COM303: Digital Signal Processing

Lecture 6: DFS and DTFT

Overview

- periodicity in the DFT
- the DFS
- the DTFT

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- the DFS
- the DTFT

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- the DTFT

DFT formulas

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *“time” domain*

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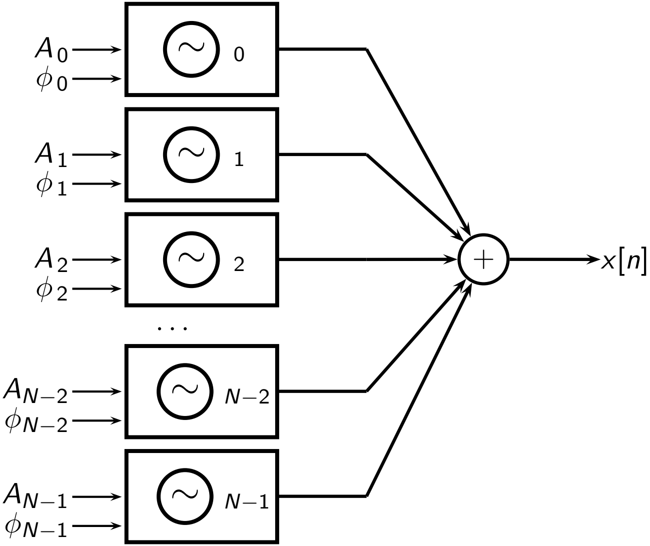
DFT synthesis formula



Running the machine too long...

$$x[n+N] = x[n]$$

output signal is N-periodic!

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

produces an N-point signal in the time domain

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Discrete Fourier Series (DFS)

DFS $=$ DFT with periodicity explicit

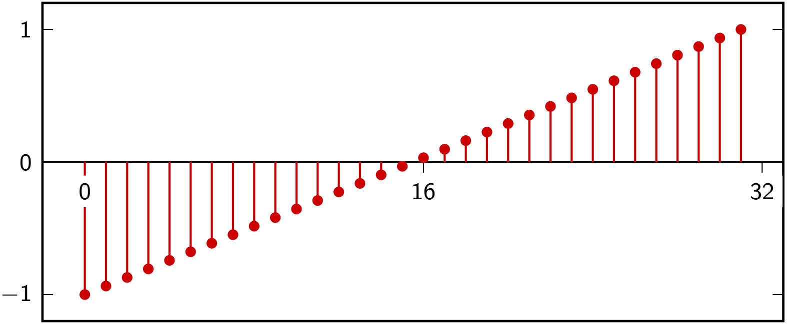
- the DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- the inverse DFS maps an N-periodic sequence of Fourier coefficients a set onto an N-periodic signal
- the DFS of an N-periodic signal is mathematically equivalent to the DFT of one period

the Fourier transform for periodic signals

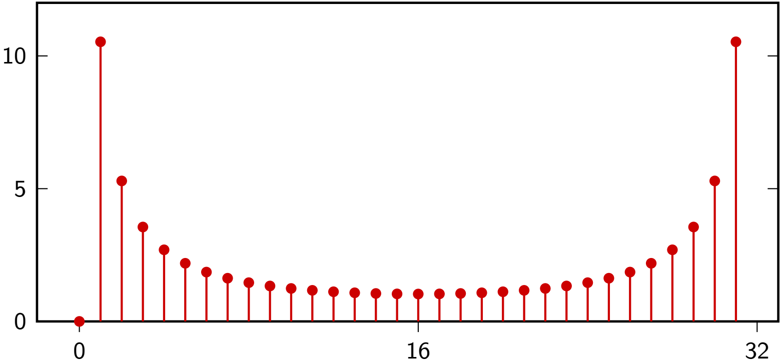
Periodic sequences: a bridge to infinite-length signals

- N-periodic sequence: N degrees of freedom
- DFS: only N Fourier coefficients capture all the information

