COM-303 - Signal Processing for Communications

Homework #5

Exercise 1. LTI Systems

Consider the transformation $\mathscr{H}\{x[n]\} = n\,x[n]$. Does $\mathscr{H}$ define an LTI system?

Exercise 2. Convolution

Let $x[n]$ be a discrete-time sequence defined as

$$x[n] = \begin{cases} M-n & 0 \leq n \leq M, \\ M+n & -M \leq n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

for some odd integer M.

(a) Show that $x[n]$ can be express as the convolution of a sequences $t[n]$ with itself. Check your result numerically (using Python or any other numerical package) for $M = 11$.

(b) Using the results found in (a), compute the DTFT of $x[n]$.

Exercise 3. System Properties

For the systems whose input-output relationship is described below, say whether the system is linear, time-invariant, BIBO stable, causal; when applicable, find the impulse response.

(a) $y[n] = x[-n]$

(b) $y[n] = e^{-j\omega n}x[n]$

(c) $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

(d) $y[n] = n\,y[n-1] + x[n]$, so that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$. (*Hint*: Since the system is causal and satisfies initial-rest conditions, we can recursively find the response to any input as, for instance, $\delta[n]$.)

Exercise 4. Ideal Filters

Derive the impulse response of a bandpass filter with center frequency ω_0 and passband ω_b:

$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \omega_0 - \omega_b/2 \le \omega \le \omega_0 + \omega_b/2, \\ 1 & -\omega_0 - \omega_b/2 \le \omega \le -\omega_0 + \omega_b/2, \\ 0 & \text{elsewhere.} \end{cases}$$

(*Hint*: consider the following ingredients: a cosine at frequency ω_0, a lowpass filter with bandwidth ω_b and the Modulation Theorem.)

Exercise 5. Signals and Systems

Consider the system $\mathscr{H}$ implementing the input-output relation $\mathscr{H}\{x[n]\} = x^2[n]$.

(a) Prove by example that the system is nonlinear.

(b) Prove that the system is time-invariant.

Now consider the following cascade

$$x[n] \longrightarrow \boxed{\mathscr{H}} \longrightarrow \boxed{\mathscr{G}} \longrightarrow y[n]$$

where $\mathscr{G}$ is the ideal highpass filter:

$$G(e^{j\omega}) = \begin{cases} 0 & \text{for } |\omega| < \pi/2, \\ 2 & \text{otherwise} \end{cases}$$

(as per usual, $G(e^{j\omega})$ is 2π-periodic (i.e. prolonged by periodicity outside of $[-\pi, \pi]$)). The output of the cascade is therefore $y[n] = \mathscr{G}\{\mathscr{H}\{x[n]\}\}$.

(c) Compute $y[n]$ when $x[n] = \cos(\omega_0 n)$ for $\omega_0 = 3\pi/8$. How would you describe the transformation operated by the cascade on the input?

(d) Compute $y[n]$ as before, with now $\omega_0 = 7\pi/8$.

Exercise 6. Linear Phase FIR Filters

The frequency response of an M-tap FIR filter with real-valued impulse response $h[n]$

is

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h[n]e^{-jn\omega}$$

where we have assumed that $h[n] = 0$ for $n < 0$ and $n \geq M$.

Show that if M is odd and the impulse response is symmetric, then the filter has a linear phase response.

Exercise 7. The Gibbs Phenomenon

In this exercise you will verify by yourself the existence of the Gibbs phenomenon using Python (or any other numerical package). The idea is to plot a zoomed-in version of the frequency response of a truncated ideal lowpass filter with cutoff frequency $\pi/2$:

$$\hat{H}(e^{j\omega}) = \sum_{n=-N}^{N} (1/2)\text{sinc}(n/2)\, e^{-j\omega n}$$

where we are interested in plotting the transform over a small interval around the cutoff frequency.

(a) Plot $\hat{H}(e^{j\omega})$ over 2000 points in the interval $1.4 \leq \omega \leq 1.7$ for $N = 20$.

(b) Repeat the above point for $N = 100$ and $N = 200$ and verify that the peak of the magnitude is still approximately 9% of the value of the discontinuity.
