Introduction to Game Theory

Boi Faltings

Laboratoire d'Intelligence Artificielle
boi.faltings@epfl.ch
http://moodle.epfl.ch/

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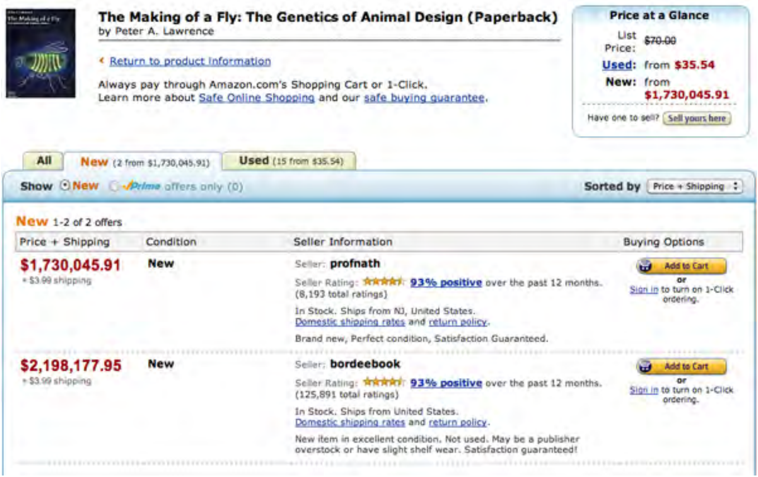
Self-interested Agents

- What happens if multiple agents optimize their strategy at the same time?

Example: Algorithmic Pricing



What went wrong?

- Profnath: adjust price to other sellers: $0.9983 \cdot sup(price)$
- Bordeebooks: use reputation to obtain higher price (buy from other seller and resell), set to $1.270589 \cdot inf(price)$

Date	profnath	bordeebook
8.4.11	\$1’730’045.91	\$2’198’177.95
9.4.11	\$2’194’443.04	\$2’788’233.00
10.4.11	\$2’783’493.00	\$3’536’675.57
11.4.11	\$3’530’663.65	\$4’486’021.69
12.4.11	\$4’478’395.76	\$5’690’199.43
13.4.11	\$5’680’526.66	\$7’217’612.38
...	...	...
18.4.11	\$18’651’708.72	\$23’698’655.93

Game Theory

- Game = multiple agents receive payoffs depending on their *combined* actions.
- Game theory: understand behavior of the agents
- Assumption: self-interest
  Rational action = maximize own payoff

$\Rightarrow$ theory of self-interested multi-agent systems

Elements of a game:

- **players**, the agents playing the game.
- **actions** that change the state of the game.
- **states** of the game.
- **knowledge (beliefs)** of the state and actions.
- **outcome** of the players' actions, in particular **payoffs** for each agent.

Assumption: every agent acts rationally so as to maximize its own payoff.

Example: Placing Stores

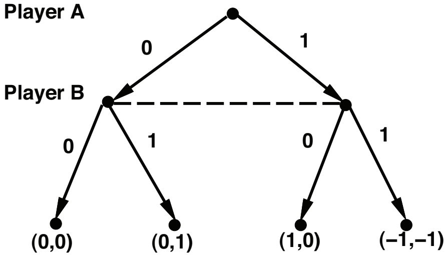
- Convenience store chains A and B decide whether to open a new store in Lausanne.
- Classical: each chain decides if expected revenue $>$ cost.
- Game theoretic: consider the action of the other chain.
- If both open a store, revenue will be only half.

Formalization as a game

- 2 players A and B.
- Actions: player A and B choose between 0 and 1 in sequence.
- States: 7 states: initial + 2 states for actions of player A and 4 states for each combination of actions.
- Knowledge: A and B do not know the other's choice.
- Payoff: If neither A and B open, they gain nothing, if both open, they loose 1, if just one of them opens, it gains 1 and the other nothing.