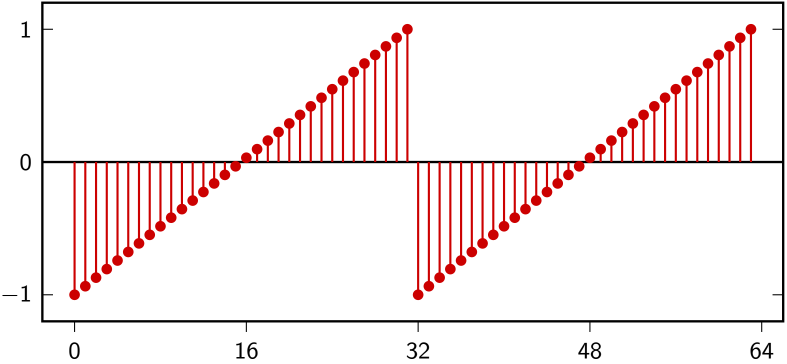
Example: 32-tap sawtooth wave



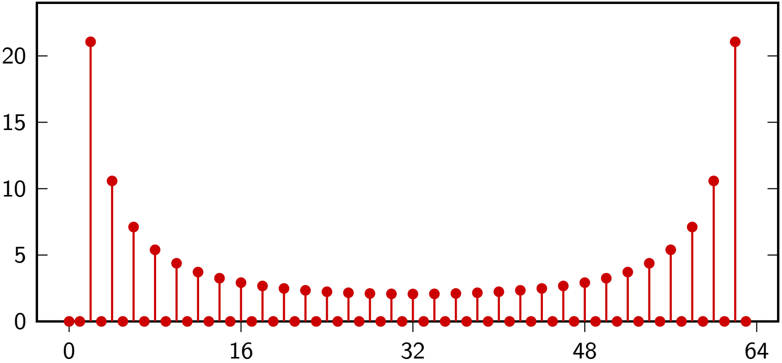
Example: DFT of 32-tap sawtooth wave



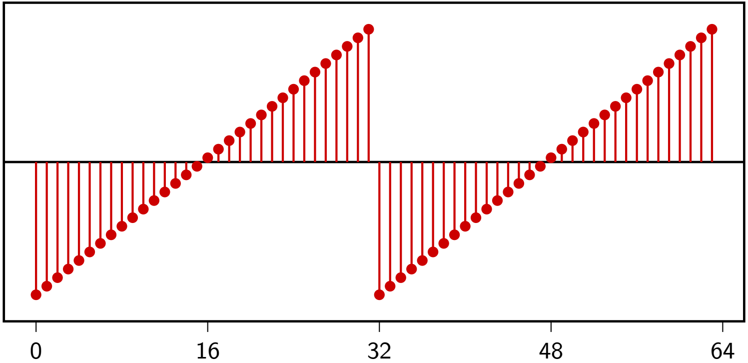
What if we take the DFT of two periods?



Example: 64-point DFT of two periods



DFT of two periods: intuition



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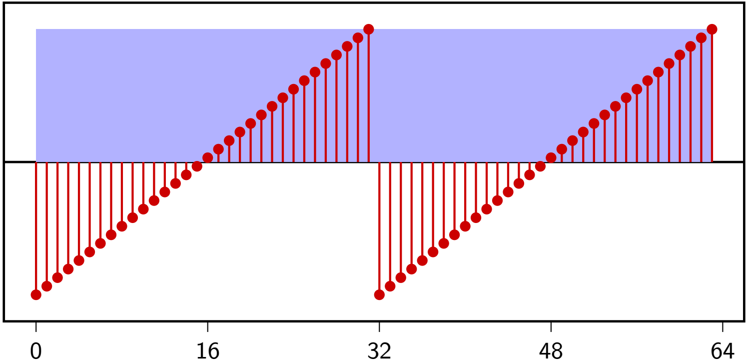
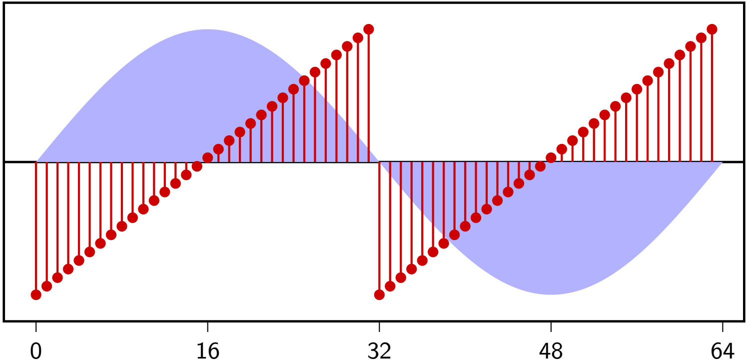


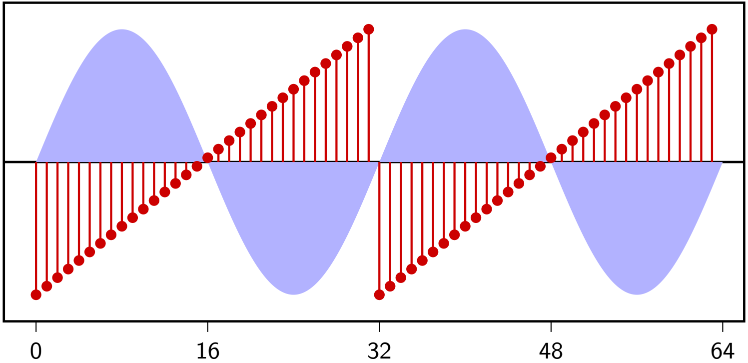
$k = 0$

DFT of two periods: intuition



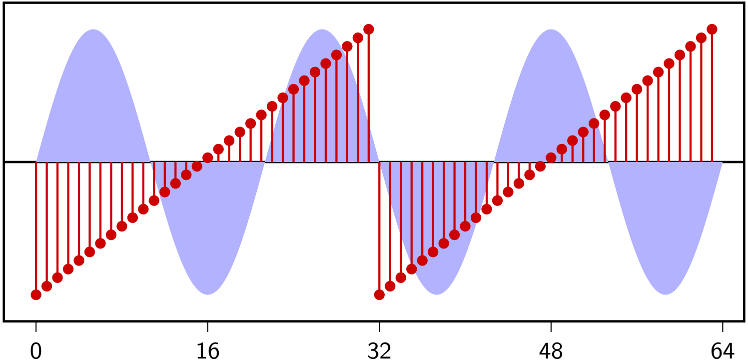
$k = 1$

DFT of two periods: intuition



$k = 2$

DFT of two periods: intuition



$k = 3$

DFT of L periods

ingredients:

- finite-length signal $x[n]$, $n = 0, 1, \ldots, N-1$
- periodic signal: $\tilde{x}[n] = x[n \mod N]$
- obviously $\tilde{x}[n] = \tilde{x}[n + pM]$ for all $p \in \mathbb{Z}$

DFT of L periods

$$X_L[k] = \sum_{n=0}^{LM-1} \tilde{x}[n] e^{-j\frac{2\pi}{LM}nk} \qquad k = 0, 1, 2 \ldots, LM-1$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} \tilde{x}[n+pM] e^{-j\frac{2\pi}{LM}(n+pM)k}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} \tilde{x}[n] e^{-j\frac{2\pi}{LM}nk} e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{M-1} x[n]\, e^{-j\frac{2\pi}{LM}nk}$$

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DFT of L periods

$$
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&= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} \tilde{x}[n + pM] e^{-j\frac{2\pi}{LM}(n+pM)k} \\
&= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} \tilde{x}[n] e^{-j\frac{2\pi}{LM}nk} e^{-j\frac{2\pi}{L}pk} \\
&= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{M-1} x[n]\, e^{-j\frac{2\pi}{LM}nk}
\end{aligned}
$$

