Solutions 1

1. Using Stirling's approximation for $\binom{2n}{n} = \frac{2n!}{n!n!}$, we obtain

$$\binom{2n}{n} p^n\, q^n \sim \frac{\sqrt{2\pi(2n)}\,\left(\frac{2n}{\mathrm{e}}\right)^{2n}}{2\pi n\,\left(\frac{n}{\mathrm{e}}\right)^{2n}}\,(pq)^n = \frac{(4pq)^n}{\sqrt{\pi n}}$$

2. a) Both X and Y are random walks with probability $1/4$ to go in either direction, and probability $1/2$ to stay in place.

b) No, they are not independent: when X makes a move, Y does not, and vice-versa.

c) Both U and V are simple symmetric random walks with probability $1/2$ to go in either direction.

d) Yes, they are independent. Denote $U_n = \eta_1 + \ldots + \eta_n$, $V_n = \chi_1 + \ldots + \chi_n$. Then one can check e.g. that (and similarly for all ± 1 combinations)

$$\mathbb{P}(\eta_n = +1, \chi_n = +1) = \mathbb{P}\left(\overrightarrow{\xi_n} = (+1, 0)\right) = \frac{1}{4} = \mathbb{P}(\eta_n = +1) \cdot \mathbb{P}(\chi_n = +1)$$

e) Note that $\overrightarrow{S_{2n}} = (0,0)$ if and only if $U_{2n} = V_{2n} = 0$, so by the independence shown above, we obtain

$$\begin{aligned} &\mathbb{P}\left(\overrightarrow{S_{2n}} = (0,0)\,|\,\overrightarrow{S_0} = (0,0)\right) = \mathbb{P}(U_{2n} = 0,\ V_{2n} = 0\,|\,U_0 = 0,\ V_0 = 0) \\ = \ \ &\mathbb{P}(U_{2n} = 0\,|\,U_0 = 0) \cdot \mathbb{P}(V_{2n} = 0\,|\,V_0 = 0) = \left(\binom{2n}{n} 2^{-2n}\right)^2 \sim \frac{1}{\pi n} \end{aligned}$$

by Exercise 1.

The solutions of exercises 3-5 will be given in 4 weeks from now.