aligned}
Y(z) &= D_2\{H(z)G(z)U_2\{X(z)\}\} \\
&= D_2\{H(z)G(z)X(z^2)\} \\
&= \frac{1}{2}\big(H(z^{1/2})G(z^{1/2})X(z) + H(-z^{1/2})G(-z^{1/2})X(z)\big) \\
&= \frac{1}{2}\big(H(z^{1/2})G(z^{1/2}) + H(-z^{1/2})G(-z^{1/2})\big)X(z) \\
&= X(z).
\end{aligned}$$

System 1 is thus unity.

System 2 is described by the following equation

$$\begin{aligned}
Y(z) &= D_2\{H(z)F(z)U_2\{X(z)\}\} \\
&= D_2\{H(z)F(z)X(z^2)\} \\
&= \frac{1}{2}\big(H(z^{1/2})F(z^{1/2})X(z) + H(-z^{1/2})F(-z^{1/2})X(z)\big) \\
&= \frac{1}{2}\big(H(z^{1/2})F(z^{1/2}) + H(-z^{1/2})F(-z^{1/2})\big)X(z) \\
&= 0.
\end{aligned}$$

System 2 is thus zero.

Solution 3. Quantization

The output of the filter is

$$\begin{aligned}
y[n] &= \hat{x}[n] * h[n] \\
&= h[n] * (x[n] + e[n]) \\
&= y[n] + e_y[n].
\end{aligned}$$

The second moment of the filtered noise is

$$\begin{aligned}
\sigma_{e_y}^2 &= \mathrm{E}\left[\sum_k h[k]e[n-k]\sum_l h[l]e[n-l]\right) \\
&= \sum_k\sum_l h[k]h[l]\underbrace{E(e[n-k]e[n-l]}_{\sigma_e^2\delta[k-l]} \\
&= \sigma_e^2\sum_k h^2[k] \\
&= \sigma_e^2\sum_{k=0}^{\infty}\frac{1}{4}(a^k+(-a)^k)^2 = \frac{\sigma_e^2}{4}\sum_{k=0}^{\infty}(a^{2k}+2a^k(-a)^k+(-a)^{2k}) \\
&= \frac{\sigma_e^2}{2}\left(\sum_{k=0}^{\infty}a^{2k}+\sum_{k=0}^{\infty}(-a^2)^k\right) \\
&= \frac{\sigma_e^2}{2}\left(\frac{1}{1-a^2}+\frac{1}{1+a^2}\right) \\
&= \sigma_e^2\left(\frac{1}{1-a^4}\right) = \frac{\Delta^2}{12(1-a^4)}
\end{aligned}$$

The same derivation can be carried out for $x[n]$ (since we assumed the input white) so that the input SNR does not change by filtering:

$$\mathrm{SNR}_{y[n]} = \frac{12\sigma_x^2}{\Delta^2}$$

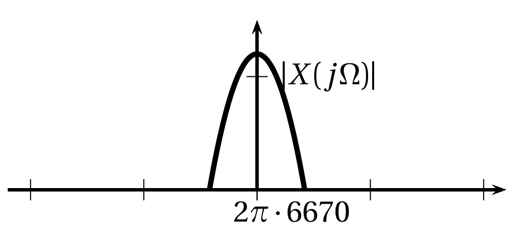
Solution 4. Digital processing of continuous-time signals

(a) Playing the record at lower rpm slows the signal down by a factor 33/78. Therefore

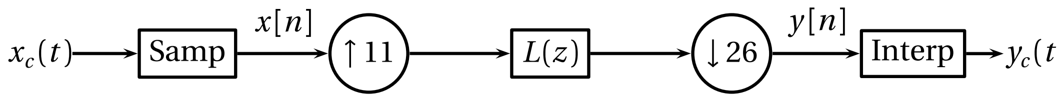
$$x(t) = s(\frac{33}{78}\ t) = s(\frac{11}{26}\ t)$$

(b) From the rescaling property of the Fourier transform

$$X(j\Omega) = \frac{26}{11}\ S(j\frac{26}{11}\Omega)$$



(c) We need to change the sampling rate so that, when $y[n]$ is interpolated at 44.1 KHz its spectrum is equal to $S(j\Omega)$. The rational sampling rate change factor is clearly 33/78 which is simply 11/26 after factoring. The processing scheme is as follows:



where $L(z)$ is a lowpass filter with cutoff frequency $\pi/26$ and gain $L_0 = 11/26$; both the sampler and interpolator work at $T_s = 1/44100$. We have:

$$
\begin{aligned}
X_c(j\Omega) &= \frac{26}{11}\, S(j\frac{26}{11}\Omega) \\
X(e^{j\omega}) &= \frac{1}{T_s}\, X_c(j\frac{\omega}{T_s}) \\
Y(e^{j\omega}) &= L_0\, X(e^{j\frac{11}{26}\omega}) \\
&= \frac{11}{26}\frac{1}{T_s}\, X_c(j\frac{11}{26}\frac{\omega}{T_s}) \\
&= \frac{1}{T_s}\, S(j\frac{\omega}{T_s}) \\
Y_c(j\Omega) &= T_s\, Y(e^{j\Omega T_s}) \\
&= S(j\Omega)
\end{aligned}
$$

(d) The sampling rate change scheme stays the same except that now $45/78 = 15/26$. Therefore, the final upsampler has to compute more samples than in the previous scheme. The computational load of the sampling rate change is entirely dependent on the filter $L(z)$. If we upsample more before the output, we need to compute more filtered samples and therefore at 45rpm the scheme is less efficient.

Solution 5. Oversampled sequences

Given that $X(e^{j\omega}) = 0$ for $\frac{\pi}{3} \le |\omega| \le \pi$, $x[n]$ can be thought of as a signal that has been sampled at 3 times the Nyquist frequency. Therefore, we can downsample the signal without losing information at least by a factor of 3.

(a) Assume n_0 is odd; we can then downsample $x[n]$ by 2, without loss of information and the corrupted sample will be discarded in the downsampling operation. We can then upsample by 2 and recover the original signal eliminating the error. If n_0 is even, simply shift the signal by 1 and perform the same operation.

(b) If the value of n_0 is not known, we need to determine whether n_0 is odd or even. We can write

$$\hat{x}[n] = x[n] - \epsilon\delta[n - n_0]$$

and therefore

$$\hat{X}(e^{j\omega}) = X(e^{j\omega}) - \epsilon e^{-j\omega n_0}$$

Now, if we compute the DTFT at $\omega = \frac{\pi}{2}$ we have:

$$\hat{X}(e^{j\frac{\pi}{2}}) = X(e^{j\frac{\pi}{2}}) + \epsilon(-j)^{n_0} = \epsilon(-j)^{n_0}$$

since, by hypothesis, $X(e^{j\frac{\pi}{2}}) = 0$. Therefore, If $\hat{X}(e^{j\frac{\pi}{2}})$ is real, n_0 is even and if it is imaginary, n_0 is odd.

(c) If there are k corrupted samples, the worst case is when the corrupted samples are consecutive. In that case we need to downsample $\hat{x}[n]$ by a factor of k and then upsample it back. To do so without loss of information it must be:

$$X(e^{j\omega}) = 0 \quad \text{for} \quad \frac{\pi}{k} \leq |\omega| \leq \pi.$$