Representation of the game

State graph (*extensive* form):



Matrix (*normal* form):

		B	
		0	1
	0	(0,0)	(0,1)
A	1	(1,0)	(-1,-1)

A is the *row* player
B is the *column* player
Payoff = (row, column)

Games in normal form

- Normal form: 2 or n players, each player P_i selects one of a *finite* set of actions $\mathcal{A}_i$.
- Players do not know each others' choices.
- Combination of choices determines a payoff to each player.
- Quite general: action can be a complex algorithm for choosing actions depending on beliefs.
- First focus on 2 players only.

Questions answered by Game Theory

- What actions should rational agents take?
- How can we modify the game to ensure better outcomes?
- How can we design games with certain properties?

Applications of game theory

- Effects of regulations and laws: what strategies will agents adopt?
- Finding best strategies in agent interaction.
- Determining stable solutions for negotiation and group decision.
- Design of auctions and cost sharing mechanisms.
- Design of decentralized algorithms, e.g. internet routing.

Classification of games

- 2/n players: first focus on 2 players
- zero-sum/general-sum
- finite/infinite number of states: assume finite
- deterministic/random (lottery)

Zero-sum vs. general-sum

- Zero-sum game: for every outcome, sum of rewards $= 0$.
- Models pure competition.
- Much stronger results than for general-sum games
- First assume zero-sum games, then generalize.

Solution Concepts

Game specifies players, actions and payoffs.
Solution to a game:

> *What actions will rational players select?*

Solution concept: rules for selecting actions for all players in a consistent way.

Strategies

- Strategy = recipe by which each player chooses its actions
- Pure strategy: for each state, the action is chosen in a deterministic way:

S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
a1	a3	a7	a1	a1	a2	a5	a7	a3	a6	a4	a3

Dominant, pure strategies

Dominant strategy = strategy which is best for every action of the other player:

		B	
		0	1
	0	(-1,1)	(1,-1)
A	1	(2,-2)	(4,-4)

$\Rightarrow$ 1 is *always* better for A
$\Rightarrow$ 0 is *always* better for B

- does not require model of the other player
- but does not always exist

Types of dominant strategies

- (strictly) dominant strategy: for every action of the other player, the strategy is strictly better than any other strategy.
- weakly dominant strategy: for every action of the other player, the strategy is at least as good as any other, and it is strictly better for at least one action of the other player.
- very weakly dominant strategy: for every action of the other player, the strategy is at least as good as any other.

Minimax, pure strategy

Game with no dominant strategy for B:

		B: 0	B: 1
	0	(-1,1)	(1,-1)
A	1	(5,-5)	(2,-2)

Minimax strategy = strategy which maximizes gains supposing that the opponent will minimize its losses (as in game-playing programs).

- A maximizes his minimal gain and plays 1
- B minimizes his maximal loss and plays 1

Minimax strategies are in equilibrium:
no agent gains from deviating

Game with no pure strategy equilibrium

		B	
		0	1
	0	(-1,1)	(1,-1)
A	1	(0,0)	(-2,2)

- Pure minimax strategies: A plays 0, B plays 0

Game with no pure strategy equilibrium

		B: 0	B: 1
	0	(-1,1)	(1,-1)
A	1	(0,0)	(-2,2)

- Pure minimax strategies: A plays 0, B plays 0
- If B always plays 0, A can do better by playing 1!
- If A always plays 1, B can do better by playing 1!
- If B always plays 1, A can do better by playing 0!
- ...

