Exercise Set 12

Saliba, May 29, 2019

Exercise 1. (a) Prove Blackwell's theorem which states that, for a recurrent renewal process,

$$\lim_{t\to\infty}(R(t+h)-R(t)) = \frac{h}{E[W]}.$$

(b) We have seen already that, for a recurrent and irreducible Markov chain $(X_n)_{n\geq 0}$,

$$\lim_{n\to\infty}\mathbb{P}(X_n=j \mid X_0=j) = \begin{cases} 0 & \text{if the states are null recurrent,} \\ \pi_j > 0 & \text{if the states are positive recurrent.} \end{cases}$$

Prove this result using Blackwell's theorem.

Exercise 2. Consider a machine installed at time $S_0 = 0$. When it breaks down, it is replaced by a new identical one and so on. Suppose that the lifetimes the machines are $U_1, U_2, \ldots$ and the replacement times are $V_1, V_2, \ldots$ unit of time. Hence, the machines start working successively at times $S_0 = 0$, $S_1 = U_1 + V_1$, $S_2 = S_1 + U_2 + V_2$ and so on. It is reasonable to suppose that the U_i's are i.i.d as well as the V_i's. Moreover, we can suppose that U_i and V_i are independent. So $W_i = U_i + V_i$ are i.i.d and $\{S_n\}$ is a renewal process. Write $\varphi(\cdot)$ for the distribution of the U_i's and $\psi(\cdot)$ for the distribution of the V_i's. Let Z_t be the indicator of the event stating that the machine is working at time t, and $f(t) = \mathbb{P}(Z_t = 1)$.

(a) Show that $f(t)$ satisfies a renwal eqution and deduce the value of $\mathbb{P}(Z_t = 1)$.

(b) What is the asymptotic probability that the machine is working?

Exercise 3. Let U_t be the time since the last renewal before t in a recurrent renewal process S with interval distribution F, that is, $U_t = t - S_{N_t-1}$.

(a) Show that $f(t) = \mathbb{P}(U_t > x)$ satisfies, for a fixed x, the following renewal equation

$$f(t) = (1-F(t))\mathbb{1}_{\{(x,\infty)\}} + \int_{[0,t]} f(t-s)dF(s).$$

(b) Show that for all $t > x$,

$$\mathbb{P}(U_t > x) = \int_{[0,t-x)}(1-F(t-s))dR(s).$$

(c) Show that

$$\lim_{t\to\infty}\mathbb{P}(U_t > x) = \lim_{t\to\infty}\mathbb{P}(Z_t > x),$$

where $Z_t = S_{N_t} - t$ is the survival time of the process at time t.

Exercise 4. Recall the context of exercise 2. Suppose that the replacements of the machines are immediate (do not take any time), and instead of replacing only the damaged machines, suppose that the rule is to replace any damaged machine *or* any machine used for τ units of time. We now distinguish two cases.

(a) Show that the times S_n of successive replacements form a renewal process, and compute the distribution F of the time between two replacements.

(b) Show that if $N(t)$ is the number of replacements in the time interval $[0, t]$, then

$$\lim_{t\to\infty} \frac{N(t)}{t} = 1 \Big/ \int_0^\tau [1 - \varphi(u)] du.$$

(c) Show that the successive moments of breakdowns $T_0, T_1, \ldots$ form a renewal process with distribution G between two moments of breakdown given by

$$1 - G(t) = [1 - \varphi(\tau)]^k [1 - \varphi(t - k\tau)], \; k\tau \le t \le k\tau + \tau.$$

(d) Show that the mean number of breakdowns per unit of time, $M(t)/t$, had the following limit when $t \to \infty$,

$$\varphi(\tau) \Big/ \int_0^\tau [1 - \varphi(u)] du.$$

Exercise 5. Let S be a renewal process with interval distribution F, and let $T_1, T_2, \ldots$ be the number of successive trials to get a success for a Bernouilli random variable with parameter p. As usual, we define $T_0 = 0$. Show that $\hat{S}_n = S_{T_n}$ form a renewal process with interval distribution between two renewals given by

$$\hat{F} = \sum_{n=1}^{\infty} pq^{n-1} F^{(n)}.$$