COM303: Digital Signal Processing

Lecture 10: Ideal Filters and Approximations

Overview:

- ideal filters
- approximating ideal filters

ideal filters

Overview:

- Filter classification in the frequency domain
- Ideal filters
- Demodulation revisited

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Filter types according to magnitude response

- Lowpass
- Highpass
- Bandpass
- Allpass

Moving Average and Leaky Integrator are lowpass filters

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Filter types according to phase response

- Linear phase
- Nonlinear phase

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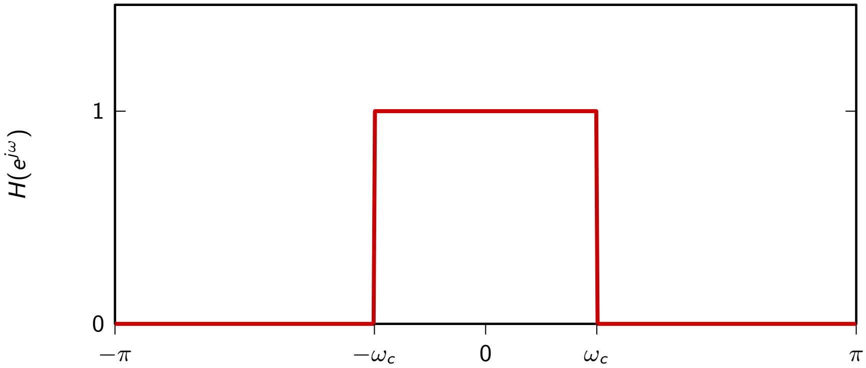
What is the best lowpass we can think of?



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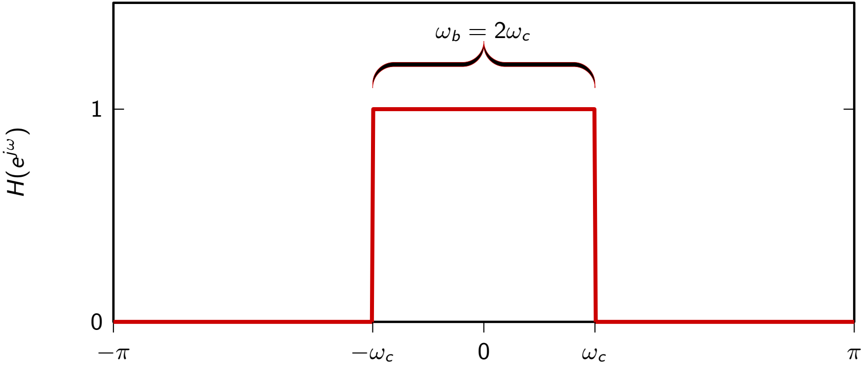

Ideal lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad \text{(}2\pi\text{-periodicity implicit)}$$

- perfectly flat passband
- infinite attenuation in stopband
- zero-phase (no delay)

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Ideal lowpass filter: impulse response

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\begin{aligned}
h[n] &= \text{IDTFT}\left\{H(e^{j\omega})\right\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \\
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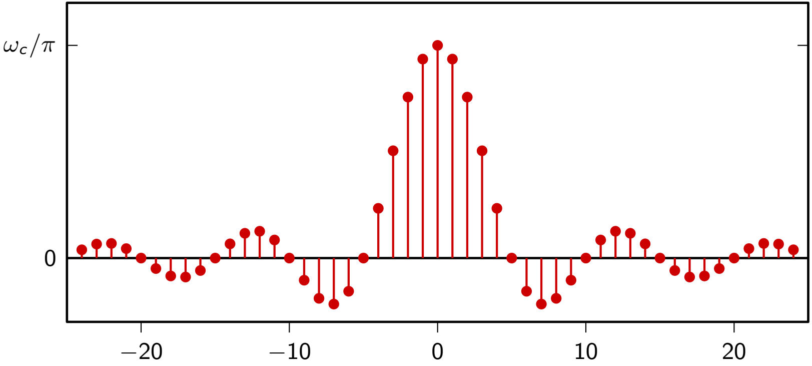

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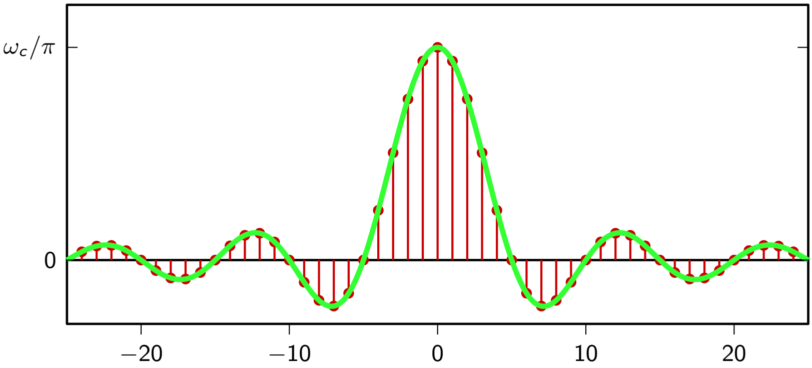

The bad news

- impulse response is infinite support, two-sided
  $\Longrightarrow$ cannot compute the output in a finite amount of time
  $\Longrightarrow$ that's why it's called "ideal"
- impulse response decays slowly in time
  $\Longrightarrow$ we need a lot of samples for a good approximation

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Nevertheless...

The sinc-rect pair:

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

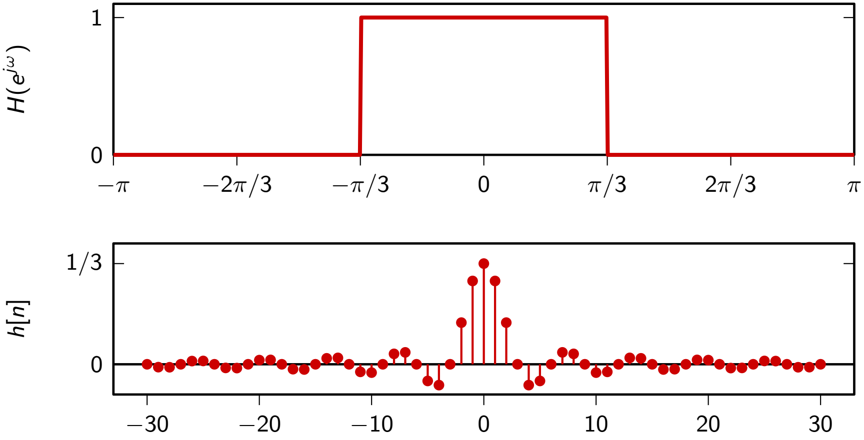
(note that $\mathrm{sinc}(x) = 0$ when x is a nonzero integer)

The ideal lowpass in canonical form

$$\boxed{\mathrm{rect}\left(\frac{\omega}{2\omega_c}\right)} \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad \boxed{\frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{\pi}n\right)}$$

Example

$$\omega_c = \pi/3\text{:}\;\; H(e^{j\omega}) = \mathrm{rect}(3\omega/2\pi),\;\; h[n] = (1/3)\mathrm{sinc}(n/3)$$



Little-known fact

- the sinc is not absolutely summable
- the ideal lowpass is not BIBO stable!
- example for $\omega_c = \pi/3$: $h[n] = (1/3)\,\mathrm{sinc}(n/3)$
- take $x[n] = \mathrm{sign}\{\mathrm{sinc}(-n/3)\}$ and

$$y[0] = (x * h)[0] = \frac{1}{3}\sum_{k=-\infty}^{\infty} |\mathrm{sinc}(k/3)| = \infty$$

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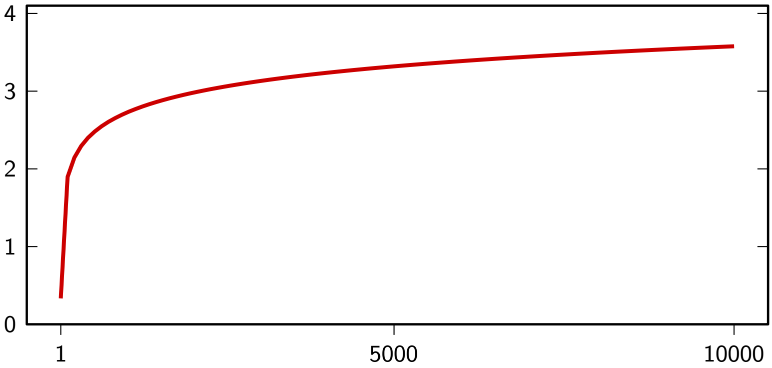
Divergence is however very slow...

$$s(n) = (1/3) \sum_{k=-n}^{n} |\operatorname{sinc}(k/3)|$$



for the mathematically-oriented:

integral criterion for convergence:

$$\sum_{n\in\mathbb{N}} f(n) < \infty \Leftrightarrow \int_1^\infty f(t)dt < \infty$$

so:

$$\sum_{n\in\mathbb{N}} \left|\frac{\sin n}{n}\right| < \infty \Leftrightarrow \int_1^\infty \left|\frac{\sin t}{t}\right| dt < \infty$$

for the mathematically-oriented:

$$
\begin{aligned}
\int_1^\infty \left|\frac{\sin t}{t}\right| dt &\geq \int_\pi^\infty \left|\frac{\sin t}{t}\right| dt \\
&= \sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin t}{t}\right| dt \\
&\geq \sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{(k+1)\pi} dt \\
&= \frac{1}{\pi}\sum_{k=1}^\infty \frac{1}{k+1}\int_{k\pi}^{(k+1)\pi} |\sin t| dt \\
&= \frac{2}{\pi}\sum_{k=1}^\infty \frac{1}{k+1} = \infty
\end{aligned}
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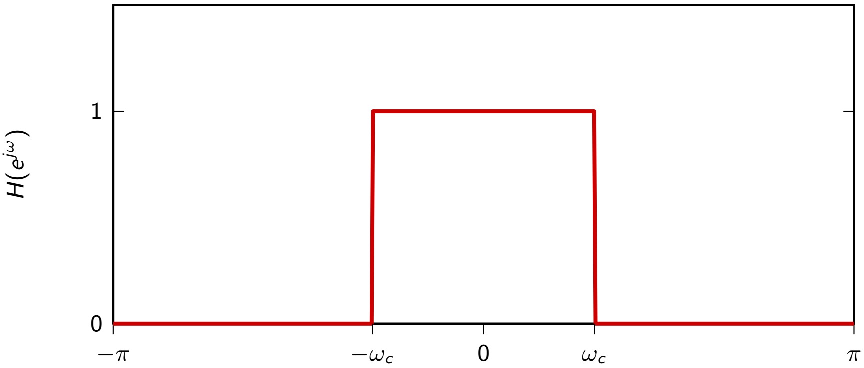
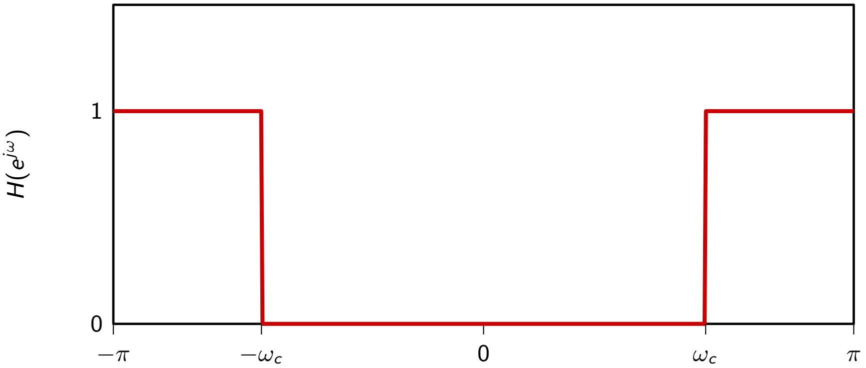
From the ideal lowpass...



