COM303: Digital Signal Processing

Lecture 4: Introduction to Fourier Analysis

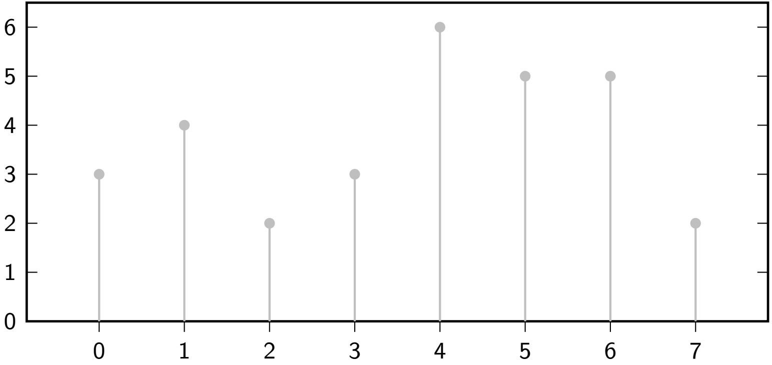
Overview

- Fourier analysis: concept and motivation
- the complex exponential
- the Fourier basis
- the DFT

The time domain

signals are often expressed as a linear combination of "atomic" time units:

$$x[n] = \sum_{k=0}^{N-1} x[k]\delta[n-k]$$



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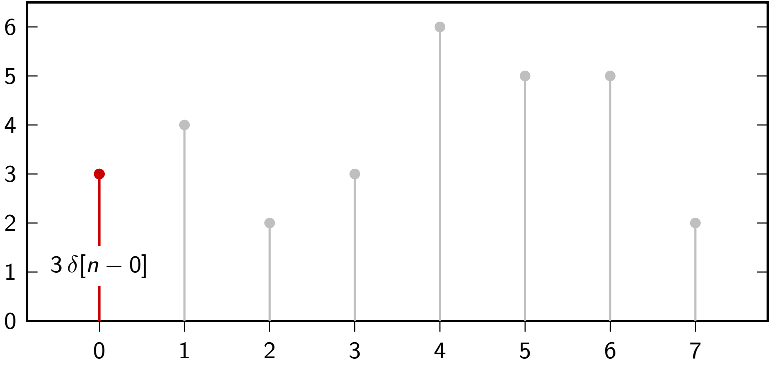

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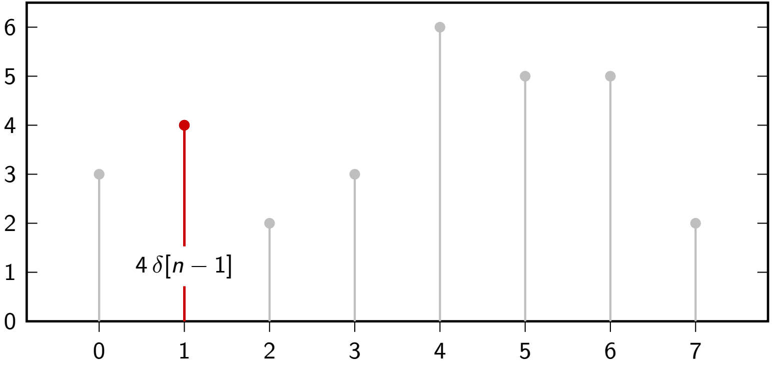

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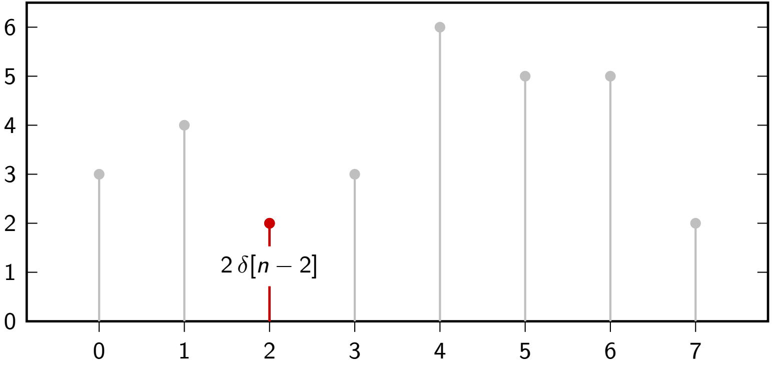

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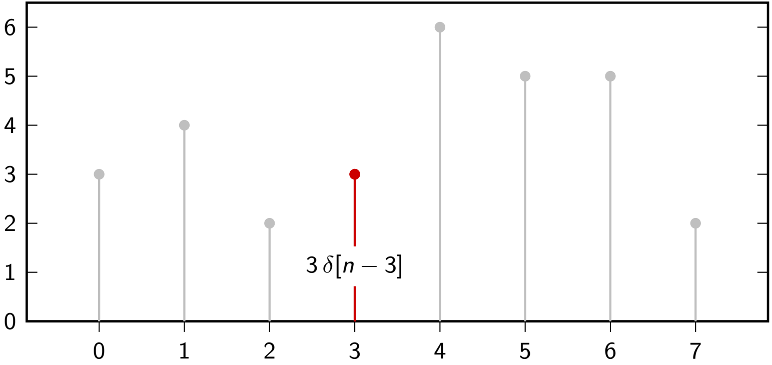

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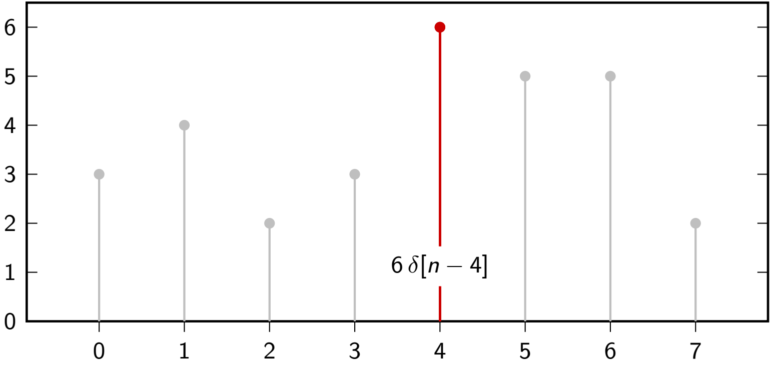

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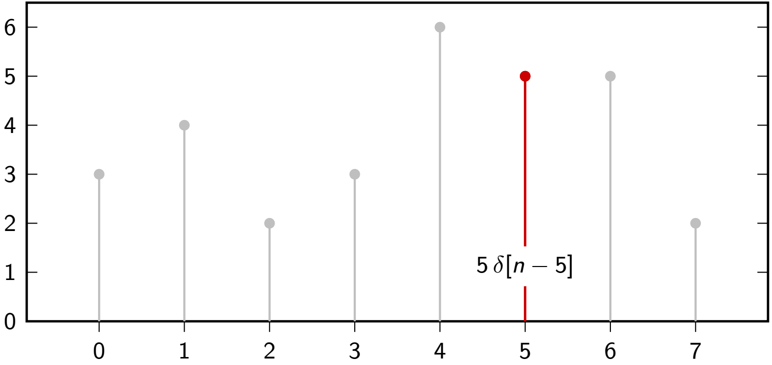

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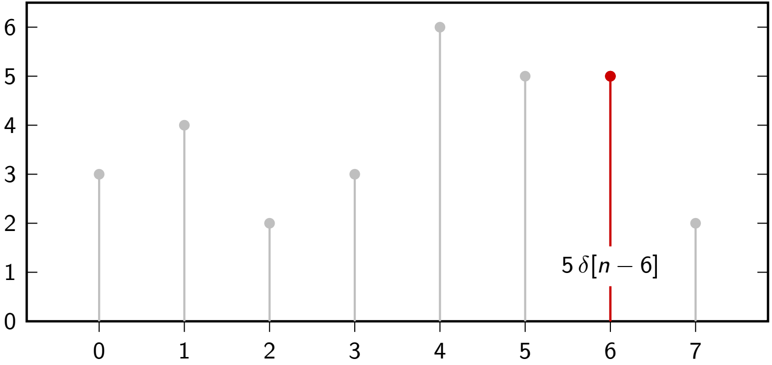

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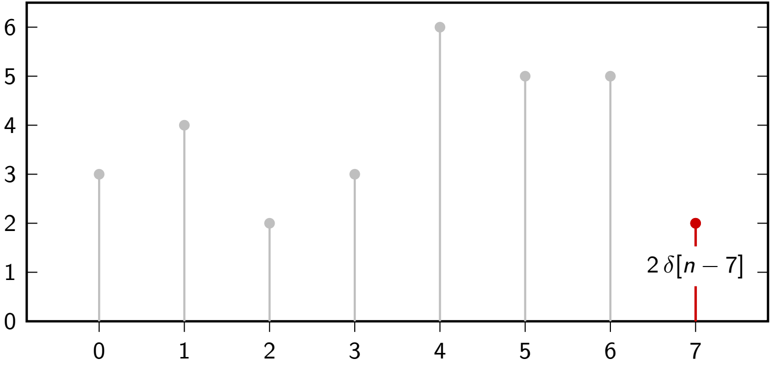

The time domain

in vector notation:

$$\mathbf{x} = \sum_{k=0}^{N-1} x_k \boldsymbol{\delta}^{(k)}$$

where $\{\boldsymbol{\delta}^{(k)}\}$ is the canonical basis for $\mathbb{C}^N$; e.g.:

$$\boldsymbol{\delta}^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

The frequency domain

Fourier analysis: express a signal as a combination of periodic oscillations:

$$\mathbf{x} = \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)}$$

where $\{\mathbf{w}^{(k)}\}$ is the Fourier basis.

Fourier transform: a change of basis in the space of discrete-time signals

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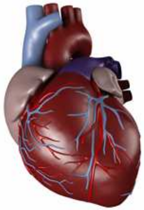
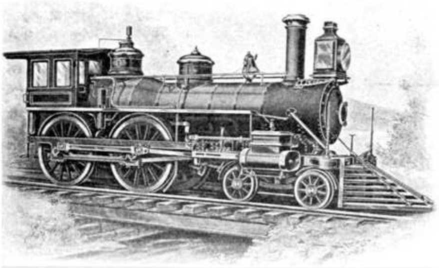
Oscillations are everywhere!





Oscillations are everywhere

- sustainable dynamic systems exhibit oscillatory behavior
- intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings...

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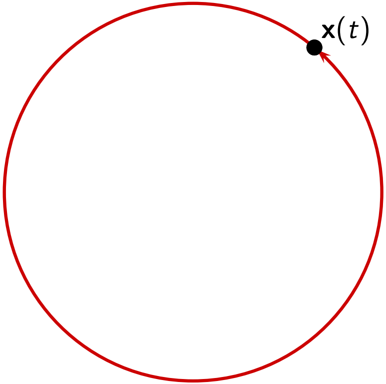

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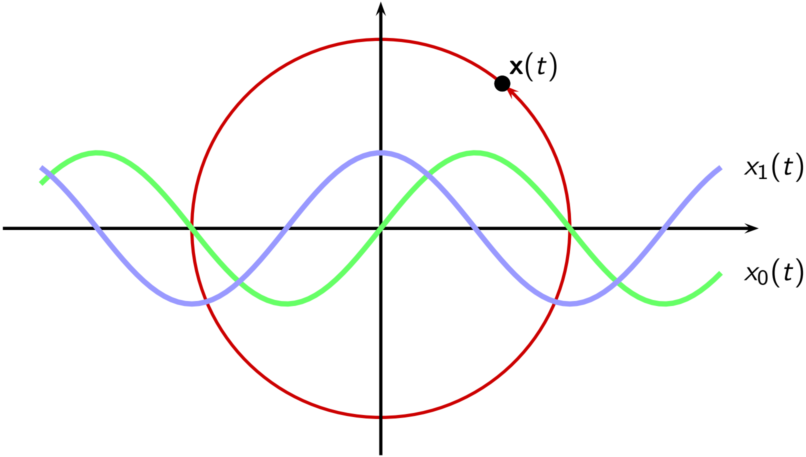
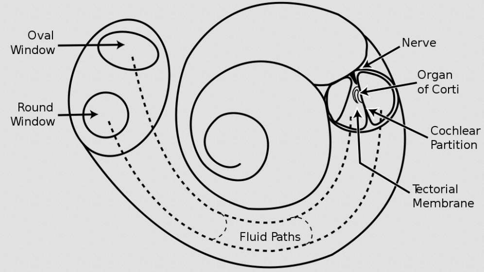


You too can detect sinusoids!

