COM303: Digital Signal Processing

Lecture 9: Linear Systems

Overview

- linear systems
- filtering by example
- stability

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Overview:

- Linearity and time invariance
- Convolution

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- Linearity and time invariance
- Convolution

A generic signal processing device

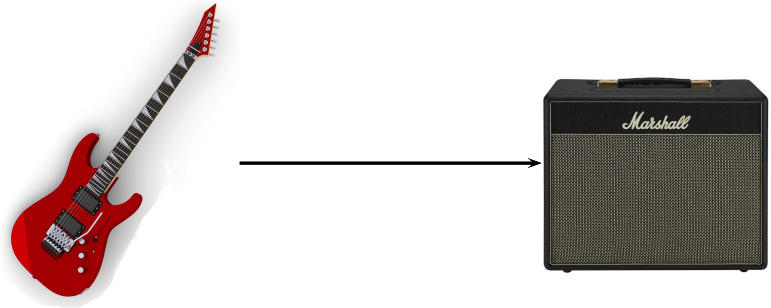
$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$y[n] = \mathcal{H}\{x[n]\}$$

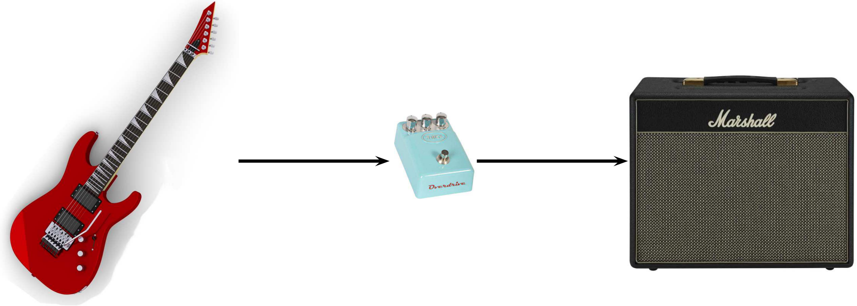
Linearity

$$\mathcal{H}\{\alpha\, x_1[n] + \beta\, x_2[n]\} = \alpha\, \mathcal{H}\{x_1[n]\} + \beta\, \mathcal{H}\{x_2[n]\}$$

Linearity



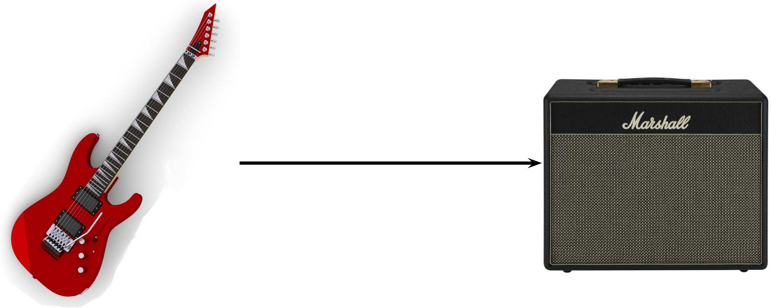
(Non) Linearity



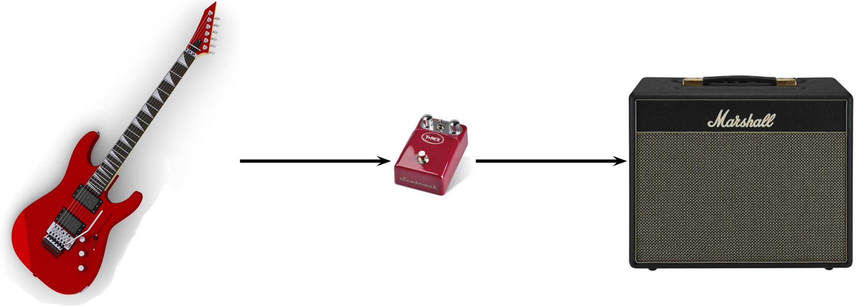
Time invariance

$$y[n] = \mathcal{H}\{x[n]\} \quad \Longleftrightarrow \quad \mathcal{H}\{x[n-n_0]\} = y[n-n_0]$$

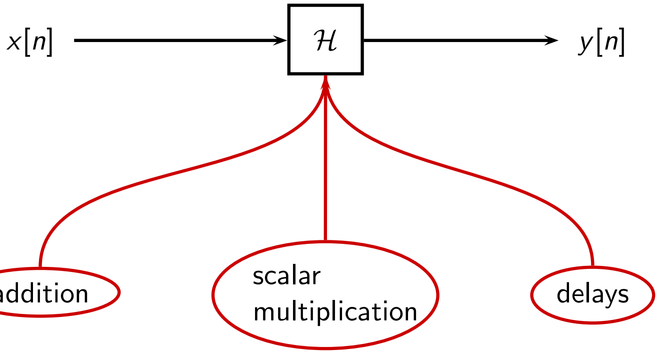
Time invariance



Time (in)variance



Linear, time-invariant systems



Linear, time-invariant systems

$$y[n] = H(x[n], x[n-1], x[n-2], \ldots, y[n-1], y[n-2], \ldots)$$

with $H(\cdot)$ a linear function of its arguments

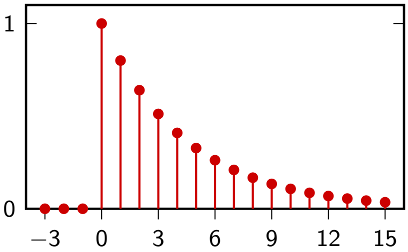
Impulse response

$$h[n] = \mathcal{H}\{\delta[n]\}$$

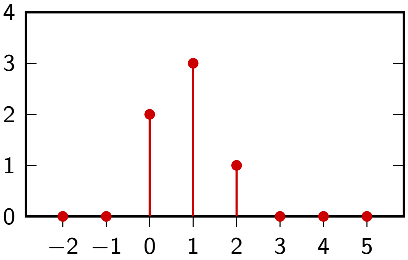
Fundamental result: impulse response fully characterizes the LTI system!

Example

$h[n]$



$x[n]$



$$h[n] = \alpha^n \, u[n]$$

$$x[n] = \begin{cases} 2 & n = 0 \\ 3 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Example

- $x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$
- we know the impulse response $h[n] = \mathcal{H}\{\delta[n]\}$;
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Example

$$y[n] = \mathcal{H}\{2\delta[n] + 3\delta[n-1] + \delta[n-2]\}$$

$$= 2\mathcal{H}\{\delta[n]\} + 3\mathcal{H}\{\delta[n-1]\} + \mathcal{H}\{\delta[n-2]\}$$

$$= 2h[n] + 3h[n-1] + h[n-2]$$

Example

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\begin{aligned}
y[n] &= \mathcal{H}\{2\delta[n] + 3\delta[n-1] + \delta[n-2]\} \\
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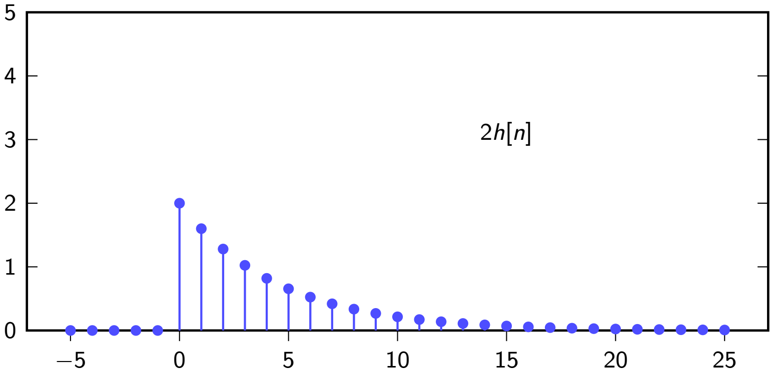

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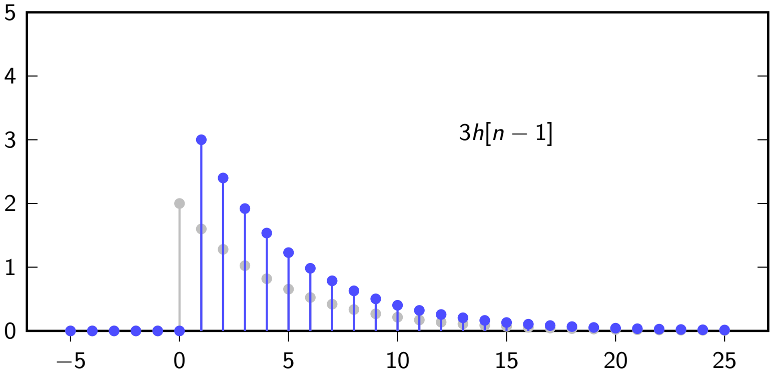

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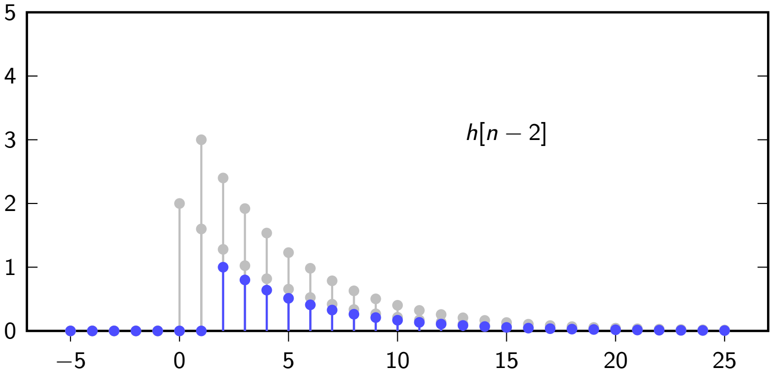

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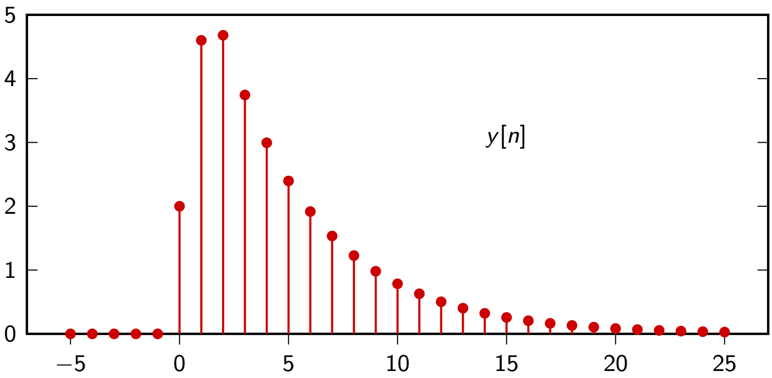

Convolution

We can always write a canonical-base decomposition:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

by linearity and time invariance:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= x[n] * h[n]$$

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$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[n] * h[n] \end{aligned}$$

Performing the convolution algorithmically

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Ingredients:

- a sequence $x[m]$
- a second sequence $h[m]$

The recipe:

- time-reverse $h[m]$
- at each step n (from $-\infty$ to ∞):
  - center the time-reversed $h[m]$ in n (i.e. shift by $-n$)
  - compute the inner product

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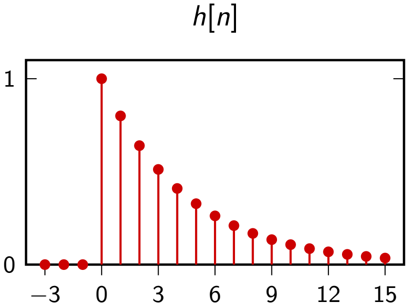
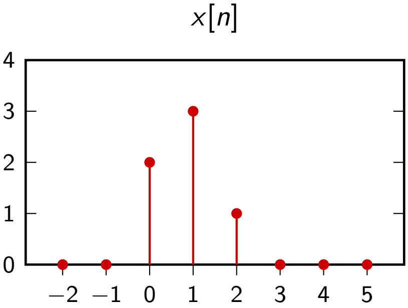
Same example, different perspective





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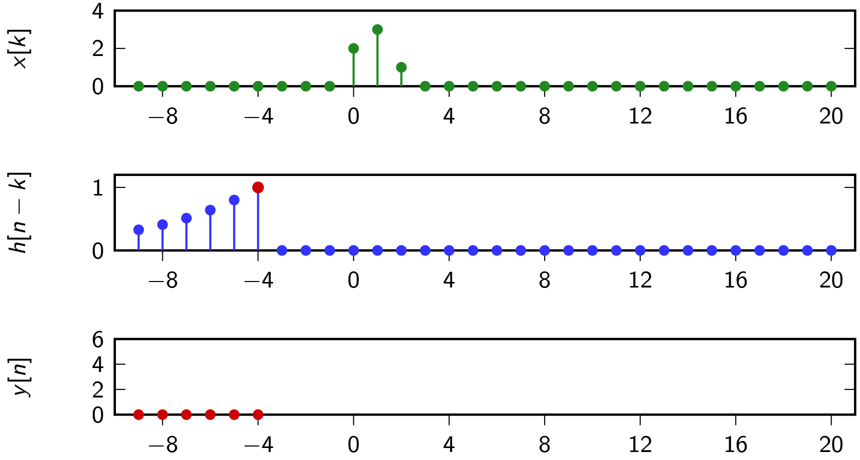

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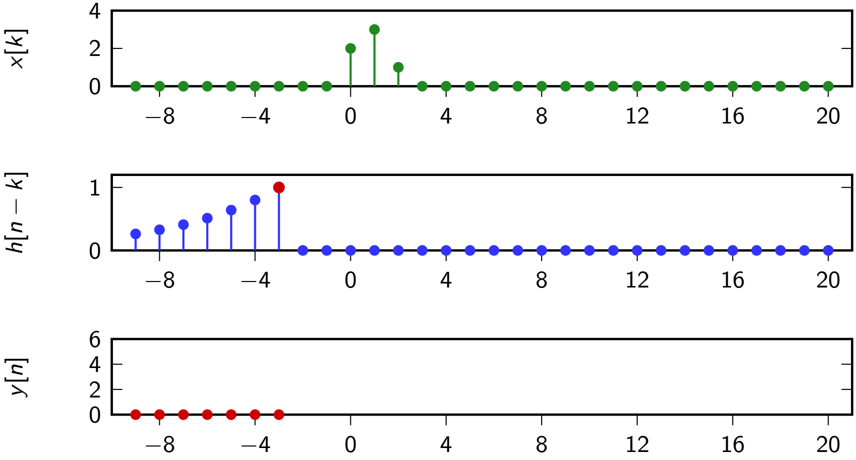