... to the ideal highpass



Ideal highpass filter

$$H_{hp}(e^{j\omega}) = \begin{cases} 1 & \text{for } \pi \geq |\omega| \geq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad (2\pi\text{-periodicity implicit})$$

$$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$$

$$h_{hp}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}n)$$

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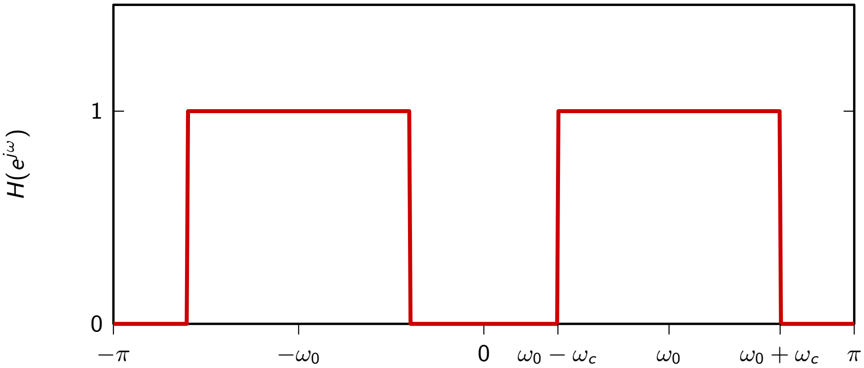
Ideal bandpass filter



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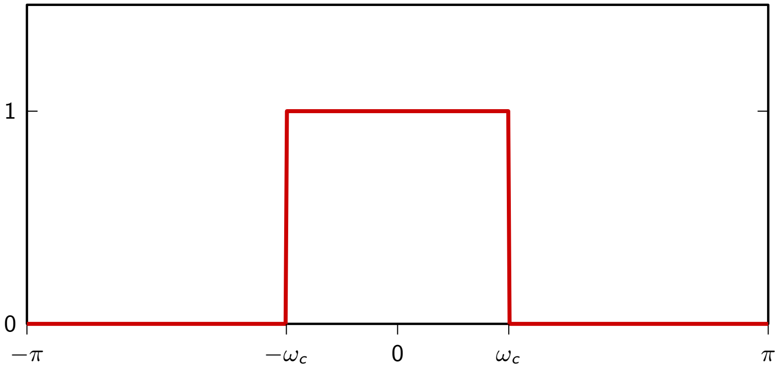

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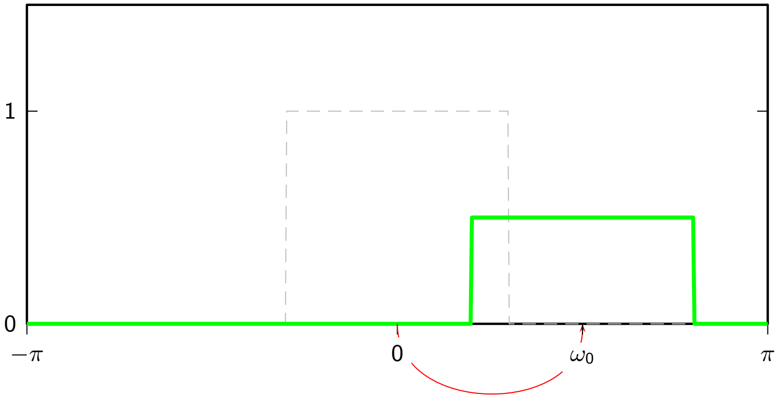

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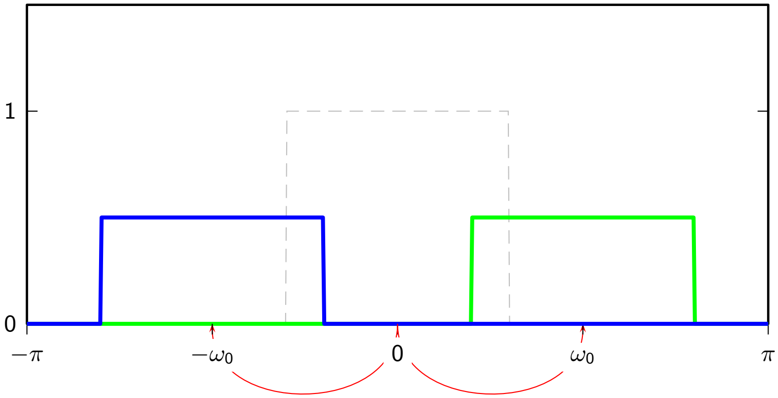

Ideal bandpass filter

$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega \pm \omega_0| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad (2\pi\text{-periodicity implicit})$$

$$h_{bp}[n] = 2\cos(\omega_0 n)\frac{\omega_c}{\pi}\mathrm{sinc}(\frac{\omega_c}{\pi}n)$$

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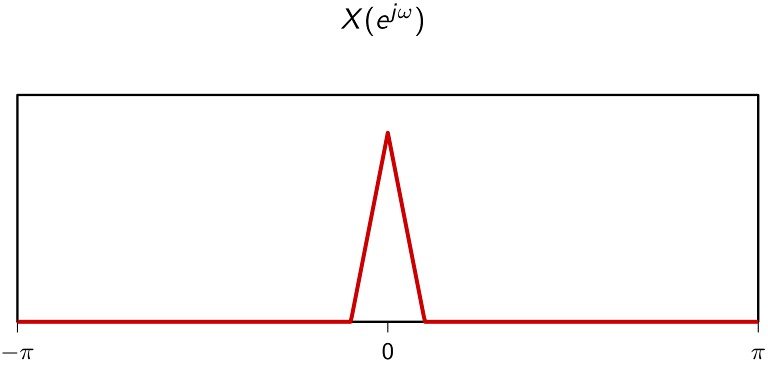
Demodulation revisited

remember the classic demodulation scheme:

- apply sinusoidal modulation to $x[n]$: $y[n] = x[n] \cos \omega_0 n$
- demodulate by multiplying by the carrier $x'[n] = y[n] \cos \omega_0 n$
- demodulated signal contains unwanted high-frequency components

Demodulation revisited



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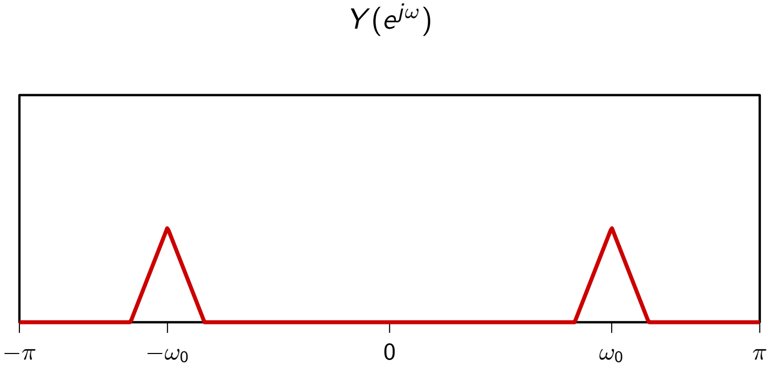

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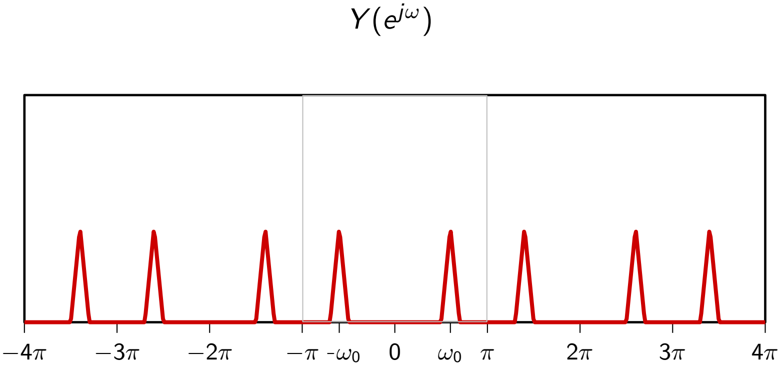

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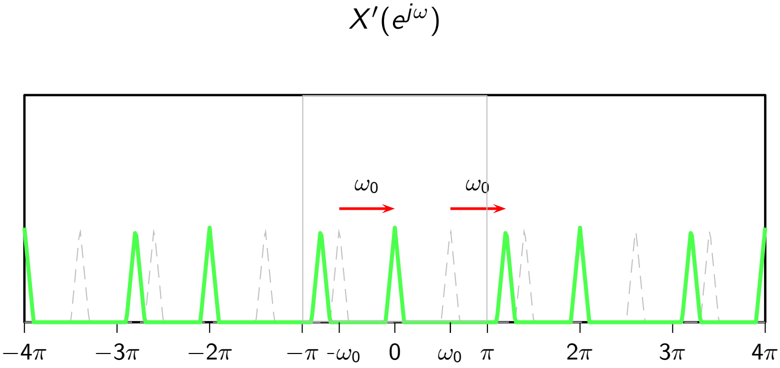

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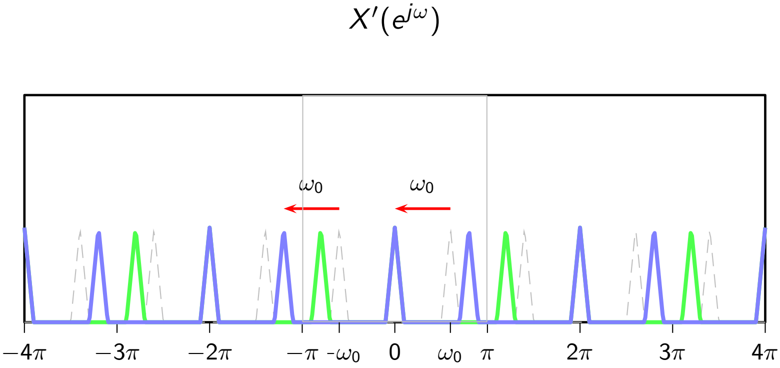

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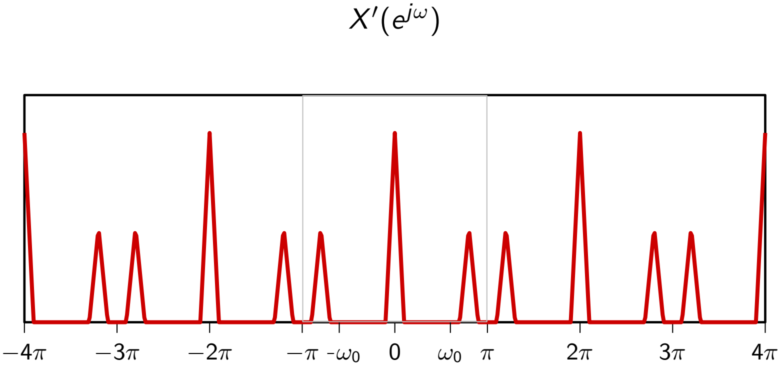
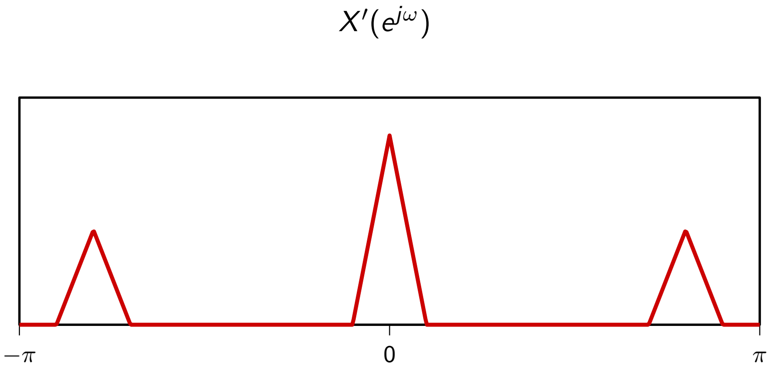


Solution: lowpass filtering



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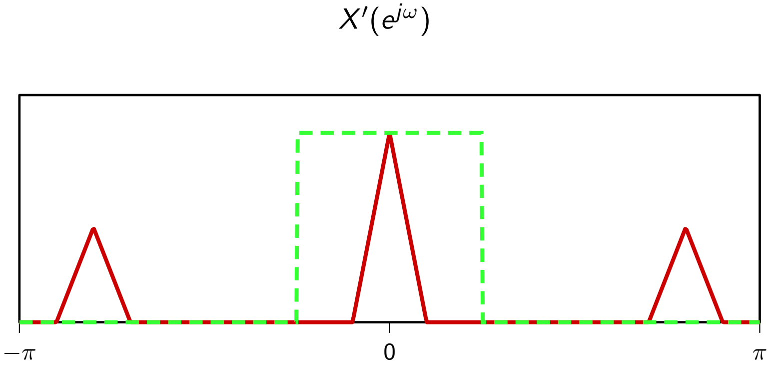

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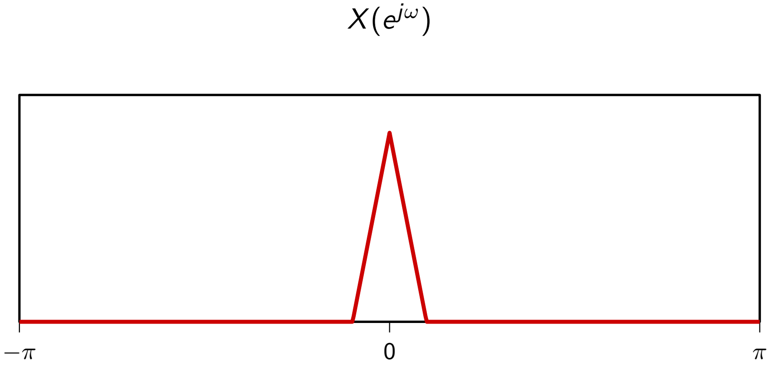

approximations of ideal filters

Overview:

- Impulse truncation
- Window method
- Frequency sampling

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How can we approximate an ideal lowpass?