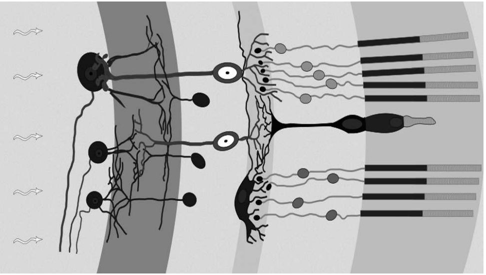
the human body has two receptors for sinusoidal signals:

cochlea (inner ear)



- air pressure sinusoids
- frequencies from 20Hz to 20KHz

rods and cones (retina)



- electromagnetic sinusoids
- frequencies from 430THz to 790THz

The intuition

- humans analyze complex signals (audio, images) in terms of their sinusoidal components
- we can build instruments that "resonate" at one or multiple frequencies (tuning fork vs piano)
- the "frequency domain" seems to be as important as the time domain

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Fundamental question

can we decompose any signal into sinusoidal elements?

yes, and Fourier showed us how to do it *exactly*!

analysis

- from time domain to frequency domain
- find the contribution of different frequencies
- discover "hidden" signal properties

synthesis

- from frequency domain to time domain
- create signals with known frequency content
- fit signals to specific frequency regions

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The mathematical setup

- let's start with finite-length signals (i.e. vectors in $\mathbb{C}^N$)
- Fourier analysis is a simple change of basis
- a change of basis is a change of perspective
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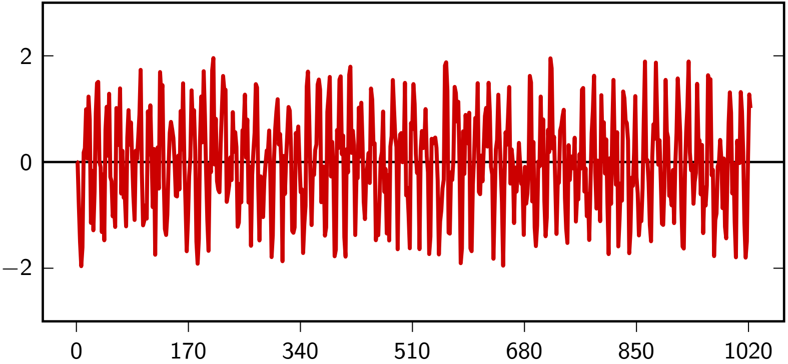
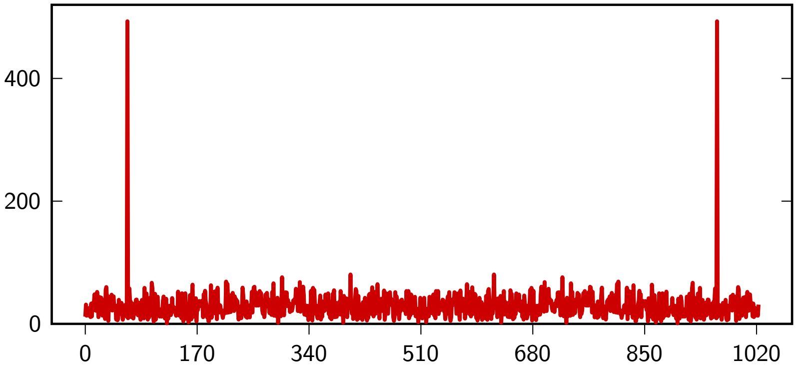
Mystery signal



Mystery signal in the Fourier basis



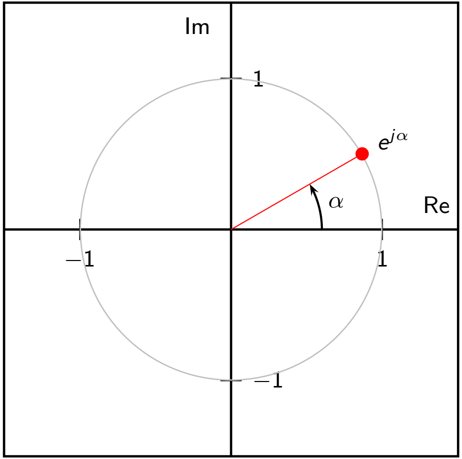
the complex exponential

Prerequisite Warning!

$$e^{j\alpha}$$

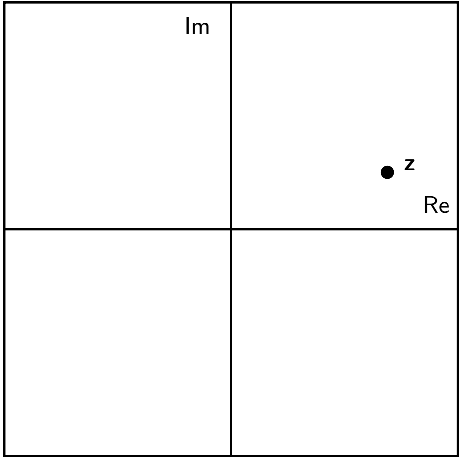
The complex exponential

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$



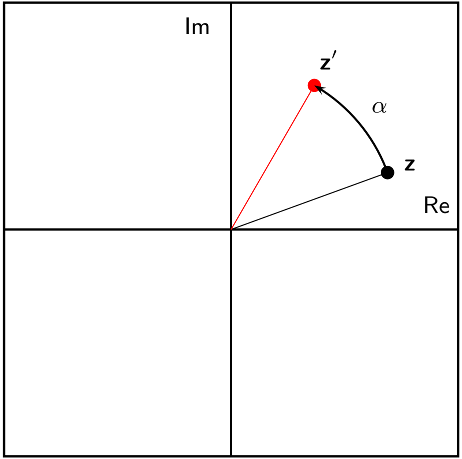
The complex exponential

$\mathbf{z}$: point on the complex plane



The complex exponential

rotation: $\mathbf{z}' = \mathbf{z}\, e^{j\alpha}$



The discrete-time oscillatory heartbeat

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A

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\begin{aligned}
x[n] &= Ae^{j(\omega n + \phi)} \\
&= A[\cos(\omega n + \phi) + j \sin(\omega n + \phi)]
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Why complex exponentials?

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- it makes sense: every sinusoid can always be written as a sum of sine and cosine
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The advantages of complex exponentials

Example: change the phase of a cosine the “old-school” way

$$\cos(\omega n + \phi) = a\cos(\omega n) - b\sin(\omega n), \qquad a = \cos\phi, \;\; b = \sin\phi$$

- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

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$$\cos(\omega n + \phi) = \mathrm{Re}\{e^{j(\omega n + \phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine "live" together
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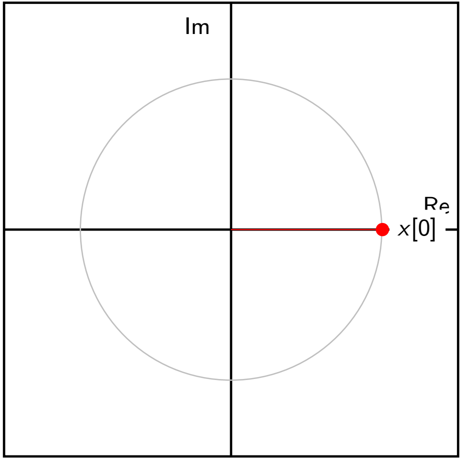

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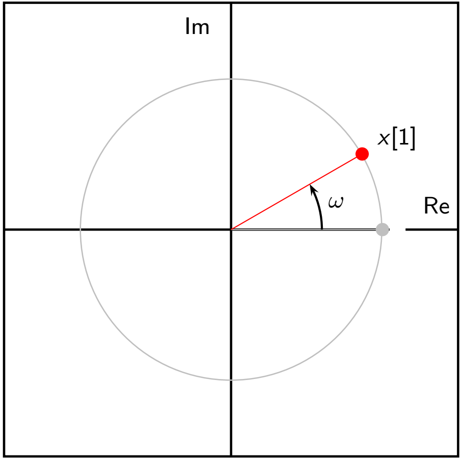

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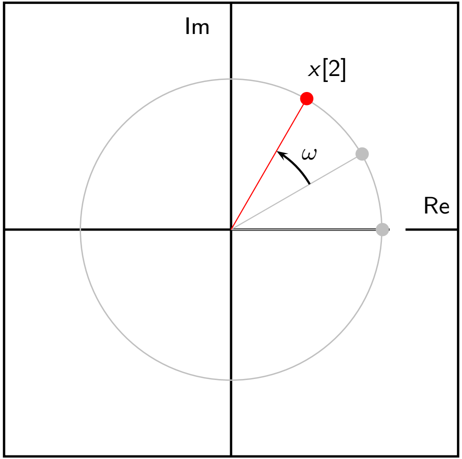

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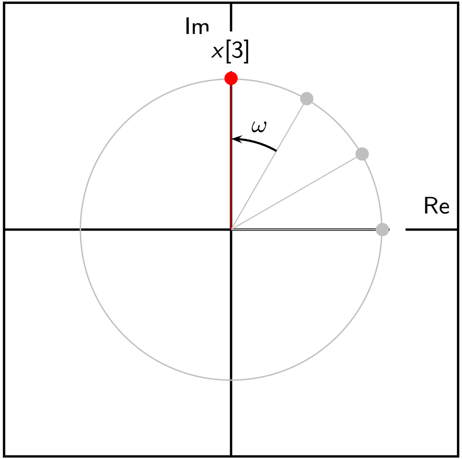

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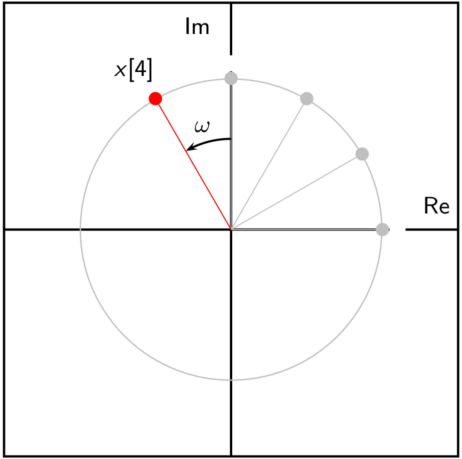

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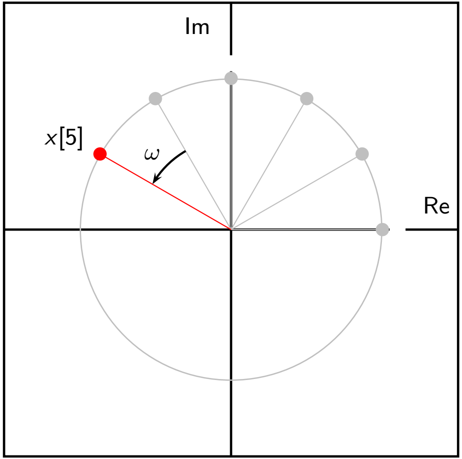

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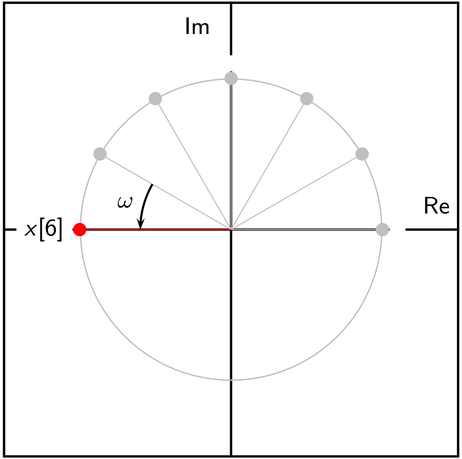

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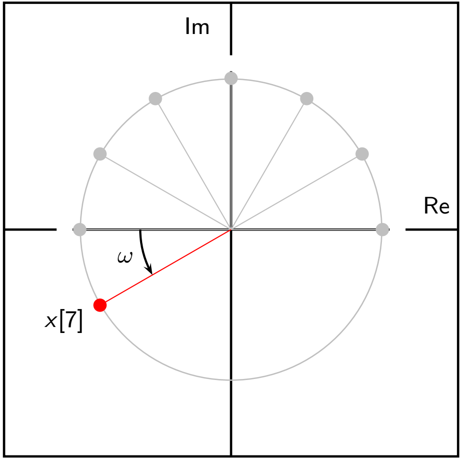

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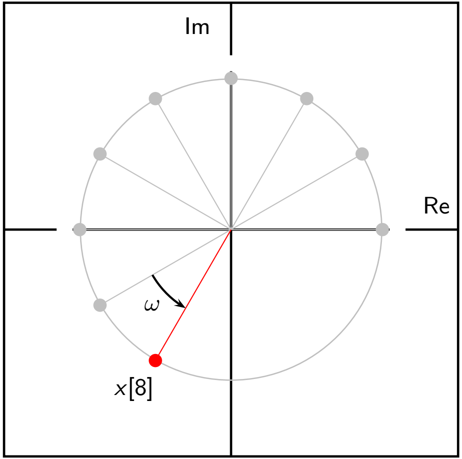

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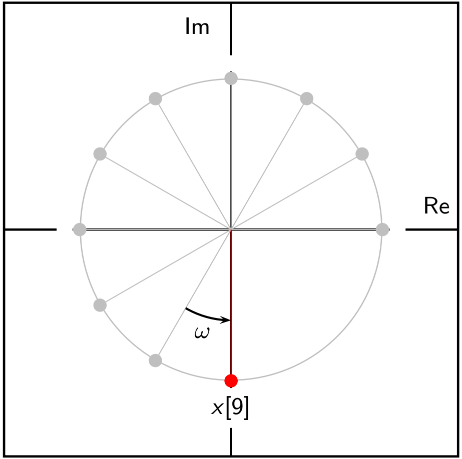

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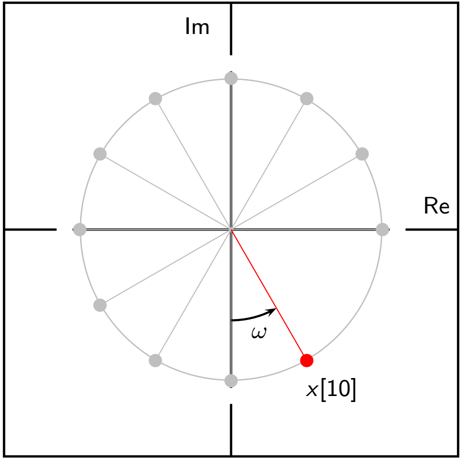

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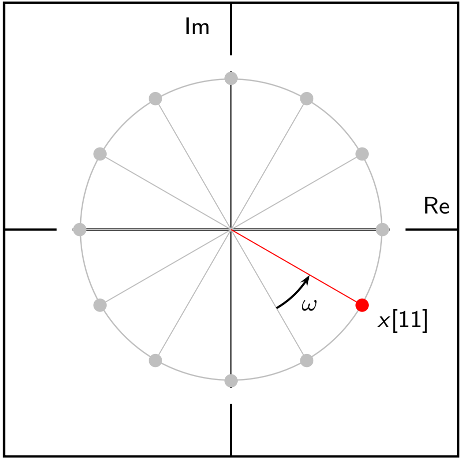

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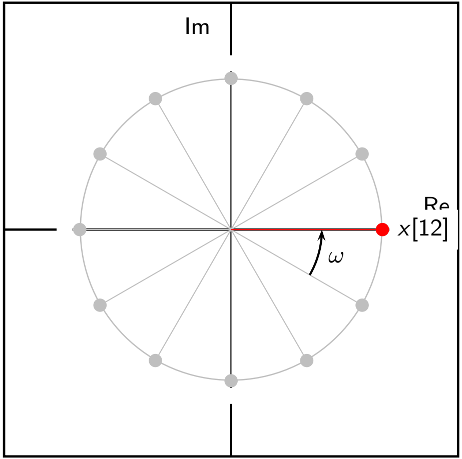

