Exercise Set 10

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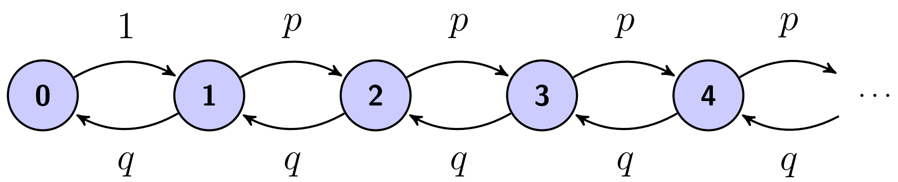
Exercise 1. Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with transition matrix P. For a fixed state k, let us denote $\bar{P}^{(k)}$ the matrix obtained from P by suppressing the line k and colon k. Then the states (X_n) are recurrent if and only if

$$\begin{cases} \bar{P}^{(k)}\boldsymbol{x} = \boldsymbol{x} \\ 0 \leq x_i \leq 1 \ \forall i \end{cases}$$

has only the null vector as a solution.
Use this result to discuss the nature of the states of the reflected random walk represented by the graph $(p+q=1)$:



Exercise 2. $M/M/1/\infty$ queue
Let us suppose that the arrivals at the EPFL service desk follow a Poisson process with parameter λ. When a client arrives, its service starts immediately if the desk is free. Otherwise, he waits for his turn. A queue with infinite length is allowed.
We assume that the service time of one customer follow an exponential distribution μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
Let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of customer in the system (while waiting or while being served) at time t.

1) Show that $(X(t))_{t\geq 0}$ is a Markov process (homogeneous).

2) Compute the generator of the process.

3) Find the transition matrix of the jump chain and deduce again the generator of the chain.

4) Determine the probability distribution of the time that the chain spends in each state.

5) Compute the asymptotic distribution of the number of customer in the queue.

6) Discuss the nature of the states with respect to the parameters μ and λ.

Exercise 3. $M/M/1/m$ queue
We consider again the queuing system of last exercise, except that the waiting room as a maximum capacity of $m-1$ customer. So that if the system starts with less than m clients, the number of clients in the system will never be larger than m, because a client who arrives while m clients are already in the system goes away and never comes back. Nevertheless it is possible that the initial state is larger than m.

1) Compute the generator, and the transition matrix of the jump Markov chain.

2) Determine the nature of the state for the associated Markov chain.

3) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 4. $M/M/\infty$ queue
Let us assume that the arrival times of clients in a system follow a Poisson process with parameter λ. The system is built from a countable collection of servers so that when a client arrives, his service starts immediately.
We assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of client in the system at time t.

1) Compute the generator and the transition matrix of the jump Markov chain.

2) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 5. $M/M/m/\infty$ queue
Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from m servers and a waiting room of infinite capacity (when a client arrives, his service starts immediately if one of the m desk is free, otherwise he goes in the waiting room).
Let us assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the jump Markov chain.

2) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 6. Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from one server and a waiting room of infinite capacity (when a client arrives, his service starts immediately if the desk is free, otherwise he goes in the waiting room).
Let us assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
Clients are busy: a client that cannot be served immediately waits a random time exponentially distributed with parameter γ, then if his service has not already started he goes away and never comes back (there s however no restriction on the duration of service).
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the jump Markov chain.

2) Gives the stationary distribution for every state $i > 0$ of this process in term of the stationary probability that the system is empty.

Exercise 7. The following theorem characterizes the limiting distribution $\boldsymbol{\pi}$ of a positive recurrent Markov process as a solution of some linear equations systems.

Theorem. The following assertions are equivalent:

(i) The states of a conservative irreducible Markov process are positive recurrent.

(ii) There exists $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} P(s) = \boldsymbol{\pi}$ for all s,

(iii) There exists $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} Q = \mathbf{0}$,

In (ii) and (iii), $\boldsymbol{\pi}$ is such that $\pi_j = \lim_{t\to\infty} \mathrm{P}(X(t) = j \mid X(0) = j)$, with $\boldsymbol{\pi} \geq \mathbf{0}$, and $\boldsymbol{\pi}\mathbf{1} = 1$.

The goal of this exercise is to prove $(ii) \Rightarrow (iii)$. For this,

a) Using the relation between Q and $\widehat{P}$, show first that $\widehat{\boldsymbol{\pi}}$ is the stationary distribution of the jump chain if and only if $\widehat{\boldsymbol{\pi}}\Lambda^{-1}Q = \mathbf{0}$, where Λ =diag(δ_i).

b) It remains to show that $\widehat{\boldsymbol{\pi}}\Lambda^{-1} = \boldsymbol{\pi}$ up to a multiplicative constant. To prove this, show first the following equality by conditioning on the last change of state before time t:

$$P_{ij}(t) = e^{-\delta_i t}\delta_{ij} + \int_0^t \sum_k P_{ik}(t-v)\delta_k \widehat{P}_{kj} e^{-\delta_j v} dv,$$

or in a matrix form,

$$P(t) = e^{-\Lambda t} + \int_0^t P(t-v)\Lambda \widehat{P} e^{-\Lambda v} dv.$$

Use this equality, and the fact that $\boldsymbol{\pi} P(t) = \boldsymbol{\pi}$ for all t and $\lim_{t\to\infty} e^{-\Lambda t} = 0$, to show that $\widehat{\boldsymbol{\pi}}\Lambda^{-1} = \boldsymbol{\pi}$ up to a multiplicative constant.

Exercise 8 (Birth and Death process)**.** We let $X(t)$ represents a population size at time t, where the death rate of one individual is μ (i.e. the individuals have an exponential lifetime of parameter μ), and the reproduction rate of an individual is λ (i.e. individuals give birth to one new individual at a time, according to the arrival moments of a Poisson process of parameter λ). We suppose that the individuals behave independently from each other. Let V_t be the remaining time, starting from t, until the next birth of an individual, and W_t the remaining time, starting from t, until the next death of an individual. We add the following hypothesis: If $X(t) = i$, there exist positive numbers p_i and q_i such that:

$$P(V_t > u; W_t > u | X(s), 0 \leq s \leq t) = e^{-q_i u},$$

$$P(V_t \leq W_t | X(s), 0 \leq s \leq t) = p_i.$$

These hypothesis imply that $X := \{X(t) : t \in \mathbb{R}\}$ is a Markov process.

(i). Find the constants p_i and q_i, and compute the generator and the transition matrix of the jump chain corresponding to X.

(ii). Write in details the Kolmogorov backward and forward equations, supposing that $X(0) = 1$.

(iii). Write the resulting differential equations that you get (ordinary for the backward and with partial derivatives for the forward) for the generating function of the population size at time t, defined as

$$F(t,s) = \sum_{k\geq 0} \mathbb{P}(X(t) = k | X(0) = 1) s^k.$$

Hint: for the backward equation, notice that $F^{(2)}(t,s) := \sum_{k\geq 0} \mathbb{P}(X(t) = k | X(0) = 2) s^k = F(t,s)^2$ by independence.

(iv). Verify that these equations have the following solution

$$F(t,s) = \begin{cases} 1 + \dfrac{(\lambda-\mu)(s-1)}{(\lambda s - \mu)\, e^{(\mu-\lambda)t} - \lambda(s-1)}, & \text{if } \lambda \neq \mu \\ 1 + \dfrac{(s-1)}{1 - \lambda t(s-1)}, & \text{if } \lambda = \mu. \end{cases}$$