Exercise Set 4

Saliba, March 13, 2019

Exercise 1. (**Markov Chain in a library**) In a library with n books, the ith book has probability p_i to be chosen at each request. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state of a Markov chain at any time to be the list of books we see as we examine the shelf from left to right. Since all the books are distinct, the state space E is the set of all permutations of the set $\{1, 2, \ldots, n\}$. Show that

$$\pi(i_1, \cdots i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots p_{i_{n-1}}}$$

is a stationary distribution.

Exercise 2. Let P be a transition matrix on a finite state space E.

(a) Prove the following linear algebra result: Given a matrix Q, Q and Q^t have the same eigenvalues.
Use this to prove that P has a stationary distribution π(i.e. a probability measure $\pi P = \pi$).

(b) Find an example, when E is an infinite state space, for which P doesn't have any stationary distribution.

Exercise 3. (**Gambler's ruin**) Assume that a gambler is making bets for 1 dollar on fair coin flips, and that she will abandon the game when her fortune falls to 0 or reaches n dollar. Let X_t be the Markov chain on $\{0, \ldots, n\}$ describing the gambler's fortune at time t, that is, $\mathbb{P}(X_{t+1} = k + 1 \mid X_t = k) = \mathbb{P}(X_{t+1} = k - 1 \mid X_t = k) = 1/2$, $k = 1, \cdots, n - 1$, and $\mathbb{P}(X_{t+1} = 0 \mid X_t = 0) = \mathbb{P}(X_{t+1} = n \mid X_t = n) = 1$. Let T be the time required to be absorbed at one of 0 or n. Assume that $X_0 = k$, where $0 \leq k \leq n$.

(i). Find the probability $\mathbb{P}_k(X_T = n)$ for the gambler to reach n dollars with initial capital k.

(ii). Compute $\mathbb{E}_k[T]$, the expected time to reach n or 0 starting from k.
Hint: Writing $f_k := \mathbb{E}_k[T]$ and $\Delta_k = f_k - f_{k-1}$, show that $\Delta_k = \Delta_{k+1} + 2$ and use this to compute f_k.

Exercise 4. A company issues n different types of coupons. A collector desires a complete set. We suppose each coupon he acquires is equally likely one of the n types. Let X_l denote the number of different types represented among the collector's first l coupons. Clearly $X_0 = 0$. Let T_k be the total number of coupons accumulated when the collection first contains k distinct coupons.

(i). Find the distribution of $T_k - T_{k-1}$ for $k \leq n$, that is, the time it takes to obtain the kth new coupon.

(ii). Find the expected value of the (random) time T representing the number of coupons needed to collect in order to have all coupon types.

Exercise 5. (a) A transition matrix P defined on a state space E and a distribution λ have the *detailed balance property* if

$$\lambda_j P_{ji} = \lambda_i P_{ij}, \ \forall i, j \in E.$$

Show that in this case, λ is a stationary distribution for P.

(b) Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is at state i if the first urn contains i black balls and $m - i$ white balls while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain.

(1) Compute its transition probability.

(2) Verify (using (a)) that the stationary distribution is given by

$$\pi(i) = \frac{\binom{b}{i}\binom{2m-b}{m-i}}{\binom{2m}{m}}.$$

(3) Can you give a simple intuitive explanation why the formula in (2) gives the right answer?