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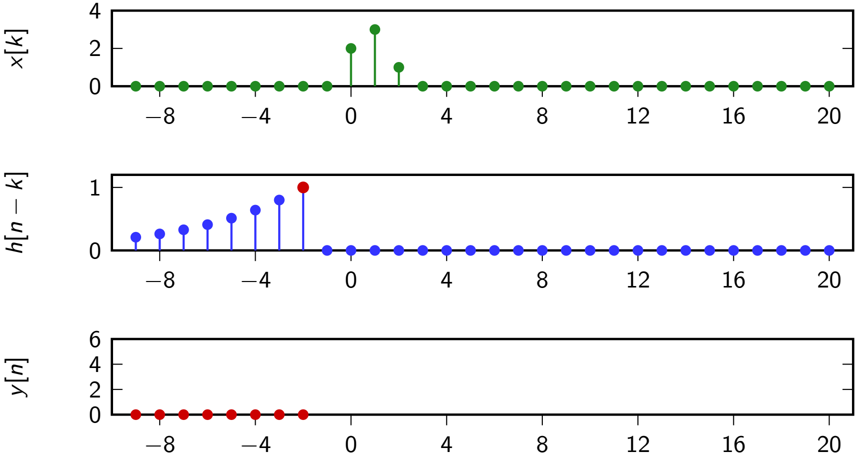

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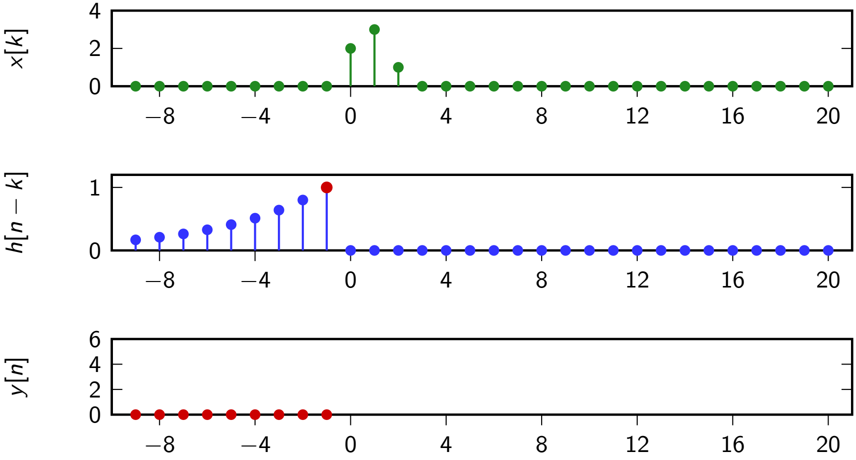

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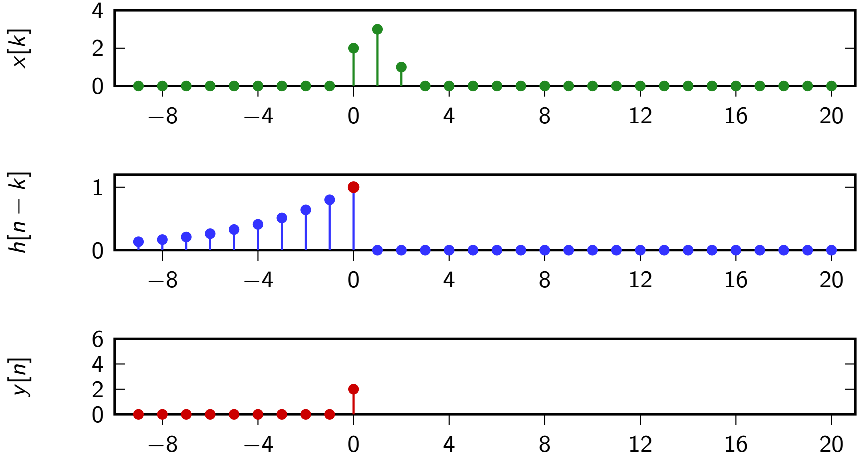

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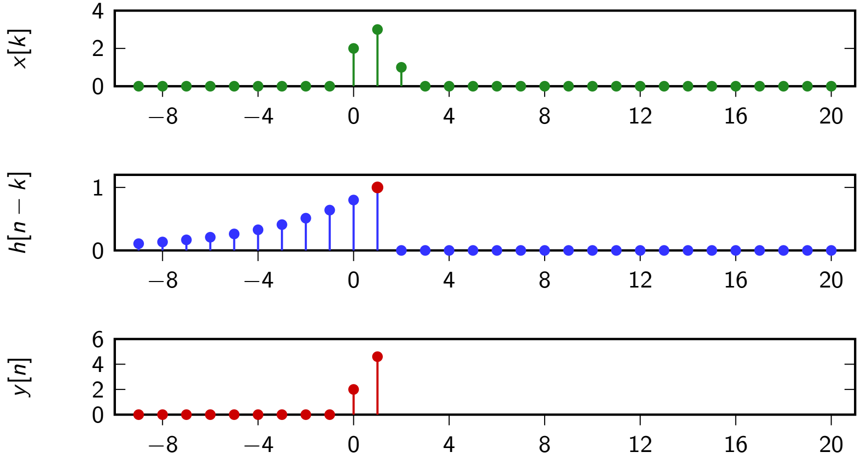

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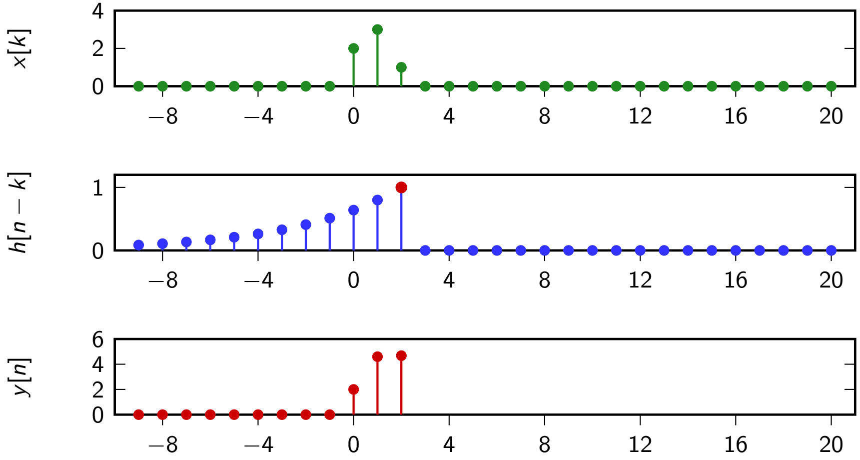

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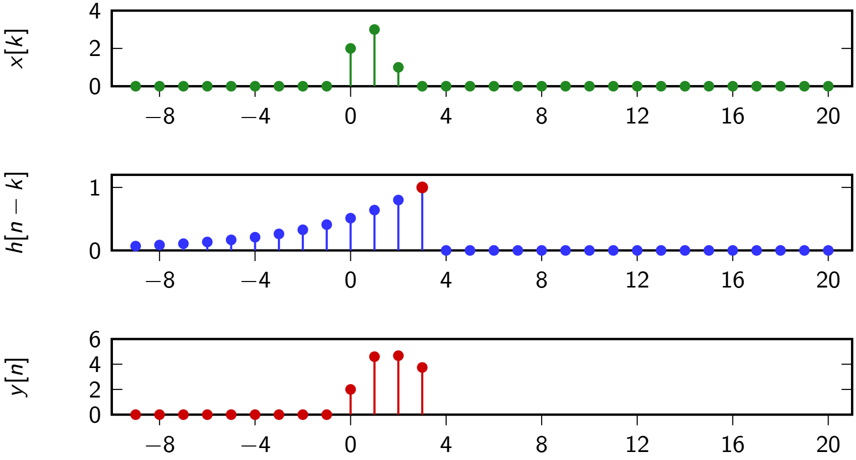

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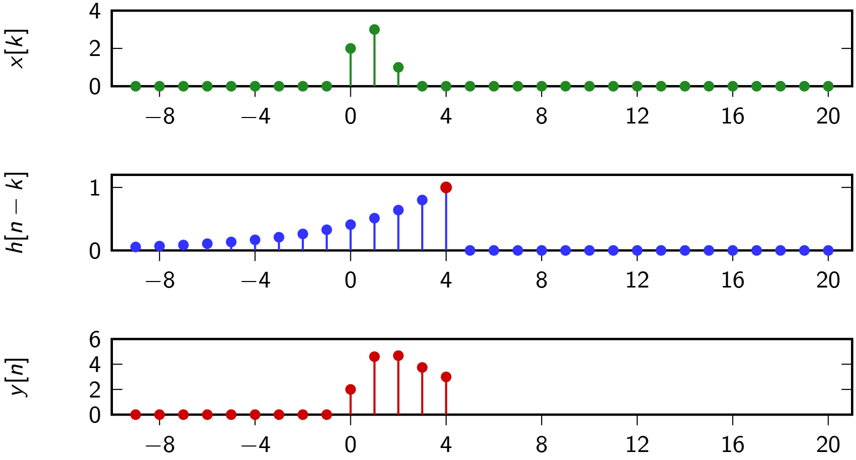

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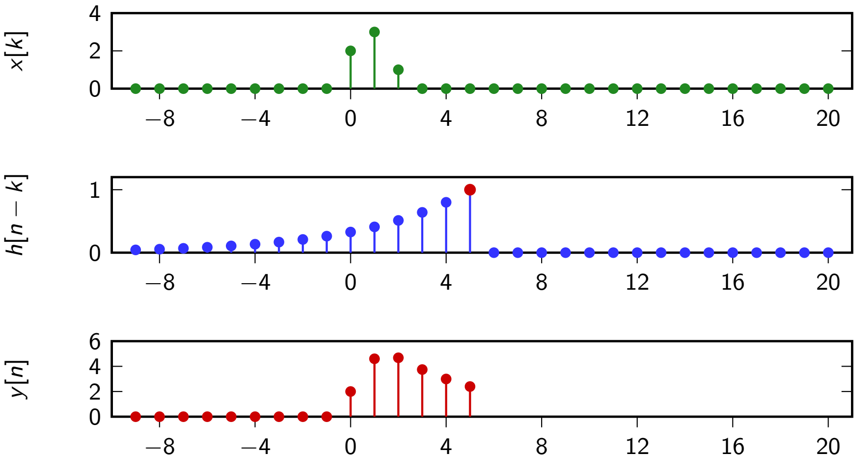

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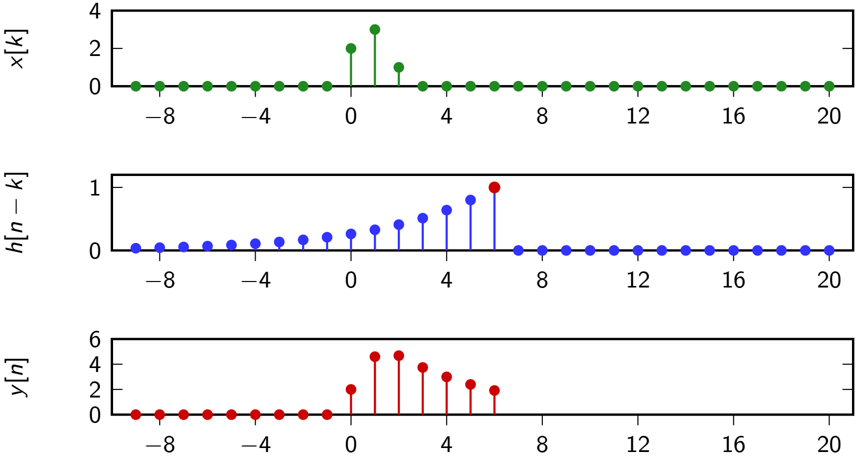

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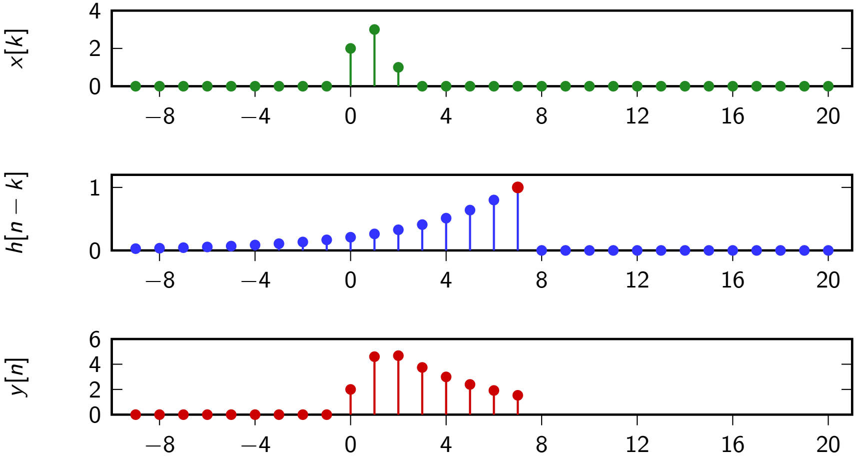