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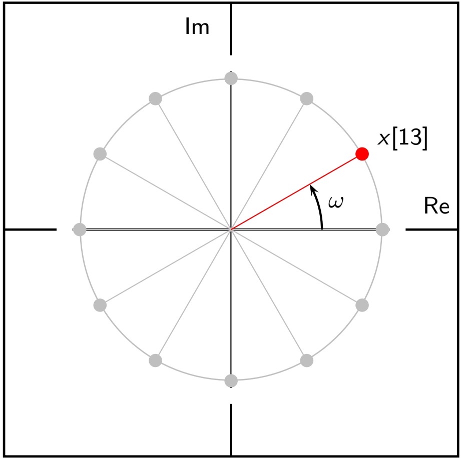

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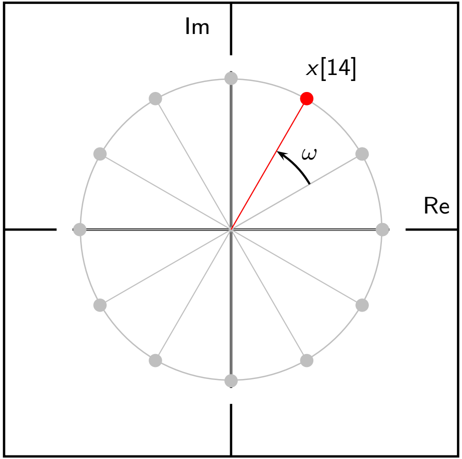

Initial phase

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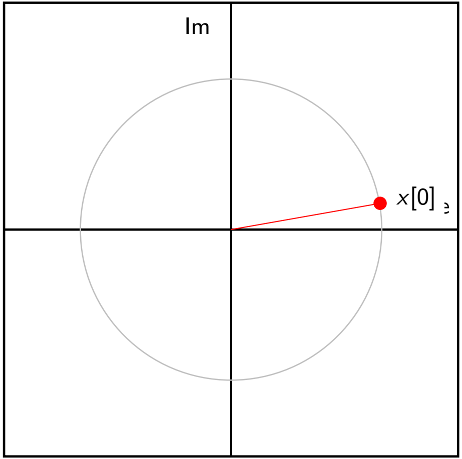

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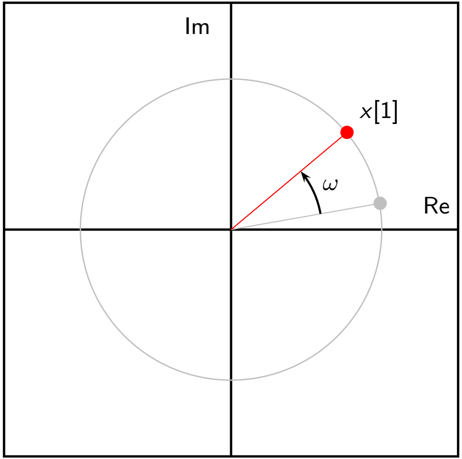

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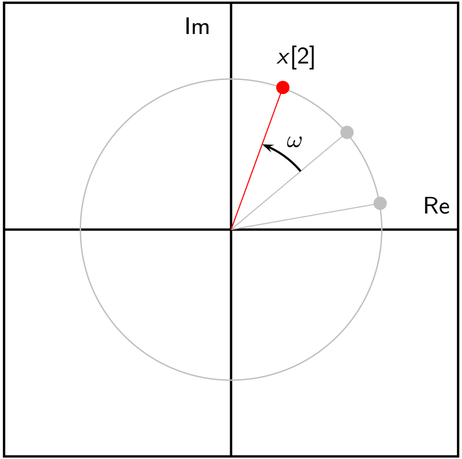

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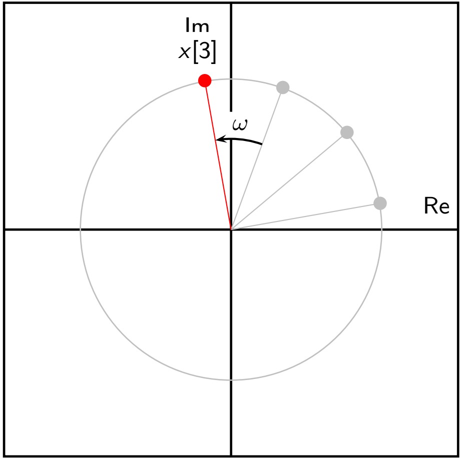

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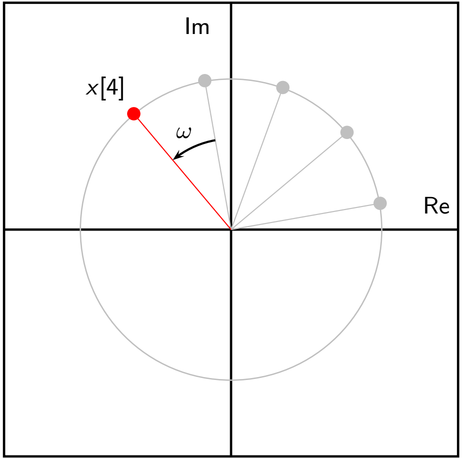

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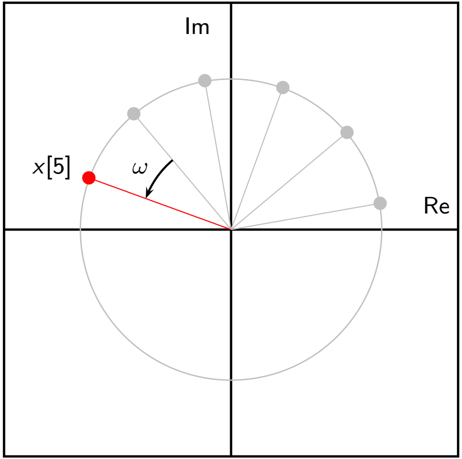

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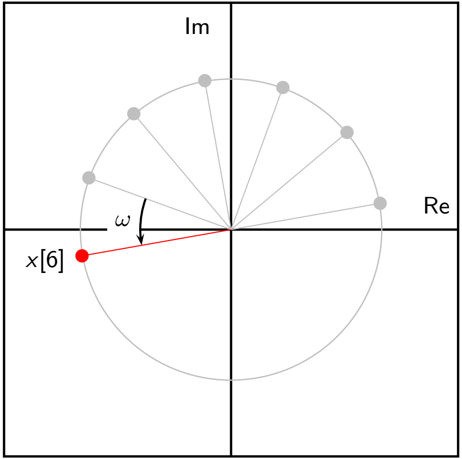

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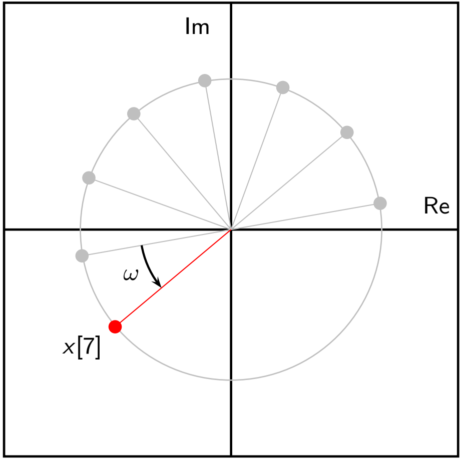

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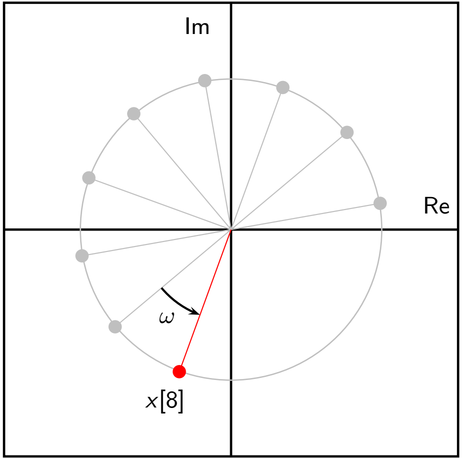

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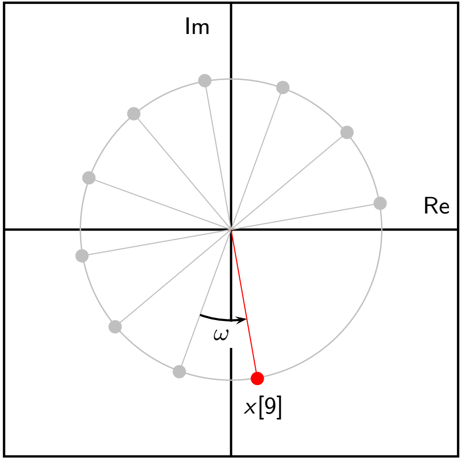

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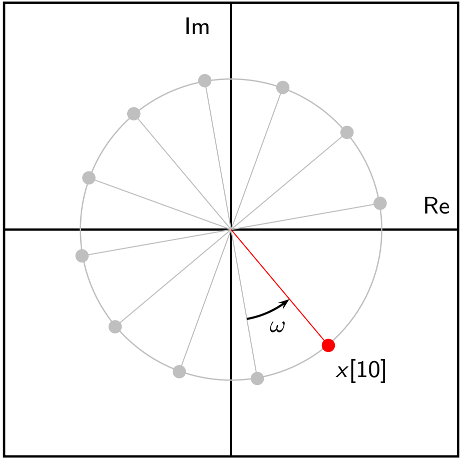

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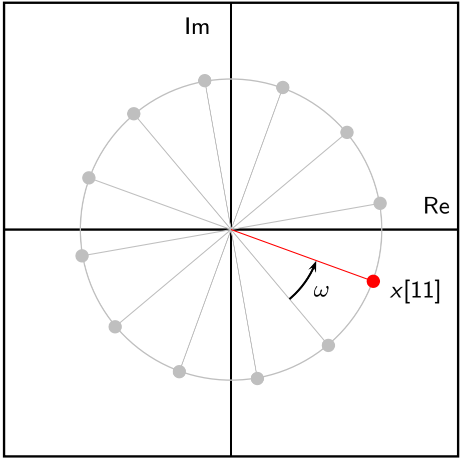

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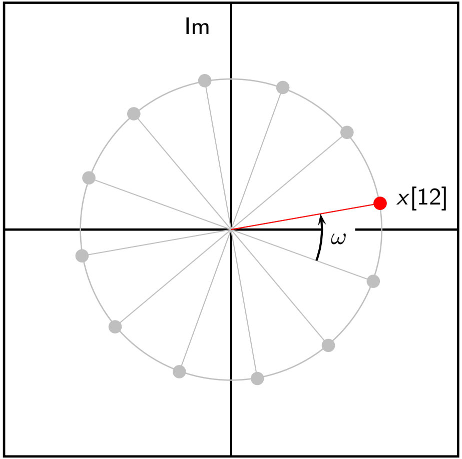

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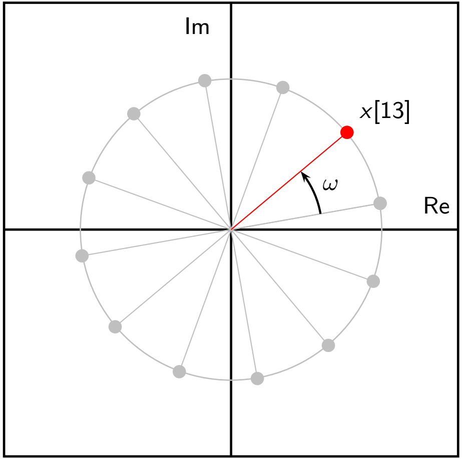

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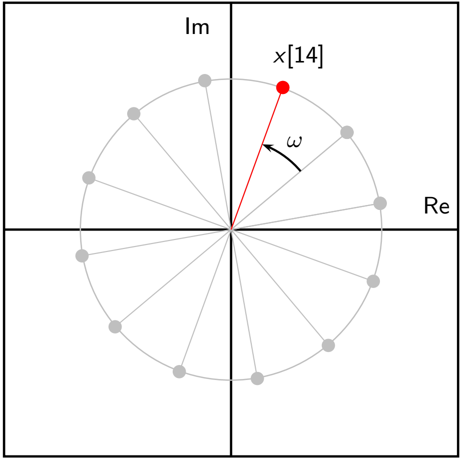

Careful: not every discrete-time sinusoid is periodic!

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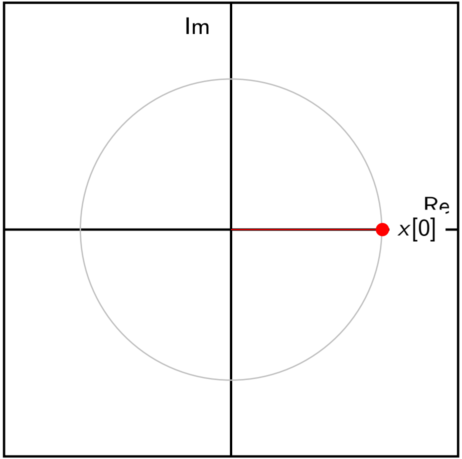

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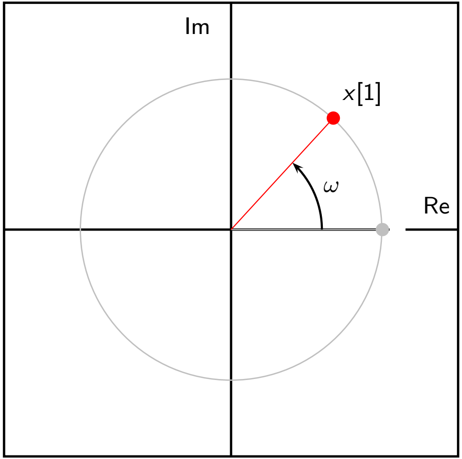

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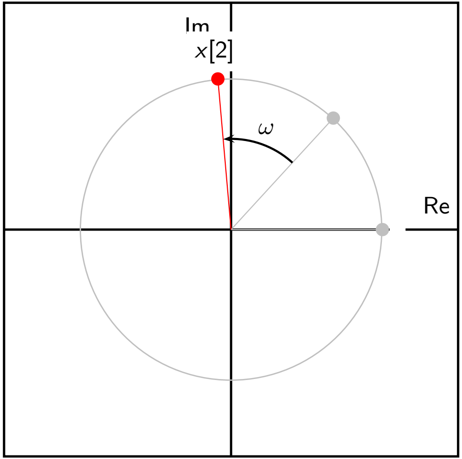

