COM-303 - "Mock" Midterm Exam

- This mock midterm exam is not graded and it is designed to test your understanding of the material so far: try to work on the problems as if taking a real exam.
- There are 5 problems with different scores for a total of 100 points; the scores indicate the difficulty of the problem.
- The solution will be discussed during the exercise session next Thursday.

Exercise 1. (20 points)
Consider the finite-support sequence

$$x[n] = \begin{cases} 1/6 & \text{for } 0 \le n < 6 \\ 0 & \text{otherwise} \end{cases}$$

Next, consider the family of complex-valued finite-support sequences

$$x_k[n] = x[n]\, e^{-j\omega_k n}$$

where $\omega_k = (2\pi/6)k$.

(a) sketch $|X(e^{j\omega})|$, the magnitude of the DTFT of $x[n]$; be as precise as possible

(b) sketch $|X_k(e^{j\omega})|$, the magnitude of the DTFT of $x_k[n]$, for $k = 1$ and $k = 4$

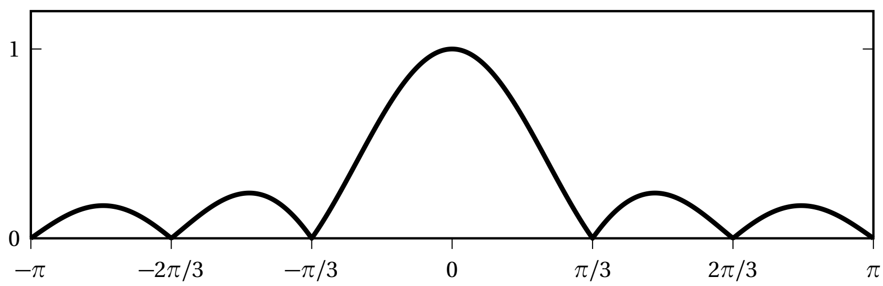
(c) prove that $\displaystyle\sum_{k=0}^{5} X_k(e^{j\omega}) = 1$

Solution:

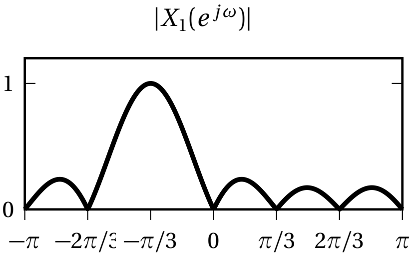
(a) the sequence corresponds to the impulse response of a moving average filter of length six; the magnitude response is

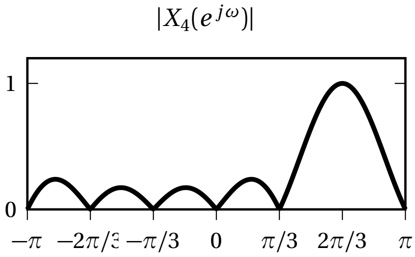
$$|X(e^{j\omega})| = \left| \frac{1}{6} \frac{\sin(3\omega)}{\sin(\omega/2)} \right|$$

so it will be equal to zero for $\omega = \pm\pi/3, \pm 2\pi/3, \pm\pi$ and equal to 1 (by continuity) for $\omega = 0$:



(b) multiplication by $e^{-j\omega_k n}$ in time corresponds to a left shift by $\omega_k = k(\pi/3)$ in frequency. Because of the 2π-periodicity of the spectrum, the shift appears as a circular shift over the $[-\pi, \pi]$ range:





(c)

$$
\begin{aligned}
\sum_{k=0}^{5} X_k(e^{j\omega}) &= \sum_{k=0}^{5} \mathrm{DTFT}\{x_k[n]\} \\
&= \mathrm{DTFT}\left\{\sum_{k=0}^{5} x_k[n]\right\} \qquad \textit{(by linearity)} \\
&= \mathrm{DTFT}\left\{\frac{1}{6}\sum_{k=0}^{5} e^{-j\frac{2\pi}{6}nk}\right\} \\
&= \mathrm{DTFT}\left\{\frac{1}{6}\mathrm{DFT}\{1\}\right\} \qquad \textit{(DFT in } \mathbb{C}^6\textit{)} \\
&= \mathrm{DTFT}\{\delta[n]\} = 1
\end{aligned}
$$

Exercise 2. (15 points)
Show that absolute summability implies finite energy, that is:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Solution: *If $\sum_n |x[n]| < \infty$, then necessarily the sequence $x[n]$ tends to zero:*

