Solutions 3

1. a) First observe that the transition probabilities do not depend on the particular shape of the (convex) polygon, but just on the number of edges. Consider next a polygon with $j+3$ edges initially. After the transition, the smallest possible polygon will have 3 edges and the largest possible polygon will have $j+4$ edges. Thus the resulting polygons have $k+3$ edges with $0 \leq k \leq j+1$. Since the transition is uniformly random, $p_{jk} = \frac{1}{j+2}$, for $0 \leq k \leq j+1$.

b) Thus,

$$\mathbb{E}(X_n | X_{n-1} = j) = \sum_{k=0}^{j+1} k\, p_{jk} = \frac{1}{j+2} \sum_{k=0}^{j+1} k = \frac{(j+1)(j+2)}{2(j+2)} = \frac{j+1}{2}$$

and

$$\mathbb{E}(X_n) = \sum_{j \geq 0} \mathbb{E}(X_n | X_{n-1} = j)\, \mathbb{P}(X_{n-1} = j) = \frac{1 + \mathbb{E}(X_{n-1})}{2}$$

Repeating this, we obtain $\mathbb{E}(X_n) = 1 - (1/2)^n + (1/2)^n\, \mathbb{E}(X_0)$.

c) Consider $(X_n, n \geq 0)$ initialized with some initial distribution for X_0. Repeating the above computation, we obtain

$$\mathbb{E}(s^{X_n} | X_{n-1} = j) = \frac{1}{j+2} \sum_{k=0}^{j+1} s^k = \frac{1}{j+2} \frac{1 - s^{j+2}}{1-s}$$

This implies that $G_n(s) = \dfrac{1}{1-s}\, \mathbb{E}\left(\dfrac{1 - s^{(X_{n-1}+2)}}{X_{n-1}+2} \right)$.

d) Now consider the process $(X_n,\, n \geq 0)$ initialized with $X_0 \sim \pi$, where π is the stationary distribution. Since $\pi = \pi P$ by definition, we have $X_n \sim \pi$ and $X_{n-1} \sim \pi$, so $G(s) = \mathbb{E}(s^{X_n}) = \mathbb{E}(s^{X_{n-1}})$ for all $n \geq 1$, and by part 3, we also have

$$G(s) = \frac{1}{1-s}\, \mathbb{E}\left(\frac{1 - s^{(X_{n-1}+2)}}{X_{n-1}+2} \right)$$

where $\mathbb{E}$ is taken with respect to π.

Differentiating with respect to s, we obtain

$$\begin{aligned} G'(s) &= \frac{1}{(1-s)^2}\, \mathbb{E}\left(\frac{1 - s^{X_{n-1}+2}}{X_{n-1}+2} \right) - \frac{1}{1-s} \mathbb{E}\left(\frac{X_{n-1}+2}{X_{n-1}+2}\, s^{X_{n-1}+1} \right) \\ &= \frac{1}{1-s}\, G(s) - \frac{1}{1-s}\, s\, G(s) = G(s) \end{aligned}$$

One checks also that $G(1) = 1$ (using Bernoulli-L'Hospital's rule), so the solution of this differential equation is $G(s) = e^{s-1}$.

e) $G_Y(s) = \sum_{k \geq 0} \lambda^k\, e^{-\lambda}\, s^k / k! = e^{-\lambda} \sum_{k \geq 0} (s\lambda)^k / k! = e^{\lambda(s-1)}$. Hence the stationary distribution of the Markov chain is Poisson with parameter 1.

2. a) Clearly, all states i s.t. i is not a power of 2 are transient. Indeed, if i is not a power of 2, then $\mathbb{P}(X_n = i \text{ for some } n > 0 \mid X_0 = i) = 0$.

Let us consider now the state 1. Then,

$$f_{11}(n) = c^{n-1}(1-c), \qquad \text{for } n \geq 1.$$

Hence,

$$f_{11} = \sum_{n\geq 1} n f_{11}(n) = (1-c)\sum_{n\geq 1} n c^{n-1} = \frac{1}{1-c} < +\infty,$$

which implies that the state 1 is positive-recurrent. As concerns the states $\{2^k\}_{k\geq 1}$, they form together with state 1 an equivalence class of the Markov chain. Therefore, they are also positive-recurrent.

b) Let π be the stationary distribution (in case it exists). Then, by solving $\pi = \pi P$, we obtain

$$\begin{cases} \pi_1 = (1-c)\sum_{i\in\mathbb{N}} \pi_{2^i} = 1-c & \\ \pi_{2^k} = c \cdot \pi_{2^{k-1}} & k \geq 1 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

Hence, the stationary distribution exists, is unique and is given by

$$\begin{cases} \pi_{2^k} = (1-c)\cdot c^k & k \geq 0 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

c) In general, by solving $\pi = \pi P$, we obtain

$$\begin{cases} \pi_1 = \sum_{k\in\mathbb{N}}(1-p_{2^k})\pi_{2^k} & \\ \pi_{2^k} = p_{2^{k-1}} \cdot \pi_{2^{k-1}} & k \geq 1 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

Therefore, $\pi_{2^k} = \prod_{j=0}^{k-1} p_{2^j}\, \pi_1$, so the stationary distribution exists and is unique if and only if

$$\sum_{k\in\mathbb{N}} \prod_{j=0}^{k-1} p_{2^j} < +\infty. \tag{1}$$

(otherwise it would imply that $\pi_1 = 0$).

d) Consider now the case $c_k = p_{2^k} = 1 - \dfrac{1}{2^k+1}$. We will show that $\lim_{k\to\infty}\prod_{j=0}^{k} c_j \neq 0$, which implies, through condition (1), that the stationary distribution does not exist. Note first that

$$\lim_{k\to\infty}\prod_{j=0}^{k} c_j = 0 \qquad \Longleftrightarrow \qquad \lim_{k\to\infty}\sum_{j=0}^{k}\log\frac{1}{c_j} = +\infty.$$

In addition,

$$\lim_{k\to\infty}\sum_{j=0}^{k}\log\frac{1}{c_j} = \lim_{k\to\infty}\sum_{j=0}^{k}\log\left(1+2^{-j}\right) \leq \lim_{k\to\infty}\sum_{j=0}^{k} 2^{-j} = 2 < +\infty,$$

where we used the fact that $\log(1+x) \leq x$ for any $x \in [0,1]$. As a result, $\sum_{k\in\mathbb{N}}\prod_{j=0}^{k-1} c_j = +\infty$, so the stationary distribution does not exist in this case.