COM-303 - Signal Processing for Communications

Solutions for Homework #7

Solution 1. Interleaving

(a) We have clearly:

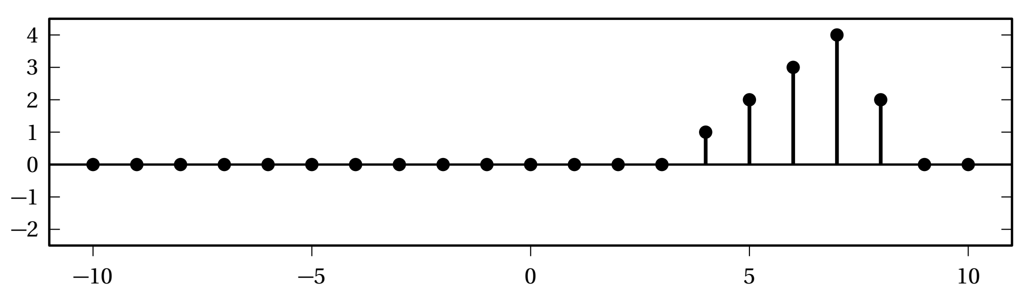
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

(b) The ROC is determined by the zeros of the transform. Since the sequence is two-sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z_0^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $\sqrt{|z_L|} < |z| < \sqrt{|z_R|}$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so:

$$\text{ROC} = 0.8 < |z| < 2.$$

Solution 2. Impulse Response

(a) $\mathscr{H}\{x[n]\}$ is like so:



(b) We have:

$$\begin{aligned} h[n] &= h_1[n]*(h_2[n]-h_3[n]*h_4[n]) \\ &= h_1[n]*(u[n+2]-u[n+2]*\delta[n-1]) \\ &= h_1[n]*(u[n+2]-u[n+1]) \\ &= h_1[n]*\delta[n+2] \\ &= h_1[n+2] \\ &= 3(-1)^n(\frac{1}{4})^{n+2}u[n]. \end{aligned}$$

(c) A discrete system is BIBO stable if the impulse response is absolutely summable. We have:

$$\sum_{n=-\infty}^{\infty}|h[n]| = 3\frac{1}{1-1/4} - 3\left(1+\frac{1}{4}\right) = \frac{1}{4},$$

which means the system is BIBO stable.

Solution 3. Generalized Linear Phase Filters

(a) $H(e^{j\omega})$ can be easily factored:

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= e^{-j\omega/2}[e^{j\omega/2} - e^{-j\omega/2}] \\ &= e^{-j\omega/2}\, 2j\sin(\omega/2) \\ &= 2\sin(\omega/2)\, e^{-j(\omega/2-\pi/2)}. \end{aligned}$$

Thus $H(z)$ is a generalized linear phase filter with fractional delay $d = 1/2$ and phase factor $\alpha = \pi/2$.

(b) The filter is of type IV since it has an even number of taps (2), it is antisymmetric, has a fractional group delay and a $\pi/2$ phase factor.

(c) The filter impulse response is given by

$$h[n] = \delta[n] - \delta[n-1].$$

Thus,

$$\sum_n h[n]\sin(\omega(n-d)+\alpha) = \sin(-\omega/2+\pi/2) - \sin(\omega/2+\pi/2) = 0$$

for all ω.

Solution 4. Zero-Phase Filtering

(a) Consider the sequence $x[n] = \delta[n-1]$. We should have $\mathscr{R}\{x[n]\}[n] = \mathscr{R}\{\delta[n]\}[n-1]$, but instead it is:

$$\mathscr{R}\{x[n]\}[n] = x[-n] = \delta[-(n+1)] = \delta[n+1]$$
$$\mathscr{R}\{\delta[n]\}[n-1] = \delta[n-1]$$

(b) First of all recall that the DTFT of $x[-n]$ is $X(e^{-j\omega})$; if $x[n]$ is real, we also have $X(e^{j\omega}) = X^*(e^{-j\omega})$. In the frequency domain we therefore have:

(a) $S(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

(b) $R(e^{j\omega}) = S(e^{-j\omega}) = H^*(e^{j\omega})X(e^{-j\omega})$ since $h[n]$ is real.

(c) $W(e^{j\omega}) = H(e^{j\omega})R(e^{j\omega}) = |H(e^{j\omega})|^2 X(e^{-j\omega})$

(d) $Y(e^{j\omega}) = W(e^{-j\omega}) = |H(e^{j\omega})|^2 X(e^{j\omega})$

Therefore the chain of transformations defines an LTI filter $\mathscr{G}$ with frequency response $G(e^{j\omega}) = |H(e^{j\omega})|^2$. The corresponding impulse response is simply

$$g[n] = h[n] * h[-n]$$

What is interesting to note here is that, even though $\mathscr{R}$ is not time invariant, we can combine time variant operators into an overall time-invariant transformation.