Mixed Strategy

Solution: play randomly, but choose probabilities rationally:

- optimal strategy for A: $(p_0, p_1) = (0.5, 0.5)$
- optimal strategy for B: $(p_0, p_1) = (0.75, 0.25)$

Mixed strategy: the action is chosen randomly within a set of k alternatives, following a probability distribution $(p_1, ..., p_k)$
Notation: $[p_1 : a_1; p_2 : a_2; ...; p_k : a_k]$ or simply $[p_1, ..., p_k]$

Mixed Strategy (2)

		B: 0	B: 1
	0	(-1,1)	(1,-1)
A	1	(0,0)	(-2,2)

		B: 0	B: 1
	0	3/8	1/8
A	1	3/8	1/8

average loss for A = 0.5 = average gain for B
If B changes prob. distribution, then A can loose less
If A changes prob. distribution, then B can gain more
$\Rightarrow$ equilibrium of mixed minimax strategies.

Minimax theorem (V. Neumann & Morgenstern)

In a zero-sum game with two players, the average gain (loss) v of player A using the best (mixed) minimax strategy is equal to the average loss (gain) v of player B using its best (mixed) minimax strategy. The value v is called the value *of the game (for A).*

$\Rightarrow$ a set of mixed equilibrium (minimax) strategies exists for any zero-sum game!

Properties...

In equilibrium, for every action of the row player A:

- the expected payoff is $\leq$ v.
- if expected payoff $<$ v, the action has zero probability.

$\Rightarrow$ every action with probability $\neq 0$ has expected payoff $=$ v.

- this set of actions is the *support*.

Computing minimax strategies

Strategy for B $= (p_1^B, ..., p_n^B) \Rightarrow$
solve a linear program with variables $v, p_1^B, ..., p_n^B$:

- minimize v (the maximal gain)
- for every action of A, expected payoff is no larger than v:

$$(\forall a_i^A) : \sum_{a_j^B} p_j^B R_A(a_i^A, a_j^B) \leq v$$

- where $R_A(a_i^A, a_j^B)$ is the payoff to agent A.

Symmetric problem to find strategy for A

Alternative: fictious play

- For large games, LP may be complex.
- Only a single optimal set of strategies
  $\Rightarrow$ converge to it through hill-climbing.
- Start with random strategies and iterate:
  - Player A increases probability of best response to B's strategy
  - Player B increases probability of best response to A's strategy
- Converges to optimal probabilities!

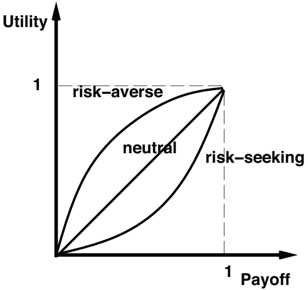
Lotteries

- With a mixed strategy, payoff is uncertain.
- This is called a *lottery* $[p_1 : o_1; p_2 : o_2; ...; p_k : o_k]$
    - k outcomes.
    - outcome o_i occurs with probability p_i and has reward $r(i)$
    - $\sum p_i = 1$
- Clearly $R = E[r(i)]$ of a lottery is not the same as receiving R for sure.

Attitudes towards risk

- People are risk averse: sure return of $\$100'000$ is preferred over \$1 Million with probability 0.1...
- ..but also risk-seeking: return of \$2.00 with probability 0.1 is preferred to \$0.20.
- Model risk attitude by mapping:
    - payoff, the reward provided by the game, to
    - utility, the subjective usefulness of that payoff to the agent.

Payoff vs. Utility



Attitude	u(10%)	u(20%)	...	u(90%)	u(0.1)	u(L)
risk-averse	40%	60%	...	99%	0.4	0.1
risk-seeking	1%	3%	...	60%	0.01	0.1

Find $u(r)$ so that $E[u(r)] = U$ is equivalent to sure return U.

