Problem Set 3 — *Not Graded!*

For the Exercise Sessions on Oct 25

Last name	First name	SCIPER Nr	Points

Rules : You are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it.

Problem 1:

Find the maximum entropy density f, defined for $x \geq 0$, satisfying $E[X] = \alpha_1$, $E[\ln X] = \alpha_2$. That is, maximize $-\int f \ln f$ subject to $\int x f(x) dx = \alpha_1$, $\int (\ln x) f(x) dx = \alpha_2$, where the integral is over $0 \leq x < \infty$. What family of densities is this?

Problem 2:

What is the maximum entropy distribution $p(x, y)$ that has the following marginals?

x \ y	1	2	3	
1	p_{11}	p_{12}	p_{12}	$\frac{1}{2}$
2	p_{21}	p_{22}	p_{23}	$\frac{1}{4}$
3	p_{31}	p_{32}	p_{33}	$\frac{1}{4}$
	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	

Problem 3:

Let $Y = X_1 + X_2$. Find the maximum entropy of Y under the constraint $E[X_1^2] = P_1$, $E[X_2^2] = P_2$:

(a) If X_1 and X_2 are independent.

(b) If X_1 and X_2 are allowed to be dependent.

Problem 4:

We learned in the course that as long as the set of feasible means is open then every such mean can be realized by an element of the exponential family. In the following verify this explicitly (by not referring to the above statement for the following scenario).

(i) Let $\phi(x) = (x^2)$.

(ii) Let $\phi(x)$ consist of all elements $x_i x_j$, where i and j go from 1 to K.

Problem 5:

What is the maximum entropy distribution, call it $p(x, i)$, on $[0, \infty] \times \mathbb{N}$, both of whose marginals have mean $\mu > 0$. (I.e., in one axis the distribution is over the positive reals, whereas in the other one it is over the natural numbers.)

Problem 6:

Let P denote the zero-mean and unit-variance Gaussian distribution. Assume that you are given N iid samples distributed according to P and let $\hat{P}_N$ be the empirical distribution.

Let Π denote the set of distributions with second moment $\mathbb{E}[X^2] = 2$. We are interested in

$$\lim_{N \to \infty} \frac{1}{N} \log \Pr\{\hat{P_N} \in \Pi\} = - \inf_{Q \in \Pi} D(Q \| P).$$

(i) Determine $-\text{arginf}_{Q \in \Pi} D(Q \| P)$, i.e., determine the element Q for which the infinum is taken on.

(ii) Determine $-\inf_{Q \in \Pi} D(Q \| P)$.