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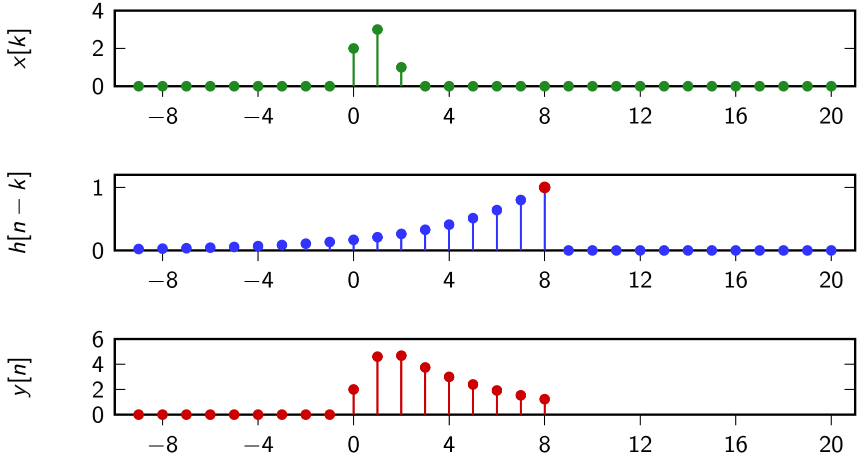

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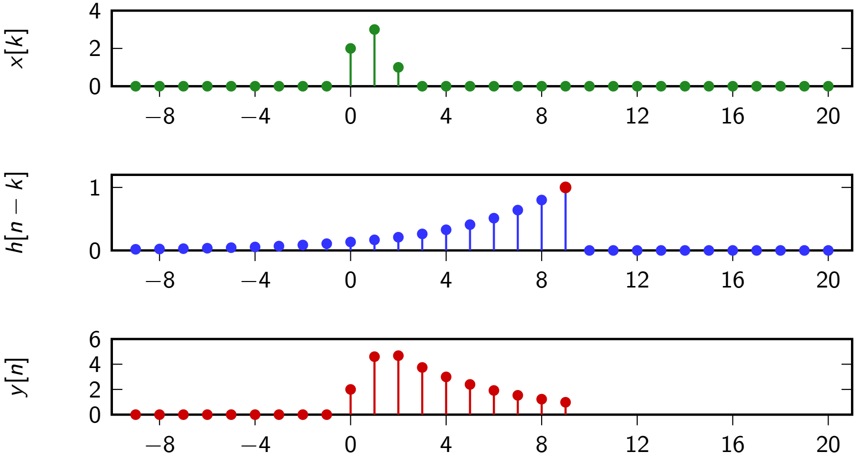

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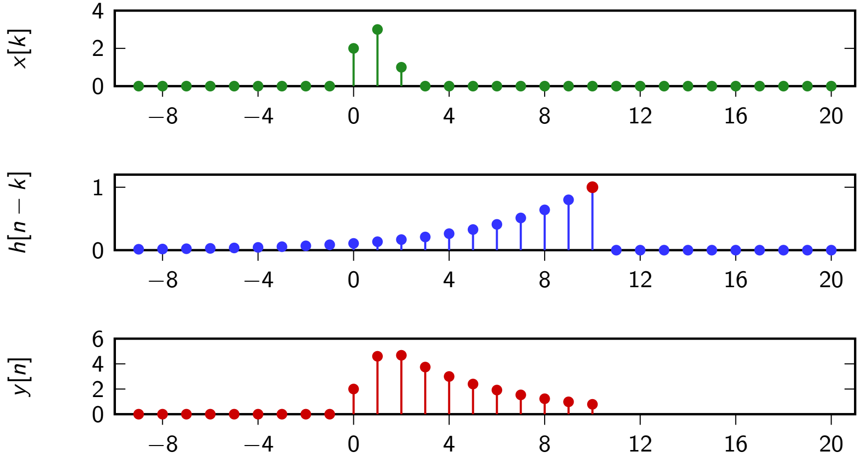

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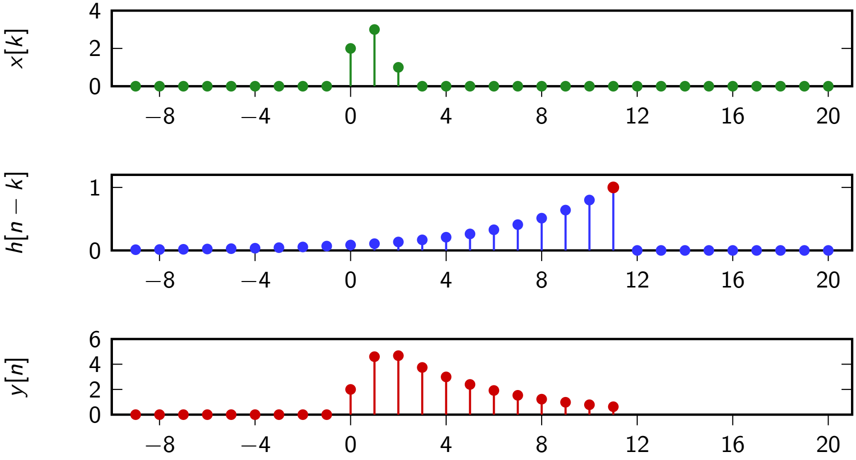

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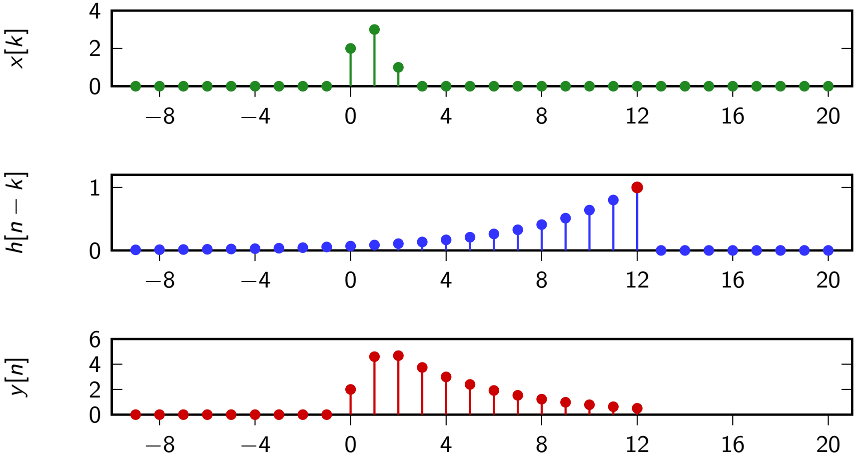

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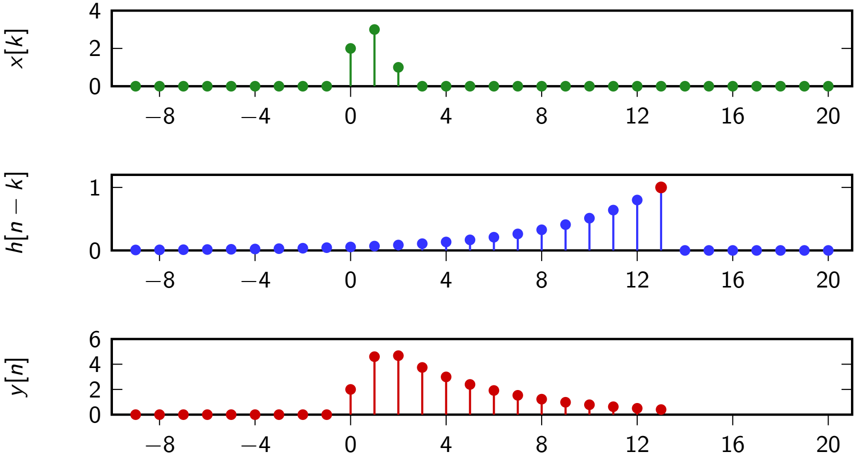

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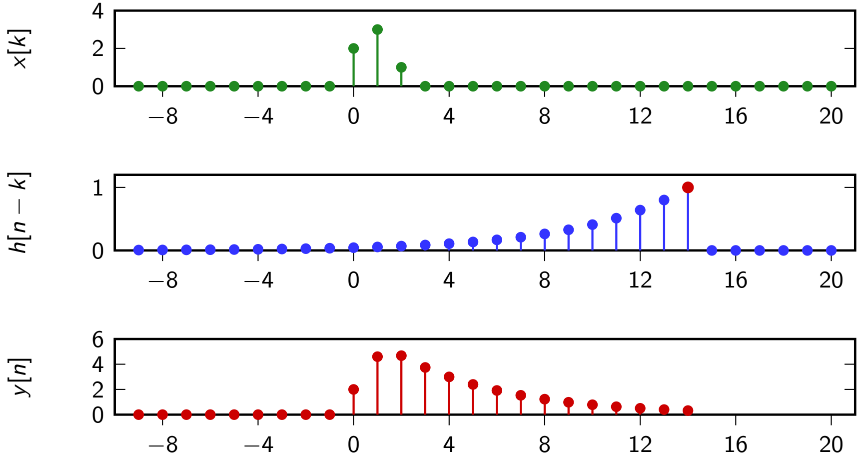

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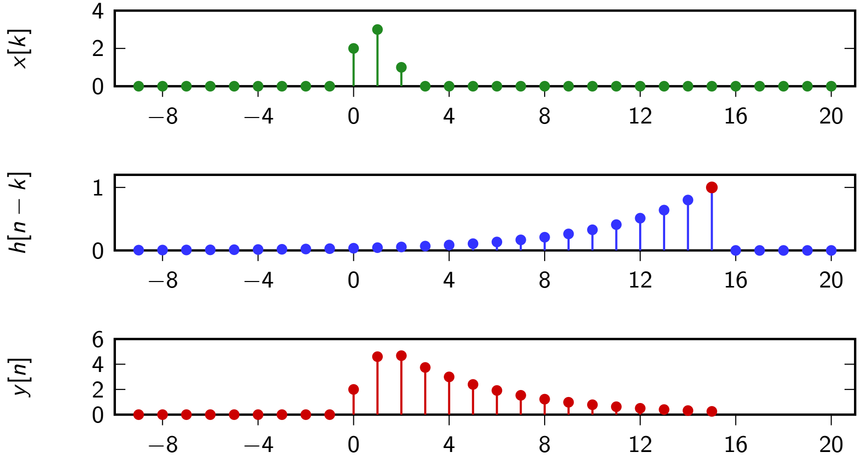

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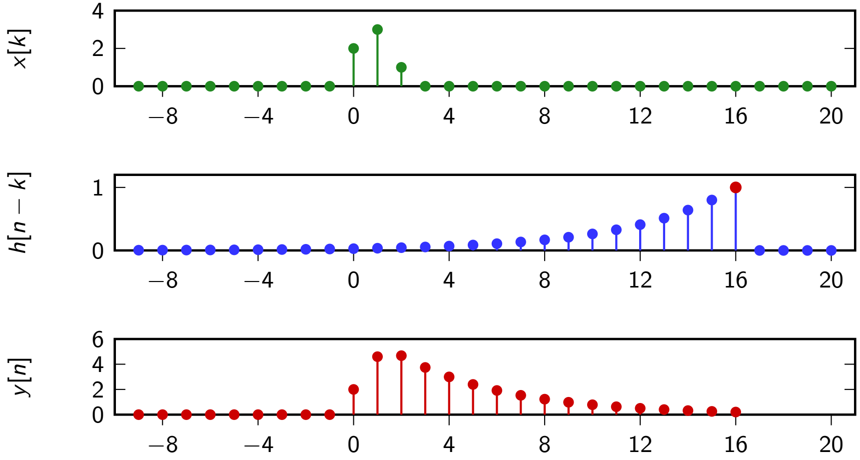

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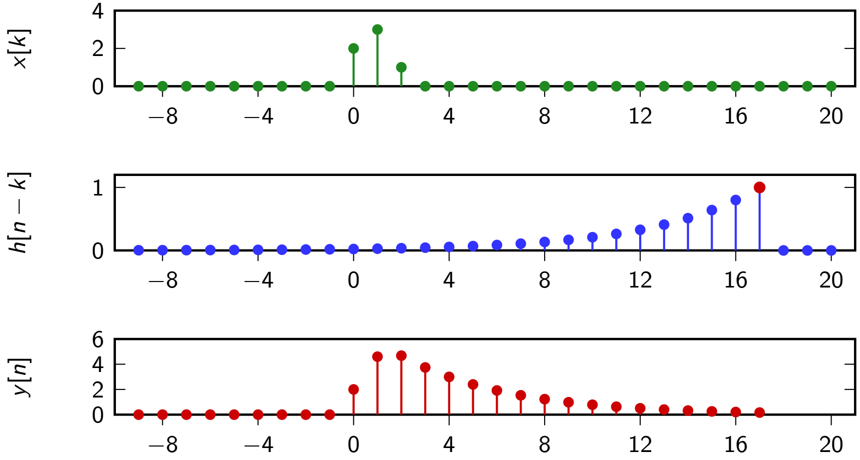

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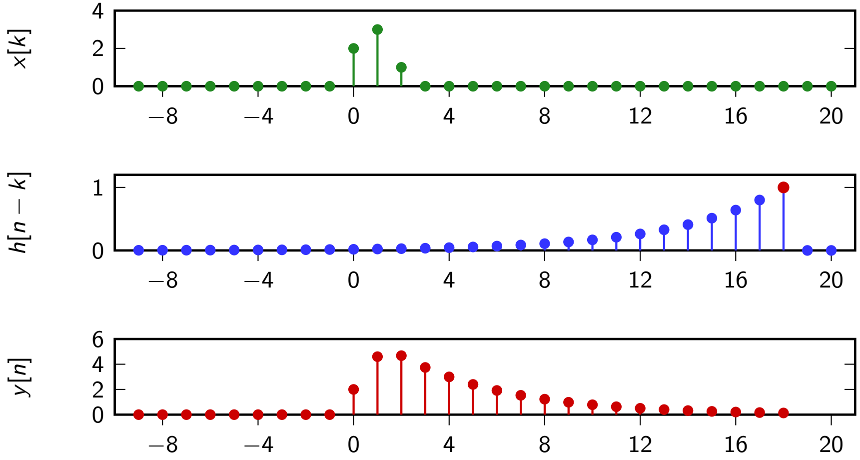