Modeling risk aversion

Consider 2 lotteries:

1. payoff of $\$100'000$ for sure
2. payoff of $\$1'000'000$ with probability 0.1

Attitude	$E[U](1)$	$E[U](2)$	prefers?
risk-neutral	$100'000$	$100'000$	-
risk-averse	$400'000$	$100'000$	1
risk-seeking	$10'000$	$100'000$	2

Utility Theory

Preference order $\succ$ should satisfy the following conditions:

- completeness: defined over any pair of outcomes.
- transitivity: if $a \succ b$ and $b \succ c$, then $a \succ c$.
- substitutability: if $a \equiv b$, then any lottery where a is substituted for b is equally preferred.
- decomposability: if two lotteries assign the same probabilities to outcomes, they are equally preferred.
- monotonicity: if $o_1 \succ o_2$ and $p > q$, then $[p : o_1; (1-p) : o_2] \succ [q : o_1; (1-q) : o_2]$
- continuity: if $o_1 \succ o_2 \succ o_3$, there exists p such that $o_2 \succ [p : o_1; (1-p) : o_3]$.

Utility Theory (2)

Van Neumann & Morgenstern:

For any preference order satisfying the 6 axioms, every outcome can be associated with a numerical utility $u(o)$ such that:

- $u(o_1) > u(o_2)$ *iff* $o_1 \succ o_2$
- $u([p_1 : o_1; ...; p_k : o_k]) = \sum_{i=1}^{k} p_i u(o_i)$

$\Rightarrow$ utility function on outcomes can represent most rational preference orders, including risk aversion.
Game payoff is a general representation of preferences.

Eliciting Utility Functions

Given k outcomes: $o_1, .., o_k$:

- elicit preference order of outcomes, assume order is $o_1 \succ o_2 \succ ... \succ o_k$
- for j = 1.. k-2:
  - elicit p_j = probability such that lottery

    $$o_j \text{ with prob. } p_i,\ o_{j+2} \text{ with prob. } (1 - p_i)$$

    is equally preferred to o_{j+1} with certainty.
  - $p_j u(o_j) + (1 - p_j) u(o_{j+2}) = u(o_{j+1})$
  - $p_j(u(o_j) - u(o_{j+2})) = u(o_{j+1}) - u(o_{j+2})$
- add equations $u(o_1) = 1, u(o_k) = 0$ and solve for the utilities.

$\Rightarrow$ utility function normalized to [0..1]

Eliciting Utility Functions (Example)

4 outcomes $o_1 \succ o_2 \succ o_3 \succ o_4$

- let $p_1 = 0.4,\ p_2 = 0.6$
- $\Rightarrow$ system of equations:

$$\begin{aligned} u(o_1) &= 1 \\ 0.4(u(o_1) - u(o_3)) &= u(o_2) - u(o_3) \\ 0.6(u(o_2) - u(o_4)) &= u(o_3) - u(o_4) \\ u(o_4) &= 0 \end{aligned}$$

- simplified:

$$\begin{aligned} 0.4(1 - u(o_3)) &= u(o_2) - u(o_3) \\ 0.6u(o_2) &= u(o_3) \end{aligned}$$

- $u(o_1) = 1, u(o_2) = 0.625, u(o_3) = 0.375, u(o_4) = 0$

General (non-zero) sum games

- zero sum: gain (A) = loss (B)
  $\Rightarrow$ pure competition, no cooperation (e.g. chess)
- general sum:
  a) no cooperation: strategies may be locally optimal, but not globally.
  b) cooperation: results may be better than a) for both players, but may require complex negotiation.

Nash Equilibrium

- Nash equilibrium: no player has an interest to change given that the other does not change.
- Theorem: every game has at least one set of mixed Nash equilibrium strategies.
- generalization of minimax strategy equilibria to general-sum games, but...
- Games can have several Nash equilibra, e.g.:

		B	
		0	1
	0	(2,1)	(0,0)
A	1	(0,0)	(1,2)

Nash Equilibria

		B	
		0	1
	0	(2,1)	(0,0)
A	1	(0,0)	(1,2)

- pure strategy equilibria, (0,0) and (1,1), but also...
- mixed strategy equilibrium: ([2/3,1/3],[1/3,2/3])

Characteristics of Nash equilibria

Properties of the i-th Nash equilibrium for player A (similar for B):