(c) $G(e^{j\omega})$ is a real function, therefore its phase is zero.

Solution 5. Optimal FIR Filters

The filter is clearly a bland lowpass with transition band approximatly between 0.3π and 0.4π. [For the record, it has been designed with the command `firpm(12, [0 .3 .35 1], [1 1 0 0])`, but this knowledge is not necessary to solve the exercise.]

(a) By elimination, it cannot be Type III or IV, because either would have a zero in $\omega = 0$ and it cannot be a Type II, since it would have a zero in π. Therefore it is Type I (odd-lenght, symmetric)

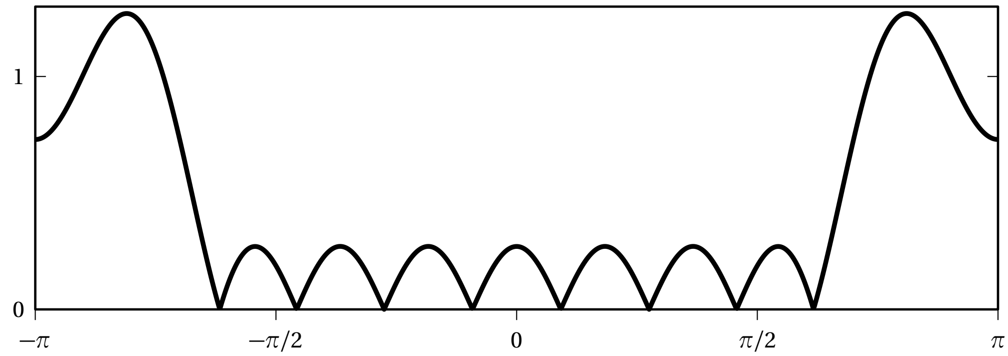
(b) The plot shows the magnitude, but it is easy to count the 8 alternations:

- 3 alternations in the passband (one at zero, one around 0.25π and one at band edge)
- 5 alternations in stopband (ont at band edge, one at π and 3 in between

We know that the number of taps will be $M = 2L + 1$ where $L + 2$ is the number of alternations. Therefore, $M = 13$.

(c) Since we don't know the phase (the plot only shows the magnitude), we can't say if the filter is causal.

(d) We're modulating the filter to π, thereby transforming the lowpass in highpass:



Note that it doesn't matter where the nonzero taps of the original impulse response are, since we're interested just in the magnitude response.

Solution 6. Demodulation

The modulated signal is given by

$$y[n] = x[n]\cos\omega_c n,$$

where $x[n]$ is the bandlimited signal. The purpose of demodulation is to extract the original signal from the modulated input signal. Remark that techniques A and B require the knowledge of the carrier frequency ω_c, while the Galena demodulator does not.

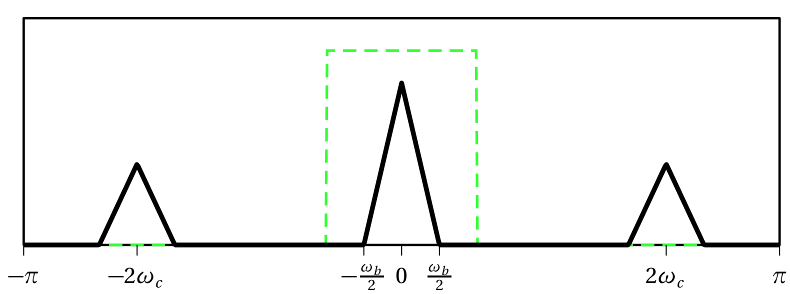
A) **Classic demodulation:** After the multiplication by a copy of the carrier, the signal is given by

$$\begin{aligned} u[n] &= y[n]\cos\omega_c n \\ &= x[n]\cos^2\omega_c n \\ &= \frac{1}{2}x[n] + \frac{1}{2}x[n]\cos 2\omega_c n \end{aligned}$$

The spectrum of the signal is

$$U(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{4}\left[X(e^{j(\omega+2\omega_c)}) + X(e^{j(\omega-2\omega_c)})\right]$$

and it is sketched here:



From the figure, it's clear that if we use a lowpass with cutoff frequency just above $\omega_b/2$ (shown with a dashed line in the picture), then the components at $\pm 2\omega_c$ will be filtered out, thereby recovering the orignal signal. Of course an ideal low-pass as in the plot is impossible to design in practice but we can use an IIR or an FIR filter with good properties instead. In particular, we can exploit the gap between the baseband spectrum and the copy at the higher frequency to allocate a gently sloping transition band for the filter; this allows us to use very simple lowpass filters. The drawback of this approach, however, is that if there is noise in the received signal, more noise will leak through into the demodulated signal if the transition band is large.