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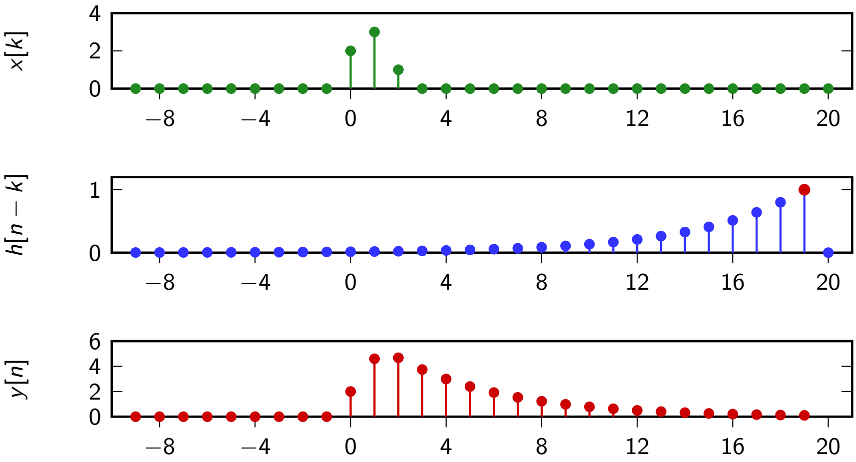

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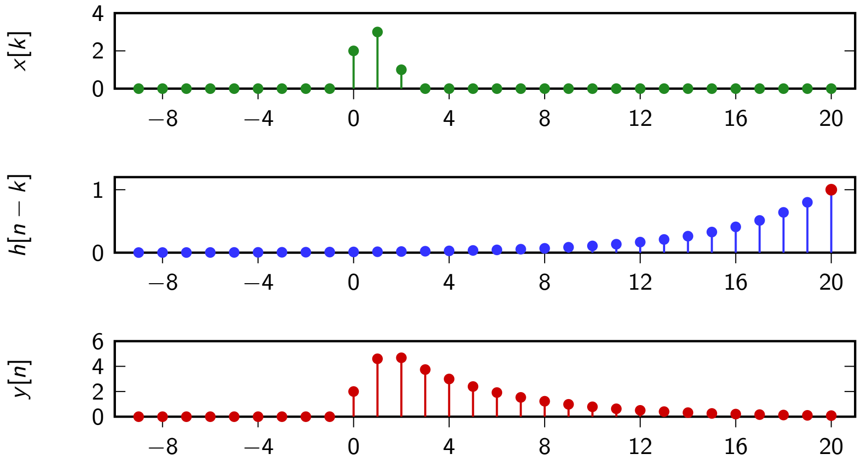

Convolution properties

- linearity and time invariance (by definition)
- commutativity: $(x * h)[n] = (h * x)[n]$
- associativity for absolutely- and square-summable sequences: $((x * h) * w)[n] = (x * (h * w))[n]$

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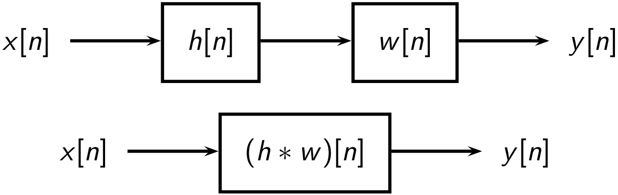

filtering by example

Overview:

- Moving average filter
- Leaky integrator

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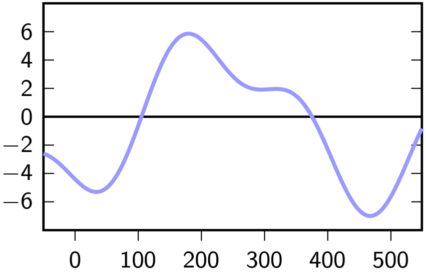
Typical filtering scenario: denoising



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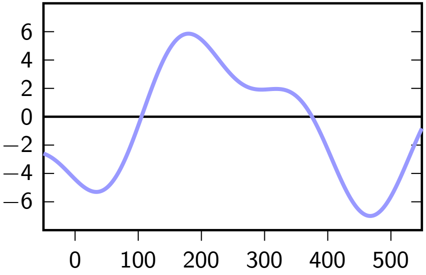

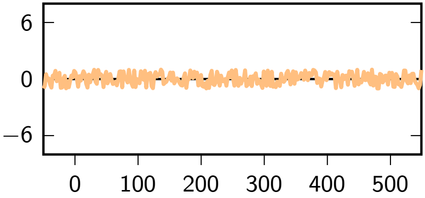

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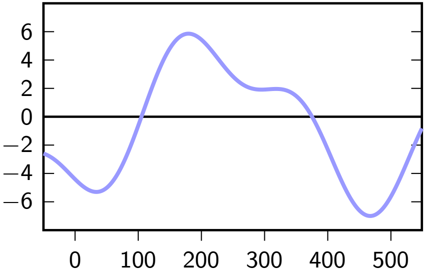

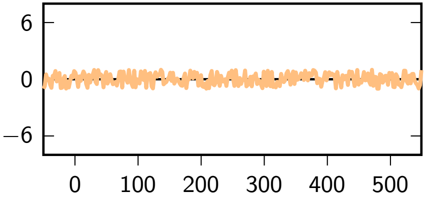

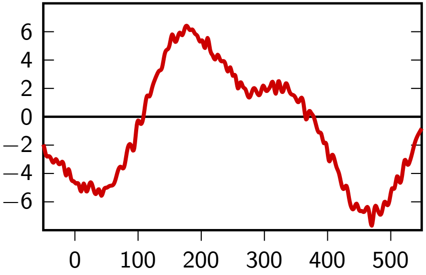

Denoising by Moving Average

- idea: replace each sample by the local average
- for instance: $y[n] = (x[n] + x[n-1])/2$
- more general:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

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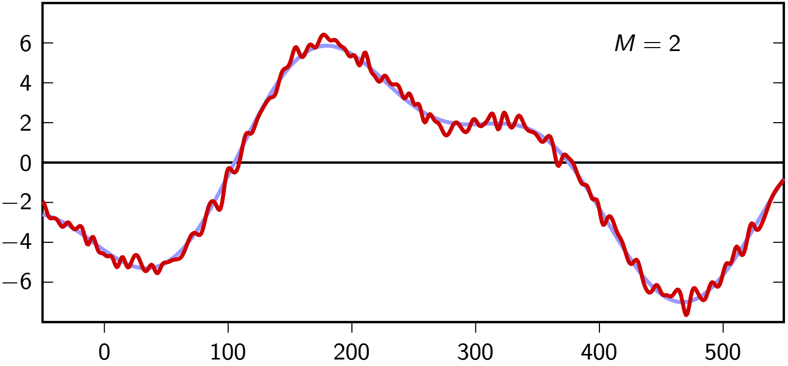

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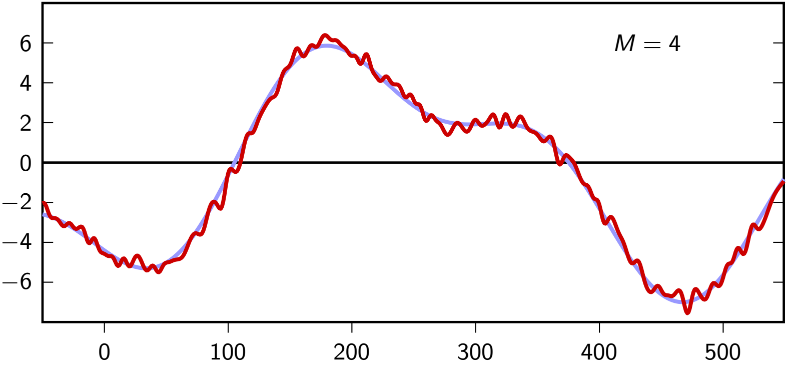

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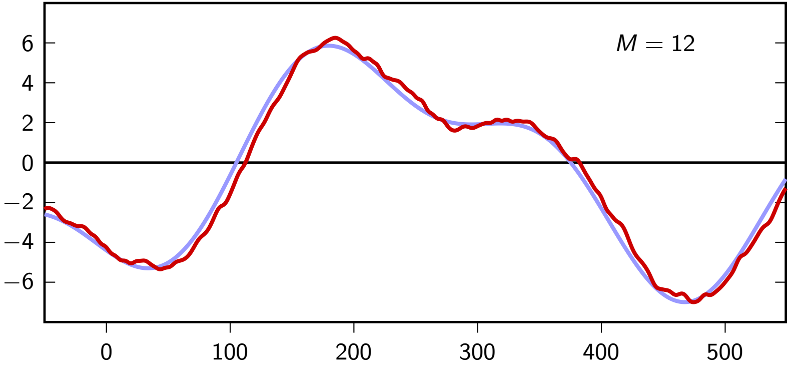

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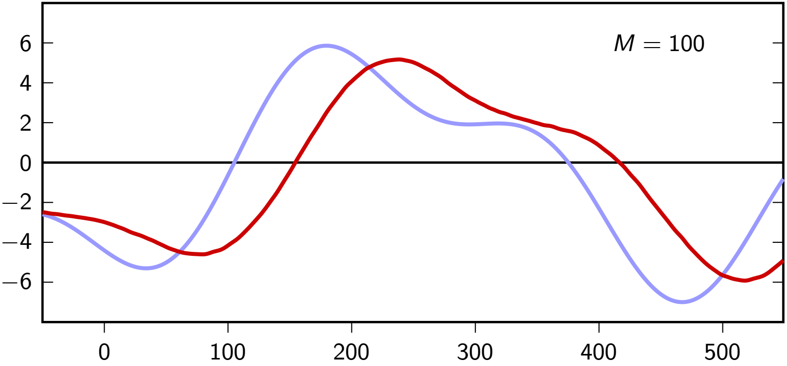

MA: impulse response

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

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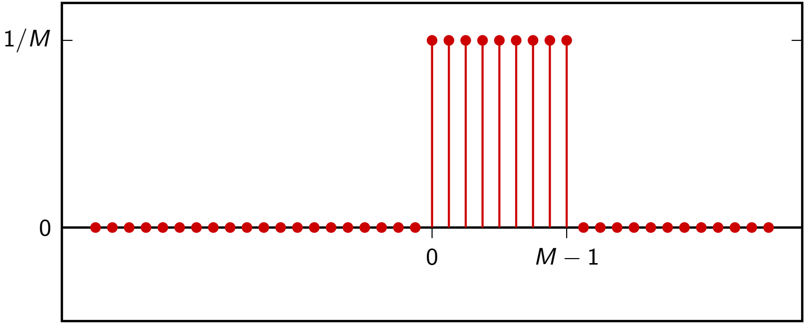

MA: analysis

- smoothing effect proportional to M
- number of operations and storage also proportional to M

From the MA to a first-order recursion

$$y_M[n] = \boxed{\frac{1}{M}\left(x[n] + x[n-1] + x[n-2] + \ldots + x[n-M+1]\right)}$$

moving average over M points

From the MA to a first-order recursion

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moving average over M points

$$y_{M-1}[n] = \frac{1}{M-1}\left(x[n] + x[n-1] + x[n-2] + \ldots + x[n-M+2]\right)$$

From the MA to a first-order recursion

$$y_M[n] = \frac{1}{M}x[n] + \frac{1}{M}\left(x[n-1] + x[n-2] + \ldots + x[n-M+1]\right)$$

From the MA to a first-order recursion

$$y_M[n] = \frac{1}{M}x[n] + \boxed{\frac{1}{M}\left(x[n-1] + x[n-2] + \ldots + x[n-M+1]\right)}$$

"almost" $y_{M-1}[n-1]$

i.e., moving average over $M-1$ points, delayed by one

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The Leaky Integrator

- when M is large, $y_{M-1}[n] \approx y_M[n]$ (and $\lambda \approx 1$)
- try the filter

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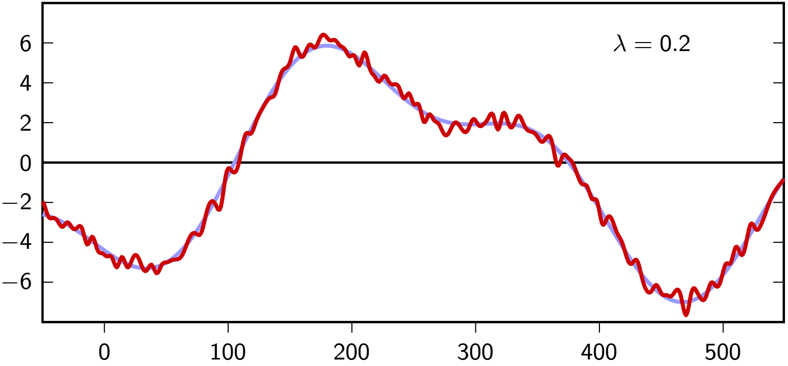
Denoising recursively with the Leaky Integrator



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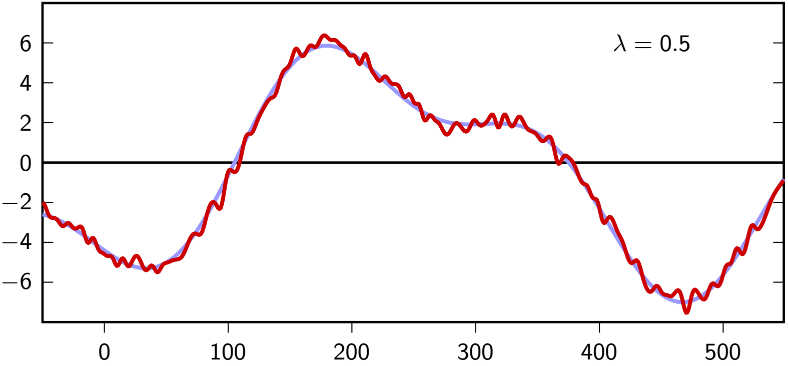