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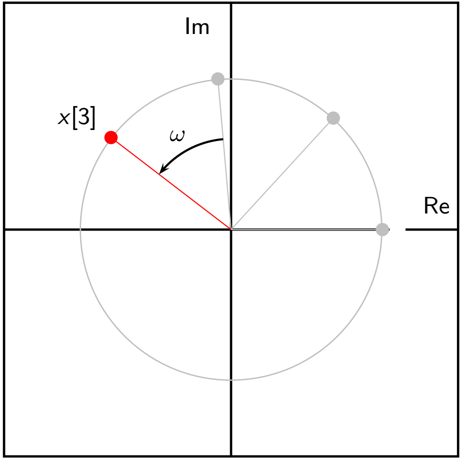

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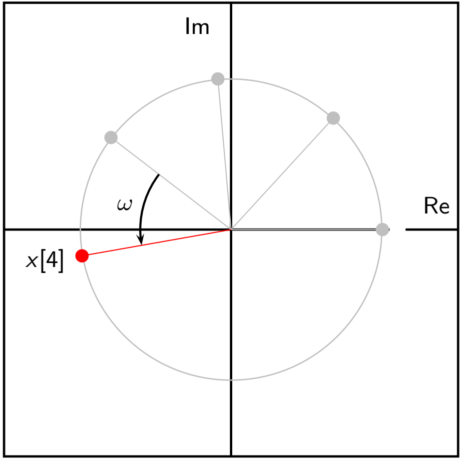

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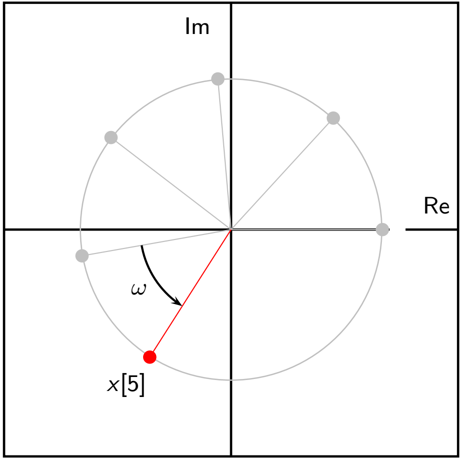

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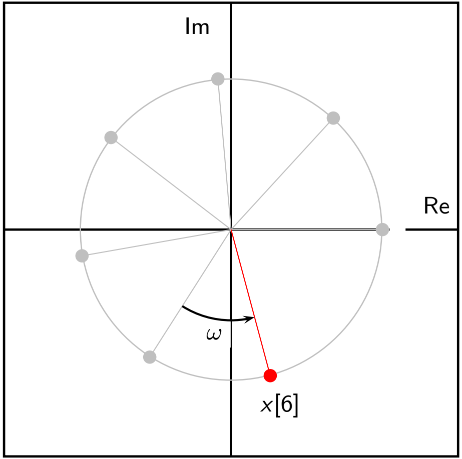

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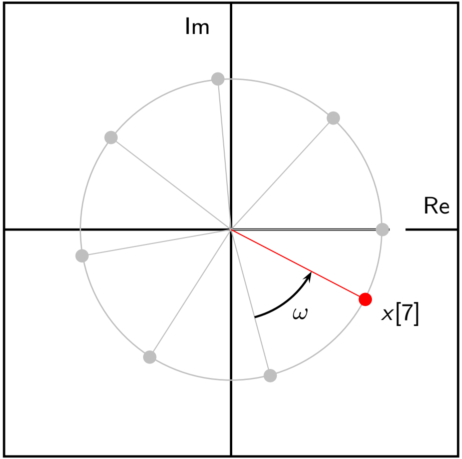

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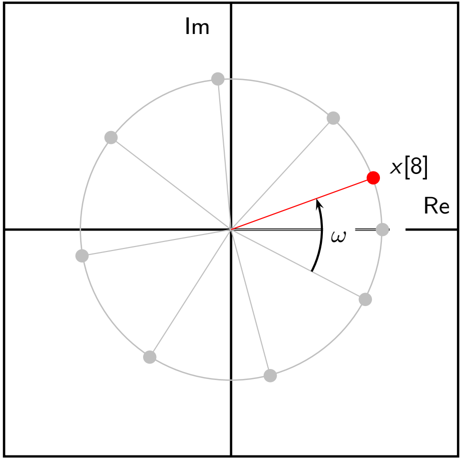

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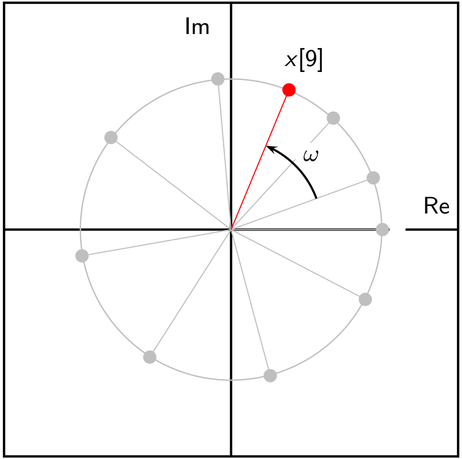

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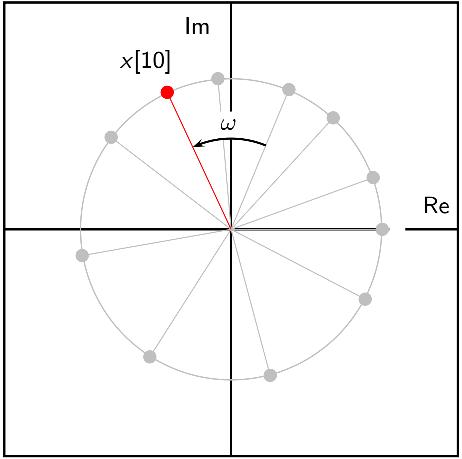

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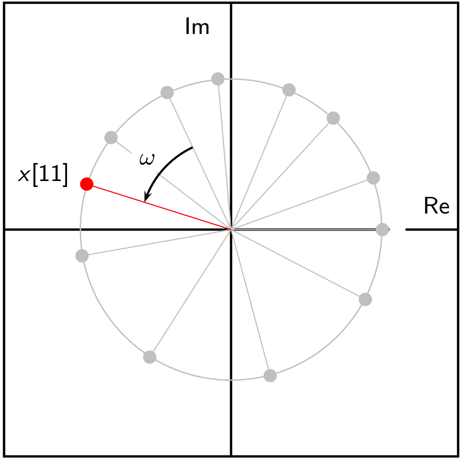

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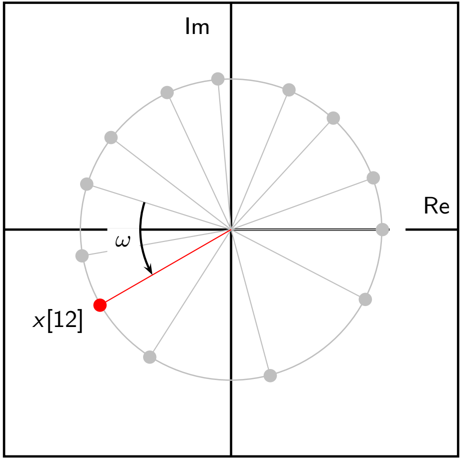

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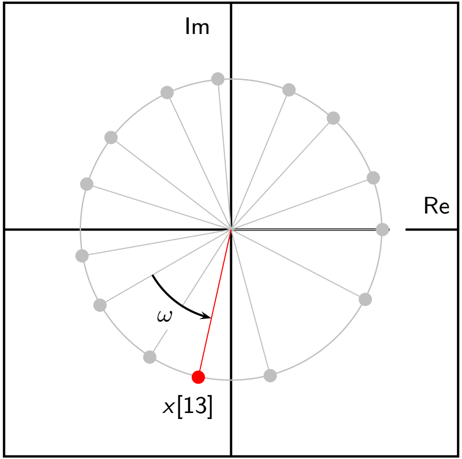

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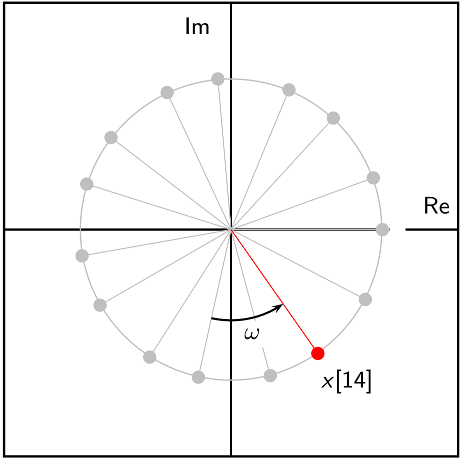

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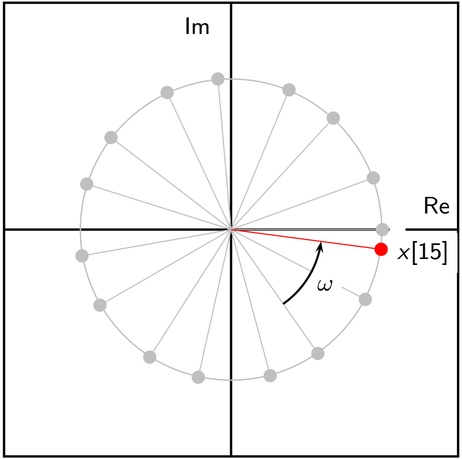

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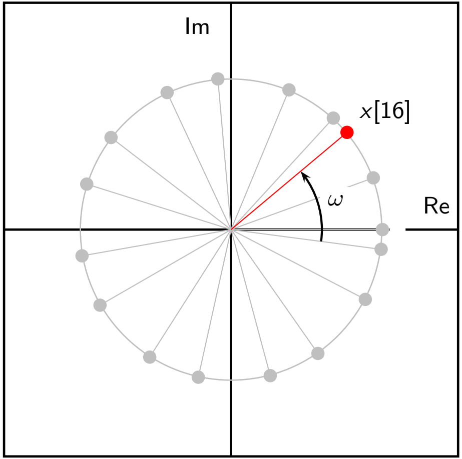

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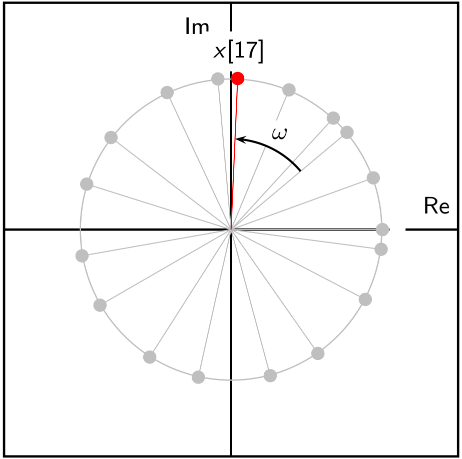

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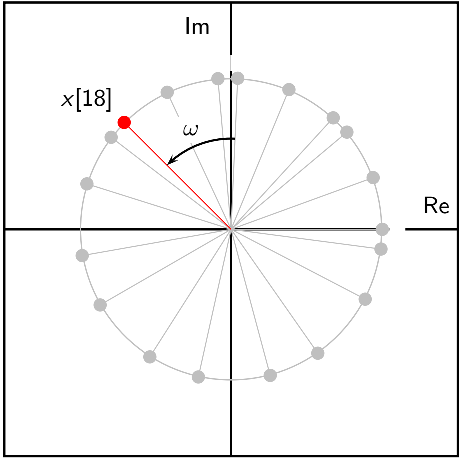

Condition for periodicity in discrete time

$$e^{j\omega n} \text{ periodic in } n \iff \omega = \frac{M}{N}2\pi, \quad M, N \in \mathbb{Z}$$

Condition for periodicity in discrete time

$$
\begin{aligned}
x[n] &= x[n+N] \\
e^{j(\omega n+\phi)} &= e^{j(\omega(n+N)+\phi)} \\
e^{j\omega n}e^{j\phi} &= e^{j\omega n}e^{j\omega N}e^{j\phi} \\
e^{j\omega N} &= 1 \\
\omega N &= 2M\pi, \quad M \in \mathbb{Z} \\
\omega &= \frac{M}{N}2\pi
\end{aligned}
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Condition for periodicity in discrete time

$$x[n] = x[n+N]$$

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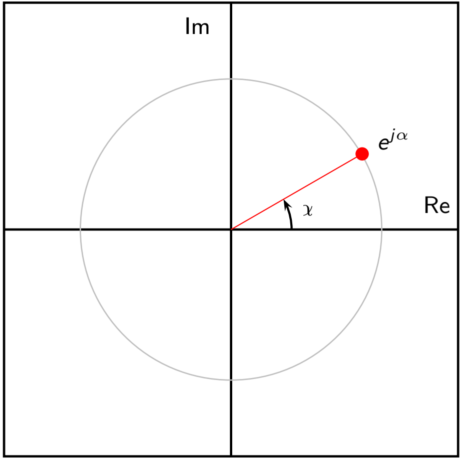
2π phase periodicity of complex exponentials

$$e^{j\alpha} = e^{j(\alpha+2k\pi)} \quad \forall k \in \mathbb{Z}$$

2π-periodicity: one point, many names



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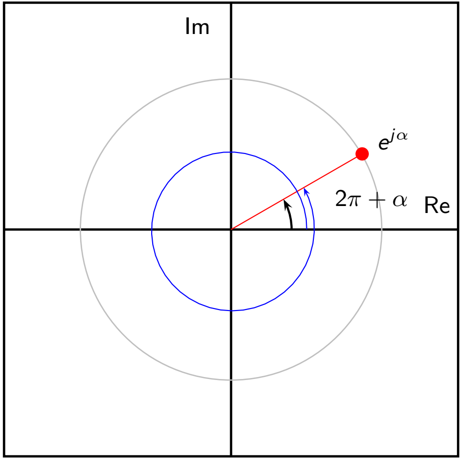

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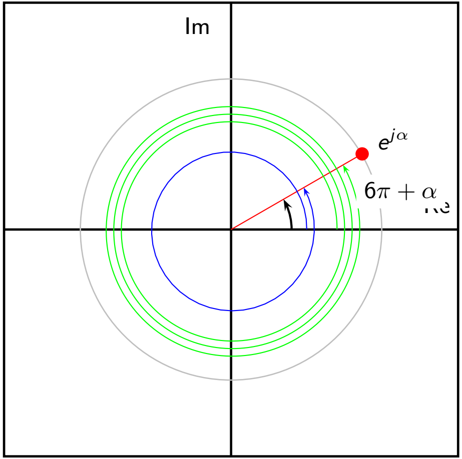

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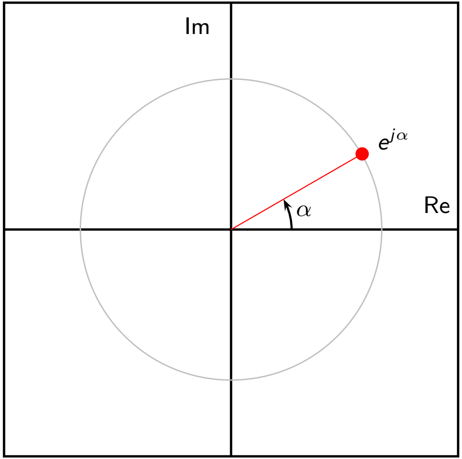

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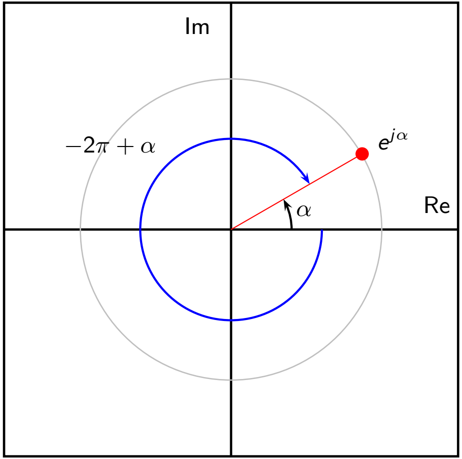

How “fast” can we go?