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \lim_{n\to\pm\infty} x[n] = 0;$$

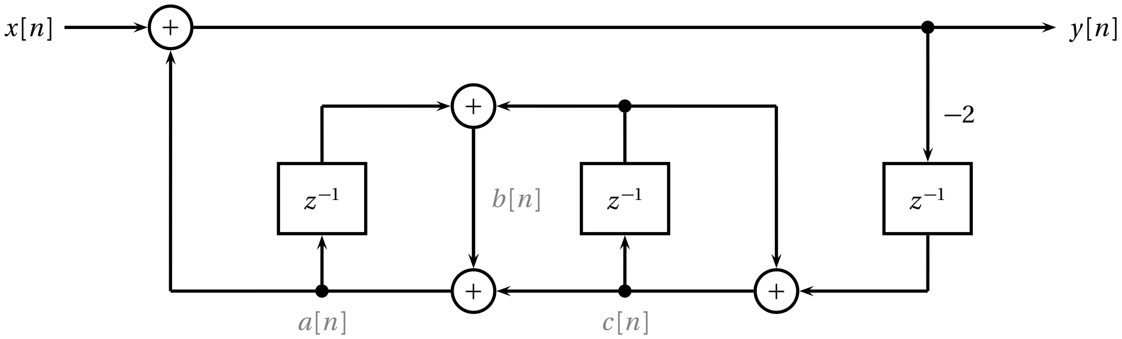
therefore there must exist an integer $n_0 > 0$ so that, for all $|n| > n_0$, $|x[n]| < 1$. Then we can write

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{|n|\le n_0} |x[n]|^2 + \sum_{|n|>n_0} |x[n]|^2;$$

the first term in the sum is necessarily finite, while for the second term, since $x^2 < x$ for $|x| < 1$, we have

$$\sum_{|n|>n_0} |x[n]|^2 \le \sum_{|n|>n_0} |x[n]| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Exercise 3. (30 points)
Consider the causal system described by the following block diagram:



Compute its transfer function $H(z) = Y(z)/X(z)$.

Solution: *Consider the intermediate signals $a[n]$, $b[n]$, $c[n]$ as in the above figure. In the z-domain we have*

$$\begin{aligned}
Y(z) &= X(z) + A(z)\\
A(z) &= B(z) + C(z)\\
B(z) &= z^{-1}A(z) + z^{-1}C(z)\\
C(z) &= z^{-1}C(z) - 2z^{-1}Y(z)
\end{aligned}$$

Using the third equation with the second

$$A(z) = z^{-1}A(z) + z^{-1}C(z) + C(z) \quad \Rightarrow \quad A(z) = \frac{1+z^{-1}}{1-z^{-1}}C(z)$$

while the fourth equation gives

$$C(z) = \frac{-2z^{-1}}{1-z^{-1}}Y(z)$$

Replacing these results in the first equation:

$$Y(z) = X(z) - 2z^{-1}\frac{1+z^{-1}}{(1-z^{-1})^2}Y(z)$$

$$\left[1+2z^{-1}\frac{1+z^{-1}}{(1-z^{-1})^2}\right]Y(z) = \left[\frac{1-2z^{-1}+z^{-2}+2z^{-1}+2z^{-2}}{(1-z^{-1})^2}\right]Y(z) = X(z)$$

so that finally

$$H(z) = \frac{(1-z^{-1})^2}{1+3z^{-2}}$$

Exercise 4. (15 points)
Show that

$$\delta[n] - \frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right) = (-1)^n\frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right).$$

Solution: *You can solve this in the time domain or in the frequency domain.*
Time-domain solution: *For $n = 0$, $\operatorname{sinc}(n/2) = 1$ so that the equality is obviously satisfied:*

$$1 - \frac{1}{2} = \frac{1}{2};$$

for n nonzero and even the argument of both sincs is an integer, so the equation becomes simply

$$\delta[n] = 0$$

which is true for all nonzero even values of n; finally, for n odd, we have the tautology

$$-\frac{1}{2}\text{sinc}\left(\frac{n}{2}\right) = -\frac{1}{2}\text{sinc}\left(\frac{n}{2}\right).$$

Frequency-domain solution: $(1/2)\text{sinc}(n/2)$ *is the impulse response of an ideal lowpass with cutoff frequency $\omega_c = \pi/2$. Therefore, the left-hand side is the impulse response of an ideal highpass with cutoff frequency $\omega_c = \pi/2$. The right-hand side is the lowpass impulse response modulated by* $\cos(\pi n)$, *which shifts the frequency response by π; therefore that too is the impulse response of an ideal highpass with cutoff frequency $\omega_c = \pi/2$.*

Exercise 5. (20 points)

In this exercise we will study a data transmission scheme known as *phase modulation* (PM). Consider a discrete-time signal $x[n]$, with the following properties:

- $|x[n]| < 1$ for all n
- $X(e^{j\omega}) = 0$ for $|\omega| < \alpha$, with α small.

