COM-303 - Signal Processing for Communications

Solutions for Homework #12

Solution 1. Chanel capacity

Using Shannon's capacity formula, we can derive the upper limit on the channel's througput as $C = W\log_2(1+SNR) = 39.949 Mbps$

Solution 2. QAM error rate

The correct answers are (b) and (d).

Solution 3. QAM demodulation

Remember that the input signal is $s[n] = b_r[n]\cos(\omega_c n) - b_i[n]\sin(\omega_c n)$, and the goal is to retrieve the baseband signals $b_r[n]$ and $b_i[n]$.

(a) For $m_1[n] = \cos(\omega_c n)$, $m_2[n] = \sin(\omega_c n)$ we can use the "double angle" trigonometric formulas to obtain

$$s[n]m_1[n] = \frac{1}{2}[b_r[n] + b_r[n]\cos(2\omega_c n) - b_i[n]\sin(2\omega_c n)]$$
$$s[n]m_2[n] = \frac{1}{2}[-b_i[n] + b_r[n]\sin(2\omega_c n) - b_i[n]\cos(2\omega_c n)];$$

all modulated terms are at double the carrier frequency so that, by filtering out frequencies over ω_c, we can retrieve the basesband signals.

(b) For $m_1[n] = 1 + \cos\omega_c n$, $m_2[n] = 1 + \sin\omega_c n$ we have

$$\begin{aligned} s[n]m_1[n] &= \frac{b_r[n]}{2} + \frac{1}{2}(b_r[n]\cos(2\omega_c n) - b_i[n]\sin(2\omega_c n)) \\ &\quad + b_r[n]\cos(\omega_c n) - b_i[n]\sin(\omega_c n) \\ s[n]m_2[n] &= \frac{b_i[n]}{2} + \frac{1}{2}(b_r[n]\sin(2\omega_c n) - b_i[n]\cos(2\omega_c n)) \\ &\quad + b_r[n]\cos(\omega_c n) - b_i[n]\sin(\omega_c n); \end{aligned}$$

here we have modulated terms at the carrier frequency but, by filtering out frequencies over ω_0 we can still retrieve the basesband signals.

(c) For $m_1[n] = \cos\frac{\omega_c n}{2}\cos\frac{3\omega_c n}{2}$, $m_2[n] = \sin\frac{\omega_c n}{2}\cos\frac{3\omega_c n}{2}$,we have

$$
\begin{aligned}
s[n]m_1[n] &= \frac{b_r[n]}{4} + \frac{b_r[n]}{4}(\cos(\omega_c n) + \cos(2\omega_c n) + \cos(3\omega_c n)) \\
&\quad - \frac{b_i[n]}{4}(-\sin(\omega_c n) + \sin(2\omega_c n) + \sin(3\omega_c n)) \\
s[n]m_2[n] &= \frac{b_i[n]}{4} + \frac{b_r[n]}{4}(\sin(\omega_c n) - \sin(2\omega_c n) + \sin(3\omega_c n)) \\
&\quad + \frac{b_i[n]}{4}(\cos(\omega_c n) - \cos(2\omega_c n) - \cos(3\omega_c n));
\end{aligned}
$$

here we have modulated terms at the 1x, 2x, and 3x the carrier frequency but, again, by filtering out frequencies over ω_0 we can still retrieve the basesband signals.

Solution 4. ADSL

(a) Throughput is proportional to the SNR, therefore the channel with the largest possible throughput is CH_7 while the channels with the lowest throughput are CH_2 and CH_5.

(b) For each channel, we need to first isolate the operational quadrant on the QAM error rate chart, and then select the curve with the highest number of bits per symbol that intersects the quadrant. The upper bound for the quadrant is given by the max probability of error: we can first draw a horizontal line on the SNR chart, intersecting the ordinate at $P_{err} = 10^{-6}$ and the operational area of the SNR chart will be *below* this line. Then, for each sub-channel we can draw a vertical line at the sub-channel SNR: the area to its *left* will completely identify the operational quadrant. Finally, once we have determined the number of bits per symbol M, the resulting throughput will be equal to $MF_s\omega_b$ where ω_b is the bandwidth of the subchannel, i.e. $\pi/8$.

- CH_3 has a max SNR of 24dB so that both the $M = 2$ and $M = 4$ curves intersect the quadrant. We can choose $M = 4$ so that the final throughput is 500Kbps.
- CH_4 has a max SNR of 32dB; we can use $M = 6$ for a max throughput of 750Kbps
- CH_7 has a max SNR of 32dB; we can use $M = 10$ for a max throughput of 1.25Mbps

---

Solution 5. Bandwidth constraint

(a) In general, the Baud rate is equal to the available bandwidth so we have $250\cdot 10^6$ symbols/s. $\mathscr{A}$ has 32 symbols, so we need $\log_2 32 = 5$ bits/symbol. Hence the throughput is $250\cdot 10^6 \cdot 5 = 1.25\cdot 10^9$ bits/s

(b) The two conditions for the trasmitter design are, firstly, $F_s > 2F_{max}$, as per the sampling theorem. With that, we should choose F_s to be an integer multiple of the bandwidth, $F_s = KW$. We can pick $K = 10$ for $F_s = 1.5$GHz or $K = 12$ for $F_s = 2.4$GHz, so both (b),(c) are valid answers.

Solution 6. Power constraint

The channel bandwidth is 3 kHz, so the Baud rate is $W = 3000$ symbols/s; in other to find the total bit rate we need to determine the maximum amount of bits per symbol that we can send given the power constraint (i.e. the max SNR) and the accepted probability of error. We know tha for QAM signaling $P_{err} = e^{-3\cdot 2^{-(M+1)}\cdot \mathrm{SNR}}$. This gives us $M \approx 6$, and so the total bitrate 18Kbit/sec.

Solution 7. Gain and probability of error

Assume each received symbol is affected by a noise sample $\eta[n] \in \mathscr{U}[-100, 100]$. Then

$$
\begin{aligned}
P_{err} &= P[|\eta[n]| > G \mid \mathrm{n}^{th}\mathrm{bit} \neq \pm 31G] \cdot P[\mathrm{n}^{th}\mathrm{bit} \neq \pm 31G] \\
&\quad + P[\eta[n] > G \mid \mathrm{n}^{th}\mathrm{bit} = -31G] \cdot P[\mathrm{n}^{th}\mathrm{bit} = -31G] \\
&\quad + P[\eta[n] < -G \mid \mathrm{n}^{th}\mathrm{bit} = 31G] \cdot P[\mathrm{n}^{th}\mathrm{bit} = 31G] \\
&= P[|\eta[n]| > G] \cdot \frac{30}{32} + P[\eta[n] > G] \cdot \frac{1}{32} + P[\eta[n] < -G] \cdot \frac{1}{32} \\
&= \frac{30}{32} \cdot 2 \cdot P[\eta[n] > G] + \frac{2}{32} \cdot P[\eta[n] > G]
\end{aligned}
$$

We know that $P_{err} = 10^{-2}$, so $P[\eta[n] > G] = 10^{-2} \cdot \frac{32}{62} = 0.0051$, and $G = 99$.