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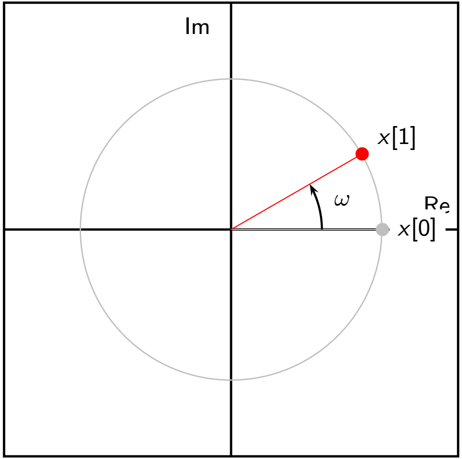
$$0 \leq \omega < 2\pi$$

Remember the complex exponential generating machine



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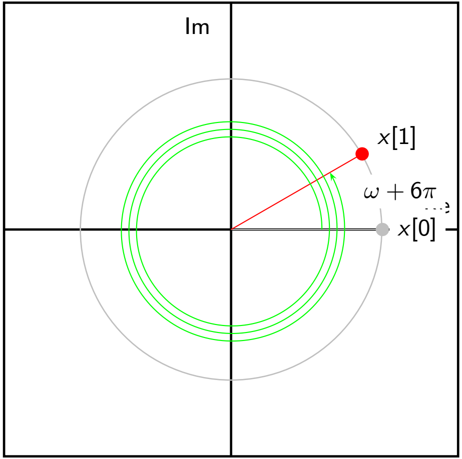
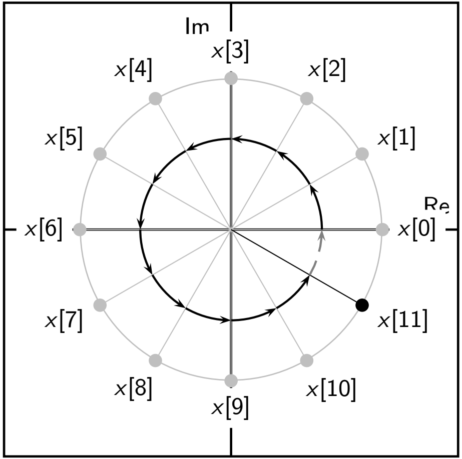


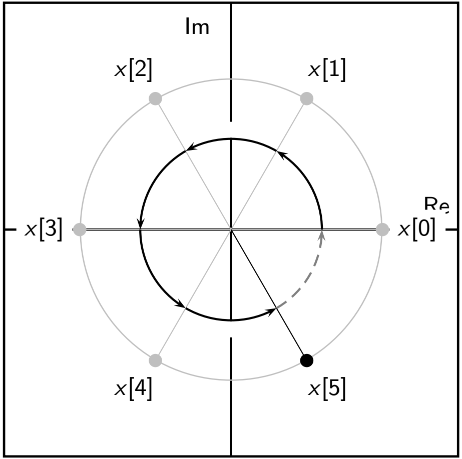
How "fast" can we go?

$$\omega = 2\pi/12$$



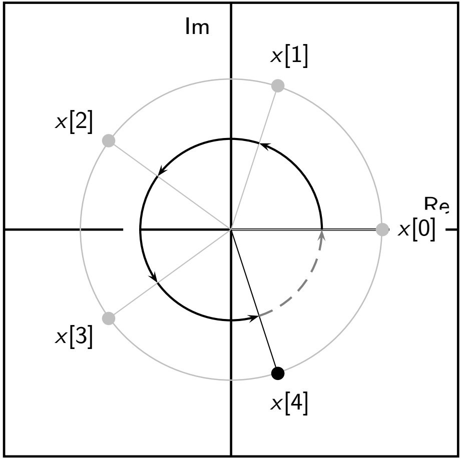
How "fast" can we go?

$$\omega = 2\pi/6$$



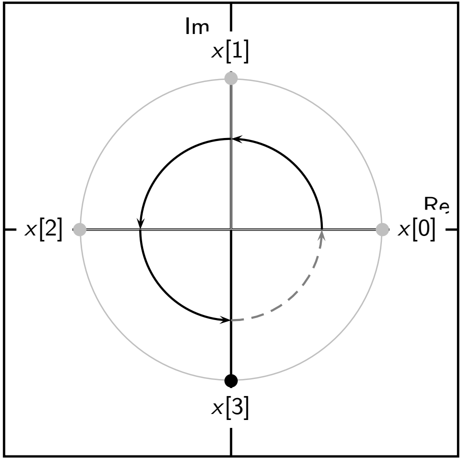
How “fast” can we go?

$$\omega = 2\pi/5$$



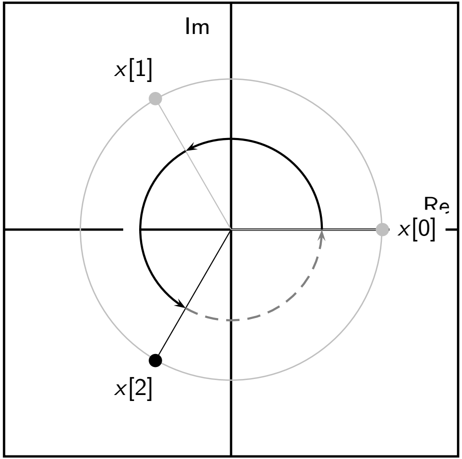
How "fast" can we go?

$$\omega = 2\pi/4$$



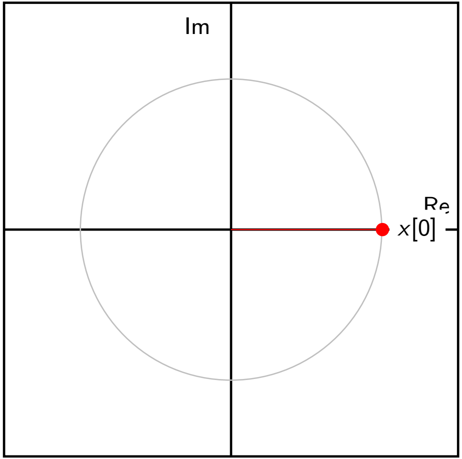
How “fast” can we go?

$$\omega = 2\pi/3$$



How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



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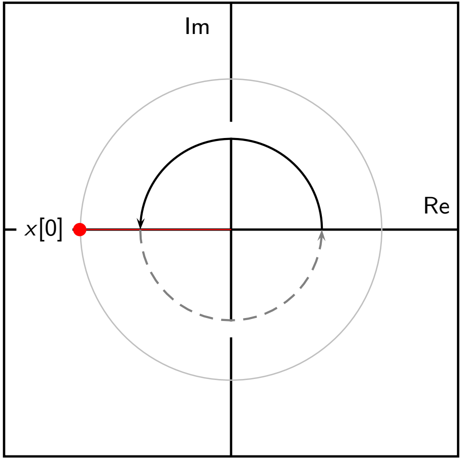

How "fast" can we go?

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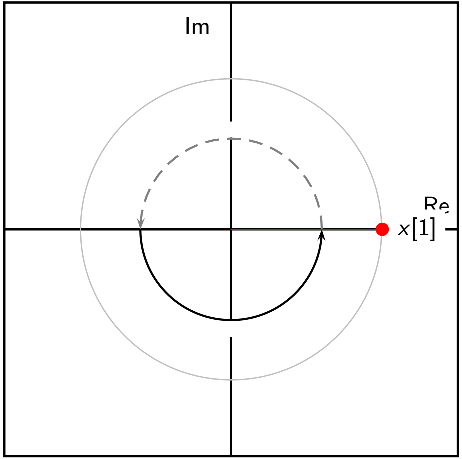

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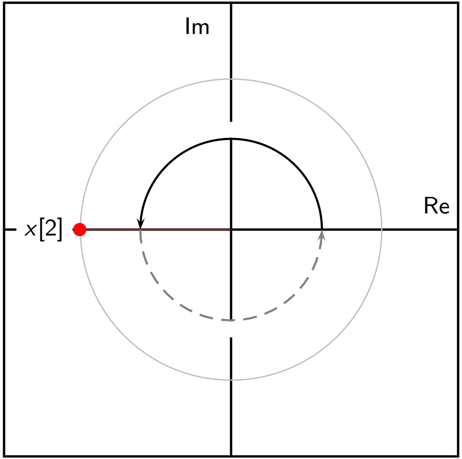

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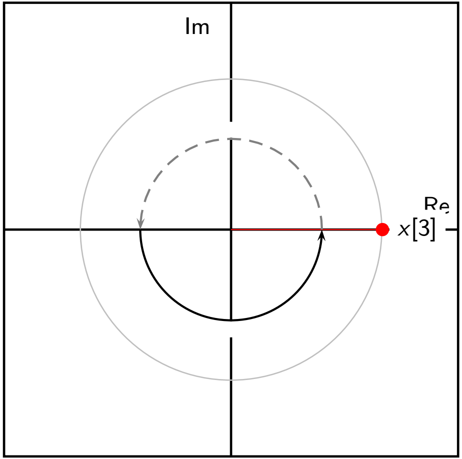

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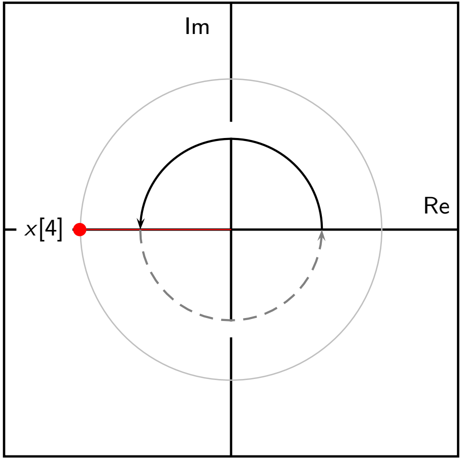

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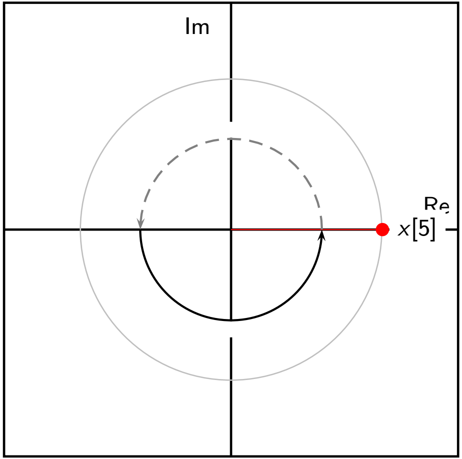
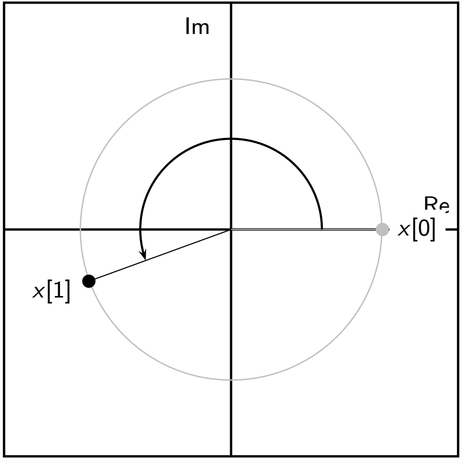


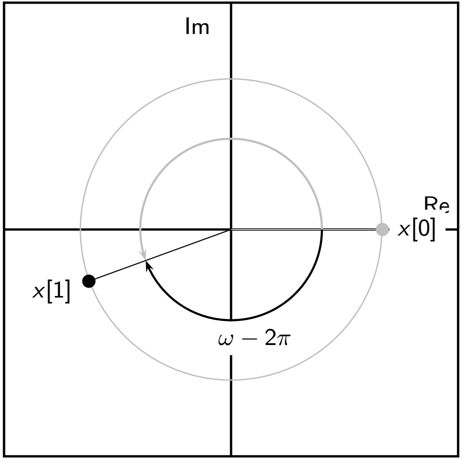
What if we go “faster”?

$$\pi < \omega < 2\pi$$



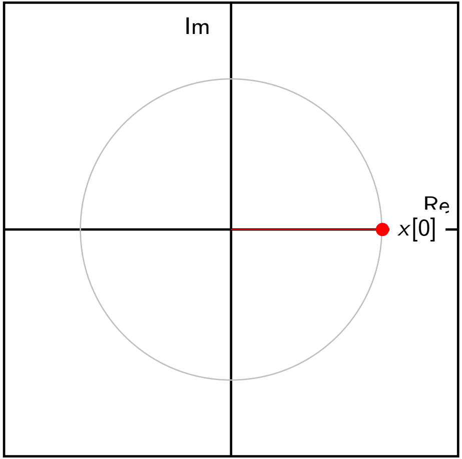
What if we go “faster”?