Idea #1:

- pick ω_c
- compute ideal impulse response $h[n]$
- truncate $h[n]$ to a finite-support $\hat{h}[n]$
- $\hat{h}[n]$ defines an FIR filter

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Approximation by truncation

FIR approximation of length $M = 2N + 1$:

$$\hat{h}[n] = \begin{cases} \dfrac{\omega_c}{\pi} \operatorname{sinc}\left(\dfrac{\omega_c}{\pi} n\right) & |n| \leq N \\ \\ 0 & \text{otherwise} \end{cases}$$

Why it could be a good idea

$$\text{MSE} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 d\omega$$

$$= \|H(e^{j\omega}) - \hat{H}(e^{j\omega})\|^2$$

$$= \|h[n] - \hat{h}[n]\|^2$$

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MSE is minimized by symmetric impulse truncation around zero

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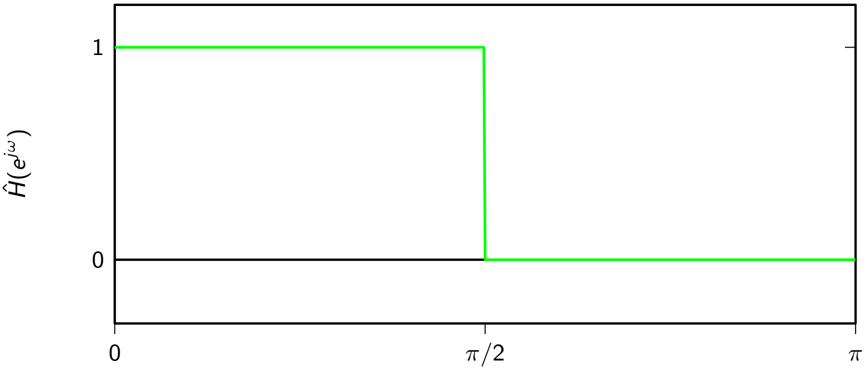

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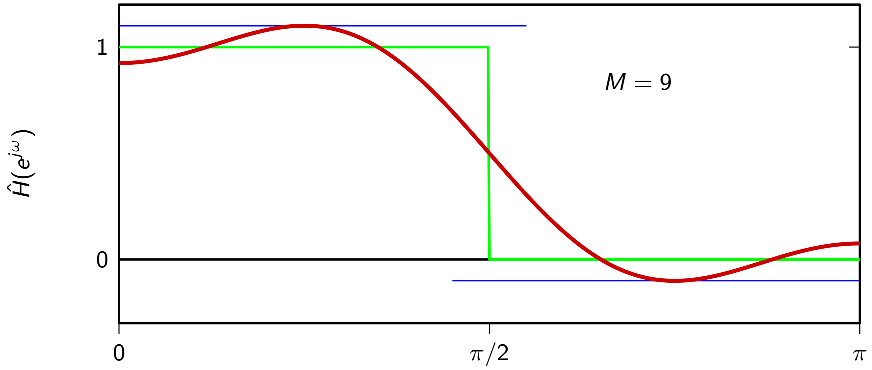

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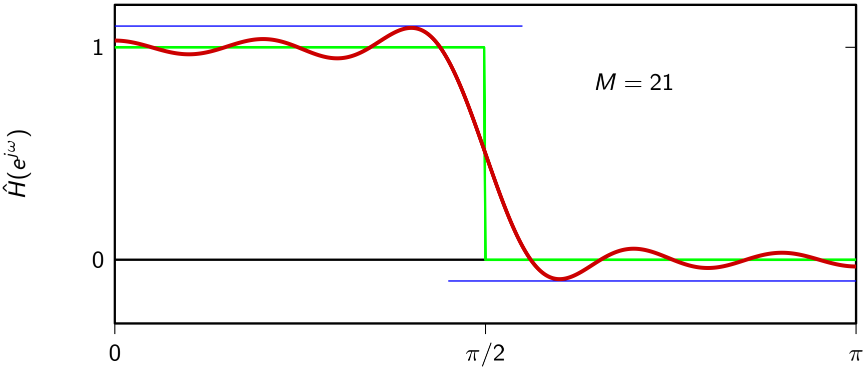

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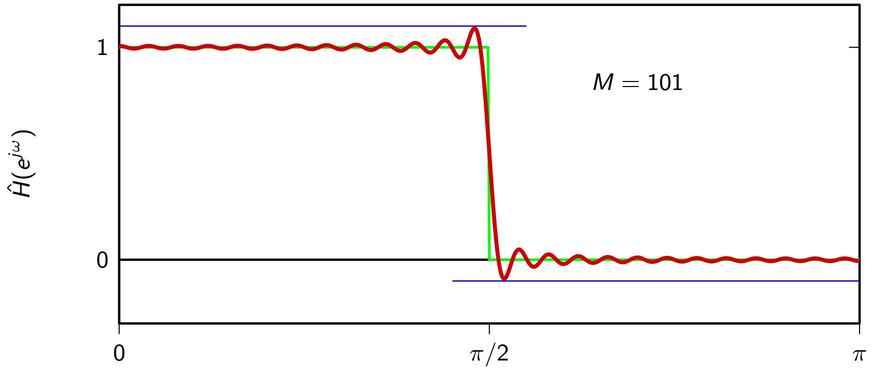

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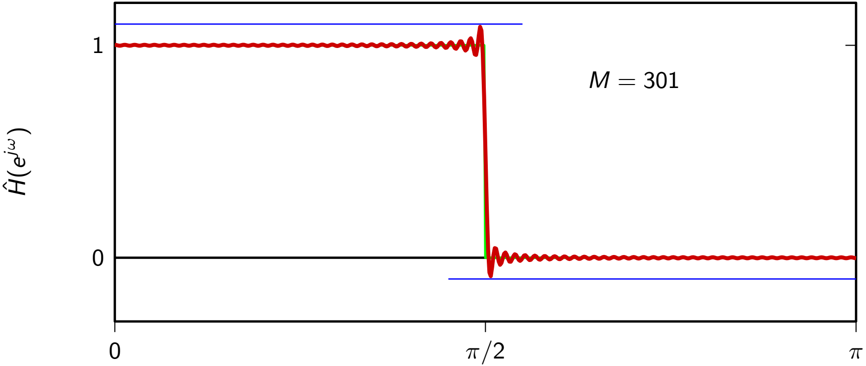

The Gibbs phenomenon

The maximum error around the cutoff frequency
is around 9% of the height of the jump
regardless of N

Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n]\, w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

Understanding the Gibbs phenomenon

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The convolution and modulation theorems

$$\mathrm{DTFT}\,\{(x * y)[n]\} = X(e^{j\omega})\,Y(e^{j\omega})$$

$$\mathrm{DTFT}\,\{x[n]\,y[n]\} = (X * Y)(e^{j\omega})$$

The convolution and modulation theorems

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Convolution of DTFTs

in $\ell_2(\mathbb{Z})$:

$$(x * y)[n] = \langle x^*[k], y[n-k] \rangle = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

in $L_2([-\pi, \pi])$:

$$(X * Y)(e^{j\omega}) = \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma$$

Convolution of DTFTs

in $\ell_2(\mathbb{Z})$:

$$(x * y)[n] = \langle x^*[k], y[n-k] \rangle$$
$$= \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

in $L_2([-\pi, \pi])$:

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$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma$$

Modulation theorem: proof

$$
\begin{aligned}
\text{IDTFT}\left\{(X * Y)(e^{j\omega})\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(X * Y)(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})Y(e^{j(\omega-\sigma)})e^{j\omega n}d\sigma d\omega \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})Y(e^{j(\omega-\sigma)})e^{j\sigma n}e^{j(\omega-\sigma)n}d\sigma d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})e^{j\sigma n}d\sigma\,\frac{1}{2\pi}\int_{-\pi}^{\pi}Y(e^{j(\omega-\sigma)})e^{j(\omega-\sigma)n}d\omega \\
&= x[n]\,y[n]
\end{aligned}
$$

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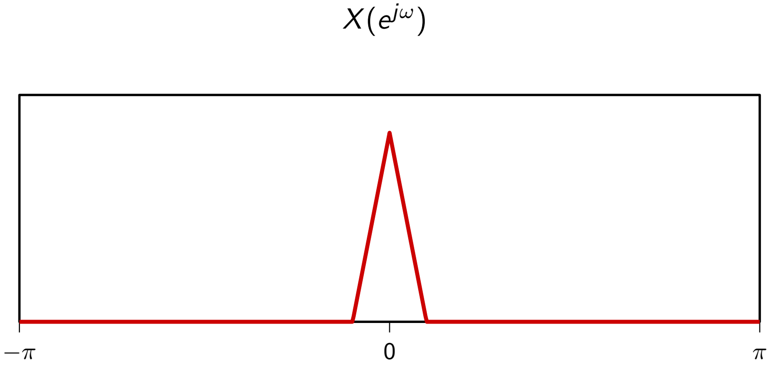
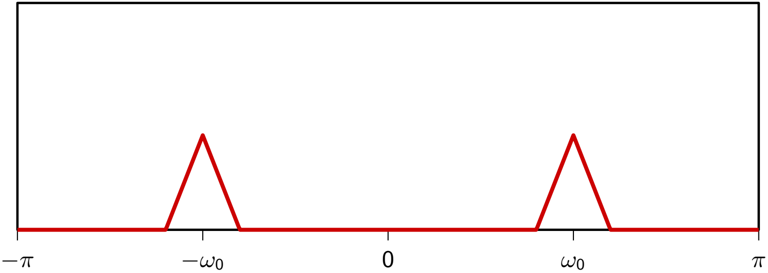
\end{aligned}$$

Aside: sinusoidal modulation revisited



Aside: sinusoidal modulation revisited

$$Y(e^{j\omega}) = \mathrm{DTFT}\left\{x[n]\cos(\omega_0 n)\right\}$$



Aside: sinusoidal modulation revisited

$$
\begin{aligned}
\text{DTFT}\{x[n]\cos\omega_0 n\} &= X(e^{j\omega}) * \frac{1}{2}[\tilde{\delta}(\omega-\omega_0)+\tilde{\delta}(\omega+\omega_0)] \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} X(e^{j(\omega-\sigma)})\tilde{\delta}(\sigma-\omega_0)d\sigma + \frac{1}{4\pi}\int_{-\pi}^{\pi} X(e^{j(\omega-\sigma)})\tilde{\delta}(\sigma+\omega_0)d\sigma \\
&= \frac{1}{2}\left[X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right]
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$$

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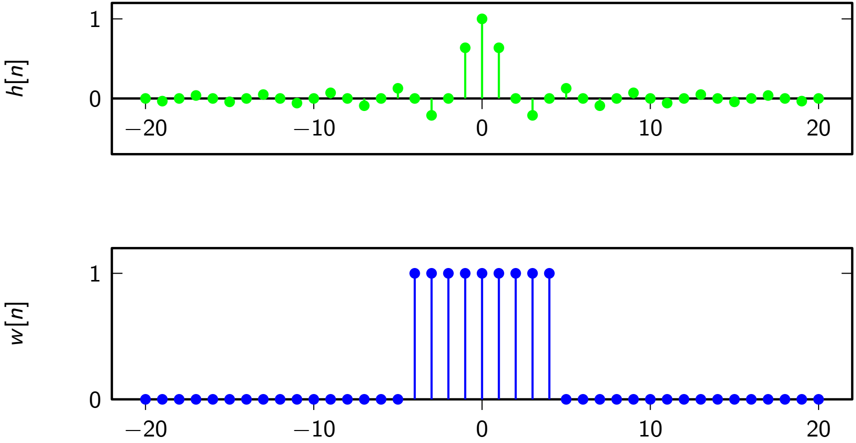
\end{aligned}
$$

Back to the Gibbs phenomenon

The maximum error around the cutoff frequency
is around 9% of the height of the jump
regardless of N

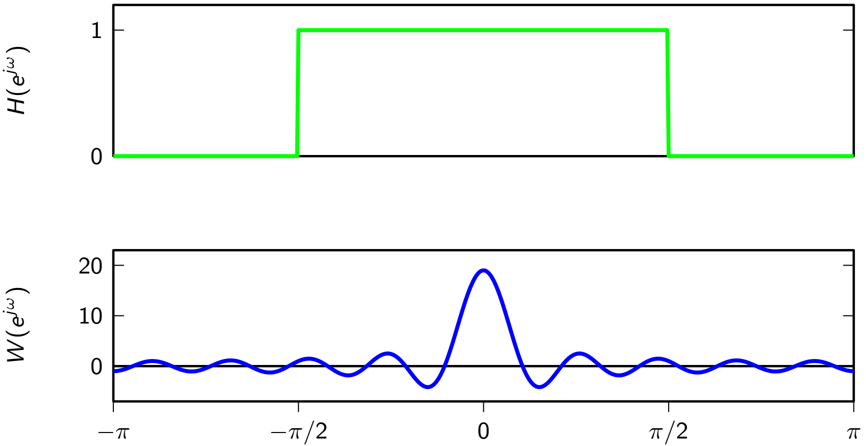
Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n]\, w[n]$$