We've seen this before

$$\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} = \begin{cases} L & \text{if } k \text{ multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

DFT of L periods

if k is a multiple of L then k/L is an integer, so:

$$\sum_{n=0}^{M-1} x[n]\, e^{-j\frac{2\pi}{M}n\frac{k}{L}} = X[k/L]$$

DFT of L periods

$$X_L[k] = \begin{cases} L\,X[k/L] & \text{if } k = 0, L, 2L, 3L, \dots \\ 0 & \text{otherwise} \end{cases}$$

DFT and DFS

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period
- to stress the periodicity of the underlying signal, we use the term DFS

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Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- DFS $\{\tilde{x}[n-M]\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- IDFS $\left\{ \tilde{X}[k] \right\} = \tilde{x}[n-M]$

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a delay in time becomes a *linear phase* factor in frequency

Finite-length time shifts revisited

For an N-point signal $x[n]$:

- $x[n-M]$ is *not* well-defined
- what is IDFT $\left\{e^{-j\frac{2\pi}{N}Mk}\ X[k]\right\}$?

Finite-length time shifts revisited

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Finite-length time shifts revisited

$$\mathrm{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}$$

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$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = \begin{cases} N & \text{if } k \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

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$$\forall L, N \in \mathbb{N}, \exists p \in \mathbb{N} : \quad L = pN + (L \mod N)$$

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shifts for finite-length signals are “naturally” circular

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The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
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the Fourier transform for infinite-length signals

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $\dfrac{2\pi}{N}k$ becomes denser in $[0, 2\pi]$...
- In the limit $\dfrac{2\pi}{N}k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \qquad \omega \in \mathbb{R}$$

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Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

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DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega + 2k\pi)n} \quad \forall k \in \mathbb{N}$
- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

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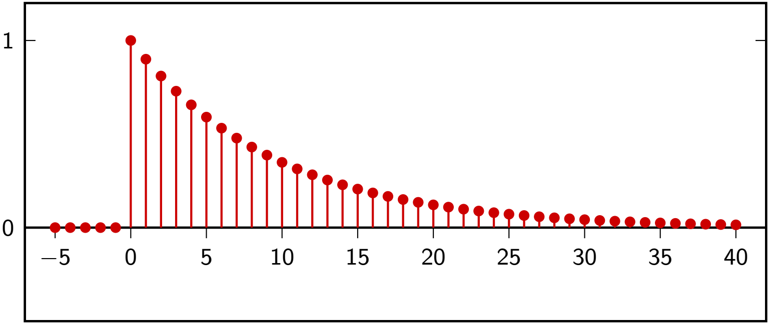
$x[n] = \alpha^n u[n], \quad |\alpha| < 1$



DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

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$$= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n}$$

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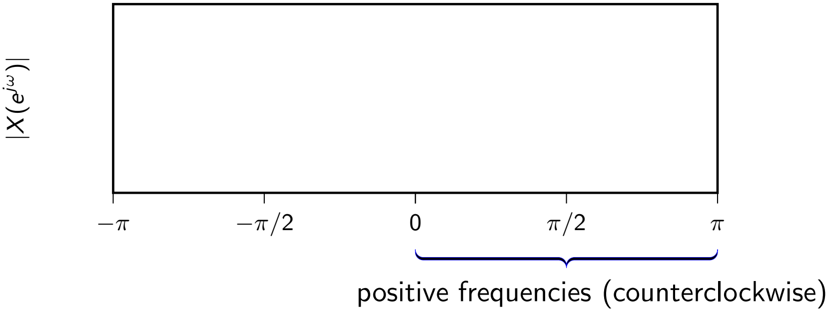
$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \\
&= \frac{1}{1 - \alpha e^{-j\omega}}
\end{aligned}
$$

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$|X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}$$

Plotting the DTFT



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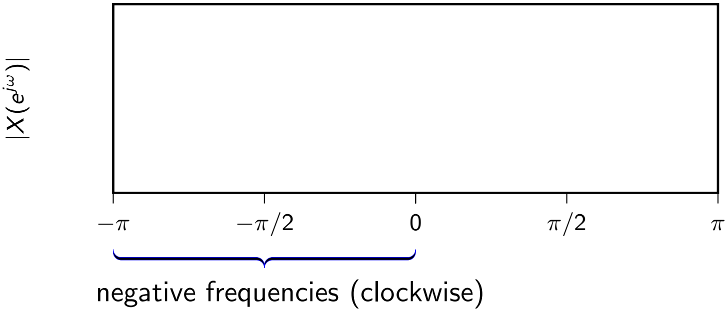

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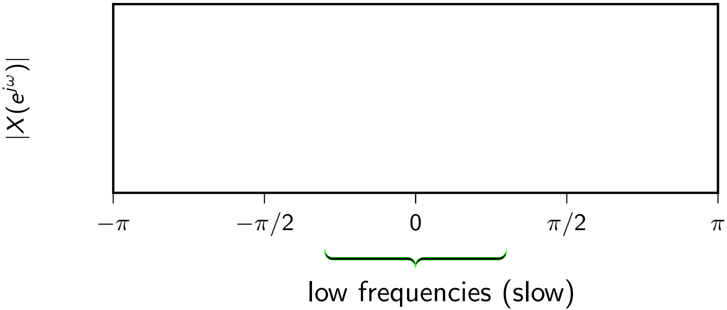