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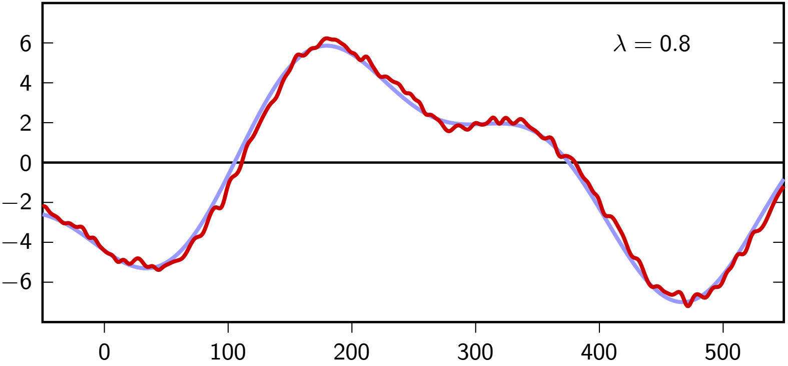

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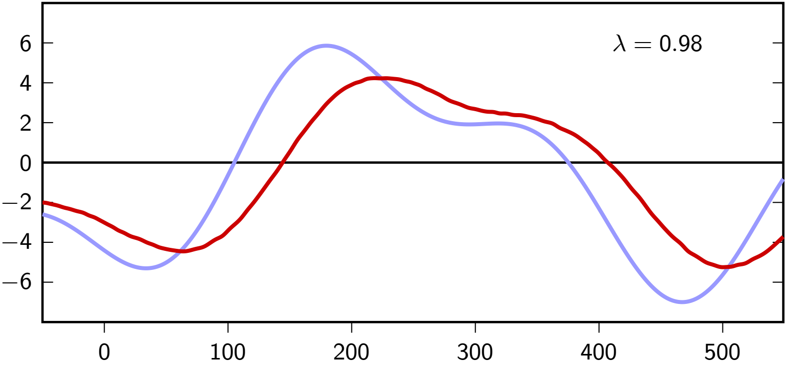

What about the impulse response?

$$y[n] = \lambda y[n-1] + (1-\lambda)\delta[n]$$

- $y[n] = 0$ for all $n < 0$
- $y[0] = \lambda y[-1] + (1-\lambda)\delta[0] = (1-\lambda)$
- $y[1] = \lambda y[0] + (1-\lambda)\delta[1] = \lambda(1-\lambda)$
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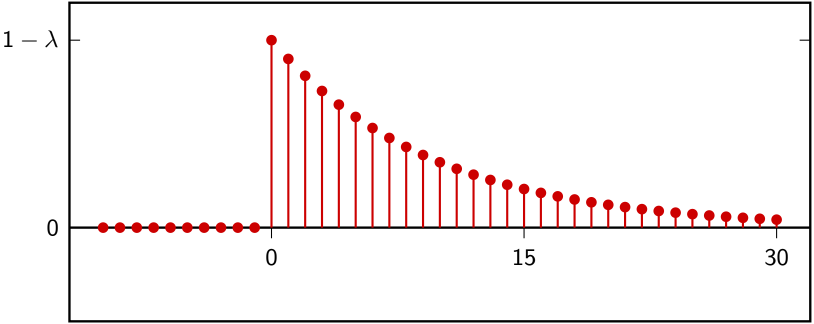
Impulse response

$$h[n] = (1-\lambda)\lambda^n\, u[n]$$



Leaky Integrator: why the name

Discrete-time integrator is a boundless accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

We can rewrite the integrator as

$$y[n] = y[n-1] + x[n]$$

Leaky Integrator: why the name

To prevent "explosion" pick $\lambda < 1$

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$

keep only a fraction λ of the accumulated value so far and forget ("leak") a fraction $1-\lambda$

add only a fraction $1-\lambda$ of the current value to the accumulator

LI: analysis

- smoothing effect dependent on λ
- number of operations and storage: *independent of* λ
- recursion generates infinite-length impulse response
- infinite-length impulse responses are computable

filter stability

Filter types according to impulse response

- Finite Impulse Response (FIR)
- Infinite Impulse Response (IIR)
- causal
- noncausal

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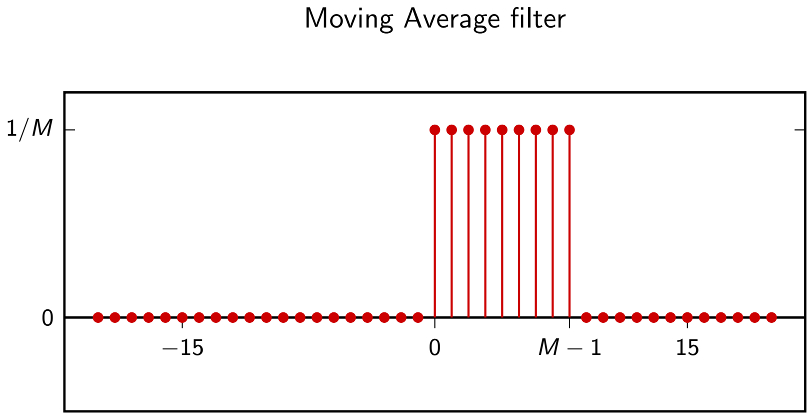
FIR

- impulse response has finite support
- only a finite number of samples are involved in the computation of each output sample

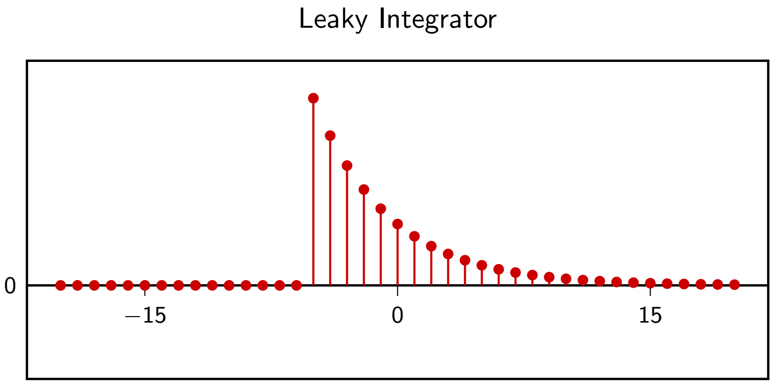
FIR (example)



IIR

- impulse response has infinite support
- a potentially infinite number of samples are involved in the computation of each output sample
- surprisingly, in many cases the computation can still be performed in a finite amount of steps

IIR (example)

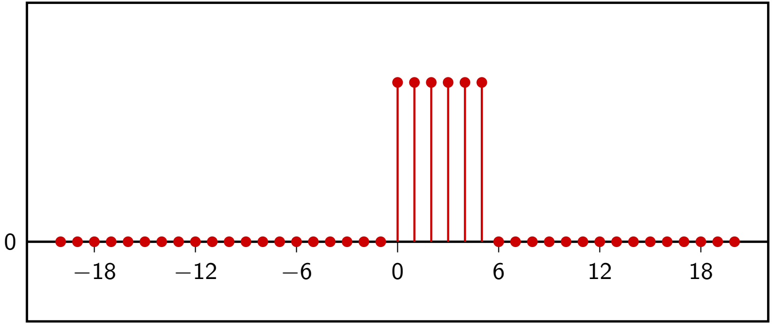


Causal vs Noncausal

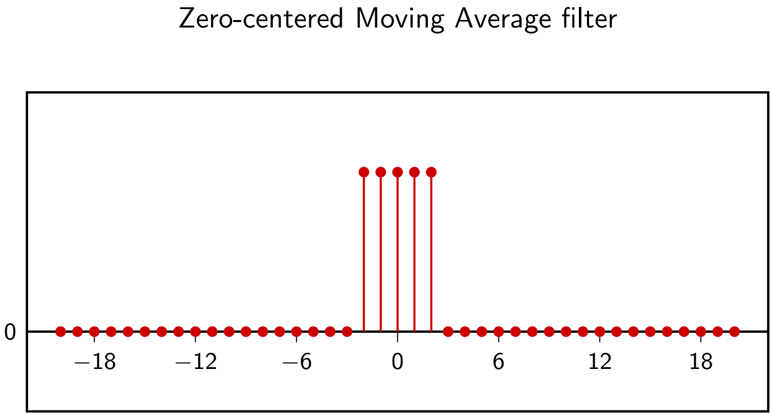
- causal:
  - impulse response is zero for $n < 0$
  - only past samples (with respect to the present) are involved in the computation of each output sample
  - causal filters can work "on line" since they only need the past
- noncausal:
  - impulse response is nonzero for some (or all) $n < 0$
  - can still be implemented in a offline fashion (when all input data is available on storage, e.g. in Image Processing)

Causal example

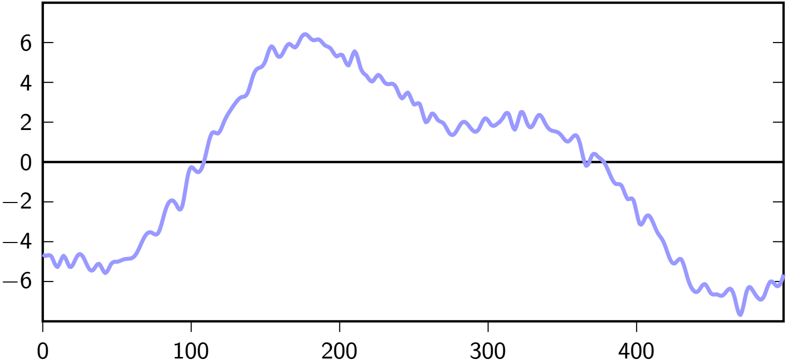
Moving Average filter



Noncausal example



Causal and Noncausal Moving Average



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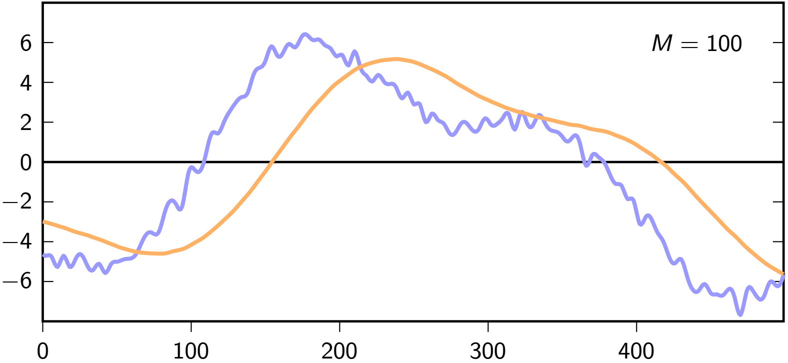

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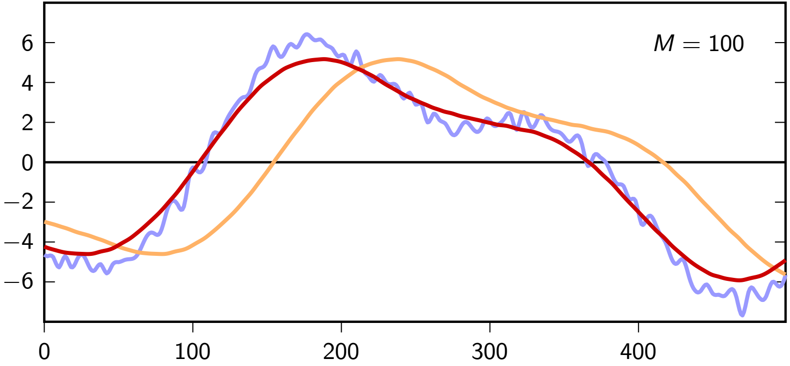

Stability

- key concept: avoid "explosions" if the input is nice
- a nice signal is a bounded signal: $|x[n]| < M$ for all n
- Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

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Fundamental Stability Theorem

A filter is BIBO stable if and only if its impulse response is absolutely summable

Proof ($\Rightarrow$)

Hypotheses:

- $|x[n]| < M$
- $\sum_n |h[n]| = L < \infty$

Thesis:

- $|y[n]| < \infty$

Proof:

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &\leq ML \end{aligned}$$

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- $\forall |x[n]| < \infty$, $|(x * h)[n]| < \infty$

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- $\sum_n |h[n]| < \infty$

Proof (by contradiction):

- assume hypothesis true, yet $\sum_n |h[n]| = \infty$
- build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$
- clearly, $|x[n]| < \infty$
- however

$$(x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

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The good news

FIR filters are always stable

Checking the stability of IIRs

Let's check the Leaky Integrator:

$$\sum_{n=-\infty}^{\infty} |h[n]| = |1-\lambda| \sum_{n=0}^{\infty} |\lambda|^n$$

$$= \lim_{n\to\infty} |1-\lambda| \frac{1-|\lambda|^{n+1}}{1-|\lambda|}$$

$$< \infty \quad \text{for } |\lambda| < 1$$

stability is guaranteed for $|\lambda| < 1$

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Checking the stability of IIRs

We will study indirect methods for filter stability later

frequency response

Overview:

- Eigensequences
- Convolution theorem
- Frequency and phase response

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A remarkable result

$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow ?$$

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&= h[n] * e^{j\omega_0 n} \\
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&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\
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$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

- complex exponentials are *eigensequences* of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
- DTFT of impulse response determines the frequency characteristic of a filter

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Magnitude and phase

If $H(e^{j\omega_0}) = Ae^{j\theta}$, then

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n+\theta)}$$

amplitude:
amplification ($A > 1$)
or attenuation ($0 \leq A < 1$)

phase shift:
delay ($\theta < 0$)
or advancement ($\theta > 0$)

The convolution theorem

In general:

$$\mathrm{DTFT}\left\{x[n] * h[n]\right\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

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$$\begin{aligned}
\text{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\
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&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\
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&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega (n-k)} \\
&= H(e^{j\omega}) X(e^{j\omega})
\end{aligned}
$$

The convolution theorem

$$\begin{aligned}
\mathrm{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\
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Frequency response

$$H(e^{j\omega}) = \mathrm{DTFT}\left\{h[n]\right\}$$

Two effects:

- **magnitude:** amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
- **phase:** overall delay and shape changes