B) **Complex demodulation:** remember that the Hilbert filter is defined as

$$H(e^{j\omega}) = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases}.$$

The DTFT of the modulated signal is

$$Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right].$$

The complex signal $c[n]$ is the sum of $a[n] = y[n]$ and $b[n] = y[n] * h[n]$. Therefore the spectra are given by:

$$\begin{aligned} A(e^{j\omega}) &= Y(e^{j\omega}), \text{and} \\ B(e^{j\omega}) &= H(e^{j\omega})Y(e^{j\omega}) = \frac{1}{2}[-jX(e^{j(\omega-\omega_c)}) + jX(e^{j(\omega+\omega_c)})]. \end{aligned}$$

The spectrum of the complex signal is

$$\begin{aligned} C(e^{j\omega}) &= A(e^{j\omega}) + jB(e^{j\omega}) \\ &= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)}) + X(e^{j(\omega-\omega_c)}) - X(e^{j(\omega+\omega_c)})\right] \\ &= X(e^{j(\omega-\omega_c)}). \end{aligned}$$

(Note that the amplitude of the spectrum of a complex signal is not necessarily a symmetric function.)

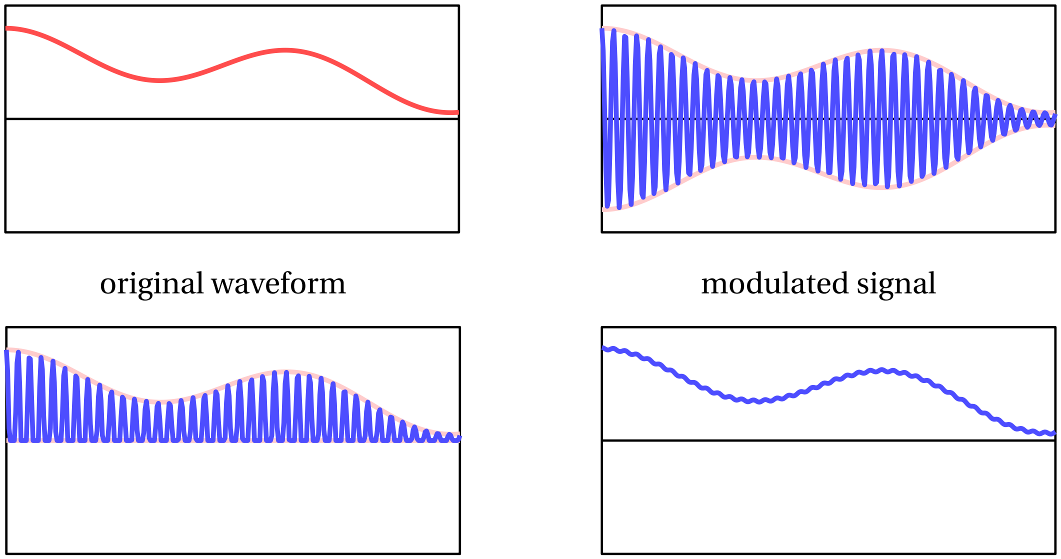
Now, we know from the properties of the Fourier transform that $\text{DTFT}\{e^{j\omega_c n}x[n]\} = X(e^{j(\omega-\omega_c)})$, and therefore

$$c[n] = e^{j\omega_c n}x[n].$$

The demodulated signal is finally

$$\hat{x}[n] = \Re\{c[n]e^{-j\omega_c n}\} = \Re\{x[n]\} = x[n].$$

C) **Galena demodulation:** The principle of operation of a Galena demodulator is best understood graphically. The first two panels in Figure **??** show the original signal and its modulated version respectively. The magnitude operation introduces as non-linearity which, electrically, corresponds to passing the signal through a diode and which, in the Galena receivers, was introduced by the electrical properties of the contact between a Galena crystal and a thin wire called the "cat's whisker". The diode kills the negative part of the modulated waveform and the lowpass filter operates a smoothing process which retains the waveform envelope as in the last panel. The beauty of this system is that it can recover the original signal with no explicit knowledge of the carrier's frequency.



original waveform

modulated signal

rectified signal

lowpass-filtered reconstruction