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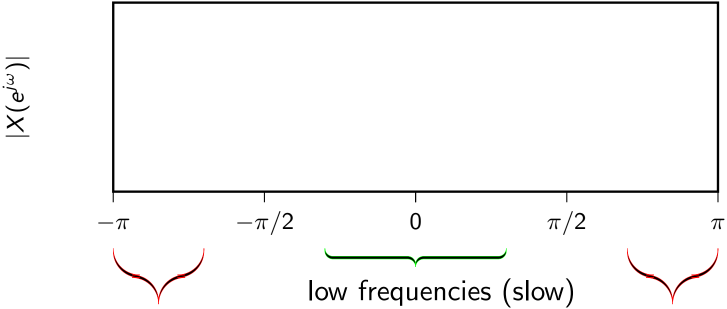
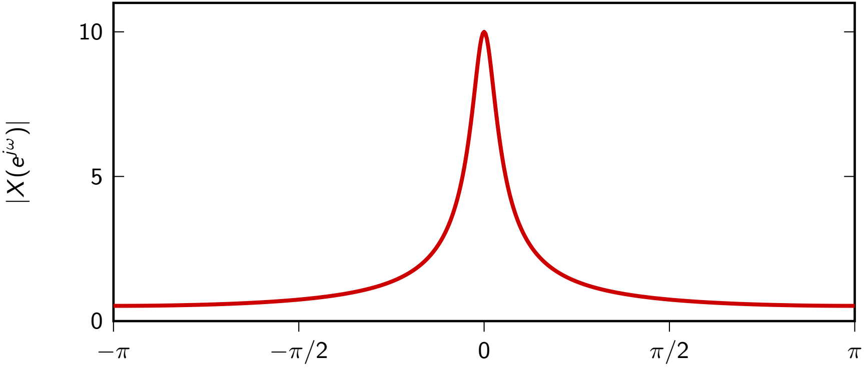


low frequencies (slow)

high frequencies (fast)

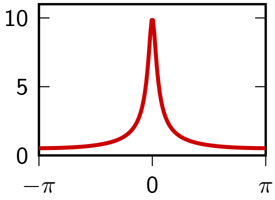
high frequencies (fast)

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$



$\alpha = 0.9$

Remember the periodicity!



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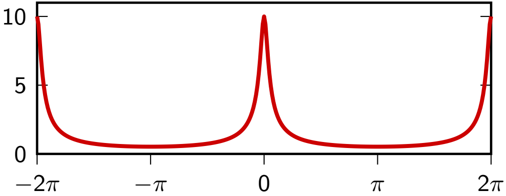

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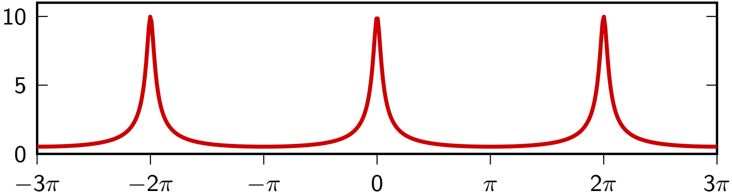

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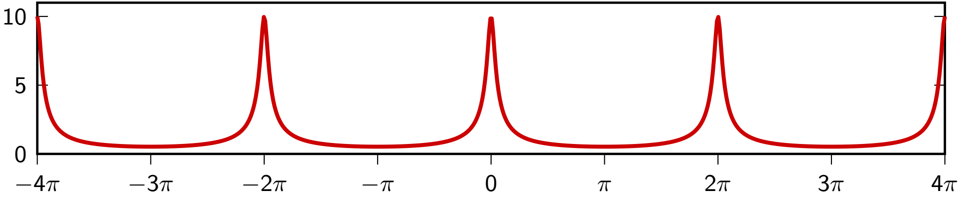

DTFT intuition and properties

Overview:

- DTFT Existence
- Properties
- DTFT as basis expansion

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

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- is it a change of basis?

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|X(e^{j\omega})| &= |\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}| \\
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Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k} \right) e^{j\omega n} d\omega$$

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&= x[n]
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the DTFT as the limit of the DFS

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
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Some intuition

With $x[n]$ absolutely summable we can build arbitrarily "periodized" sequences:

$$\tilde{x}_N[n] = \sum_{p=-\infty}^{\infty} x[n + pN]$$

clearly $\tilde{x}_N[n] = \tilde{x}_N[n + N]$

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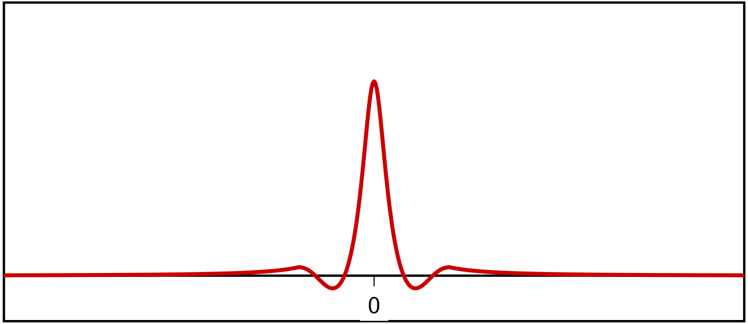
Periodization



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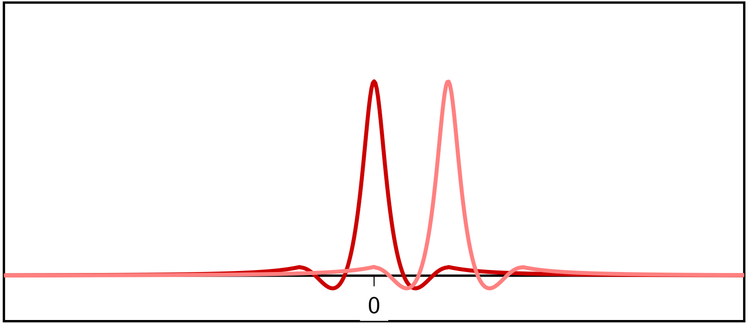

