Solutions 7

Saliba, April 17, 2019

Exercise 1 (**Countable exponential races**). Let I be a countable space and let $T_k, k \in I$, be independent exponential random variables with $T_k \sim Exp(q_k)$ with $0 < q := \sum_{k \in I} q_k < \infty$. Set $T = inf_k T_k$. Let K be the random variable with values in I that is equal to k whenever $T = T_k$ and $T_j > T_k$ for $j \neq k$. Show that T and K are independent with $T \sim Exp(q)$ and $\mathbb{P}(K = k) = q_k/q$. Deduce that $\mathbb{P}(K = k \text{ for some } k) = 1$.

Solution. We have $K = k$ if $T_k < T_j$ for all $j \neq k$. By the total probability formula, we have

$$\begin{aligned}\mathbb{P}(K = k \text{ and } T \geq t) =& \mathbb{P}(T_k \geq t \text{ and } T_j > T_k \text{ for all } j \neq k) \\ =& \int_t^\infty q_k e^{-q_k s}\, \mathbb{P}(T_j > s \text{ for all } j \neq k) ds \\ =& \int_t^\infty q_k e^{-q_k s} \prod_{j \neq k} e^{-q_j s} ds \\ =& \int_t^\infty q_k e^{-qs} ds = \frac{q_k}{q} e^{-qt}.\end{aligned}$$

Hence we have that $\mathbb{P}(K = k \text{ for some } k) = 1$ and T and K have the claimed joint distribution.

Exercise 2 (**General construction of Markov processes**). Let us consider a countable state space E and an array of positive numbers $(\lambda_{i,j})_{i,j \in E; i \neq j}$ with $\sum_{j \in E; j \neq i} \lambda_{i,j} < \infty$ for all $i \in E$. We recursively define a continuous time stochastic process $(X(t))_{t \geq 0}$ on E starting at $i_0 \in E$ as follows:

(i). Define $T_0 = 0$ and set $X(T_0) = i_0 \in E$;

(ii). For $n \in \mathbb{N}$: suppose we know T_{n-1} and $X(T_{n-1}) = i_{n-1}$. Independently of the previous steps, generate independent exponential random variables $E_1, E_2, \dots$ with $E_j \sim Exp(\lambda_{i_{n-1},j})$. Define $T_n = T_{n-1} + \inf_{j \in \mathbb{N}} E_j$ and $i_n = \text{argmin}_{j \in E} E_j$, that is, the (random) index of the exponential variable that is the smallest. Then put

$$X(t) = \begin{cases} i_{n-1} & \text{for } t \in [T_{n-1}, T_n) \\ i_n & \text{for } t = T_n. \end{cases}$$

a) What is the distribution of the time between the jumps of the process $(X(t))_{t \geq 0}$?

b) Let $\widehat{P}_{ij}$ be the probability

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1} = i)).$$

Find the matrix $\widehat{P} = (\widehat{P}_{ij})_{i,j \in E}$.

c) Show that $(X(t))_{t \geq 0}$ is a homogeneous Markov process.

Solution. a) We are looking for the distribution of the waiting time between two jumps, i.e. the distribution of $S_n = T_n - T_{n-1}$, by definition this is defined as

$$S_n = \inf_{j \in \mathbb{N}} E_j.$$

According to exercise 1, we have that $S_n \sim Exp(\sum_{j=1}^\infty \lambda_{i_{n-1},j})$.

b) We know by a) that the waiting time between two jumps of the process is a n exponential random variable arising as the minimum of an countable exponential race. The first exercise gives us additionally that

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1}) = i) = \frac{\lambda_{i,j}}{\sum_{k \neq i} \lambda_{k,j}}.$$

c) We have to show that $(X(t))_{t\geq 0}$ is a Markov process, that is

$$\mathbb{P}(X_t = j \mid \{X_r, r \leq s, X_s = i\}) = \mathbb{P}(X_t = j \mid X_s = i).$$

Since we condition on $\{X_r, r \leq s, X_s = i\}$, there exists a time m (depending on ω) such that $\{X_r, r \leq s, X_s = i\} = \{T_{m-1} < s < T_m \text{ and } X_s = i\}$. First, note that by construction the process before time T_{m-1} is irrelevant for determining this probability:

$$\begin{aligned}
&\mathbb{P}(X_t = j \mid \{X_r, r \leq s \text{ and } T_{m-1} < s < T_m \text{ and } X_s = i\}) \\
&= \mathbb{P}(X_t = j \mid \{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}).
\end{aligned}$$

Then memorylessness property of the exponential random variables implies that for $S_m = T_m - T_{m-1}$

$$S_m \sim S_m - (s - T_{m-1}) \sim Exp(\sum_{j=1}^{\infty} \lambda_{i,j}),$$

i.e. knowing that the exponential rate exceeds $s - T_{m-1}$ is irrelevant for determining the current transitions probabilities. Moreover, since $X_{T_{m-1}} = X_s$ by definition of T_{m-1} and T_m, $\{X_s = i\}$ is the only relevant information for the next evolution of the process based on information contained in $\{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}$. Thus

$$\mathbb{P}(X_t = j \mid \{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}) = \mathbb{P}(X_t = j \mid X_s = i)$$

this finishes the proof.

Definition (The Q-matrix). One way of thinking about the evolution of the Markov process $(X(t))_{t\geq 0}$ is in terms of its Q-matrix, which is known as the generator of the process. A matrix $Q = (q_{ij})_{i,j\in E}$ is a Q-matrix if it satisfies

(i). $-\infty < q_{ii} \leq 0$ for all $i \in E$;

(ii). $0 \leq q_{ij} < \infty$ for all $i \neq j$;

(iii). $\sum_{j\in E} q_{ij} = 0$ for all $i \in E$.