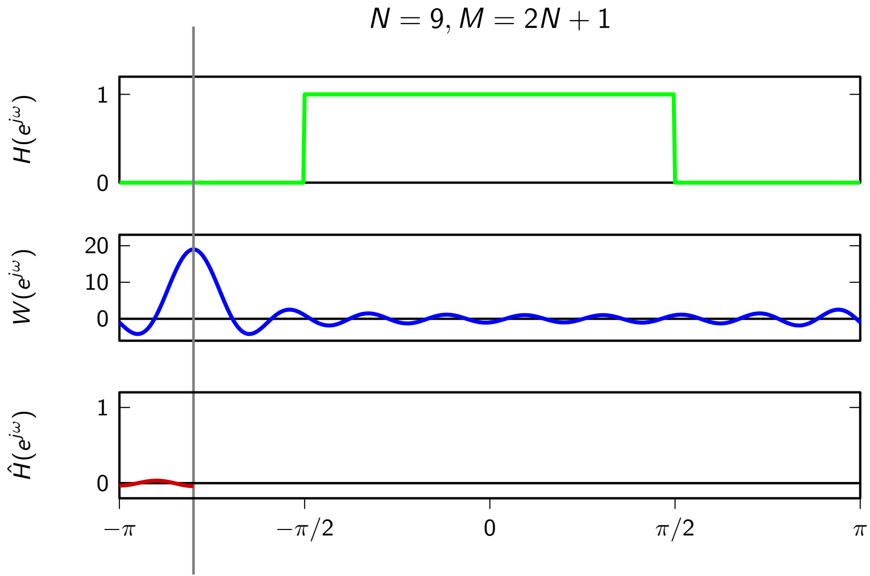


Understanding the Gibbs phenomenon

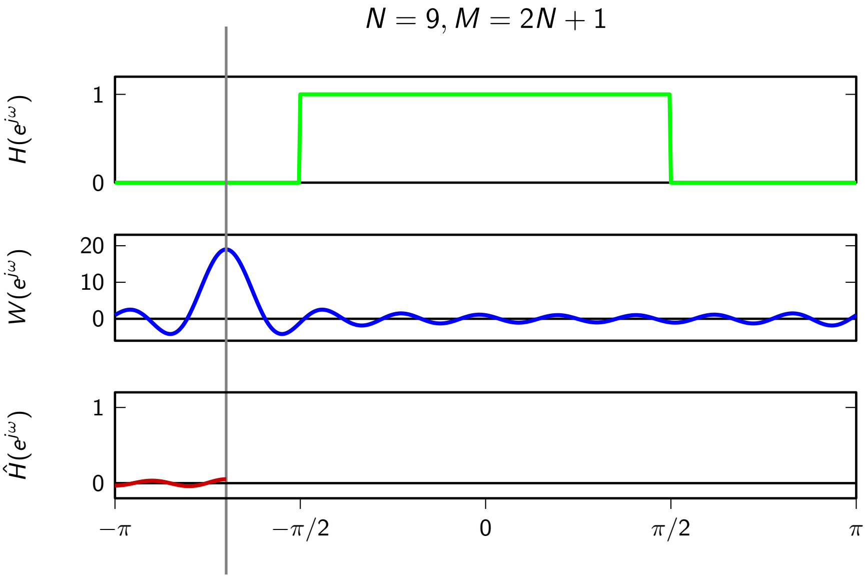
$$\hat{H}(e^{j\omega}) = (H * W)(e^{j\omega}), \qquad W(e^{j\omega}) = \sin(\omega M/2)/\sin(\omega/2)$$