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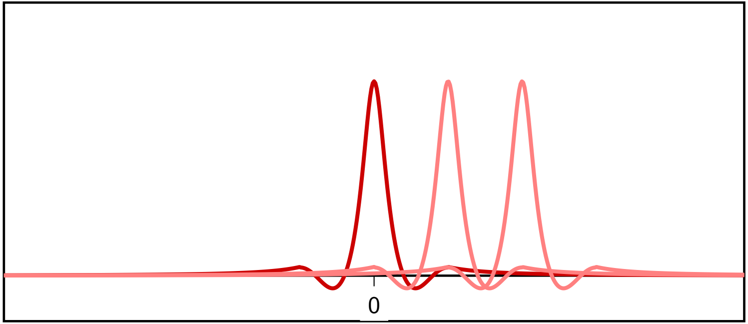

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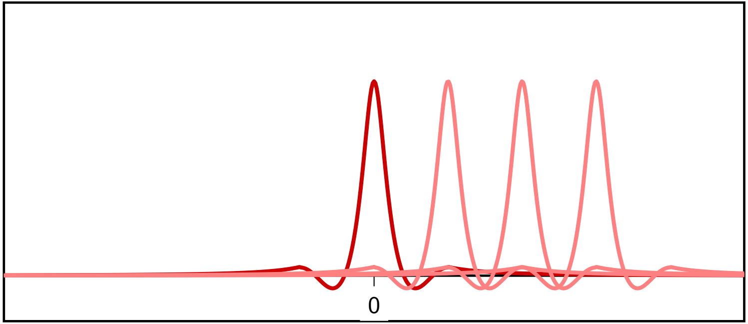

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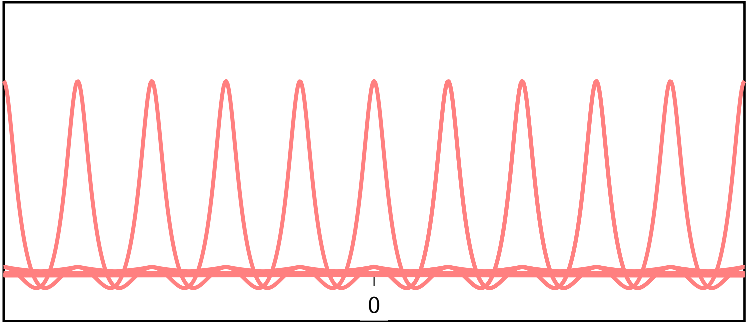

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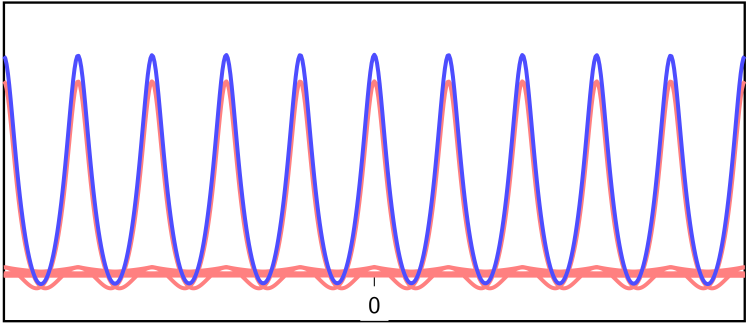

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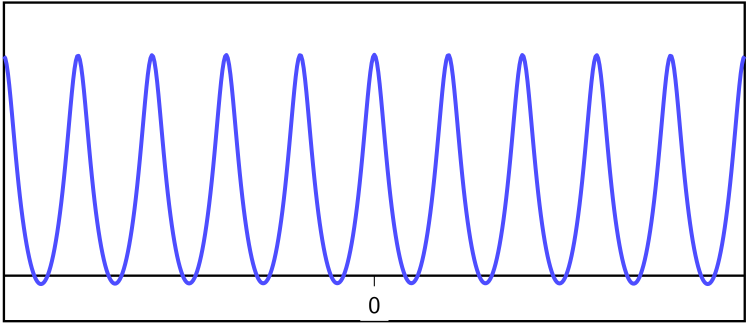
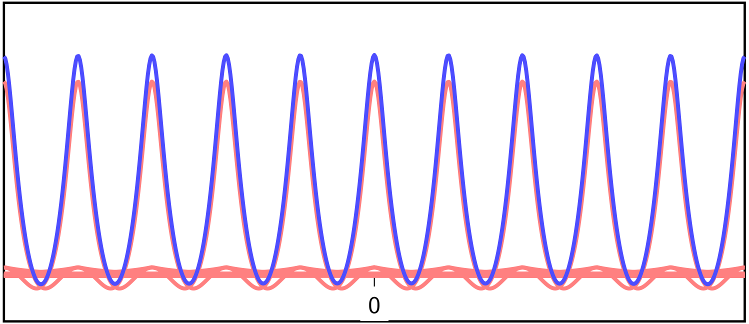


Periodization

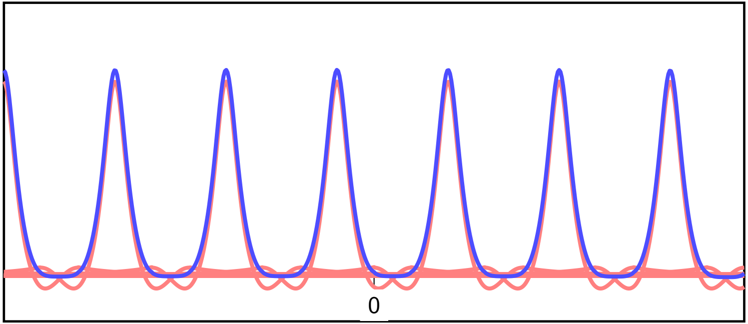
Let N grow large...



