Solutions 6

1. a) This is a direct computation:

$$(P\,\phi^{(k)})_j = \sum_{l=0}^{N-1} p_{jl}\,\exp(2\pi i l k/N) = \ldots = \left(\sum_{m=0}^{N-1} c_m\,\exp(2\pi i m k/N)\right)\exp(2\pi i j k/N) = \lambda_k\,\phi_j^{(k)}$$

b1) The eigenvalues are $\lambda_k = \frac{1}{2}\left(\exp(2\pi i k/N) + \exp(-2\pi i k/N)\right) = \cos(2\pi k/N)$

b2) Recalling that

$$\sum_{j=0}^{N-1} \exp(2\pi i j k/N) = \begin{cases} N & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

we obtain that the eigenvalues are

$$\lambda_k = \frac{1}{N-1}\sum_{j=1}^{N-1} \exp(2\pi i k j/N) = \frac{1}{N-1}\left(\sum_{j=0}^{N-1} \exp(2\pi i k j/N) - 1\right) = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-1}{N-1} & \text{otherwise} \end{cases}$$

c) Observe that if the spectral gap is determined by the second largest eigenvalue (i.e., the one closest to $+1$), then adding self-loops of weight α to every state can only reduce the spectral gap, as this addition increases all eigenvalues. Adding self-loops might only increase the spectral gap (and therefore speed up the convergence to equilibrium) when the gap is determined by the least eigenvalue (i.e., the one closest to -1).

c1) In this case, the second largest eigenvalue is given by $\cos(2\pi/N)$, while the least eigenvalue is given by $\cos(2\pi(N-1)/2N) = \cos(\pi - \pi/N) = -\cos(\pi/N)$. So the spectral gap is $\gamma = 1 - \cos(\pi/N)$. Adding self-loops of weight α changes these eigenvalues to

$$\alpha + (1-\alpha)\,\cos(2\pi/N) \quad \text{and} \quad \alpha - (1-\alpha)\,\cos(\pi/N)$$

respectively. The weight α realizing the largest spectral gap is therefore solution of the equation

$$1 - \alpha - (1-\alpha)\,\cos(2\pi/N) = 1 - |\alpha - (1-\alpha)\,\cos(\pi/N)| = 1 + \alpha - (1-\alpha)\,\cos(\pi/N)$$

so $\alpha\,(2 + \cos(\pi/N) - \cos(2\pi/N)) = \cos(\pi/N) - \cos(2\pi/N)$. For N large, the value of α (which represents approximately the increase of the spectral gap in this case) is therefore roughly given by

$$\alpha \simeq \frac{1}{2}\left(1 - \frac{\pi^2}{2N^2} - 1 + \frac{2\pi^2}{N^2}\right) = \frac{3\pi^2}{4N^2}$$

(and recall that the spectral gap of the original chain is $\simeq \frac{\pi^2}{2N^2}$)

c2) For the complete graph, the computation is (much) simpler. The only eigenvalue not equal to 1 is $-\frac{1}{N-1}$. If one therefore sets $\alpha = \frac{1}{N}$ (which one can check leads to the transition matrix $p_{ij} = 1/N$ for all i, j), this increases the value of the spectral gap from $1 - \frac{1}{N-1}$ to 1. In this case, the convergence to the uniform distribution is immediate: after one step only, the distribution of the chain is indeed uniform.

2. a) The transition matrix is doubly stochastic, so the stationary distribution is uniform (i.e., $\pi_{(ij)} = \frac{1}{KM} = \frac{1}{N}$ for every $(ij) \in S$). The chain is clearly irreducible (and finite), so also positive-recurrent. It is aperiodic (and therefore ergodic) if and only if either K or M is odd, in which case π is also a limiting distribution.

b) First observe that when both K and M are even, then one eigenvalue is equal to -1 (take $k = K/2$ and $m = M/2$), so the spectral gap is 0. In the case were K is odd and M is even, the eigenvalue which is the closest to -1 is

$$\lambda_{(\frac{K-1}{2},\frac{M}{2})} = \frac{1}{2}\left(\cos\left(\frac{\pi(K-1)}{K}\right) - 1\right) = -\frac{1}{2}\left(\cos\left(\frac{\pi}{K}\right) + 1\right) \simeq -1 + \frac{\pi^2}{4K^2}$$

while the eigenvalue which is the closest to $+1$ is

$$\text{either} \quad \lambda_{(1,0)} = \frac{1}{2}\left(\cos\left(\frac{2\pi}{K}\right) + 1\right) \simeq 1 - \frac{\pi^2}{K^2} \quad \text{or} \quad \lambda_{(0,1)} = \frac{1}{2}\left(1 + \cos\left(\frac{2\pi}{M}\right)\right) \simeq 1 - \frac{\pi^2}{M^2}$$

In the case $K \geq \frac{M}{2}$, $\lambda_{(\frac{K-1}{2},\frac{M}{2})}$ "wins" among these three (i.e., determines the spectral gap). When $K < \frac{M}{2}$, $\lambda_{(0,1)}$ wins. We have a symmetric situation for K even and M odd. Finally, when both K and M are odd, the eigenvalue which are candidates for being the closest to either $+1$ or -1 are

$$\begin{aligned}
\lambda_{(\frac{K-1}{2},\frac{M-1}{2})} &= \frac{1}{2}\left(\cos\left(\frac{\pi(K-1)}{K}\right) + \cos\left(\frac{\pi(M-1)}{M}\right)\right) = -\frac{1}{2}\left(\cos\left(\frac{\pi}{K}\right) + \cos\left(\frac{\pi}{M}\right)\right) \\
&\simeq -1 + \frac{\pi^2}{4K^2} + \frac{\pi^2}{4M^2} \\
\lambda_{(1,0)} &= \frac{1}{2}\left(\cos\left(\frac{2\pi}{K}\right) + 1\right) \simeq 1 - \frac{\pi^2}{K^2} \\
\lambda_{(0,1)} &= \frac{1}{2}\left(1 + \cos\left(\frac{2\pi}{M}\right)\right) \simeq 1 - \frac{\pi^2}{M^2}
\end{aligned}$$

for K, M large. When K is close to M, the first of these three eigenvalues (namely $\lambda_{(\frac{K-1}{2},\frac{M-1}{2})}$) wins.

c) In the case where $K = M$ is an odd (and large) number, $N = K^2 = M^2$ and the spectral gap is given approximately by

$$\gamma \simeq \frac{\pi^2}{2N}$$

so by the theorem seen in class,

$$\max_{(ij)\in S} \|P^n_{(ij)} - \pi\|_{\mathrm{TV}} \leq \frac{\sqrt{N}}{2} \exp(-\gamma n) \simeq \frac{1}{2} \exp\left(\frac{\log N}{2} - \frac{n\pi^2}{2N}\right)$$

which falls below ε for $n = \dfrac{2N}{\pi^2}\left(\frac{\log N}{2} + c\right)$ and $c = \log(1/2\varepsilon)$, so

$$T_\varepsilon \leq \frac{2N}{\pi^2}\left(\frac{\log N}{2} + \log(1/2\varepsilon)\right)$$