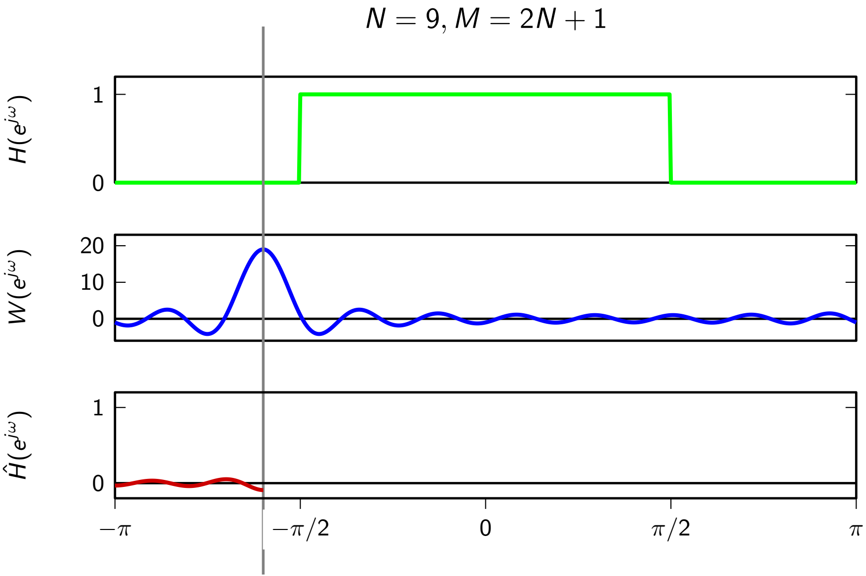
Implicit frequency-domain convolution



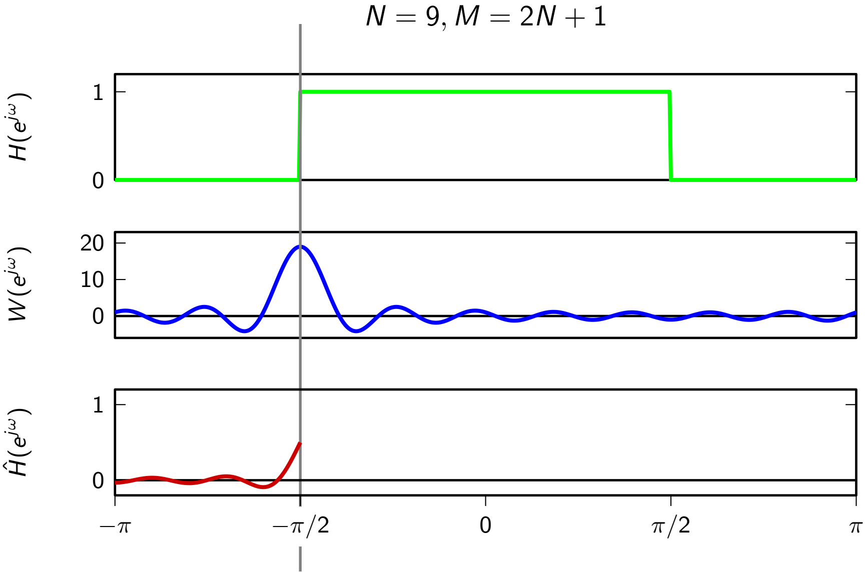
Implicit frequency-domain convolution



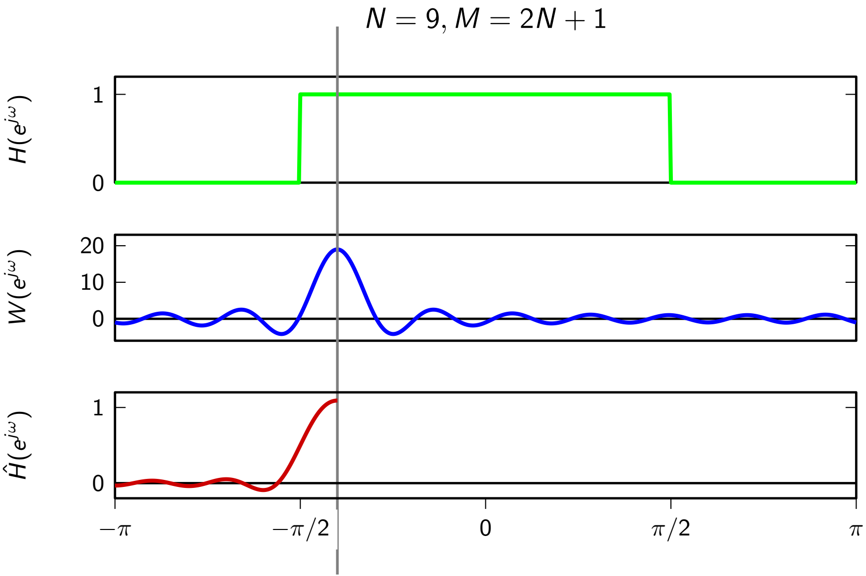
Implicit frequency-domain convolution



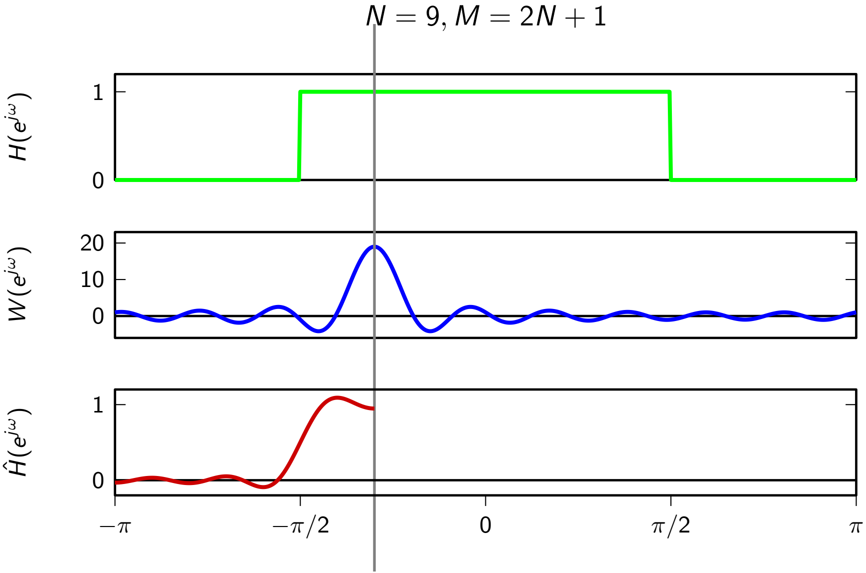
Implicit frequency-domain convolution



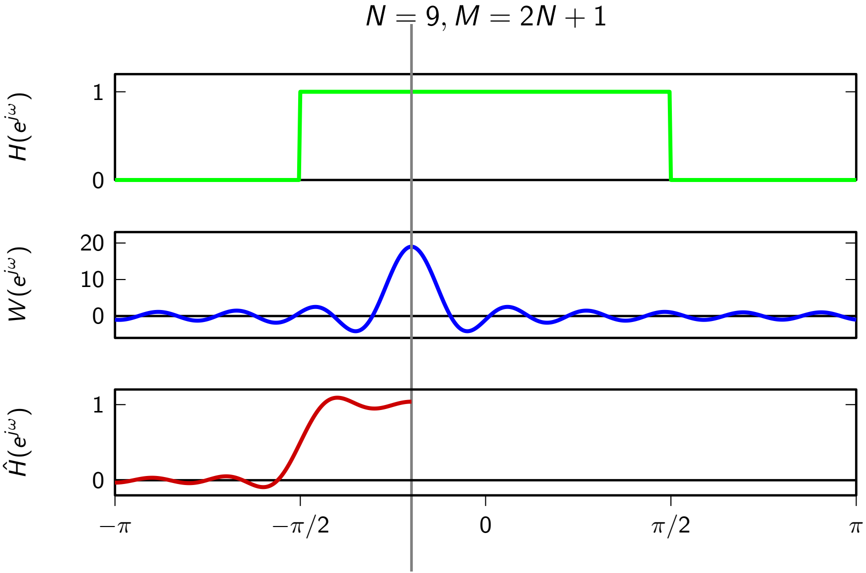
Implicit frequency-domain convolution



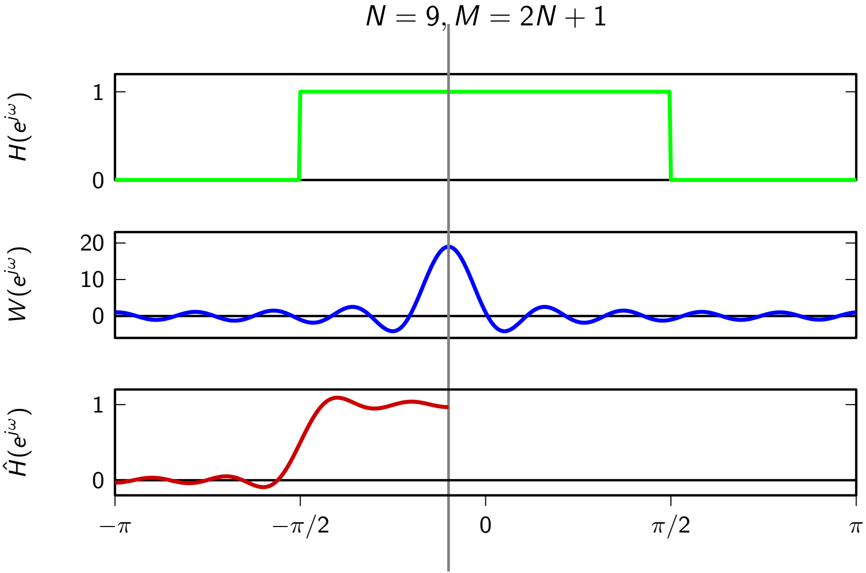
Implicit frequency-domain convolution



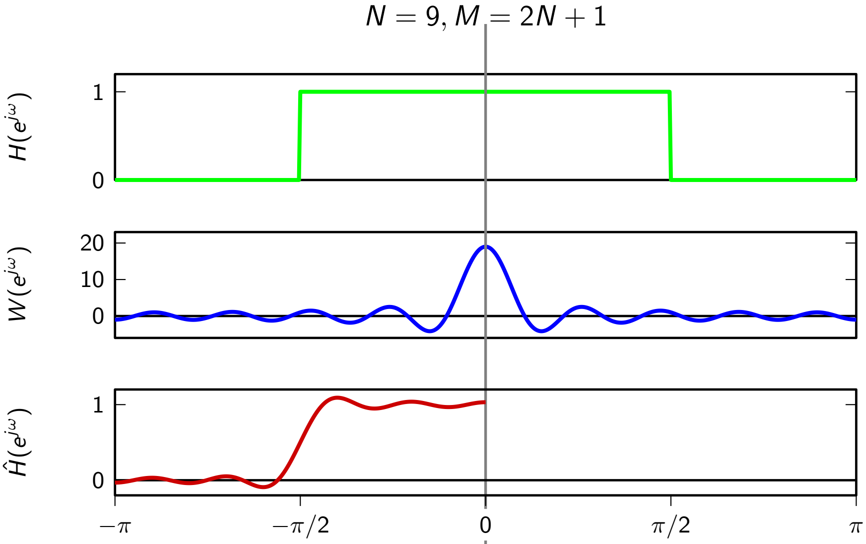
Implicit frequency-domain convolution



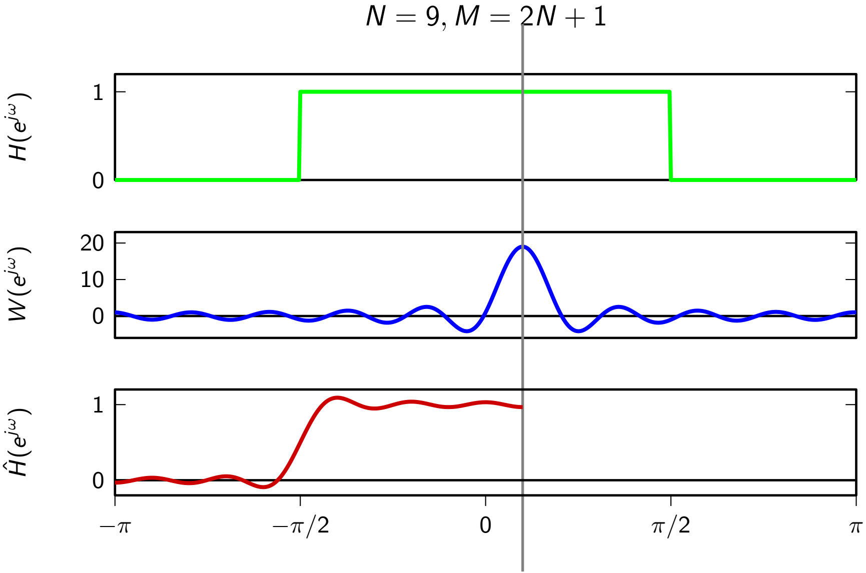
Implicit frequency-domain convolution



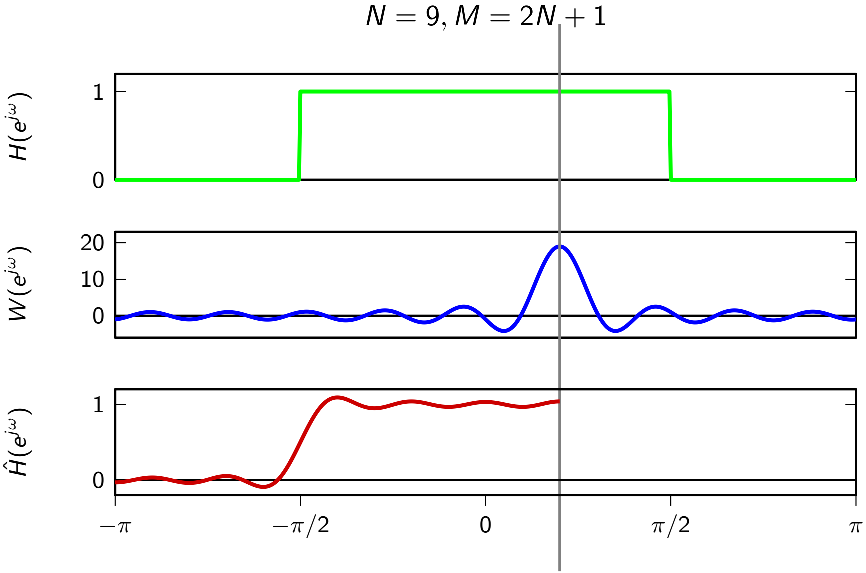
Implicit frequency-domain convolution



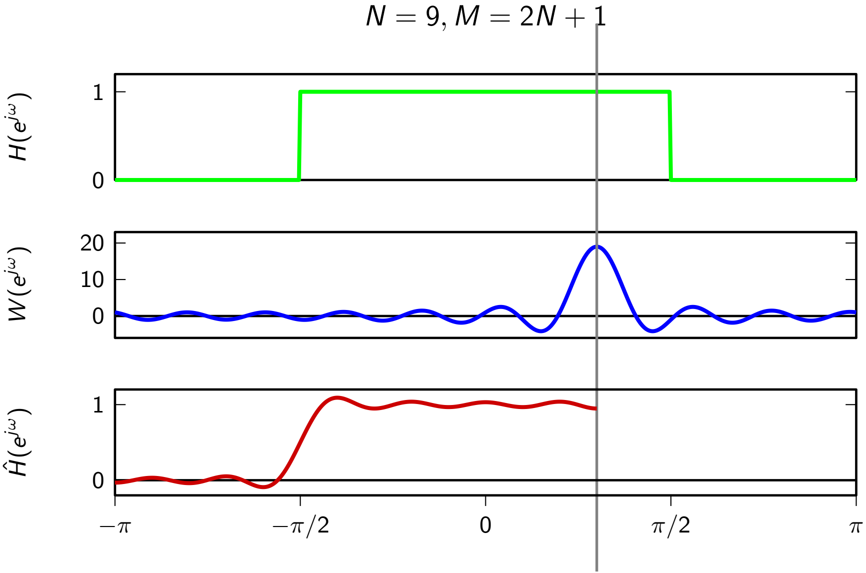
Implicit frequency-domain convolution



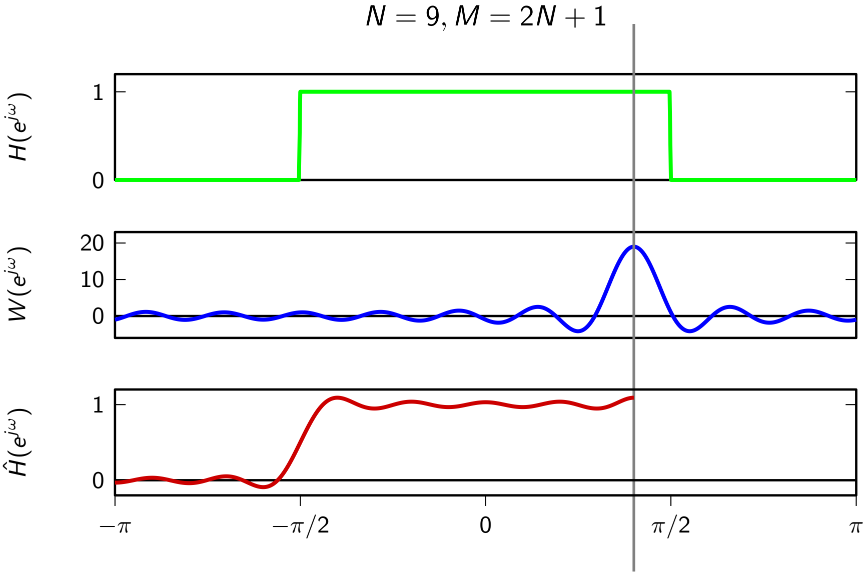
Implicit frequency-domain convolution



Implicit frequency-domain convolution



Implicit frequency-domain convolution



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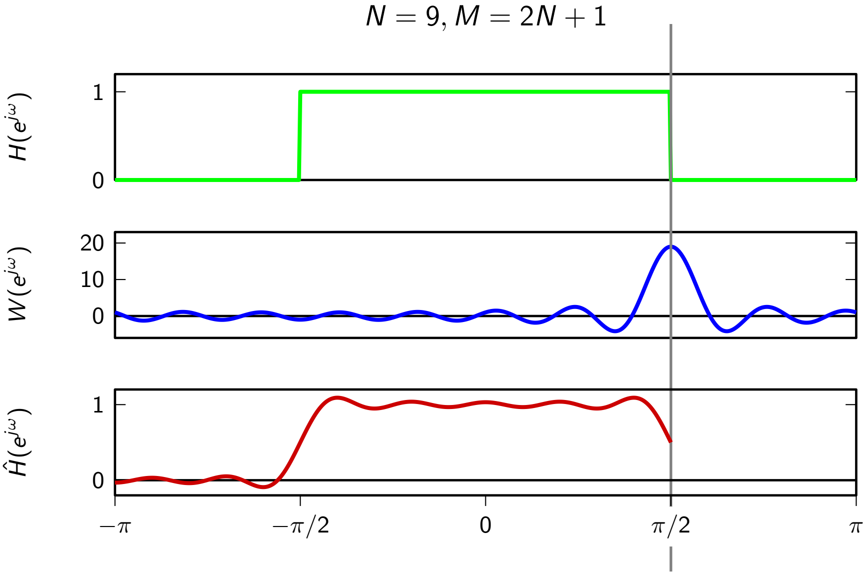

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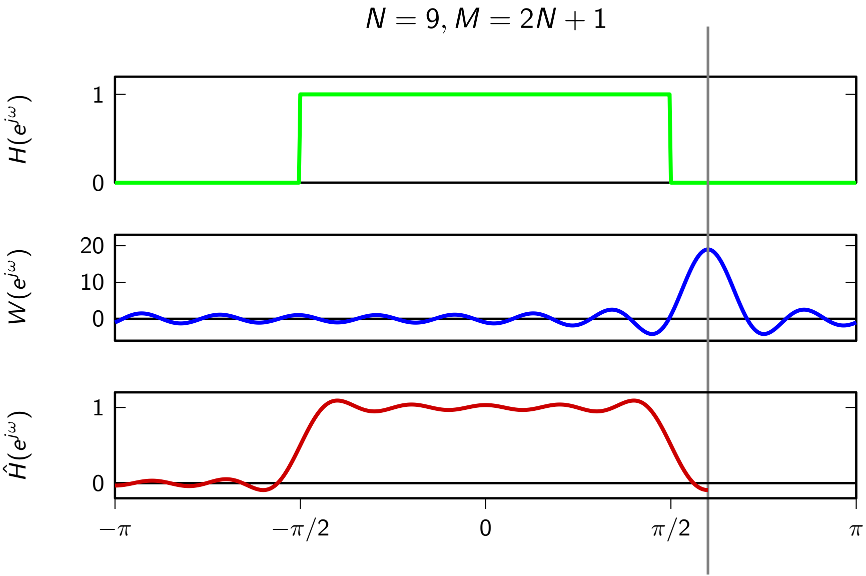