$N = 10$

Periodization

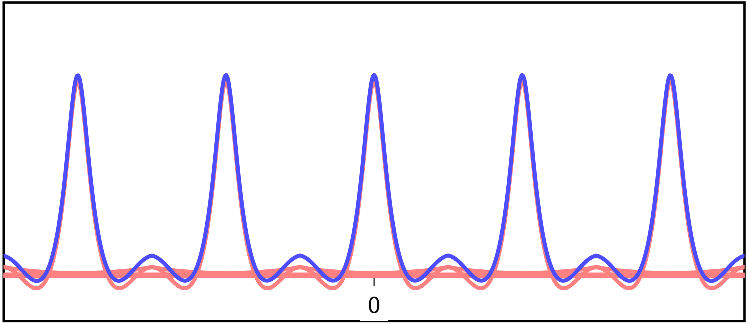
Let N grow large...



$N = 15$

Periodization

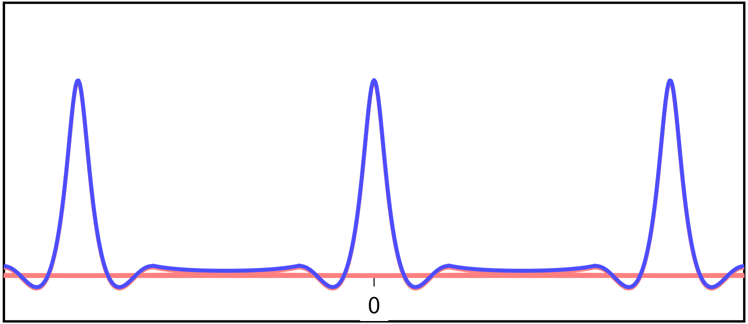
Let N grow large...



$N = 20$

Periodization

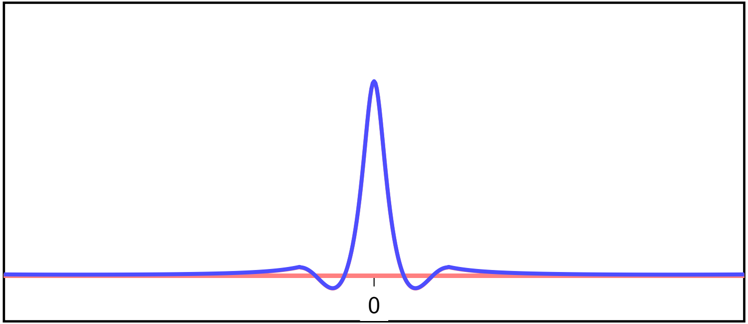
Let N grow large...



$N = 40$

Periodization

… as N grows, $\tilde{x}_N[n] \to x[n]$



$N = 80$

From DFS to DTFT

Natural spectral representation for $\tilde{x}_N[n]$ is the DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}_N[n] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \sum_{p=-\infty}^{\infty} x[n+pN] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+pN] e^{-j\frac{2\pi}{N}(n+pN)k}$$

(remember $e^{j\alpha} = e^{j(\alpha+2K\pi)} \quad \forall K$)

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From double sum to single sum

we can always write

$$\sum_{m=-\infty}^{\infty} y[m] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} y[n+pN]$$

Example (N=4)

n

p

	0	1	2	3
. . .				
-1				
0				
1				
2				
. . .				

$$m = n + 4p$$

Example (N=4)

		n			
		0	1	2	3
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		0	1	2	3
p					
	. . .				
	-1				
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	1	4	5		
	2				
	. . .				

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Example (N=4)

n

	0	1	2	3
...				
-1				
0	0	1	2	3
1	4	5	6	
2				
...				

p

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Example (N=4)

		n			
		0	1	2	3
	. . .				
p	-1				
	0	0	1	2	3
	1	4	5	6	7
	2				
	. . .				

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Example (N=4)

		m			
		0	1	2	3
	. . .				
p	-1	-4	-3	-2	-1
	0	0	1	2	3
	1	4	5	6	7
	2	8	9	10	11
	. . .				

$$n = 4p + m$$

From DFS to DTFT

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- we're comfortable with DFS: change of basis, energy conservation, etc.
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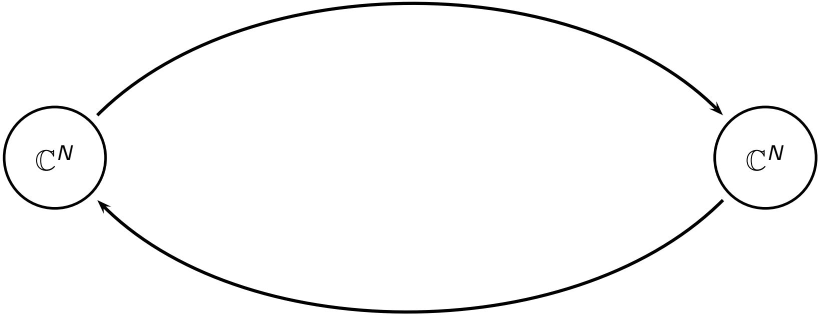
Review: DFT



Review: DFT

$$\begin{tikzcd}
\mathbb{C}^N \arrow[r, bend left, "{X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n]\rangle}"] & \mathbb{C}^N \arrow[l, bend left]
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Review: DFT

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Review: DFS

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What about the DTFT?