Frequency response

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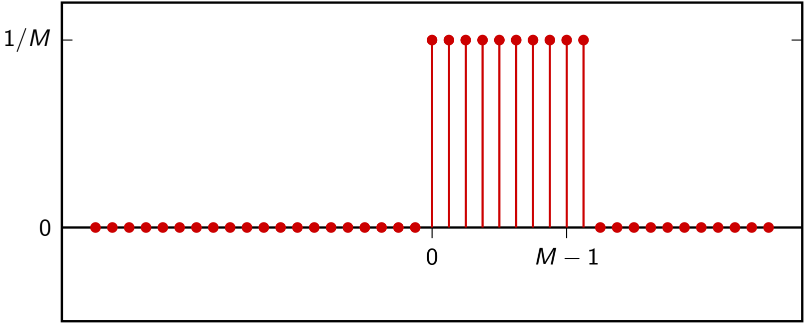
Two effects:

- magnitude: amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
- phase: overall delay and shape changes

Moving Average revisited

$$h[n] = (u[n] - u[n-M])/M$$

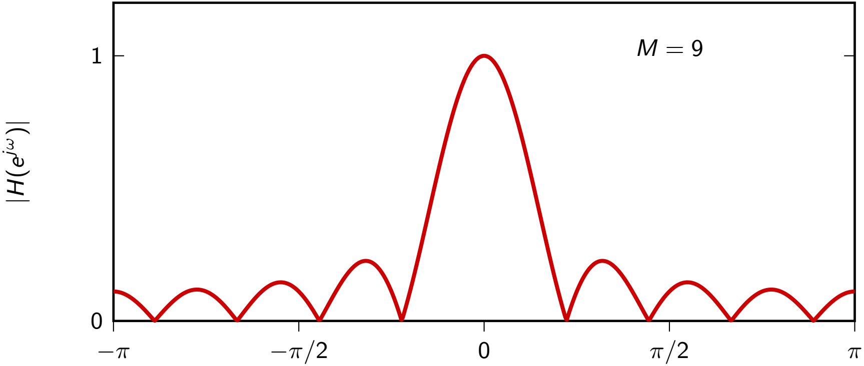


Moving Average revisited

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M-1} \frac{1}{M}\, e^{-j\omega n} \\
&= \frac{1}{M}\, \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} \\
&= \frac{1}{M}\, \frac{e^{-j\frac{\omega M}{2}}\left[e^{j\frac{\omega M}{2}} - e^{-j\frac{\omega M}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} \\
&= \frac{1}{M}\, \frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(M-1)}
\end{aligned}
$$

Moving Average, magnitude response

$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$



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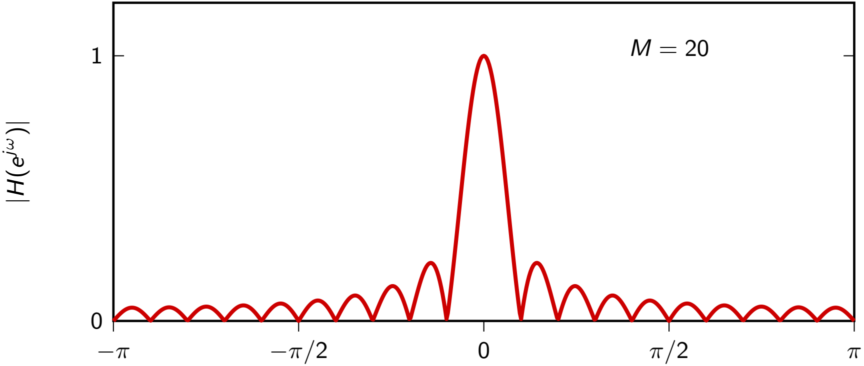

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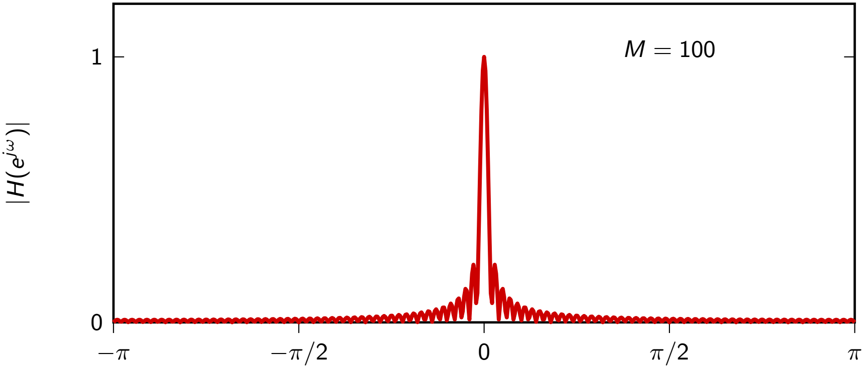
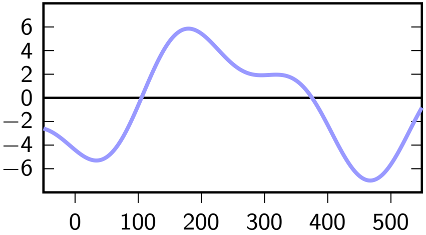
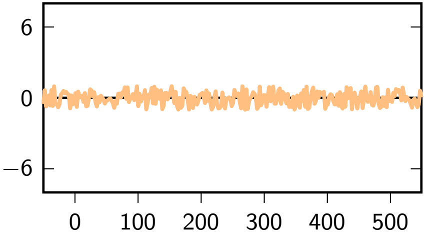
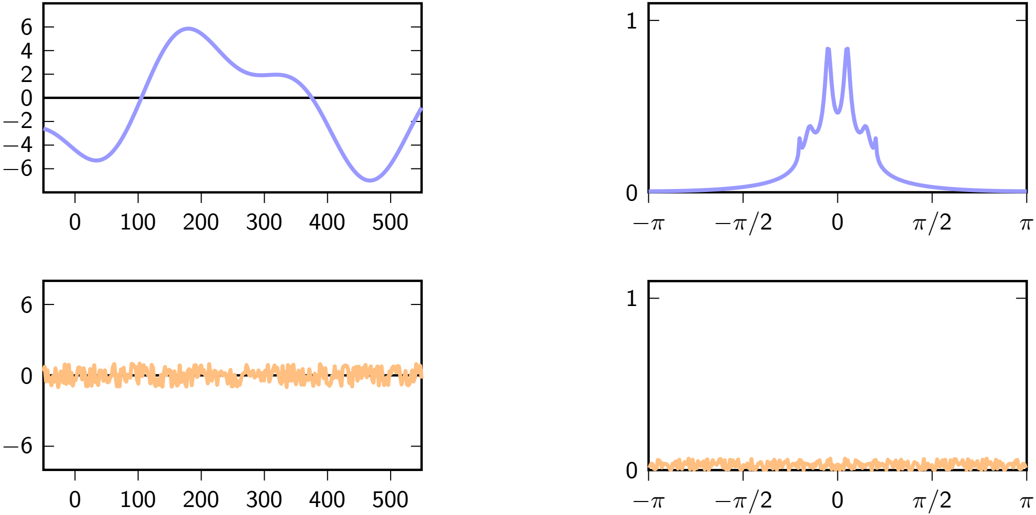


Denoising revisited





Denoising revisited



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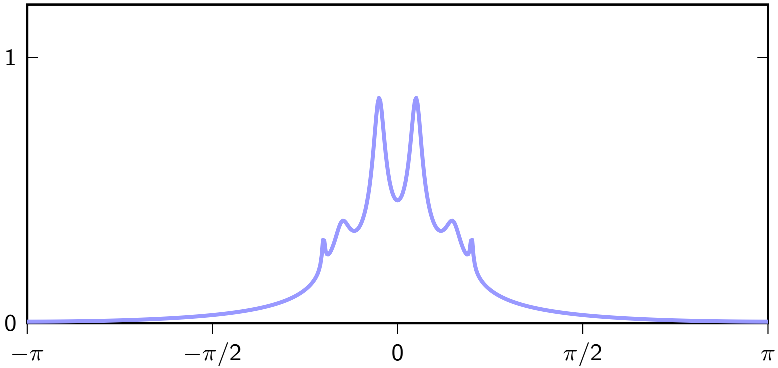
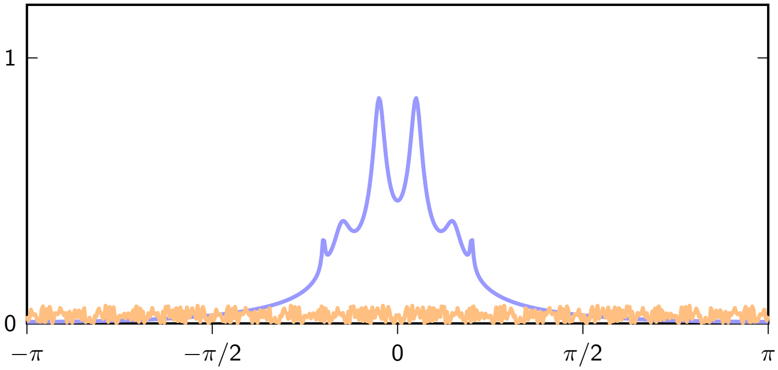


Denoising revisited



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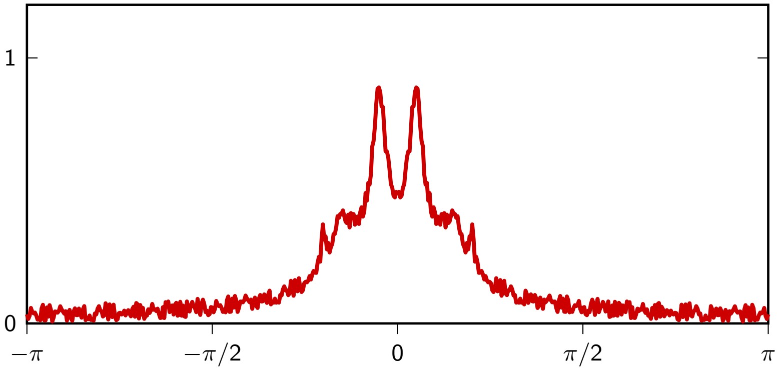

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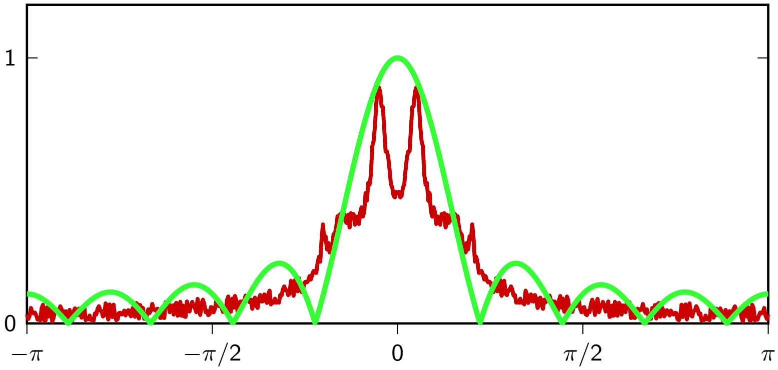
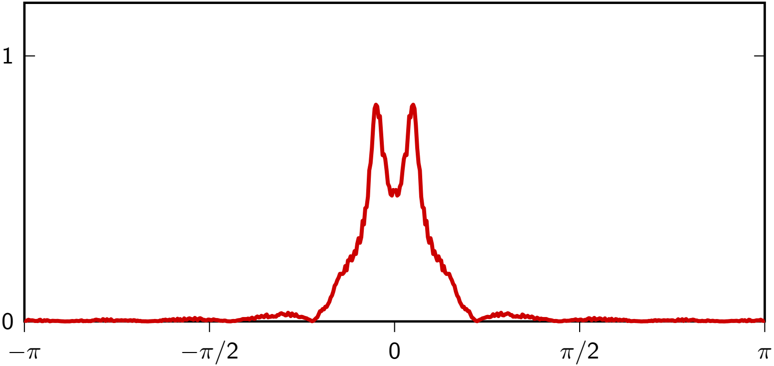
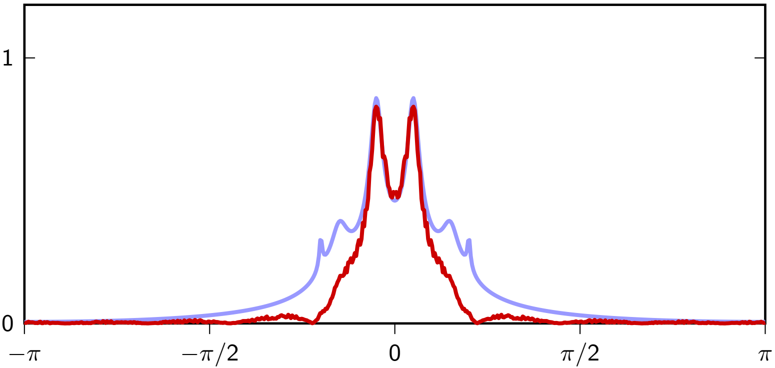


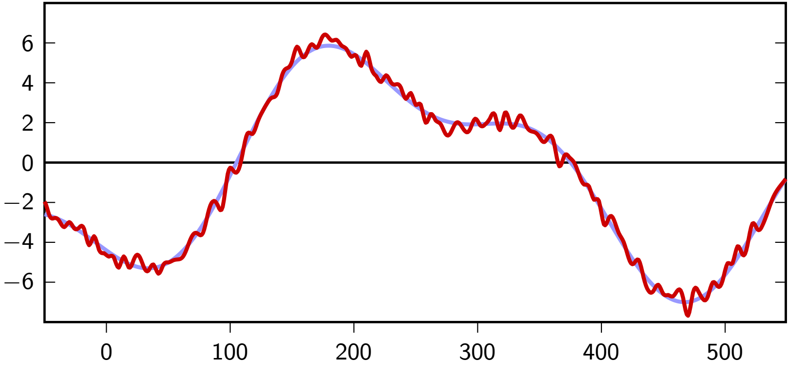
Denoising revisited



Denoising revisited



By the way, remember the time-domain analysis...