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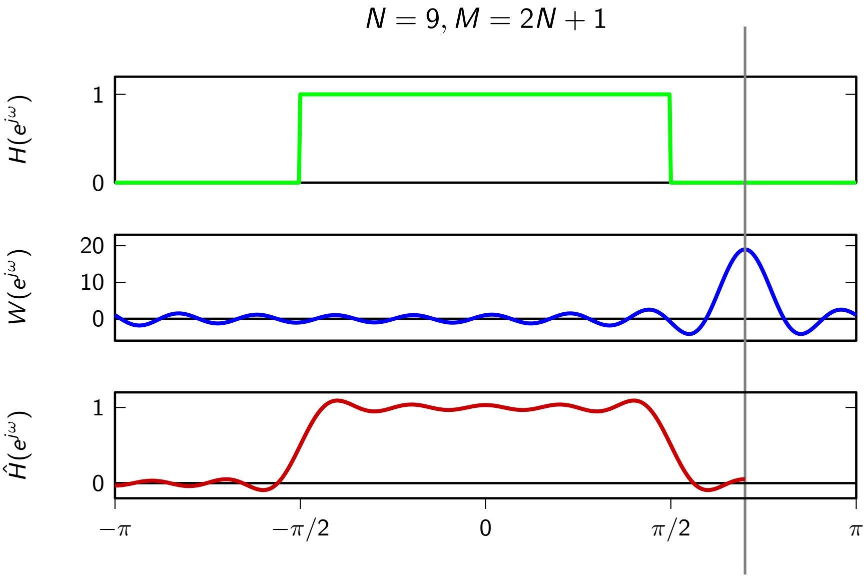

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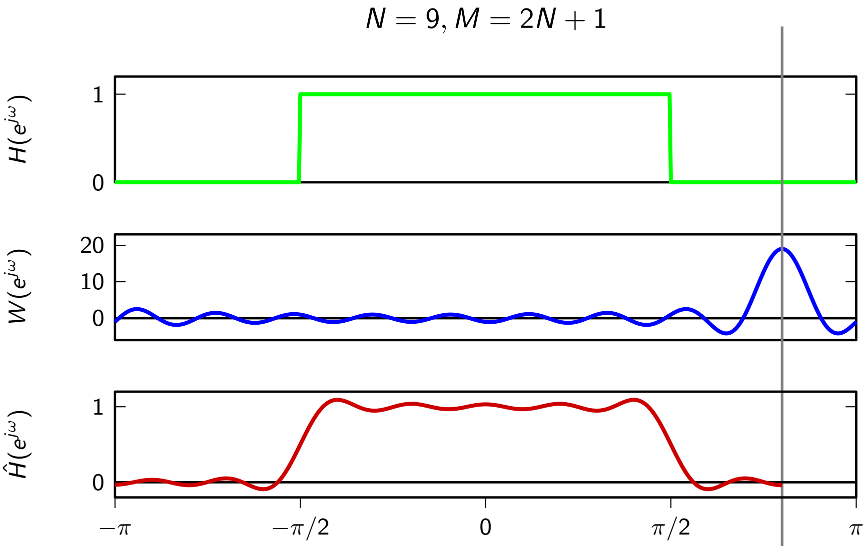
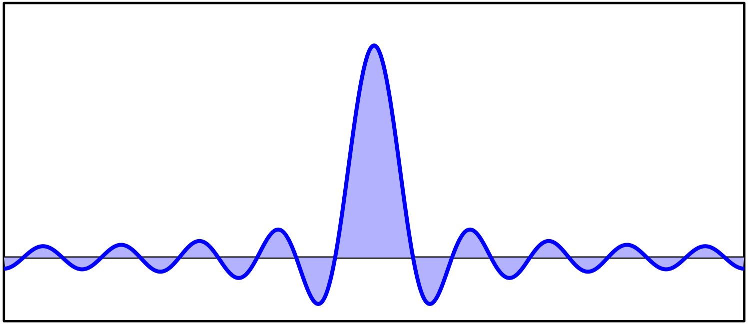


Quantifying the Gibbs overshoot – 1

Observation 1: integral of window's transform is independent of N:



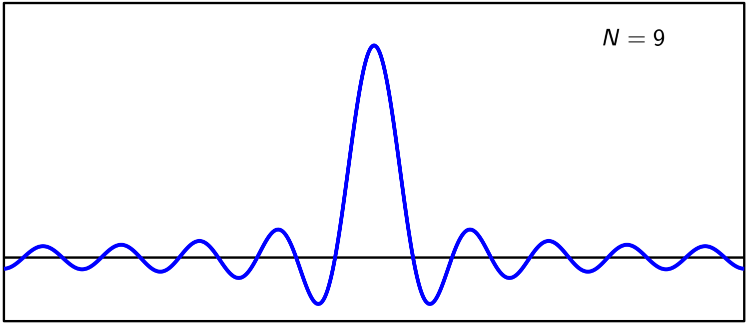
Quantifying the Gibbs overshoot – 1

Observation 1: integral of window's transform is independent of N:

$$\begin{aligned}
\int_{-\pi}^{\pi} W(e^{j\omega}) &= \int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{-j\omega n} d\omega \\
&= \sum_{n=-N}^{N} \int_{-\pi}^{\pi} e^{-j\omega n} d\omega \\
&= 2\pi
\end{aligned}$$

Quantifying the Gibbs overshoot – 2

For large N, the area is concentrated around the midpoint:



Quantifying the Gibbs overshoot – 2

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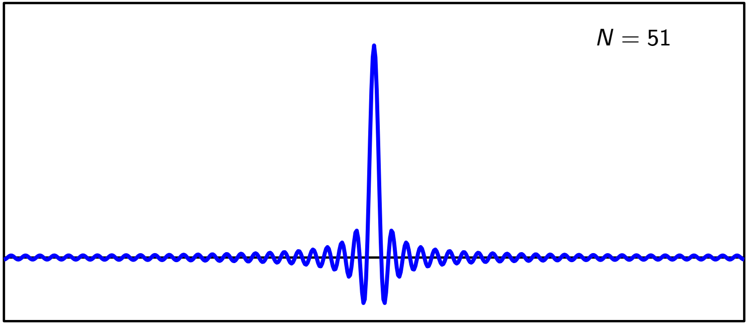

Quantifying the Gibbs overshoot – 2

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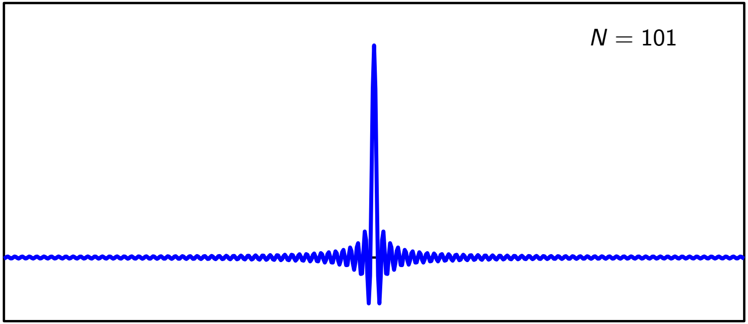
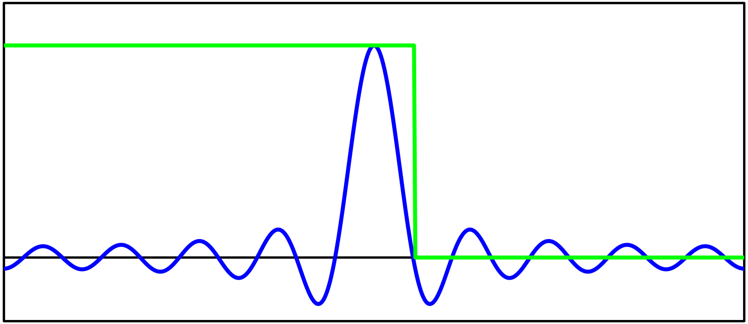


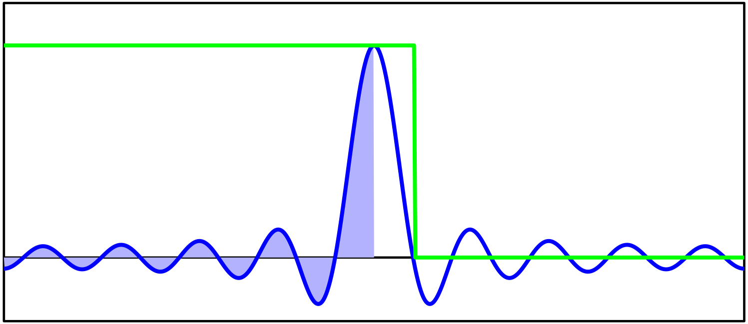
Quantifying the Gibbs overshoot – 3

maximum value of the convolution integral:



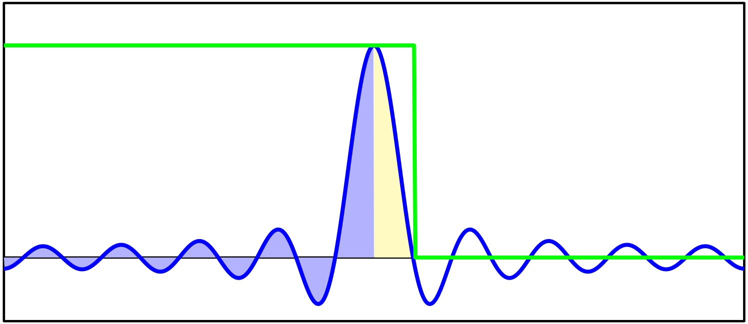
Quantifying the Gibbs overshoot – 3

maximum value of the convolution integral: A



Quantifying the Gibbs overshoot – 3

maximum value of the convolution integral: $A + B$



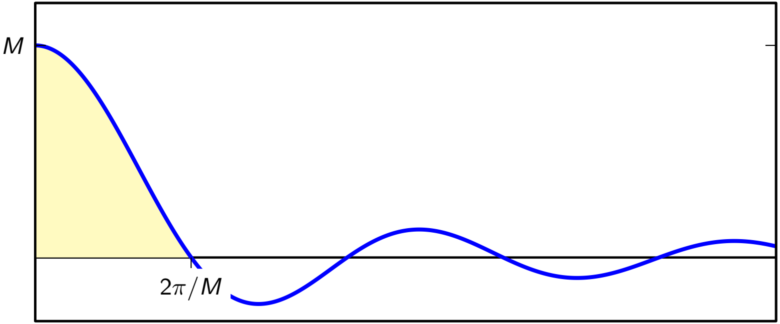
Quantifying the Gibbs overshoot – 4

For large N, A is basically half the total area:

$$A \approx \frac{1}{2} \int_{-\pi}^{\pi} W(e^{j\omega}) = \pi$$

Quantifying the Gibbs overshoot – 5

$$W(e^{j\omega}) = \sin(\omega M/2)/\sin(\omega/2), \qquad M = 2N+1$$

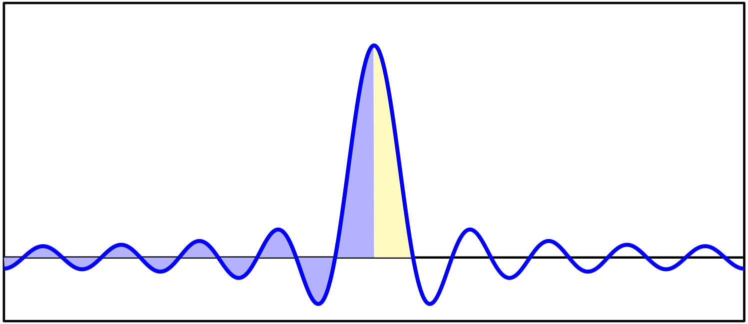


Quantifying the Gibbs overshoot – 5

$$
\begin{aligned}
\int_0^{2\pi/M} W(e^{j\omega}) &= \int_0^{2\pi/M} \frac{\sin(\omega M/2)}{\sin(\omega/2)} d\omega \\
&= \frac{2}{M} \int_0^{\pi} \frac{\sin t}{\sin(t/M)} dt \qquad (t = M\omega/2) \\
&\approx \frac{2}{M} \int_0^{\pi} \frac{\sin t}{(t/M)} dt \qquad (\sin t/M \approx t/M \text{ for } M \text{ large}) \\
&= 2 \int_0^{\pi} \frac{\sin t}{t} dt \\
&\approx 2\pi \times 0.589 \quad \text{independent of } M!
\end{aligned}
$$