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

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- “basis” is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
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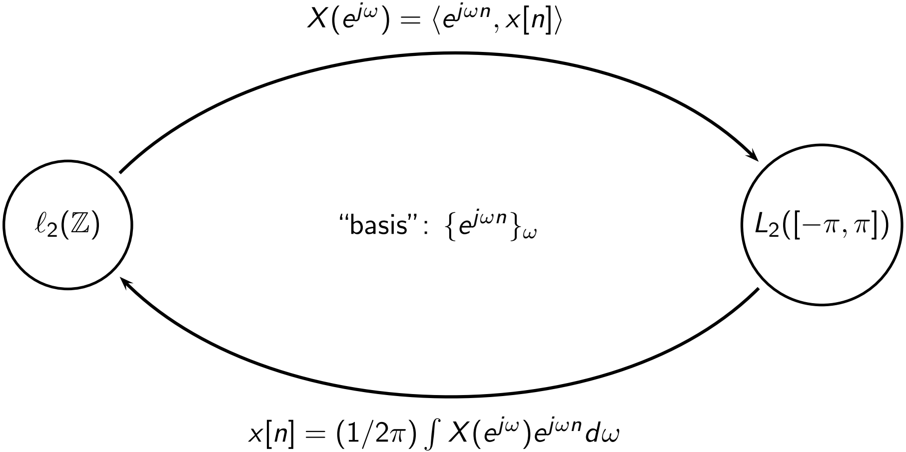
DTFT



$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$

$\ell_2(\mathbb{Z})$

“basis”: $\{e^{j\omega n}\}_\omega$

$L_2([-\pi, \pi])$

$x[n] = (1/2\pi) \int X(e^{j\omega}) e^{j\omega n} d\omega$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega - \omega_0)})$$

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Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \iff X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \iff X(e^{j\omega}) = X^*(e^{-j\omega})$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \iff |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!

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DTFT as basis expansion

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- problem: too many interesting sequences are *not* square summable!

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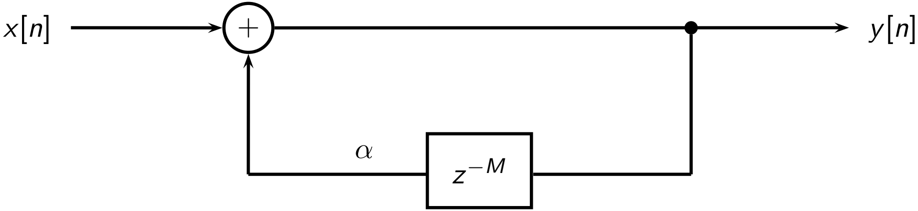
The DTFT of the Karplus-Strong output

Karplus-Strong revisited



$$y[n] = \alpha\, y[n-M] + x[n]$$

Karplus-Strong revisited

- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- generated signal is infinite-length but not periodic:

$$y[n] = \bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1], \alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1], \alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots$$

- what is the DTFT of this signal?

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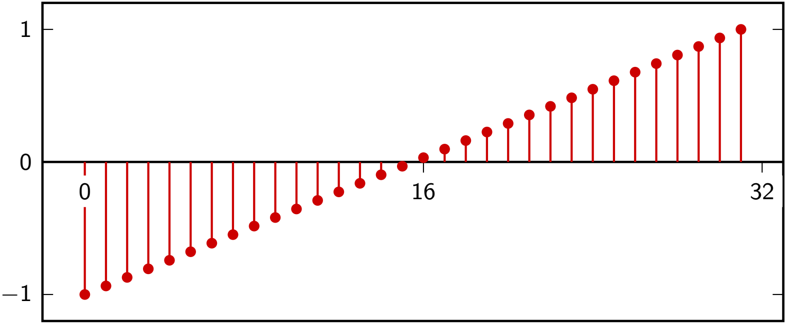
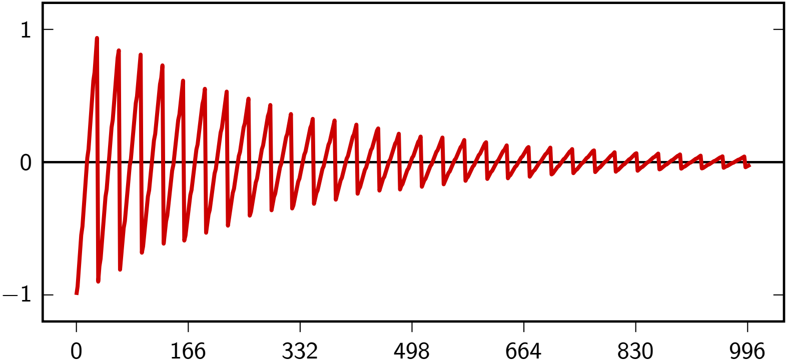
KS revisited: 32-tap sawtooth wave

$$x[n] = 2n/(M-1) - 1, \quad n = 0, 1, \ldots, M-1$$



KS revisited: decay $\alpha = 0.9$

$$y[n] = \alpha^{\lfloor n/M \rfloor} \bar{x}[n \mod M]\, u[n]$$



DTFT of KS signal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

Same trick we used before:

$$\sum_{m=-\infty}^{\infty} y[m] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} y[n + pN]$$

DTFT of KS signal

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$$= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)}$$

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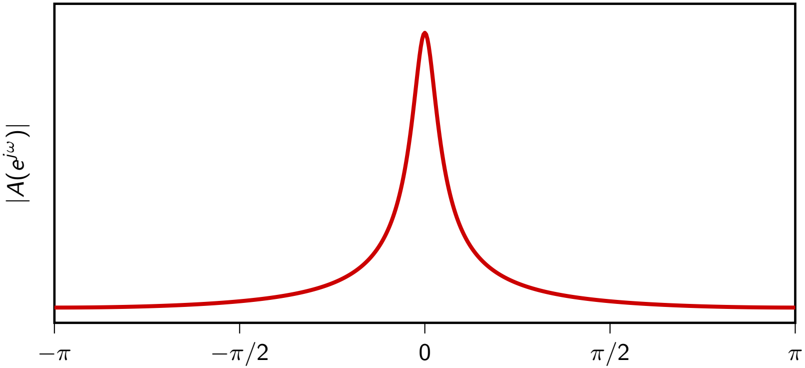
We know the first term

$$A(e^{j\omega}) = \mathrm{DTFT}\left\{\alpha^n\, u[n]\right\} = \frac{1}{1-\alpha e^{-j\omega}}$$