$$\pi < \omega < 2\pi$$



Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



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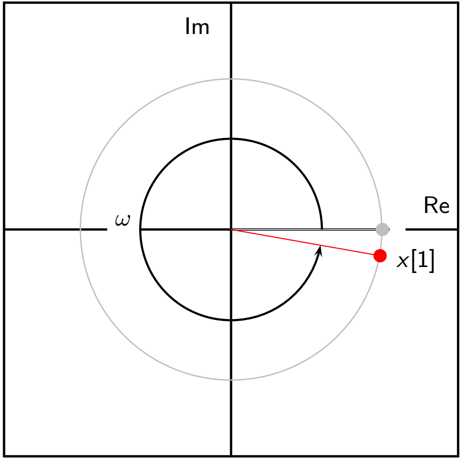

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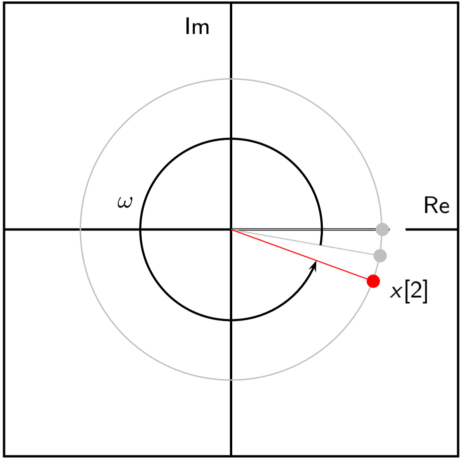

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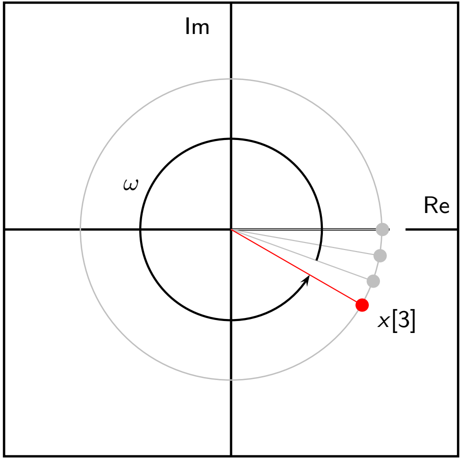

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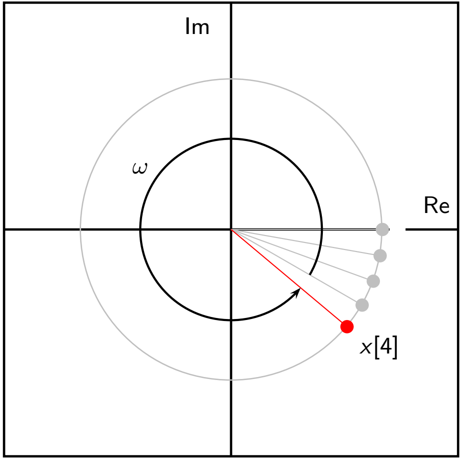

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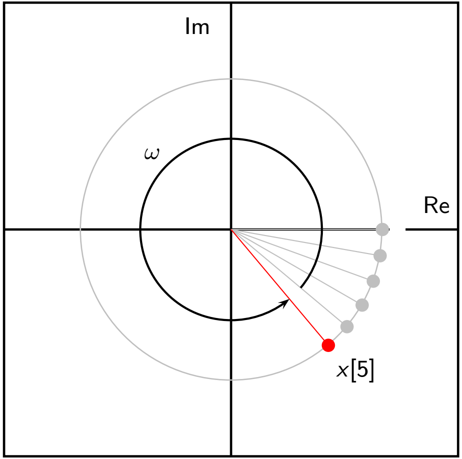

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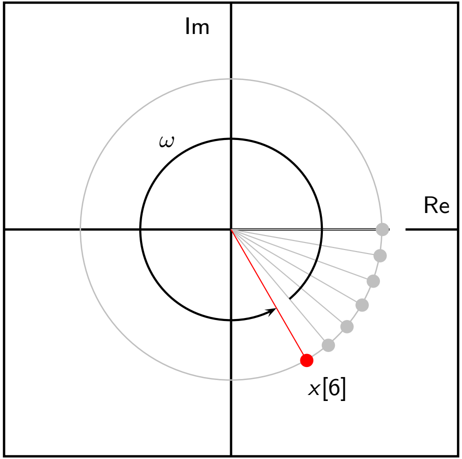

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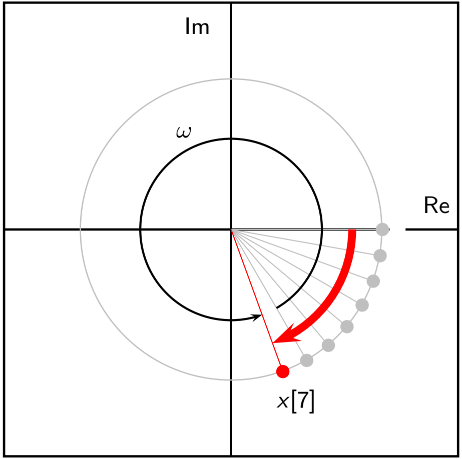

The wagonwheel effect

Summary

- $x[n] = e^{j(\omega n+\phi)}$ is the prototypical DSP oscillation
- discrete-time oscillations are periodic ONLY if frequency a rational multiple of π
- in discrete time, ω and $\omega + 2k\pi$ are indistinguishable frequencies

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

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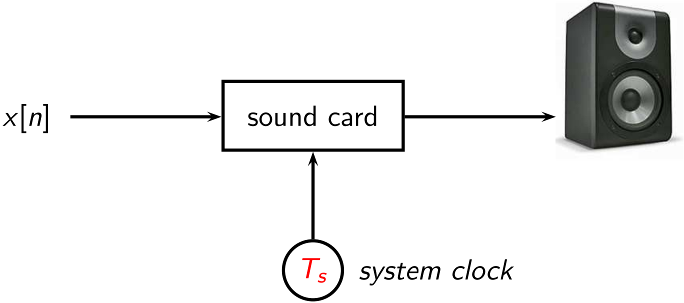
How your PC plays sounds

$$x[n] \longrightarrow \boxed{\text{sound card}} \longrightarrow$$ 

How your PC plays sounds



Digital vs physical frequency

- set T_s, time in seconds between samples
- periodicity of M samples $\longrightarrow$ periodicity of MT_s seconds
- real world frequency:

$$f = \frac{1}{MT_s}$$

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the Fourier basis

The Fourier Basis for $\mathbb{C}^N$

Claim: the set of N signals in $\mathbb{C}^N$

$$w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \ldots, N-1$$

is an orthogonal basis in $\mathbb{C}^N$.

The Fourier Basis for $\mathbb{C}^N$

In vector notation:

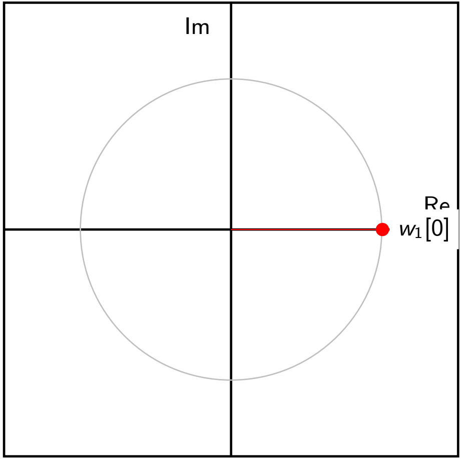
$$\{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1}$$

with

$$w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$$

is an orthogonal basis in $\mathbb{C}^N$

Recall the complex exponential generating machine...



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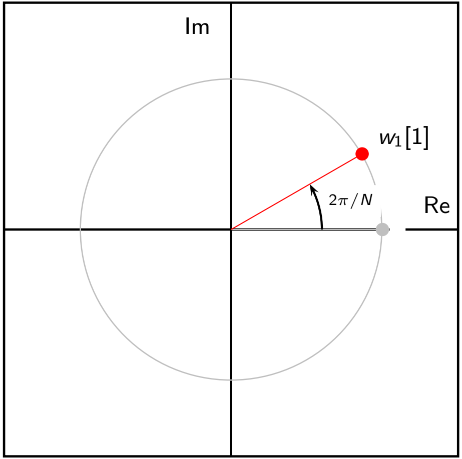

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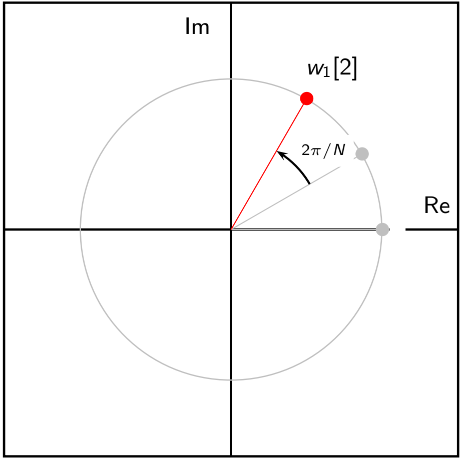

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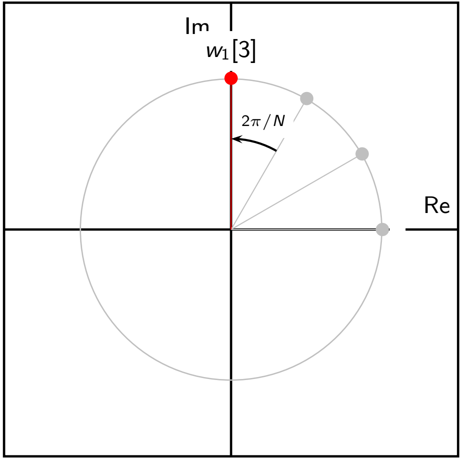

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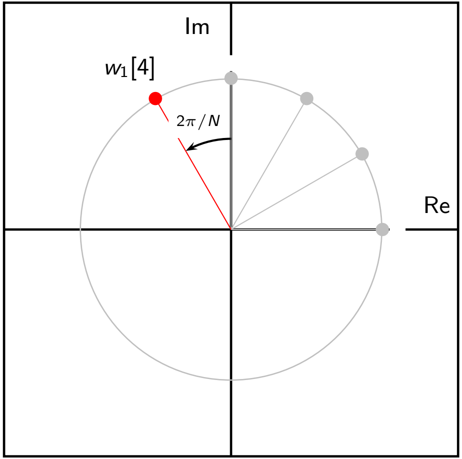

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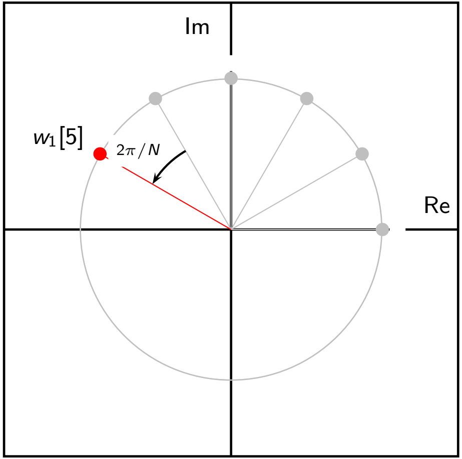
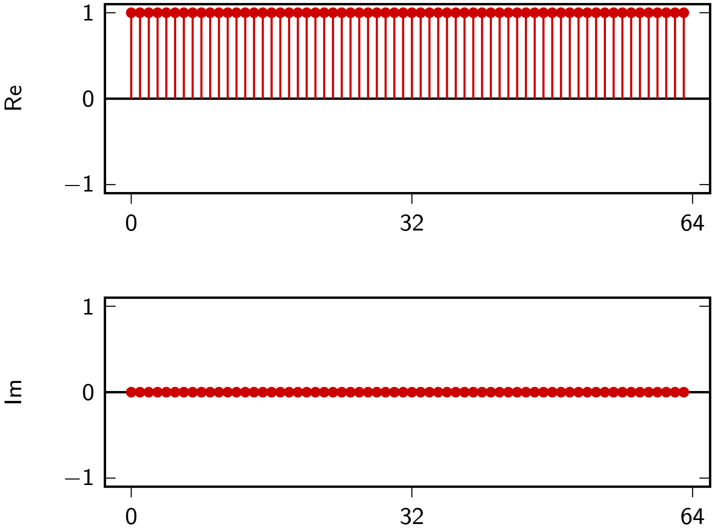
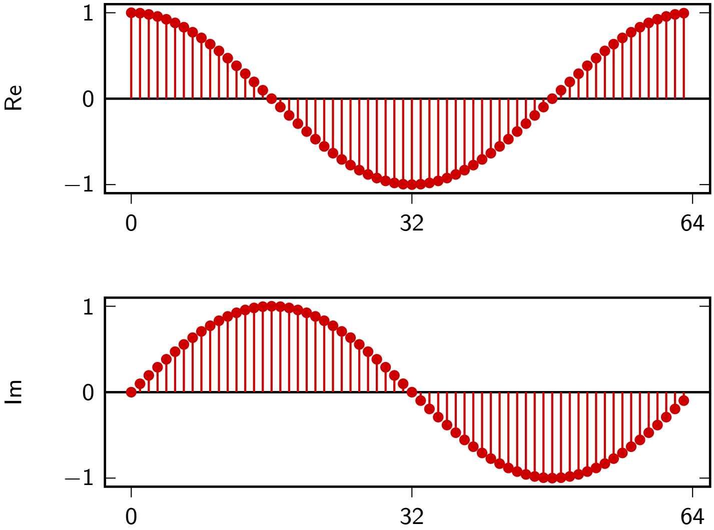