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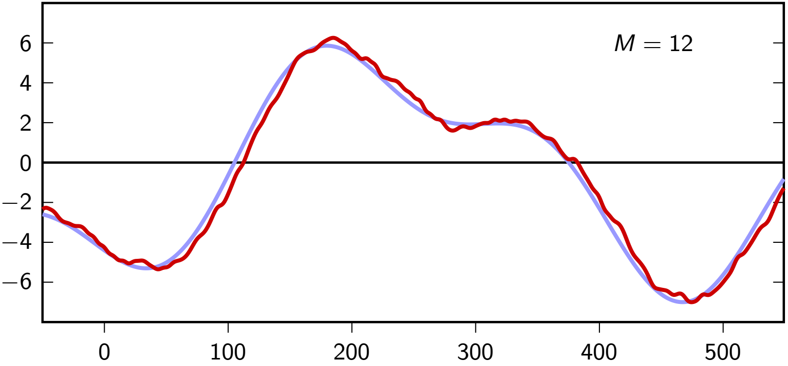

What about the phase?

Assume $|H(e^{j\omega})| = 1$

- zero phase: $\angle H(e^{j\omega}) = 0$
- linear phase: $\angle H(e^{j\omega}) = d\omega$
- nonlinear phase

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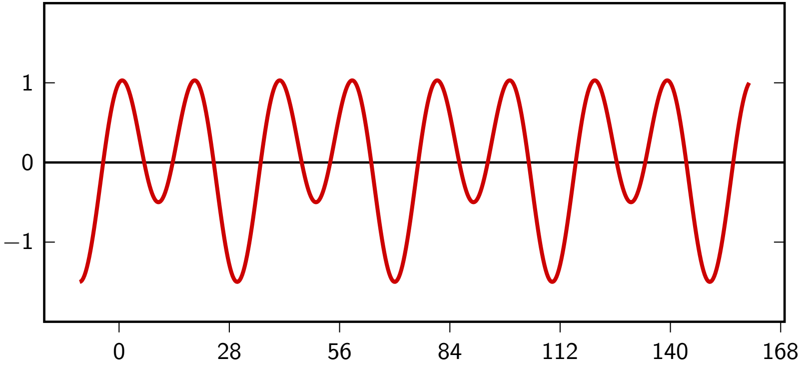
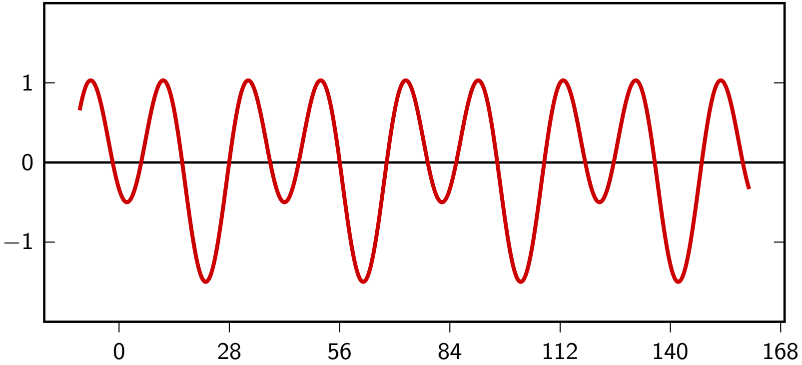
Phase and signal shape

$$x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n) \qquad \omega_0 = \frac{2\pi}{40}$$



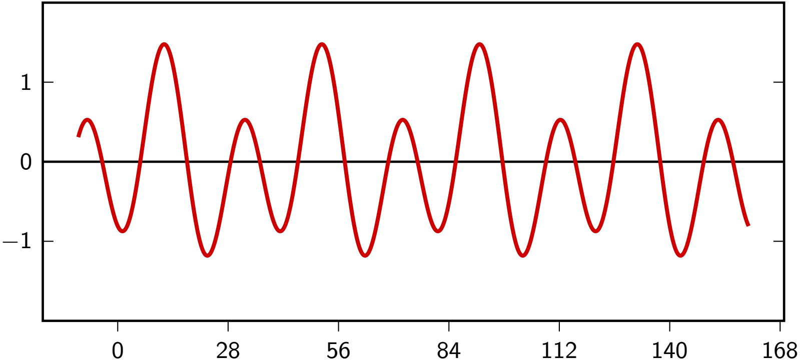
Phase and signal shape: linear phase

$$x[n] = \frac{1}{2}\sin(\omega_0 n + \theta_0) + \cos(2\omega_0 n + 2\theta_0) \qquad \theta_0 = \frac{8\pi}{5}$$



Phase and signal shape: nonlinear phase

$$x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n + 2\theta_0)$$

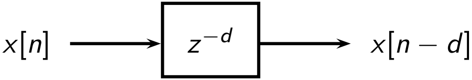


Linear phase

$$x[n] \longrightarrow \boxed{z^{-d}} \longrightarrow x[n-d]$$

- $y[n] = x[n-d]$
- $Y(e^{j\omega}) = e^{-j\omega d}\, X(e^{j\omega})$
- $H(e^{j\omega}) = e^{-j\omega d}$
- linear phase term

Linear phase



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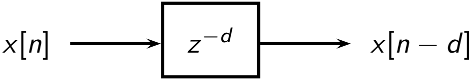


- $y[n] = x[n-d]$
- $Y(e^{j\omega}) = e^{-j\omega d} X(e^{j\omega})$
- $H(e^{j\omega}) = e^{-j\omega d}$
- linear phase term

Linear phase

In general, if $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega d}, \quad$ with $A(e^{j\omega}) \in \mathbb{R}$

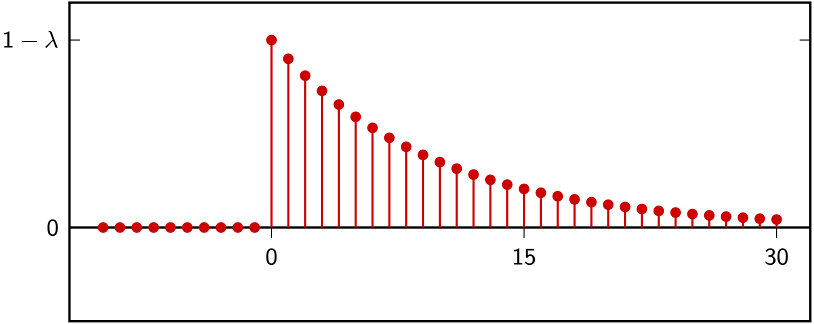
$$x[n] \longrightarrow \boxed{A(e^{j\omega})} \longrightarrow \boxed{\text{delay by } d} \longrightarrow y[n]$$

Moving Average is linear phase

$$H(e^{j\omega}) = \frac{1}{M}\,\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\frac{M-1}{2}\omega}$$

Leaky integrator revisited

$$h[n] = (1-\lambda)\lambda^n \, u[n]$$



Leaky integrator revisited

$$H(e^{j\omega}) = \frac{1-\lambda}{1-\lambda e^{-j\omega}}$$

Finding magnitude and phase require a little algebra...

Leaky integrator revisited

Recall from complex algebra:

$$\frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$$

so that if $x = 1/(a+jb)$,

$$|x|^2 = \frac{1}{a^2+b^2}$$

$$\angle x = \tan^{-1}\left[-\frac{b}{a}\right]$$

Leaky integrator revisited

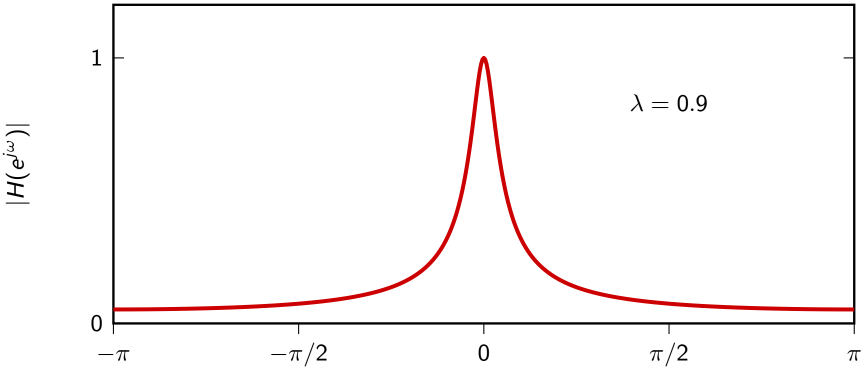
$$H(e^{j\omega}) = \frac{1-\lambda}{(1-\lambda\cos\omega) - j\lambda\sin\omega}$$

so that:

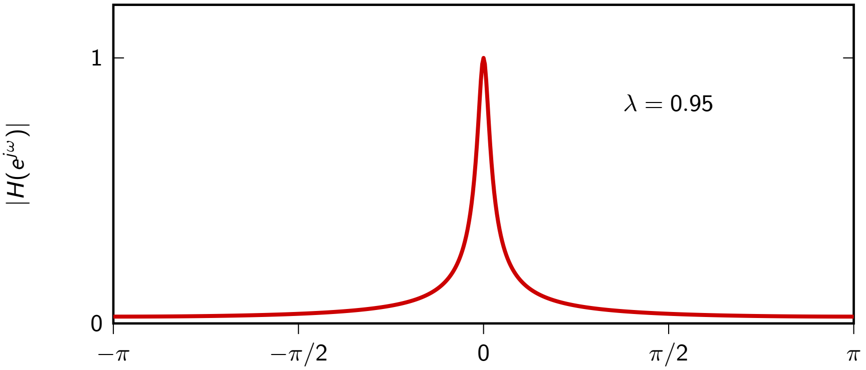
$$|H(e^{j\omega})|^2 = \frac{(1-\lambda)^2}{1-2\lambda\cos\omega+\lambda^2}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{\lambda\sin\omega}{1-\lambda\cos\omega}\right]$$

Leaky integrator, magnitude response



Leaky integrator, magnitude response



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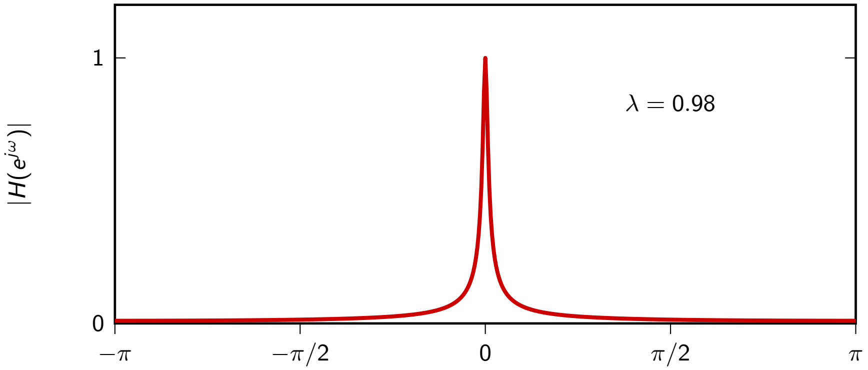
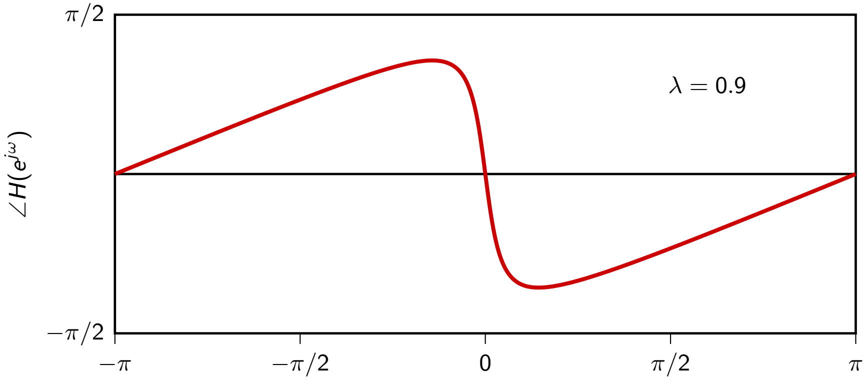
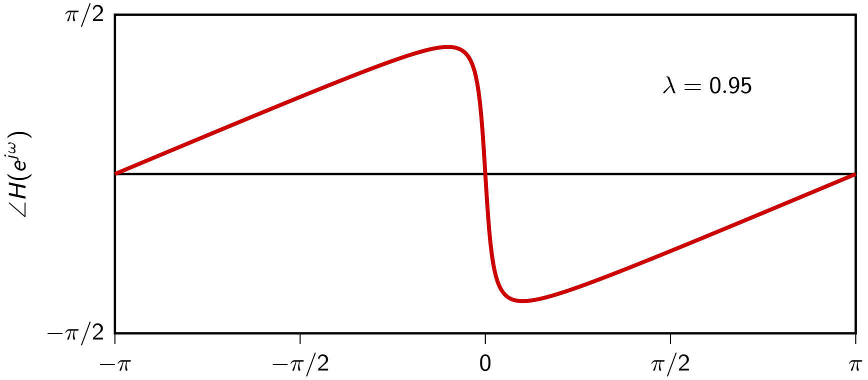