The Q-matrix of the Markov process $(X(t))_{t\geq 0}$ as constructed above is given by $q_{ii} = -\sum_{j\neq i} \lambda_{i,j}$ for $i \in E$, and $q_{ij} = \lambda_{ij}$ for $j \neq i$.

Exercise 3. In a population of size N, a rumor is begun by a single individual who tells it to everyone he meets; they in turn pass the rumor to everyone they meet, once a person has passed the rumor to somebody he exits the system. Assume that each individual meets another randomly with exponential rate $1/N$. Let $X(t)$, $t \geq 0$ be the number in $E = \{1, \dots, N\}$ of people who know the rumor at time t.

a) Draw a graph to visualize the chain. Write down the Q-matrix of the chain.

b) How long does it take in average until everyone knows the rumor if $X(0) = 1$?

Solution. a) The Q-matrix has the form

$$\begin{pmatrix} -\frac{N-1}{N} & \frac{N-1}{N} & 0 & \cdots & 0 \\ 0 & -\frac{N-2}{N} & \frac{N-2}{N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & -\frac{1}{N} & \frac{1}{N} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) We need to compute $\mathbb{E}_1(T_N)$ where $T_N = \inf\{t : X(T_N) = N\}$. Remark that T_N is just a sum of exponential random variables

$$T_N = \sum_{i=1}^{N-1} E_i,$$

where $E_i \sim \text{Exp}(\frac{N-i}{N})$, So that

$$E_1(T_N) = \sum_{i=1}^{N-1} \frac{N}{N-i} \approx N \log N.$$

You could notice that this is exactly the continuous time version of the coupon's collector model.

Exercise 4. For $i \in \mathbb{N}$, let E_i be independent copies of an exponential random variable of parameter λ. We let $T_n := E_1 + \cdots + E_n$ and

$$N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leq t\}}, \ t \geq 0.$$

The process $(N(t))_{t\geq 0}$ is called a homogeneous Poisson process with intensity λ. Let $T_0 = 0$ and we say that $T_1, T_2, T_3, \ldots$ are the successive arrival times of the Poisson process, and E_n the intervals $T_n - T_{n-1}$.

(i). Show that T_n follows an Erlang law with parameters n and λ having density:

$$f_{T_n}(t) = \frac{\lambda^n}{(n-1)!} \ t^{n-1} \ e^{-\lambda t} \ \mathbb{1}_{\{t>0\}}.$$

(ii). Show that, $\forall t > 0$, $N(t)$ follows a Poisson law with parameter λt, i.e.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Solution. (i). We proceed by induction on n. For $n = 1$, T_1 follows an exponential law with parameter λ, which is equivalent to an Erlang law of parameter 1 and λ. Suppose that $T_n \sim \text{Erlang}(n, \lambda)$. Remark that $E_{n+1} \sim Exp(\lambda)$ is independent of T_n. For $t > 0$, we have

$$\begin{aligned}F_{T_{n+1}}(t) &= \mathbb{P}(T_{n+1} \le t) = \mathbb{P}(T_n + E_{n+1} \le t) = \int_0^\infty \mathbb{P}(T_n + E_{n+1} \le t \mid T_n = u) f_{T_n}(u) du \\ &= \int_0^t \mathbb{P}(E_{n+1} \le t - u) f_{T_n}(u) du = \int_0^t F_{E_{n+1}}(t-u) f_{T_n}(u) du.\end{aligned}$$

Implying that

$$\begin{aligned}f_{T_{n+1}}(t) &= \frac{d}{dt} \int_0^t (1 - e^{-\lambda(t-u)}) f_{T_n}(u) du \\ &= \int_0^t \Big((1 - e^{-\lambda(t-u)}) f_{T_n}(u)\Big)' du + (1 - e^{-\lambda(t-t)}) f_{T_n}(t) \\ &= \int_0^t \lambda e^{-\lambda(t-u)} \frac{\lambda^n}{(n-1)!} \; u^{n-1} \; e^{-\lambda u} du \\ &= \frac{\lambda^{n+1}}{(n-1)!} \; e^{-\lambda t} \int_0^t u^{n-1} du \\ &= \frac{\lambda^{n+1}}{n!} \; t^n \; e^{-\lambda t}.\end{aligned}$$

(ii). By definition of $(N(t))_{t \ge 0}$ and of the arrival times T_i, we know that

$$\mathbb{P}(N(t) \ge n) = \mathbb{P}(T_n \le t) = F_{T_n}(t).$$

By (i), we have

$$\begin{aligned}F_{T_{n+1}}(t) &= \int_0^t (1 - e^{-\lambda(t-u)}) f_{T_n}(u) du = F_{T_n}(t) - \int_0^t e^{-\lambda(t-u)} \cdot \frac{\lambda^n}{(n-1)!} \; u^{n-1} \; e^{-\lambda u} du \\ &= F_{T_n}(t) - \frac{\lambda^n}{n!} \; t^n \; e^{-\lambda t}\end{aligned}$$

It is a recursive relation between $F_{T_{n+1}}(t)$ and $F_{T_n}(t)$. As $F_{T_1}(t) = 1 - e^{-\lambda t}$, we get

$$F_{T_{n+1}}(t) = 1 - \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \; n \in \mathbb{N}.$$

Using this result, we obtain the distribution of $N(t)$ for a fixed t

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \ge n) - \mathbb{P}(N(t) \ge n+1) \\ &= \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} - \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.\end{aligned}$$

that is $N(t) \sim Poi(\lambda t)$.