- player A gets expected payoff $v_i(A)$
- only actions a_j in support $s_i(A)$ have probability $p_i(a_j) \neq 0$.
- all actions in the support $s_i(A)$ have expected payoff $v_i(A)$:

$$(\forall a_j \in s_i(A)) \sum_{a_k \in s_i(B)} p_i(a_k) R_A(a_j, a_k) = v_i(A)$$

- no other action has greater payoff:

$$(\forall a_j \notin s_i(A)) \sum_{a_k \in s_i(B)} p_i(a_k) R_A(a_j, a_k) \leq v_i(A)$$

This is a *linear complementarity problem* and can be solved using Lemke's method (standard solvers).

Characteristics of Nash equilibria (2)

- Different from zero-sum games: $v_i(A) \neq -v_i(B)$

$\Rightarrow$ B does not necessarily play as to minimize $v_i(A)$

$\Rightarrow$ cannot just minimize $v_i(A)$ using a linear program, but must determine values using LCP.

- Only actions in the support have equal value
  $\Rightarrow$ support needs to be known exactly.

Computing Nash equilibria

- Complex problem; much interest in theoretical Computer Science.
- Search method:
  1. search through all possible supports.
  2. for each potential support, solve for NE.
  3. output all feasible solutions.
- before searching: eliminate dominated actions.

Dominated actions

- Action a_i strictly dominates a_j if for all strategies of the other player(s), the expected payoff for a_i is greater than that for a_j.
- a_i weakly dominates a_j if expected payoff is $\geq$ for all actions and greater for at least one.
- Action a_j is dominated if there exists some other action that dominates.
- Rational players would never choose a dominated action $\Rightarrow$ eliminate from the game.

Example: dominated actions

Consider the game:

		B		
		L	C	R
	U	(3,2)	(0,1)	(2,0)
A	M	(1,1)	(1,2)	(5,0)
	D	(2,1)	(4,1)	(0,0)

- player B: R is dominated by L and C $\Rightarrow$ eliminate
- player A: M is dominated by D $\Rightarrow$ eliminate
- player B: C is (weakly) dominated by L $\Rightarrow$ eliminate
- player A: D is dominated by U $\Rightarrow$ eliminate

Eliminating dominated actions

- Straightforward: check if some other action is always better for each action of the other player.
- Strictly dominated strategies can be eliminated in any order and do not reduce the set of NE.
- Eliminating weakly dominated strategies can reduce the set of NE: (D,C) is also a NE.
- But never eliminates all NE.

Conditionally dominated actions

Action a_i for player A can *conditionally* dominate a_j given a support $s(B)$ of player B

		B		
		L	C	R
	U	(3,2)	(0,1)	(5,0)
A	M	(1,1)	(1,2)	(3,0)
	D	(2,1)	(4,1)	(0,2)

Support for player $B = \{L, R\}$
$\Rightarrow$ U conditionally dominates M for A.

Algorithm for Nash Equilibria

for all $s(A) \subseteq actions(A)$ **do**
  $actions - B \leftarrow \{a_k | a_k \in actions(B)$, not conditionally dominated given s(A)$\}$
  if $\nexists a_j \in s(A)$ conditionally dominated given actions-B **then**
    for all $s(B) \subseteq actions - B$ **do**
      if $\nexists a_j \in s(A)$ conditionally dominated given s(B) **then**
        if feasibility program satisfiable for s(A) and s(B) **then**
          return the solution as a Nash equilibrium

Eliminate conditionally dominated actions to reduce effort
Optimization: consider $s(A)$, $s(B)$ in smallest-first and most similar size first order.

Example: computing Nash equilibria

		B		
		L	C	R
A	U	(3,2)	(0,1)	(5,0)
	M	(1,1)	(1,2)	(3,0)
	D	(2,1)	(4,1)	(0,2)

$s(A) = \{U\} \Rightarrow actions - B \leftarrow \{L\} \Rightarrow s(B) = \{L\}$: NE!
$s(A) = \{M\} \Rightarrow actions - B \leftarrow \{C\} \Rightarrow s(B) = \{C\}$: $(A \rightarrow D)$
...
$s(A) = \{M, D\} \Rightarrow actions - B \leftarrow \{C, R\} \Rightarrow s(B) = \{C, R\}$: NE?