Basis vector $\mathbf{w}^{(0)} \in \mathbb{C}^{64}$



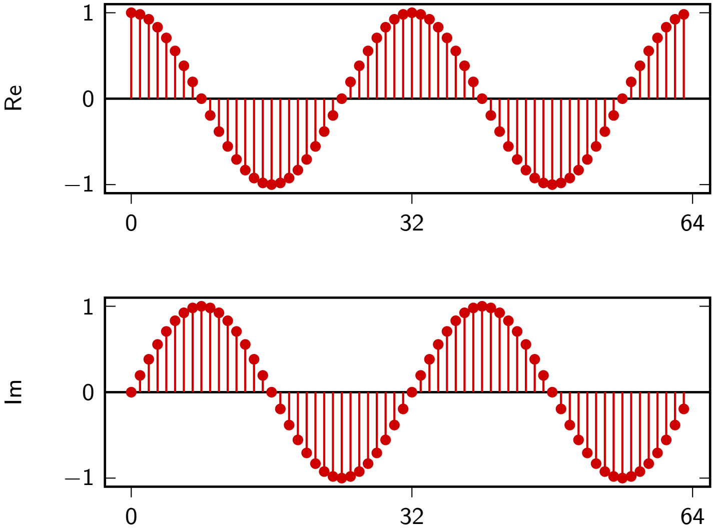
$\omega_0 = 0$

Basis vector $\mathbf{w}^{(1)} \in \mathbb{C}^{64}$



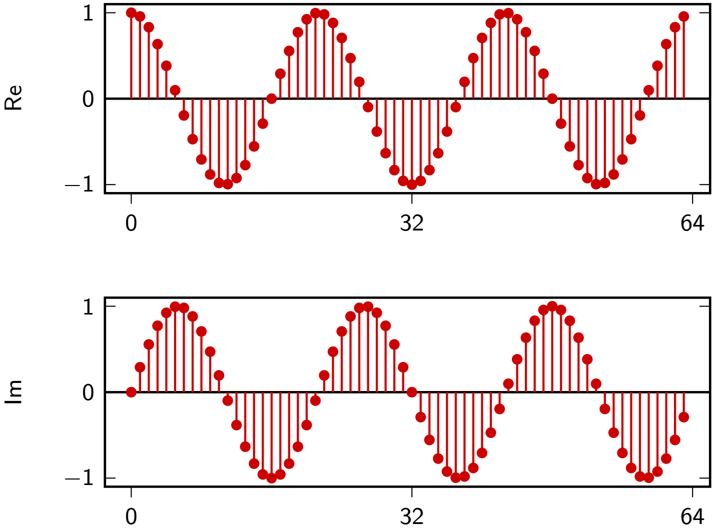
$$\omega_1 = \frac{2\pi}{64}$$

Basis vector $\mathbf{w}^{(2)} \in \mathbb{C}^{64}$



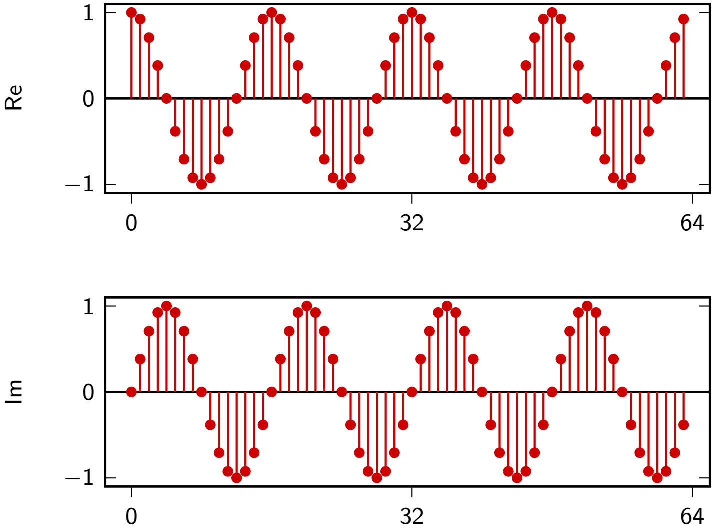
$$\omega_2 = 2\pi \, \frac{2}{64}$$

Basis vector $\mathbf{w}^{(3)} \in \mathbb{C}^{64}$



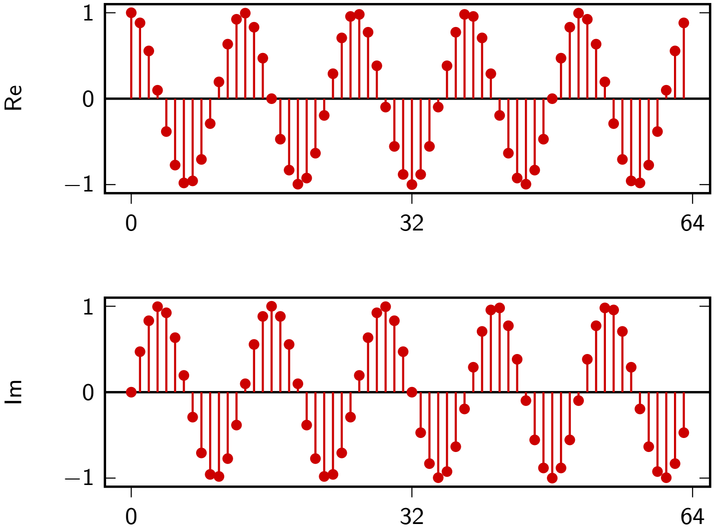
$$\omega_3 = 2\pi \, \frac{3}{64}$$

Basis vector $\mathbf{w}^{(4)} \in \mathbb{C}^{64}$



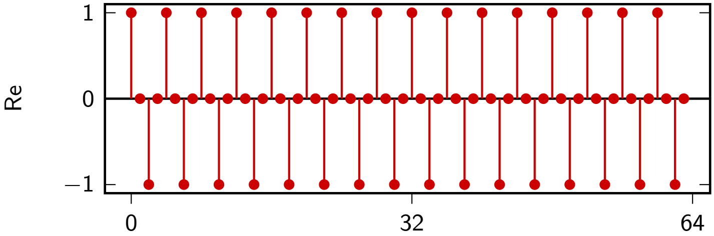
$$\omega_4 = 2\pi \, \frac{4}{64}$$

Basis vector $\mathbf{w}^{(5)} \in \mathbb{C}^{64}$

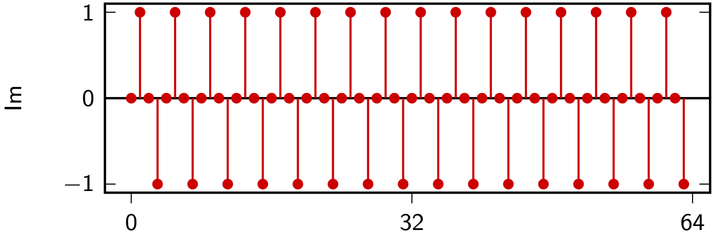


$$\omega_5 = 2\pi \frac{5}{64}$$

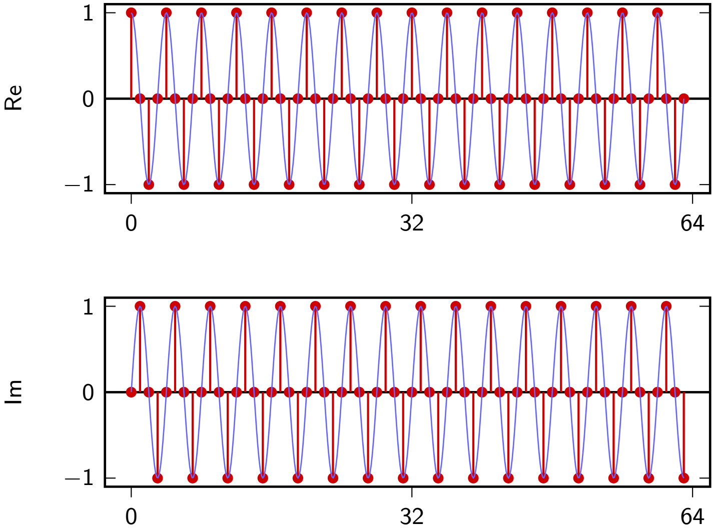
Basis vector $\mathbf{w}^{(16)} \in \mathbb{C}^{64}$



$$\omega_{16} = 2\pi \, \frac{16}{64}$$

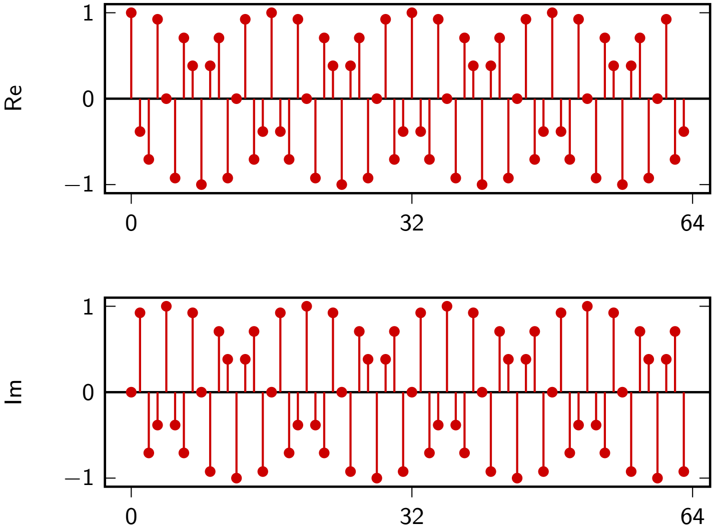


Basis vector $\mathbf{w}^{(16)} \in \mathbb{C}^{64}$



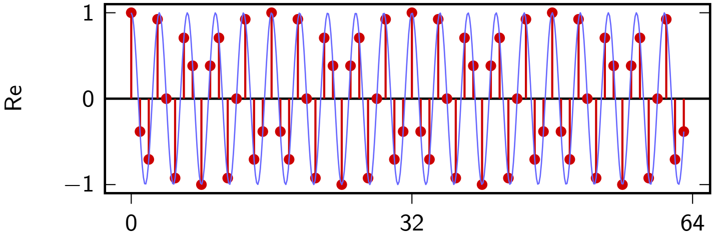
$$\omega_{16} = 2\pi\,\frac{16}{64}$$

Basis vector $\mathbf{w}^{(20)} \in \mathbb{C}^{64}$

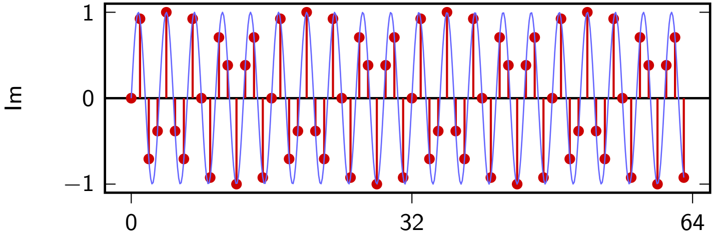


$$\omega_{20} = 2\pi \frac{20}{64}$$

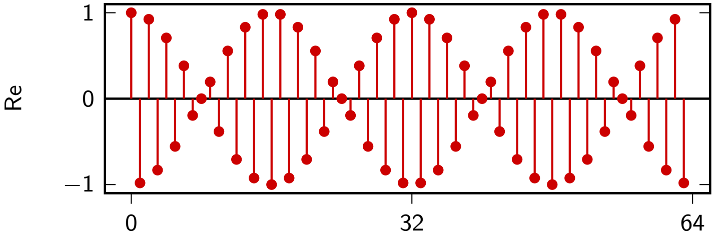
Basis vector $\mathbf{w}^{(20)} \in \mathbb{C}^{64}$



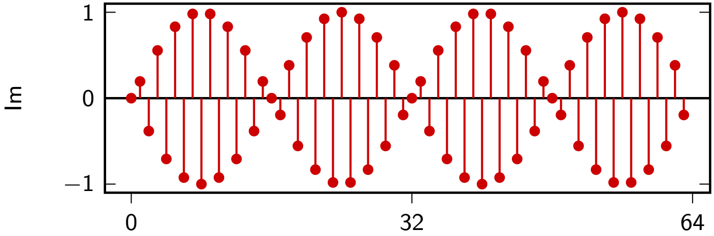
$$\omega_{20} = 2\pi \, \frac{20}{64}$$



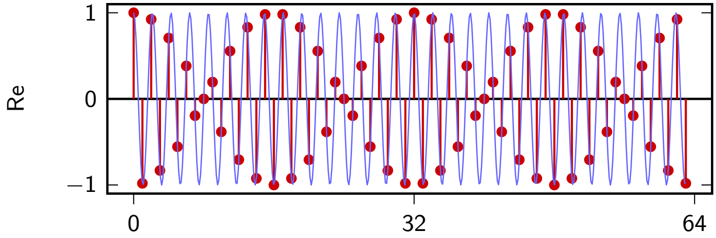
Basis vector $\mathbf{w}^{(30)} \in \mathbb{C}^{64}$



$$\omega_{30} = 2\pi \, \frac{30}{64}$$



Basis vector $\mathbf{w}^{(30)} \in \mathbb{C}^{64}$



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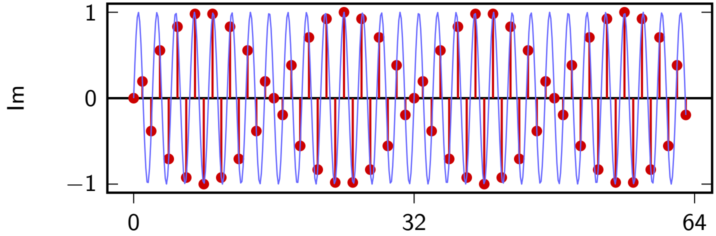
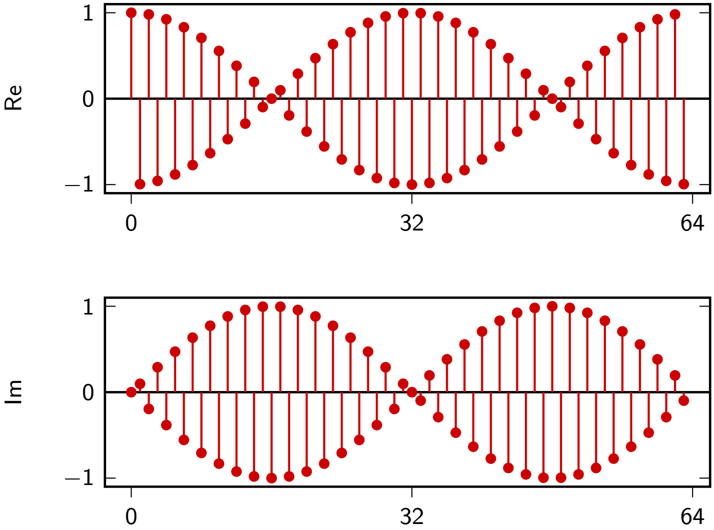
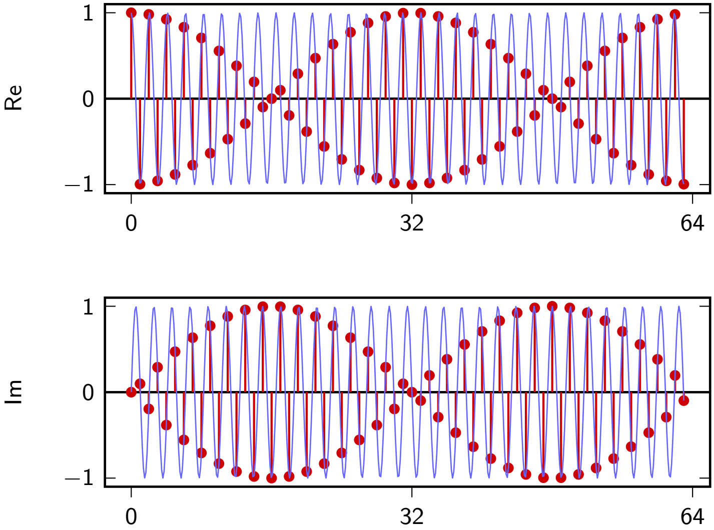