A PM transmitter with carrier frequency ω_c works by producing the signal

$$y[n] = \mathscr{P}_{\omega_c}\{x[n]\} = \cos(\omega_c n + k\, x[n])$$

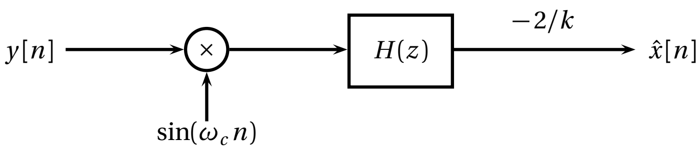
where k is a small positive constant; in other words, the data signal $x[n]$ is used to modify the instantaneous *phase* of a sinusoidal carrier. The advantage of this modulation technique is that it builds a signal with constant envelope (namely, a sinusoid with fixed amplitude) which results in a greater immunity to noise; this is the same principle behind the better quality of FM radio versus AM radio. However phase modulation is less “user friendly” than standard amplitude modulation because it is nonlinear.

(a) Show that phase modulation is *not* a linear operation.

Because of nonlinearity, the spectrum of the signal produced by a PM transmitter cannot be expressed in simple mathematical form. For the purpose of this exercise you can simply assume that the PM signal occupies the frequency band $[\omega_c - \gamma, \omega_c + \gamma]$ (and, obviously, the symmetric interval $[-\omega_c - \gamma, -\omega_c + \gamma]$) with

$$\gamma \approx 2(k+1)\alpha.$$

To demodulate a PM signal the following scheme is proposed, in which $H(z)$ is a lowpass filter with cutoff frequency equal to α:



(b) Show that $\hat{x}[n] \approx x[n]$. Assume that $\omega_c \gg \alpha$ and that k is small, say $k = 0.2$. (You may find it useful to express trigonometric functions in terms of complex exponentials if you don't recall the classic trigonometric identities. Also, remember that $\sin x \approx x$ for x sufficiently small).

Solution:

(a) Given a signal $x[n]$ fulfilling the magnitude and bandwidth requirements, if PM was a linear operation, for any scalar $\beta \in \mathbb{R}$ we should have

$$\mathscr{P}_{\omega_c}\{\beta\, x[n]\} = \beta \mathscr{P}_{\omega_c}\{x[n]\}.$$

However, irrespective of $x[n]$, $|\mathscr{P}_{\omega_c}\{\cdot\}| \leq 1$. Since we can always pick a value for β so that the right-hand side of the equality takes values larger than one, the equality cannot hold in general.

(b) *Nonlinear operators make it impossible to proceed analytically in the frequency domain. In the time domain, however, the signal after the multiplier is*

$$\begin{aligned}
d[n] &= y[n]\sin(\omega_c n) \\
&= \cos(\omega_c n + kx[n])\sin(\omega_c n) \\
&= (1/2)\sin(\omega_c n + kx[n] + \omega_c n) - (1/2)\sin(\omega_c n + kx[n] - \omega_c n) \\
&= (1/2)\sin(2\omega_c n + kx[n]) - (1/2)\sin(kx[n]) \\
&\approx (1/2)\sin(2\omega_c n + kx[n]) - (k/2)x[n]
\end{aligned}$$

where we have used the small-angle approximation for the sine since $|kx[n]| < 0.2$. *The signal* $d[n]$ *now contain a baseband component and a PM component at twice the carrier frequency, which is eliminated by the lowpass filter:*

$$\hat{x}[n] = (-2/k)h[n] * d[n] \approx x[n].$$

[Note: we used the trigonometric identity $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$. *This can be easily derived by developing the product* $(e^{j\alpha} + e^{-j\alpha})(e^{j\beta} - e^{-j\beta})/(2j)$.*]*