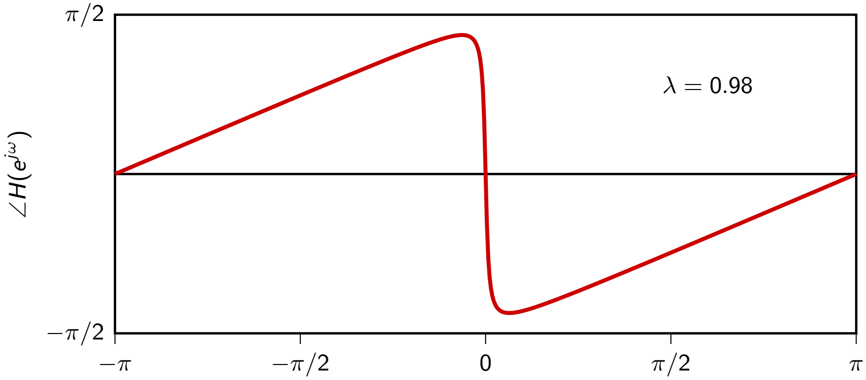
Leaky integrator, phase response



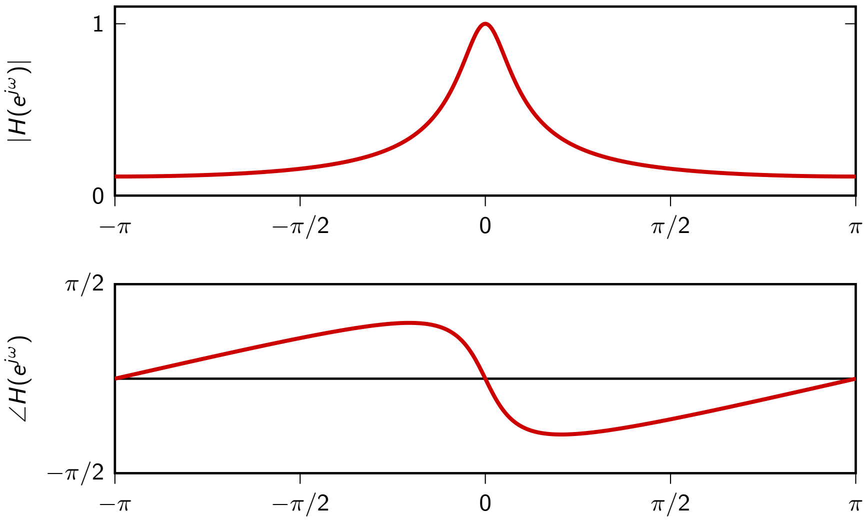
Leaky integrator, phase response



Leaky integrator, phase response



Phase is sufficiently linear where it matters



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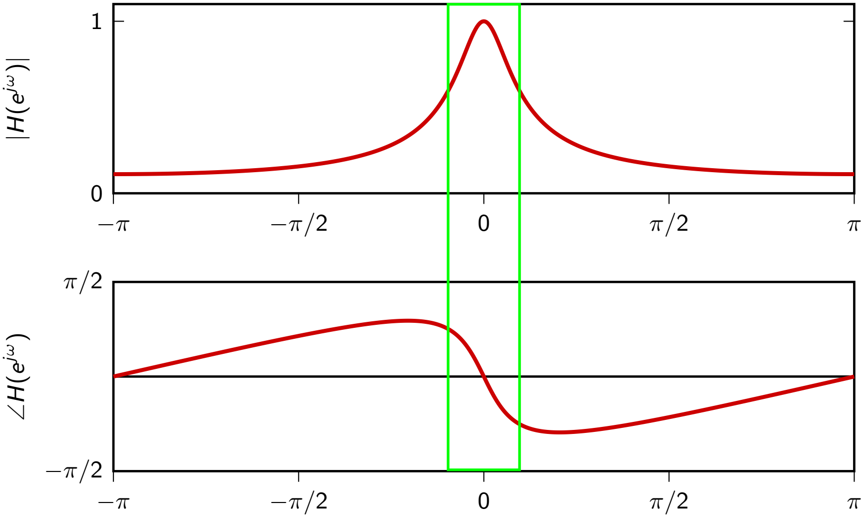
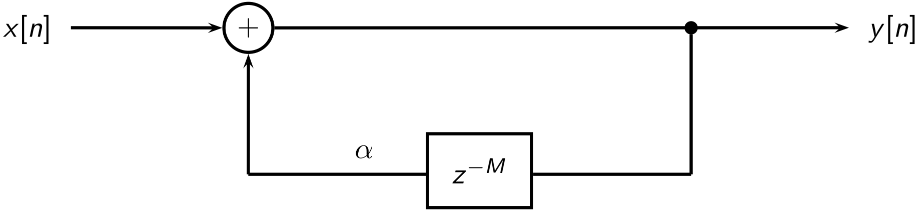


Karplus-Strong revisited, again!



$$y[n] = \alpha\, y[n-M] + x[n]$$

Karplus-Strong revisited

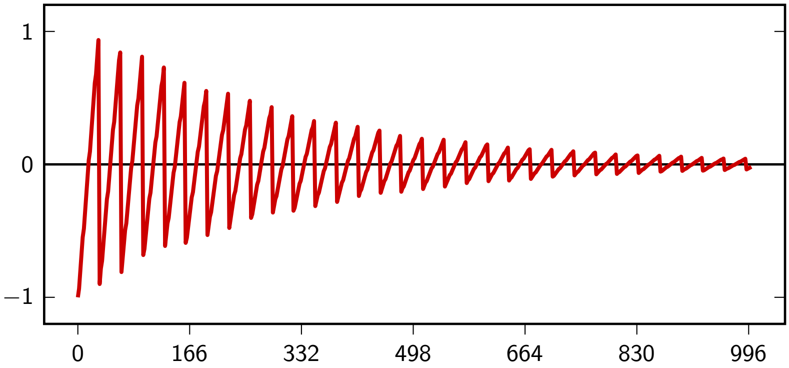
$$y[n] = \bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1], \alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1], \alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots$$

Karplus-Strong revisited

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{\text{st}}\text{ period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1]}_{2^{\text{nd}}\text{ period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots}_{\ldots}$$

KS revisited: sawtooth signal

$$y[n] = \alpha^{\lfloor n/M \rfloor} \bar{x}[n \mod M]\, u[n]$$



DTFT of KS signal, using the convolution theorem

key observation:

$$y[n] = \bar{x}[n] * w[n], \qquad w[n] = \begin{cases} \alpha^k & \text{for } n = kM \\ 0 & \text{otherwise} \end{cases}$$

KS revisited

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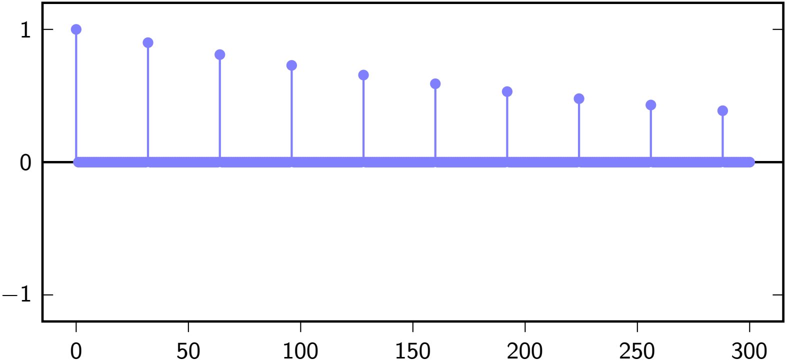

KS revisited

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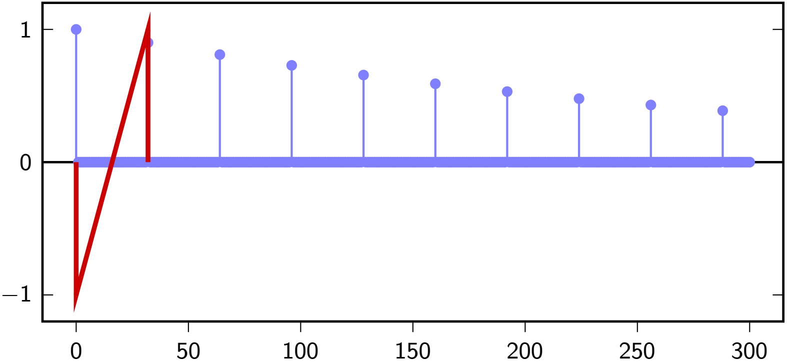

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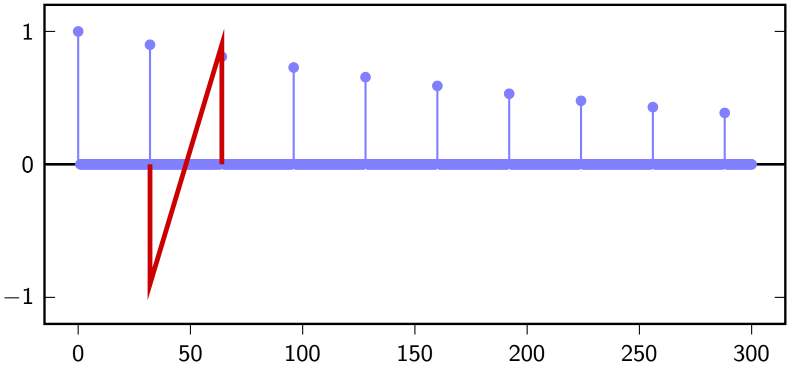

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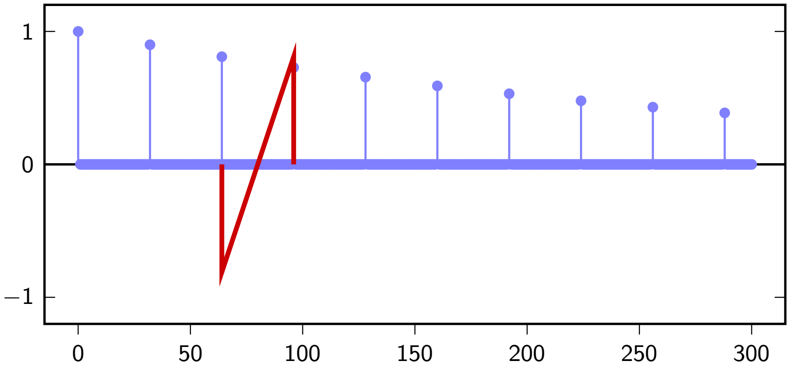

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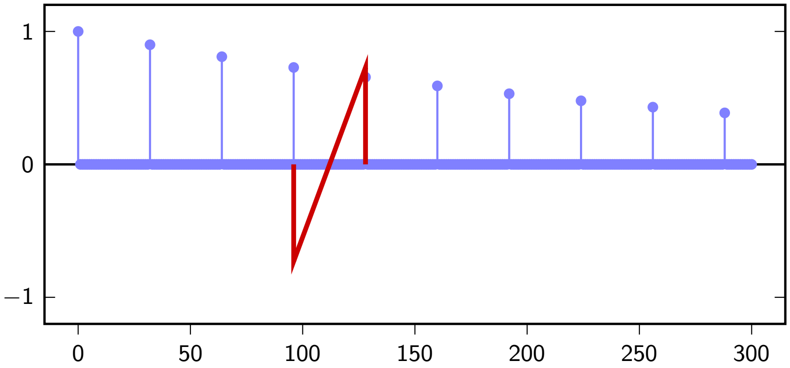

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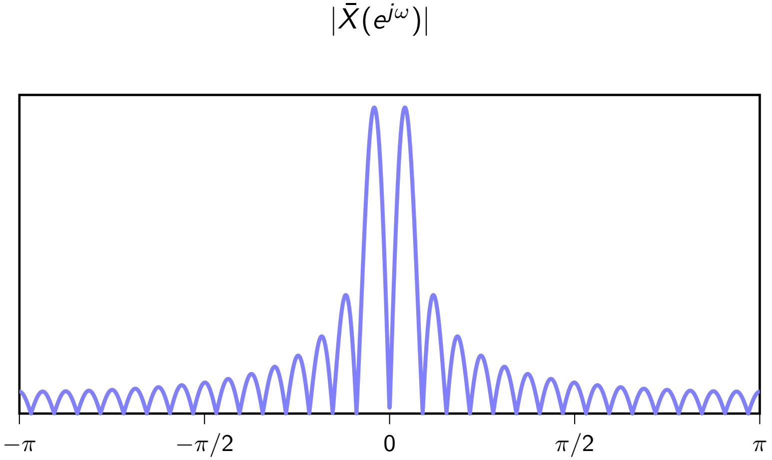
$$Y(e^{j\omega}) = \bar{X}(e^{j\omega})W(e^{j\omega})$$

DTFT of KS signal, using the convolution theorem

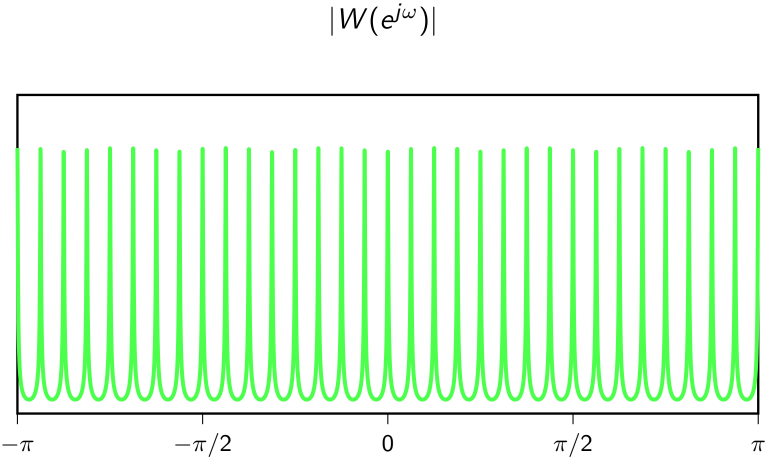
$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$

$$W(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega M}}$$

DTFT of KS



DTFT of KS



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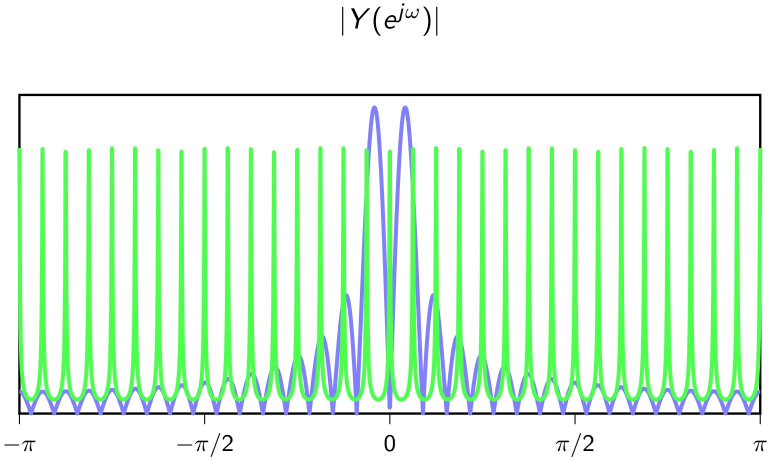

DTFT of KS