Quantifying the Gibbs overshoot – 6

maximum value of the convolution: $(A + B)/2\pi$



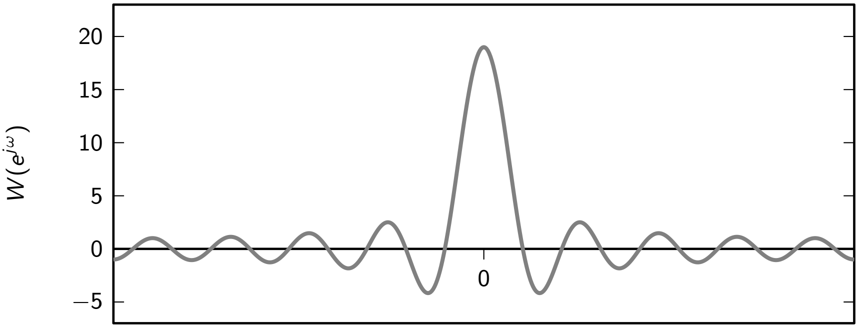
Quantifying the Gibbs overshoot – 6

$$A \approx \frac{1}{2}\int_{-\pi}^{\pi} W(e^{j\omega}) = \pi$$

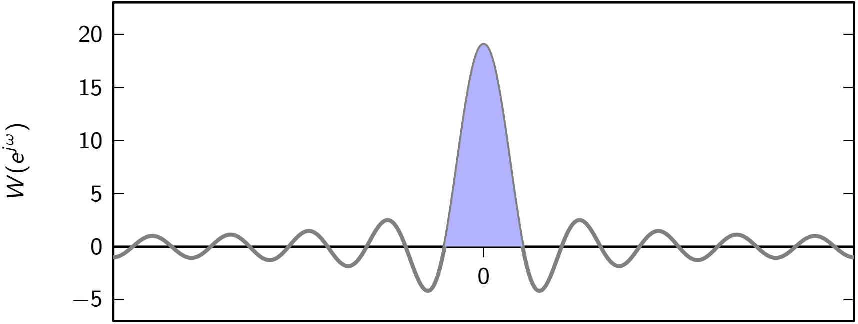
$$B \approx 2\pi \times 0.589$$

$$(A + B)/2\pi \approx 1.09$$

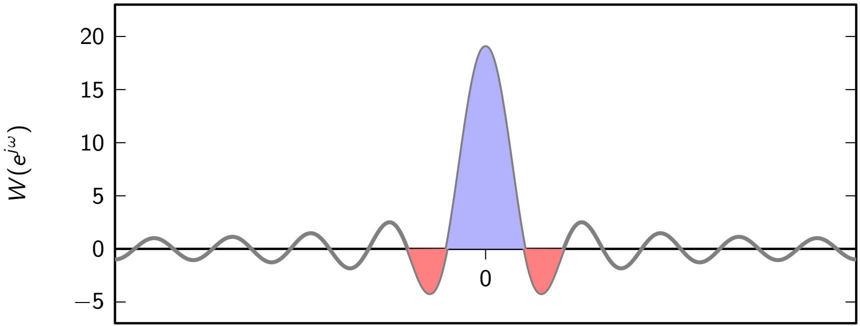
Mainlobe and sidelobes



Mainlobe and sidelobes



Mainlobe and sidelobes



What if we change the window?

We want:

- narrow mainlobe so that transition is sharp
- small sidelobe so Gibbs error is small
- short window so FIR is efficient

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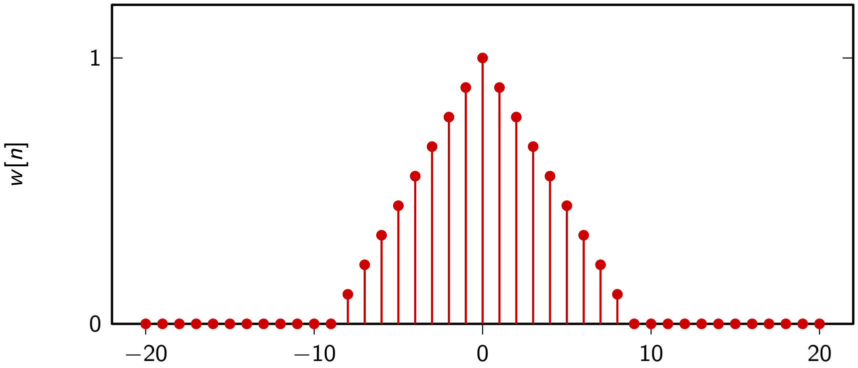
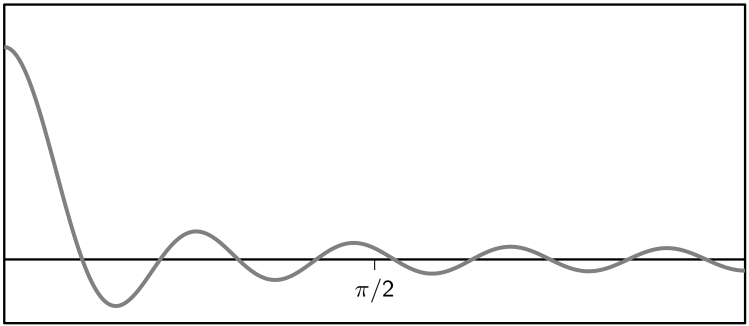
- short window so FIR is efficient

very conflicting requirements!

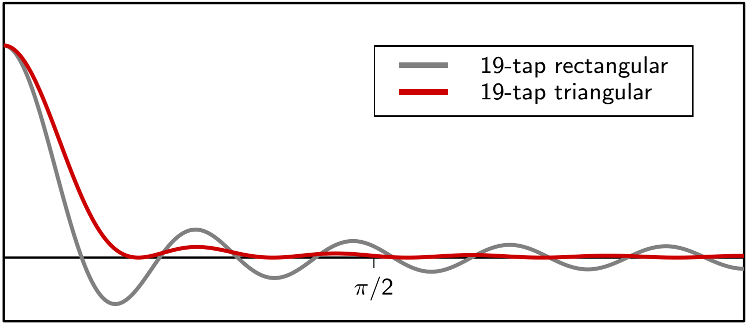
Triangular window



Rectangular vs Triangular Window



Rectangular vs Triangular Window



Window method: pros and cons

Pros:

- extremely simple
- minimizes MSE

Cons:

- can't control max error (Gibbs)
- must know the impulse response (not easy for arbitrary frequency responses)

Frequency sampling

Idea #2:

- draw desired frequency response $H(e^{j\omega})$
- take M samples $S[k] = H(e^{j\omega_k})$, $\omega_k = (2\pi/M)k$
- compute inverse DFT of the $S[k]$ values
- use $s[n]$ to build impulse response $\hat{h}[n]$

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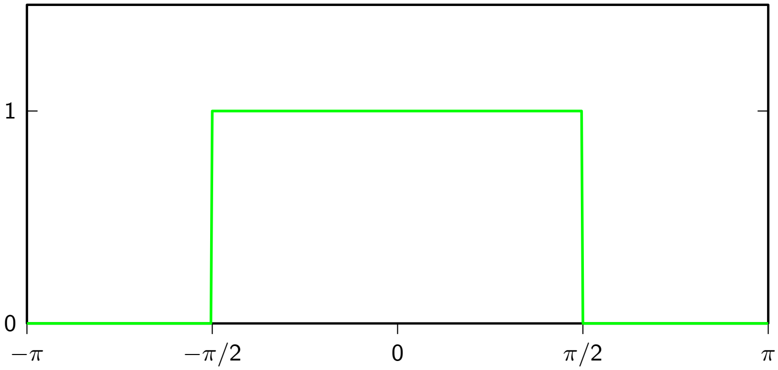
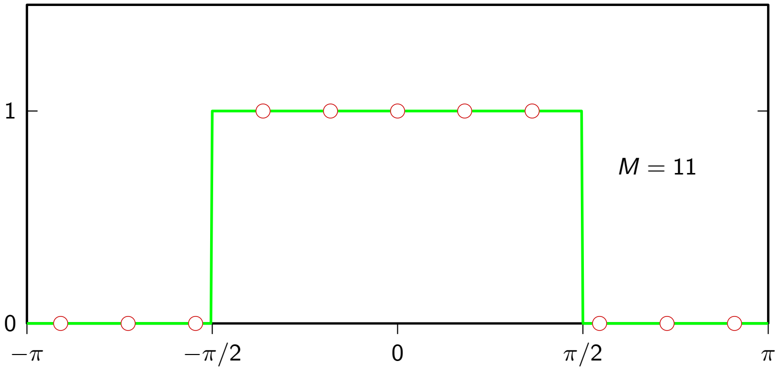
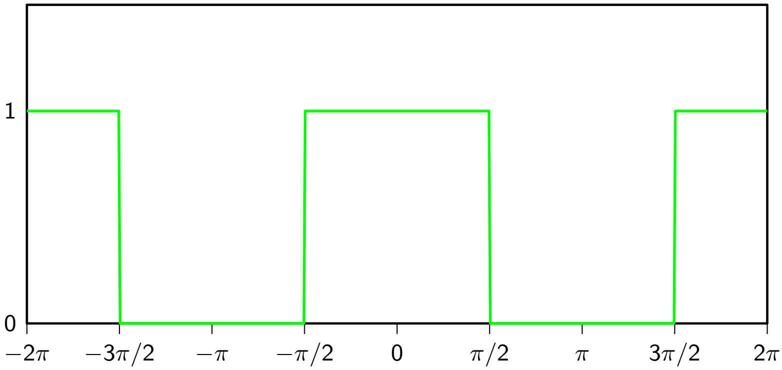
Frequency sampling: desired response



Frequency sampling: desired response



Frequency sampling: from DTFT to DFT



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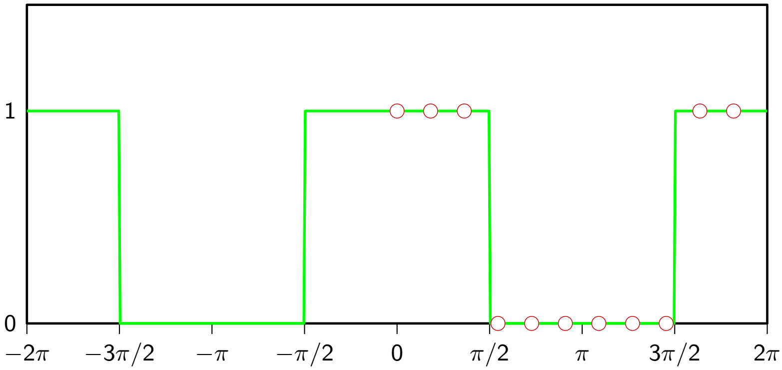
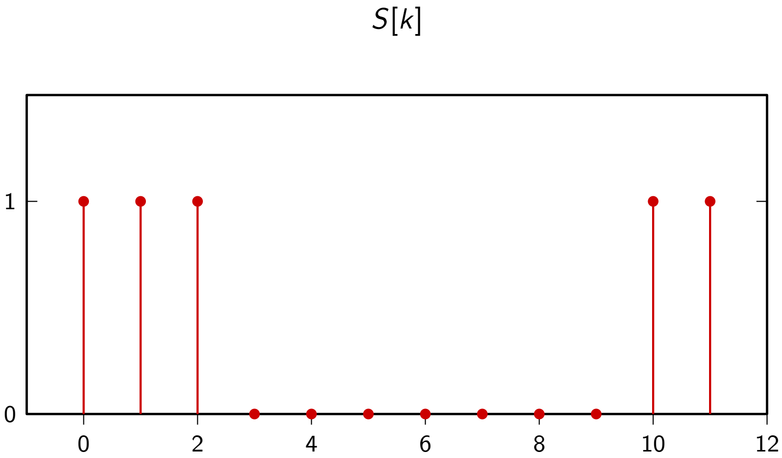
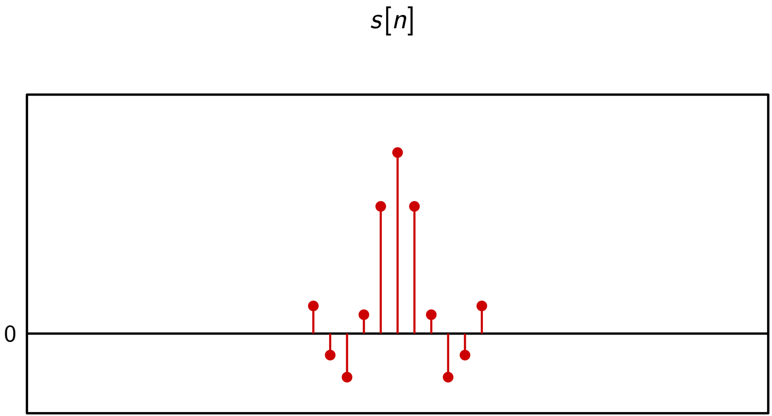


Frequency sampling: DFT samples



Frequency sampling: impulse response from IDFT



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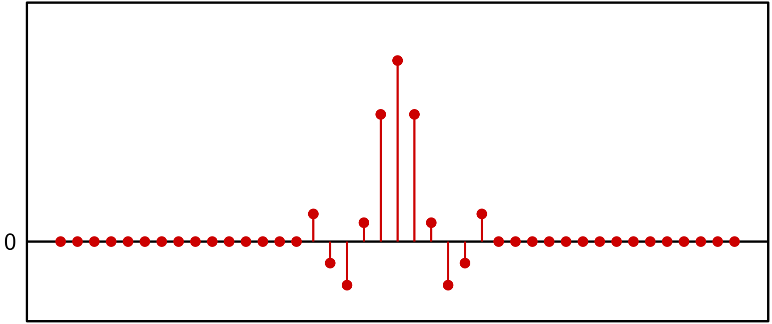
$\hat{h}[n]$