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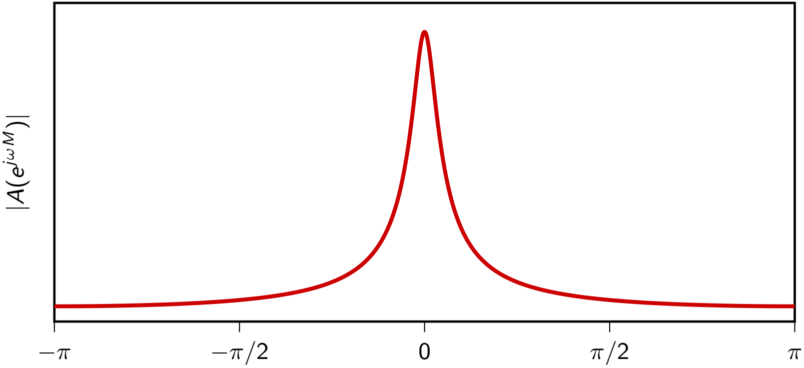
$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



$M = 1$

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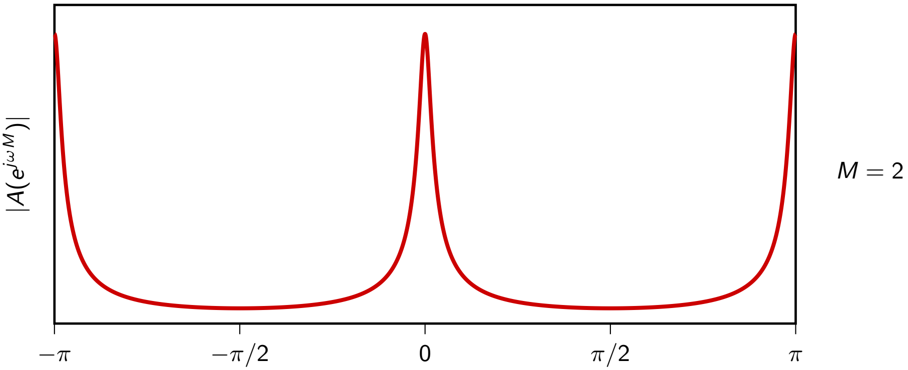

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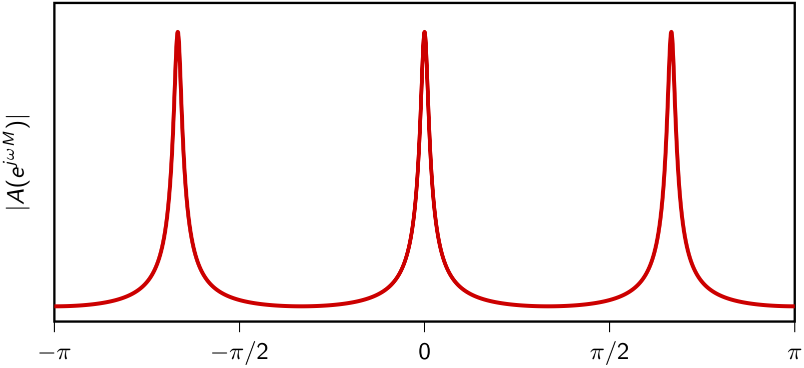

$M = 3$

We know the first term

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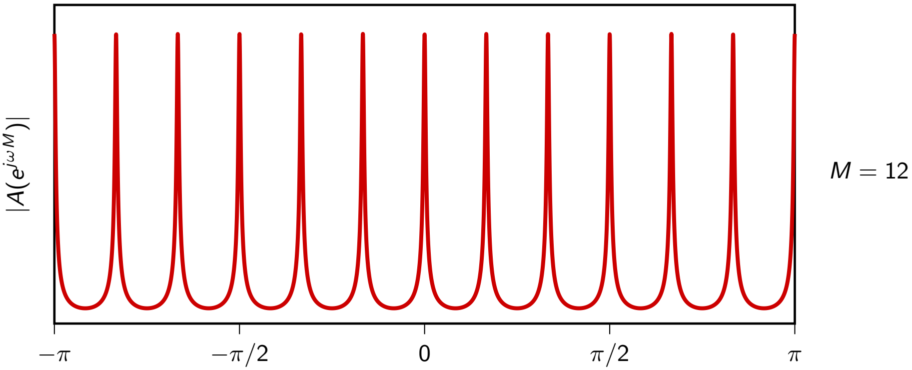
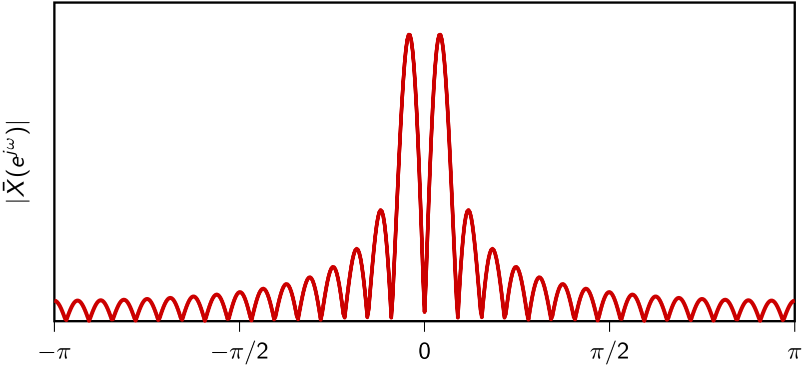


Second term is left as an exercise

$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$



DTFT of KS with decay

$$Y(e^{j\omega}) = A(e^{j\omega M})\bar{X}(e^{j\omega})$$