Information Theory and Signal Processing   Ecole Polytechnique Fédérale, Lausanne: Fall 2019
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Problem Set 5 — *Due Friday, November 15, before class starts*

For the Exercise Sessions on Nov 1 and 8

Last name	First name	SCIPER Nr	Points

Problem 1: KL Divergence (not graded)

Compute the KL Divergence of two scalar Gaussians $p(x) = \mathcal{N}(\mu_1, \sigma_1^2)$ and $q(x) = \mathcal{N}(\mu_2, \sigma_2^2)$.

Problem 2: Hoeffding's Lemma (not graded)

Prove Lemma 7.4 in the lecture notes. In other words, prove that if X is a zero-mean random variable taking values in $[a, b]$ then

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2}{2}[(a-b)^2/4]}.$$

Expressed differently, X is $[(a-b)^2/4]$-subgaussian.

Problem 3: Epsilon-Greedy Algorithm

Recall our original *explore-then-exploit* strategy. We had a fixed time horizon n. For some m, a function of n and the gaps $\{\Delta_k\}$, we explore each of the K arms m times initially. Then we pick the best arm according to their empirical gains and play this arm until we reach round n. We have seen that this strategy achieves an asymptotic regret of order $\ln(n)$ if the environment is fixed and we think of n tending to infinity but a worst-case regret of order $\sqrt{n}$ if we use the gaps when determining m and of order $n^{\frac{2}{3}}$ if we do not use the gaps in order to determine m.

Here is a slightly different algorithm. Let $\epsilon_t = t^{-\frac{1}{3}}$. For each round $t = 1, \cdots,$, toss a coin with success probability ϵ_t. If success, then explore arms uniformly at random. If not success, then pick in this round the arm that currently has the highest empirical average.

Show that for this algorithm the expected regret at *any* time t is upper bounded by $t^{\frac{2}{3}}$ times terms in t and K of lower order. This is a similar to the worst-case of the explore-then-exploit strategy but here we do not need to know the horizon a priori. Assume that the rewards are in $[0, 1]$.

Problem 4: Upper Confidence Bound Algorithm

In the course we analyzed the Upper Confidence Bound algorithm. As was suggested in the course, we should get something similar if instead we use the Lower Confidence Bound algorithm. It is formally defined as follows.

$$A_t = \begin{cases} t, & t \leq K, \\ \arg\max_k \hat{\mu}_k(t-1) - \sqrt{\frac{2\ln f(t)}{T_k(t-1)}}, & t > K. \end{cases}$$

Analyze the performance of this algorithm in the same way as we did this in the course for the UCB algorithm.

Hint: Is this algorithm well designed?

Problem 5: Thompson Sampling with Bernoulli Losses

This problem deals with a Bayesian approach to multi-arm bandits. Although we will not pursue this facet in the current problem, the Bayesian approach is useful since within this framework it is relatively easy to incorporate prior information into the algorithm.

Assume that we have K bandits, and that bandit k outputs a $\{0,1\}$-valued Bernoulli random variable with parameter $\theta_k \in [0,1]$. Let π be the uniform prior on $[0,1]^K$, i.e., the uniform prior on the set of all parameters $\theta = (\theta_1, \cdots, \theta_K)$. Let

$$T_k^1(t) = |\{\tau \leq t : A_\tau = k; Y_\tau = 1\}|,$$
$$T_k^0(t) = |\{\tau \leq t : A_\tau = k; Y_\tau = 0\}|.$$

In words, $T_k^1(t)$ is the number of times up to and including time t that we have chosen action k and the output of arm k was 1 and similarly $T_k^0(t)$ is the number of times up to and including time t that we have choses action k and the output of the arm k was 0.

The goal is to find the arm with the highest parameter, i.e., the goal is to determine

$$k^* = \text{argmax}_k \theta_k.$$

In the Bayesian approach we proceed as follows. At time time t:

1. Compute for each arm k the distribution $p(\theta_k(t)|T_k^1(t-1), T_k^0(t-1))$.
2. Generate samples of these parameters according to their distributions.
3. Pick the arm j with the largest sample.
4. Observe the output of the j-th arm, call it $Y_j(t)$, and update the counters T_j^1 and T_j^0 accordingly.

Show that this algorithm "works" in the sense that eventually it will pick the best arm. More precisely, show the following two claims.

1. Show that $p(\theta_k(t)|T_k^1(t-1), T_k^0(t-1))$ is a Beta distributed and determine α and β.
2. Show that as t tends to infinity the probability that we choose the correct arm tends to 1. [HINT: To simplify your life, you can assume that for every arm k, $T_k^1(t-1) + T_k^0(t-1) \overset{t\to\infty}{\to} \infty$.]

NOTE: Recall that the density of the Beta distribution on $[0,1]$ with parameters α and β is equal to

$$f(x;\alpha,\beta) = \text{constant } x^{\alpha-1}(1-x)^{\beta-1}.$$

Further, the expected value of $f(x;\alpha,\beta)$ is $\frac{\alpha}{\alpha+\beta}$ and its variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.