Testing for NE...

		B		
		L	C	R
A	U	(3,2)	(0,1)	(5,0)
	M	(1,1)	(1,2)	(3,0)
	D	(2,1)	(4,1)	(0,2)

- expected revenue of A:

$$\begin{aligned} E(M) &= p(C) \cdot 1 + p(R) \cdot 3 (= 2) \\ E(D) &= p(C) \cdot 4 + p(R) \cdot 0 (= 2) \\ E(M) = E(D) &\Rightarrow p(C) = p(R) = 0.5 \end{aligned}$$

- check other actions:

$$E(U) = 0.5 \cdot 0 + 0.5 \cdot 5 = 2.5$$

Not NE: A would play U!

Alternative: Fictious Play

- Ficitious play: each player observes the other's actions and chooses a strategy that is a best response.
- If player's strategies are in a NE, they will not change: absorbing state.
- However, convergence from other strategies is not guaranteed; strategies may cycle.
- Convergence is guaranteed only for zero-sum games.
- In general-sum games, converges to a weaker type of equilibrium.

Stackelberg games

- Stackelberg games have a *leader* and *follower*: decisions are made in sequence.
- Examples;
  - security measures are decided, then attackers try to circumvent them.
  - a seller sets a price, the (potential) buyer buys or not.
- If leader informs follower, the Stackelberg equilibrium can be very different from the Nash equilibrium.
- Leader can usually force a higher payoff, follower has no choice.

Example (Stackelberg)

- Consider this game:

		B	
		0	1
	0	(2,1)	(4,0)
A	1	(1,0)	(3,1.2)

Nash equilibrium: $(0,0)$ (0 is dominant for A), payoff $= (2,1)$.

- However, if A can commit to play 0 and 1 with equal probability, then the best response for B is to play 1!

$\Rightarrow$ payoff $= (3.5, 0.6)$!

- A can get significantly higher payoff.

n-player games

- Consider game with n players $P_1, .., P_n$.
- In general: game has at least one Nash equilibrium.
- Zero-sum: game has a Nash equilibrium, but not necessarily unique or minimax.

Computing n-player Nash equilibria

- 2-player game: search through combinations of 2 support sets of actions.
- n players: search combinations of n support sets.
- Similar to CSP: eliminate actions that are dominated given previous support sets.
- Feasibility program at each leaf node to determine action probabilities.
- Feasbility test is nonlinear: e.g. expected payoff for P_1 depends on probability of action combinations:

$$(\forall a_j \in s_i(P_1)) \sum_{a_{k_2} \in s_i(P_2)} \dots \sum_{a_{k_n} \in s_i(P_n)} \prod_{l=2}^{n} p_i(a_{k_l}) R(a_j, a_{k_2}, \dots, a_{k_n})$$

Graphical games

Many n-player games are structured:

- Payoff depends only on small set of other players.

$\Rightarrow$ Characterize as graph of binary relations

- Can be solved as a constraint satisfaction problem.
- Example: in land use, only care about neighbouring agents.

Implementing agents

- Agent systems are *engineered*.
- 2 components:
  - agent strategies
  - mechanism design: rules of the game
- Challenge of multi-agent systems: design mechanisms so that good agent strategies lead to good overall behavior.
- Agents can be people or computers.

Further issues

- Cooperation and negotiation: how can agents cooperate and reach agreement as a group?
- Truthfulness and mechanism design: how to prevent manipulation and lying?
- Auctions and their implementation on the internet.

Summary

- Games
- Pure and mixed strategies
- Utility theory
- Equilibrium, Nash equilibrium
- Computing Nash equilibria
- n-player games