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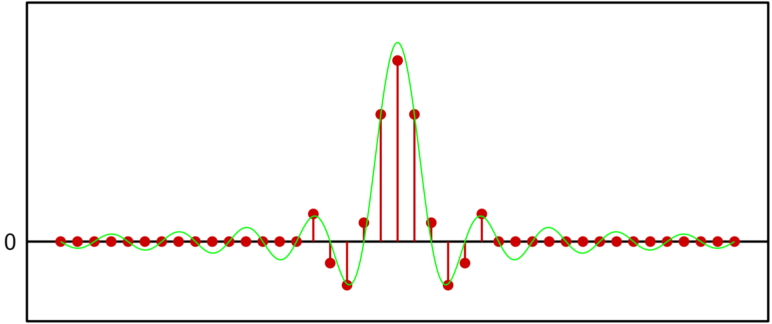

Frequency sampling: what happens in the time domain

$$\hat{h}[n] = \begin{cases} s[n] & \text{if } 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

$$s[n] = \text{IDFT}\{S[k]\}$$

$$S[k] = H(e^{j\frac{2\pi}{M}k})$$

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s[n] &= \frac{1}{M}\sum_{k=0}^{M-1} S[k]\, e^{j\frac{2\pi}{M}nk} \\
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a familiar result

$$\sum_{k=0}^{M-1} e^{-j\frac{2\pi}{M}(m-n)k} = \begin{cases} M & \text{if } m-n \text{ multiple of M} \\ 0 & \text{otherwise} \end{cases}$$

Frequency sampling: what happens in the time domain

$$s[n] = \sum_{m=-\infty}^{\infty} h[m]\,\delta[(m-n) \mod M]$$

$$= \sum_{m=-\infty}^{\infty} h[n+mM]$$

sampling in the frequency domain results in periodization in the time domain

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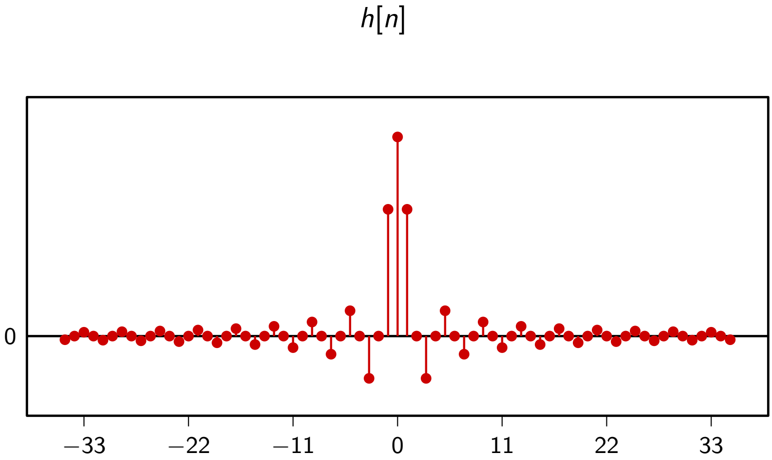
Frequency sampling: impulse response from IDFT



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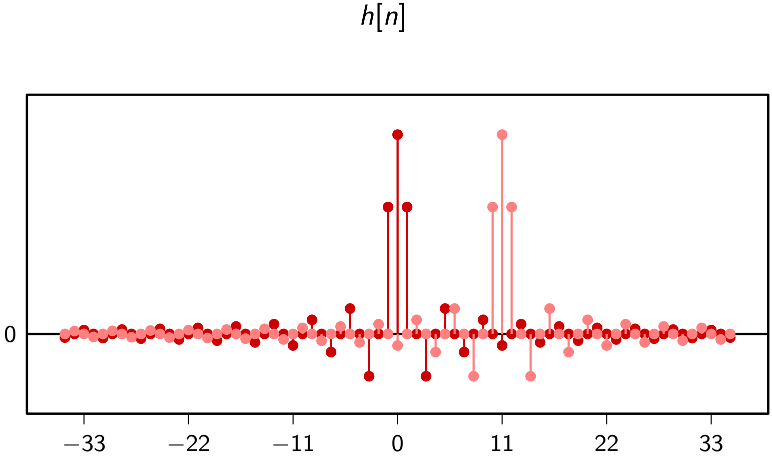

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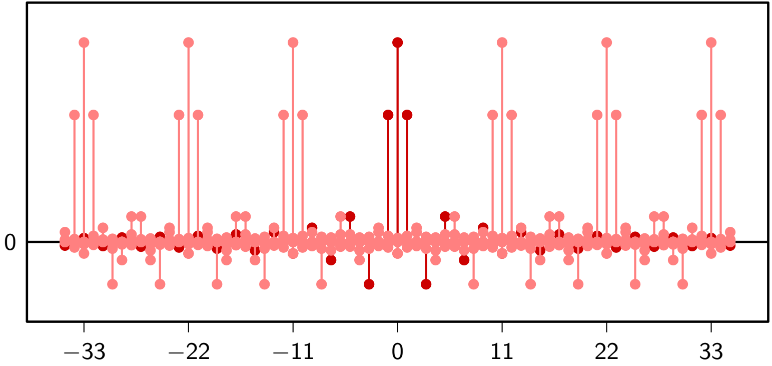

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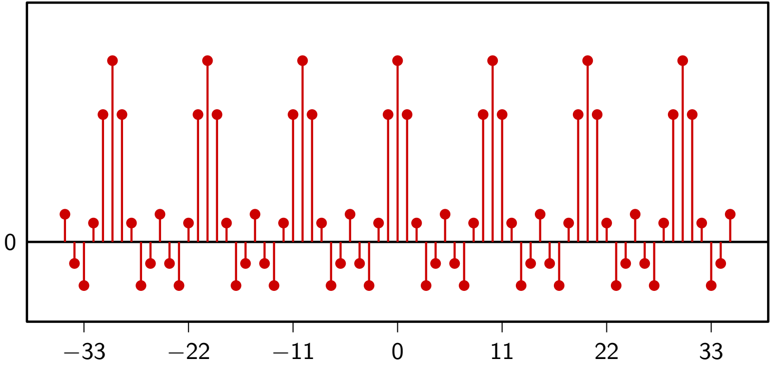

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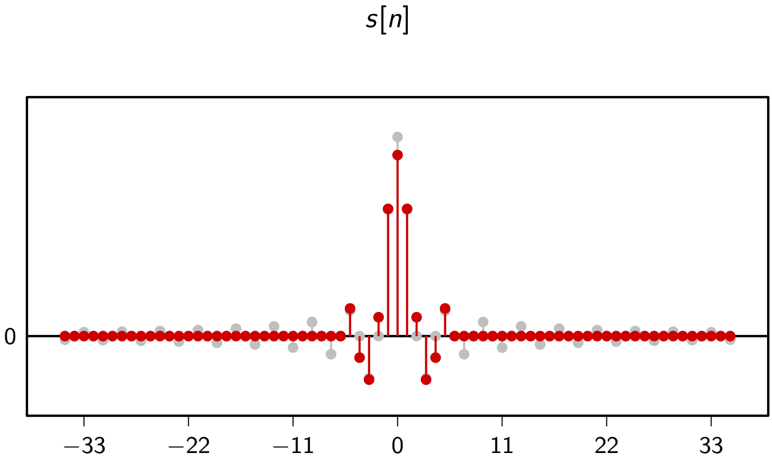

Frequency sampling: what happens in the frequency domain

what is the frequency response $\hat{H}(e^{j\omega})$?

- $s[n]$: length-M signal
- $\hat{h}[n]$: finite-support infinite sequence from $s[n]$
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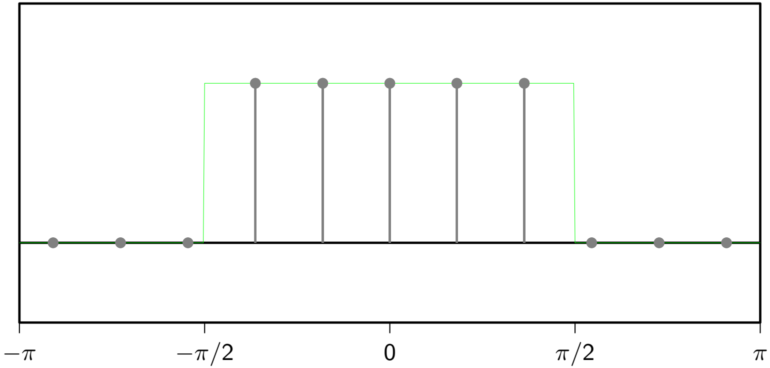
DTFT of finite-support signals

$$\hat{H}(e^{j\omega}) = \sum_{k=0}^{M-1} S[k]\Lambda(\omega - \frac{2\pi}{M}k)$$

with $\Lambda(\omega) = \frac{1}{M}\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\frac{\omega}{2}(M-1)}$: smooth interpolation of DFT values.

Frequency sampling: frequency response



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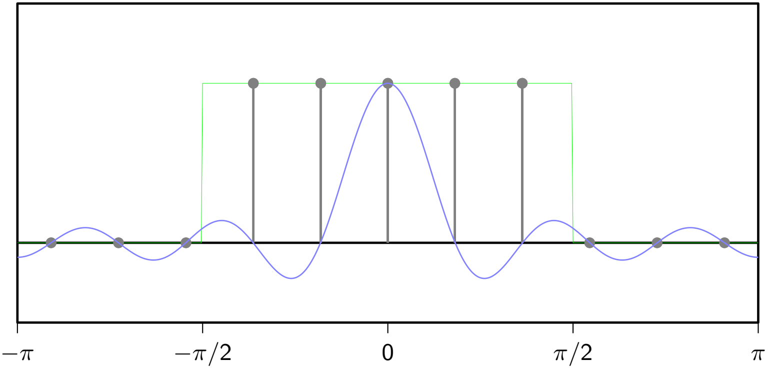

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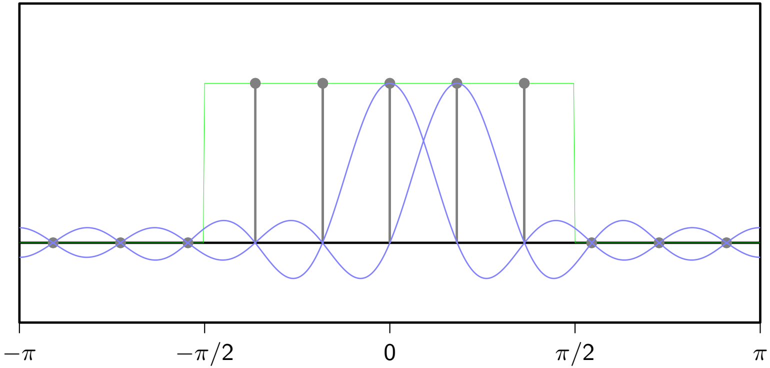

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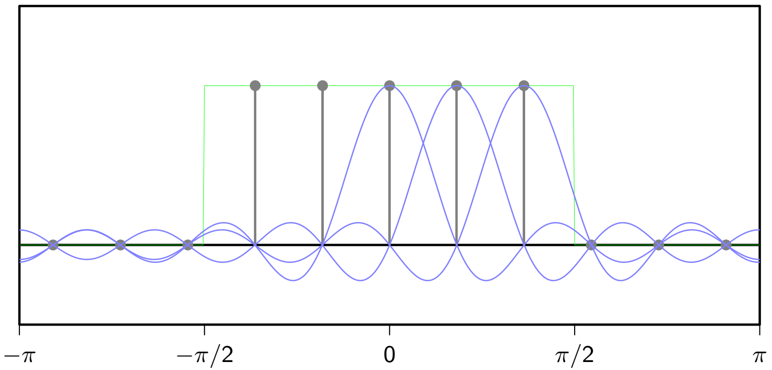

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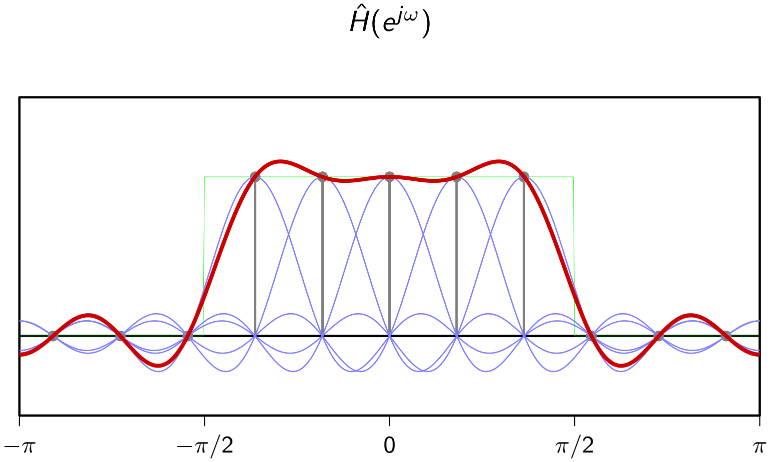

Frequency sampling: pros and cons

Pros:

- simple
- works with arbitrary frequency responses

Cons:

- can't control max error