Basis vector $\mathbf{w}^{(31)} \in \mathbb{C}^{64}$



$$\omega_{31} = 2\pi \, \frac{31}{64}$$

Basis vector $\mathbf{w}^{(31)} \in \mathbb{C}^{64}$



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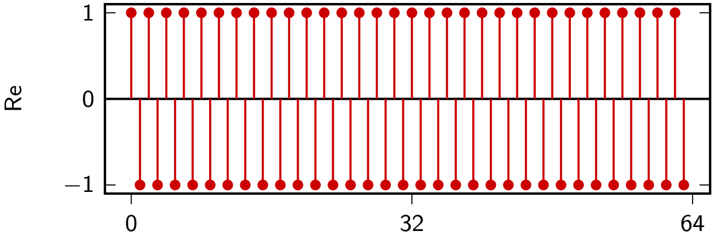
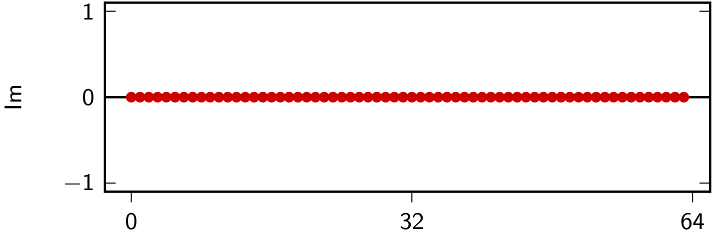
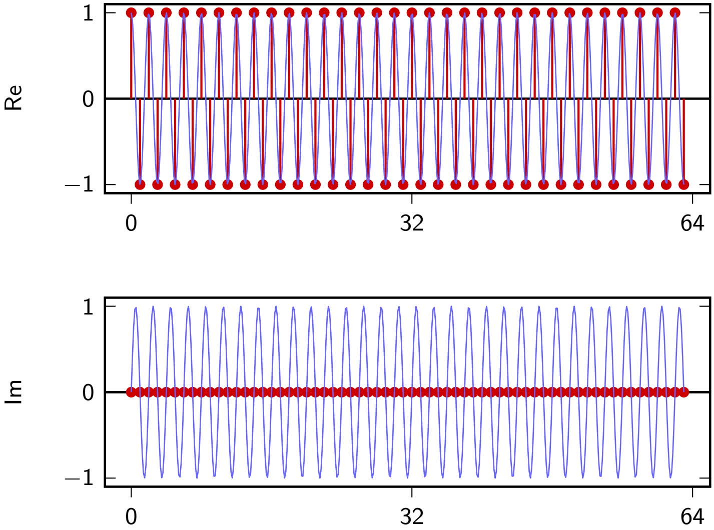
Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$



$$\omega_{32} = 2\pi \, \frac{32}{64}$$



Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$



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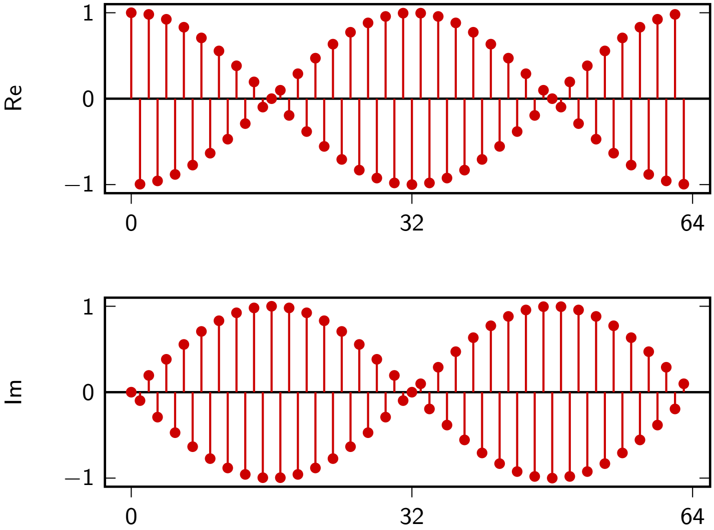

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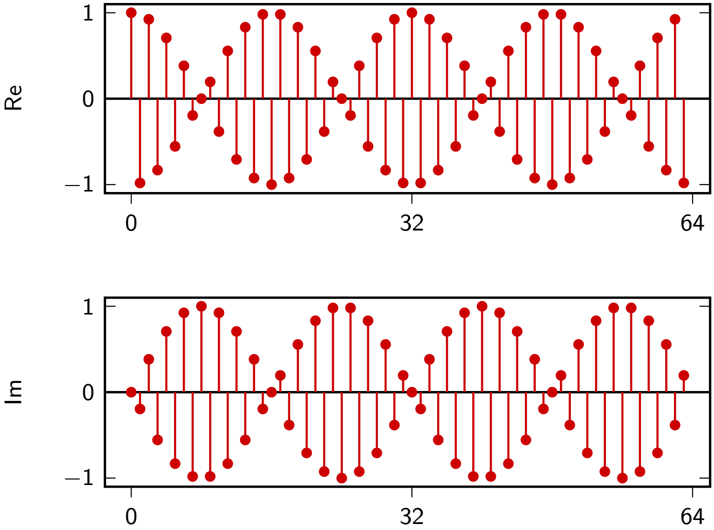
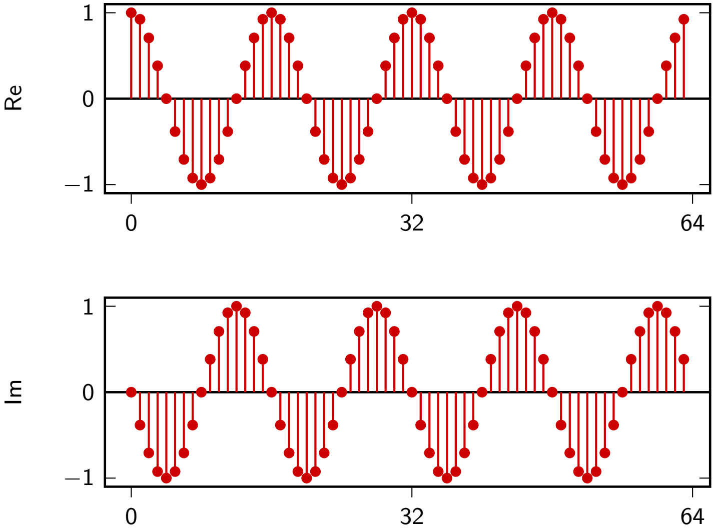
Basis vector $\mathbf{w}^{(34)} \in \mathbb{C}^{64}$



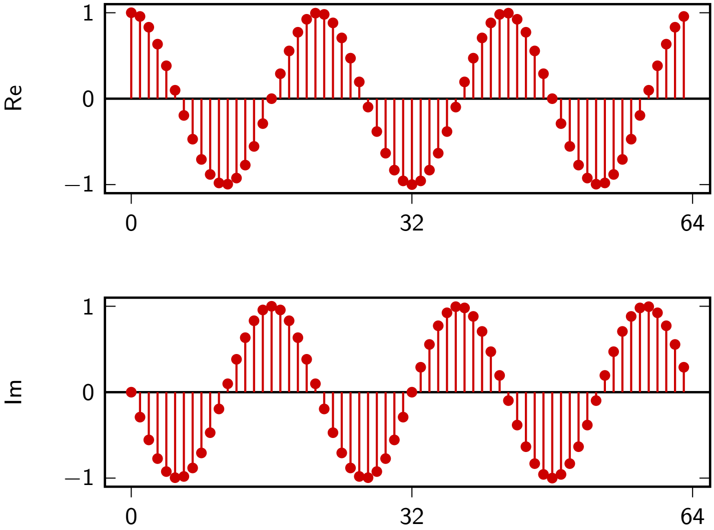
$$\omega_{34} = 2\pi\,\frac{34}{64}$$

Basis vector $\mathbf{w}^{(60)} \in \mathbb{C}^{64}$



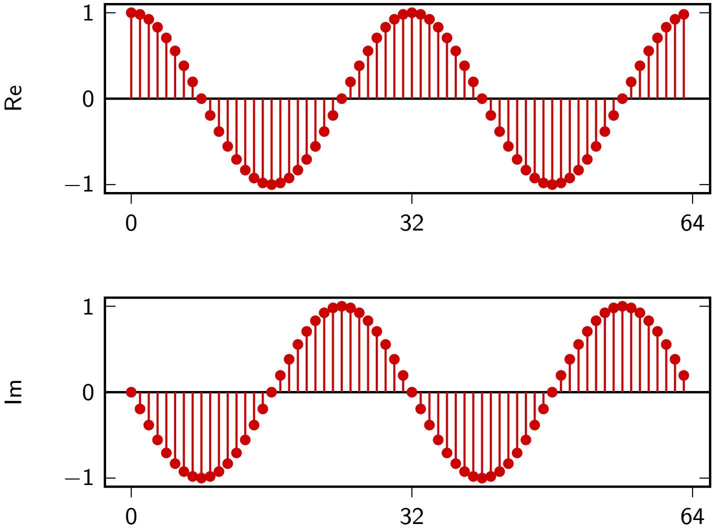
$$\omega_{60} = 2\pi \, \frac{60}{64}$$

Basis vector $\mathbf{w}^{(61)} \in \mathbb{C}^{64}$



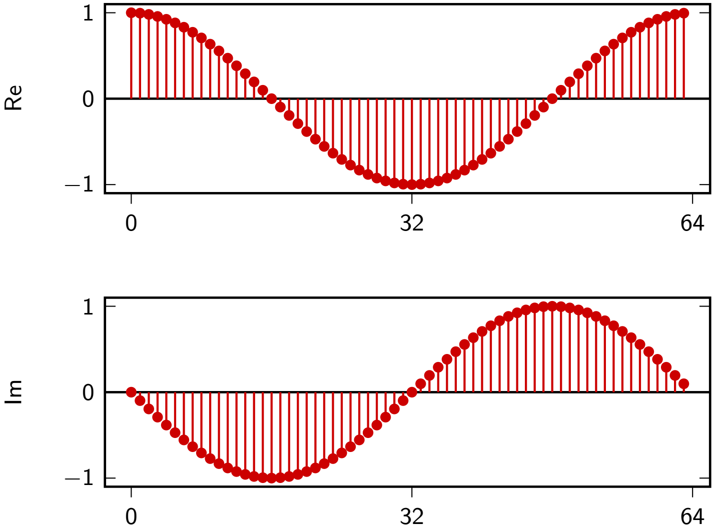
$$\omega_{61} = 2\pi \frac{61}{64}$$

Basis vector $\mathbf{w}^{(62)} \in \mathbb{C}^{64}$



$$\omega_{62} = 2\pi\,\frac{62}{64}$$

Basis vector $\mathbf{w}^{(63)} \in \mathbb{C}^{64}$



$$\omega_{63} = 2\pi \frac{63}{64}$$

Proof of orthogonality

$$
\begin{aligned}
\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle &= \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}nk}\right)^* e^{j\frac{2\pi}{N}nh} \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n} \\
&= \begin{cases} N & \text{for } h = k \\ \dfrac{1 - e^{j2\pi(h-k)}}{1 - e^{j\frac{2\pi}{N}(h-k)}} = 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Proof of orthogonality

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle = \sum_{n=0}^{N-1} (e^{j\frac{2\pi}{N}nk})^* \, e^{j\frac{2\pi}{N}nh}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n}$$

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Remarks

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*. Normalization factor would be $1/\sqrt{N}$

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the Discrete Fourier Transform

The Fourier Basis for $\mathbb{C}^N$

- in "signal" notation: $w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \ldots, N-1$
- in vector notation: $\{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1}$ with $w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$

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The Fourier Basis for $\mathbb{C}^N$

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*. Normalization factor would be $1/\sqrt{N}$
- will keep normalization factor explicit in DFT formulas

Basis expansion

Analysis formula:

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)}$$

Basis expansion (signal notation)

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *"time" domain*

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N-point signal in the *"time" domain*

Change of basis in matrix form

Define $W_N = e^{-j\frac{2\pi}{N}}$
(or simply W when N is evident from the context)

Change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n,m] = W_N^{nm}$:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ & & & \dots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix}$$

Change of basis in matrix form

Define $W_N = e^{-j\frac{2\pi}{N}}$
(or simply W when N is evident from the context)

Change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n, m] = W_N^{nm}$:

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Change of basis in matrix form

Analysis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

DFT Matrix

$$W_N^m = W_N^{(m \mod N)}$$

e.g. $W_8^{11} = W_8^3$

DFT Matrix

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Small DFT matrices: $N = 2, 3$

$$W_2 = e^{-j\frac{2\pi}{2}} = -1$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W_3 = e^{-j\frac{2\pi}{3}} = -(1 + j\sqrt{3})/2$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -(1 + j\sqrt{3})/2 & -(1 - j\sqrt{3})/2 \\ 1 & -(1 - j\sqrt{3})/2 & (1 - j\sqrt{3})/2 \end{bmatrix}$$

Small DFT matrices: $N = 4$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Small DFT matrices: $N = 5$

$$\mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W & W^3 \\ 1 & W^3 & W & W^4 & W^2 \\ 1 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

Small DFT matrices: $N = 6$

$$\mathbf{W}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & 1 & W^2 & W^4 \\ 1 & W^3 & 1 & W^3 & 1 & W^3 \\ 1 & W^4 & W^2 & 1 & W^4 & W^2 \\ 1 & W^5 & W^4 & W^3 & W^2 & W \end{bmatrix}$$