Exercise Set 3

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For a Markov chain $(X_n)_{n\geq 0}$ on the countable state space E with transition matrix P, a stationary distribution $\pi = (\pi_i)_{i\in E}$ is a vector such that

- $\pi_i \geq 0$ for all $i \in E$;
- $\sum_{i\in E} \pi_i = 1$;
- $\pi P = \pi$, that is, $\sum_{i\in E} \pi_i P_{ij} = \pi_j$, for all $j \in E$.

Exercise 1. Let $X = (X_n)_{n\geq 0}$ be a Markov chain given by the following transition matrix:

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

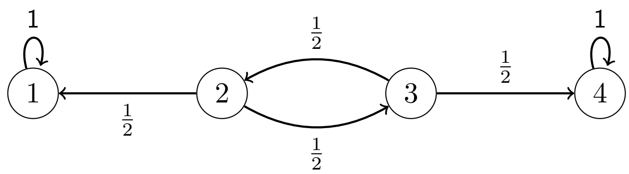
Find a stationary distribution π for X, i.e. such that $\pi P = P$.

Exercise 2. Let $(X_i)_{i\geq 0}$ be a Bernoulli process, which means that the X_i's are *iid* with a Bernoulli law of parameter p.

(a) Consider the process $(N_n)_{n\geq 0}$ of the number of successes: N_n is the number of successes of the Bernoulli process until time n included.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

(b) Consider the process $(T_n)_{n\geq 0}$ of the moment of successes: T_n is the time when the nth success happens in the Bernoulli process.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

Exercise 3. Let $(X_n)_{n\geq 0}$ be a Markov chain determined by the following diagram:



Compute for all $i = 1, 2, 3, 4$ the absorption probability

$$h_i = \mathbb{P}_i\{\exists n \geq 0 : X_n = 4\},$$

i.e. the probability that the chain is absorbed in state 4 knowing that the chain starts at $X_0 = i$. Then compute the mean absorption time knowing that the chain starts in state i

$$k_i = \mathbb{E}_i[\inf(n \geq 0 : X_n \in \{1, 4\})].$$

Exercise 4 (Random walk). Let $(X_n)_{n\geq 0}$ be a one-dimensional random walk on the state space $\mathbb{Z}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p, & y = x + 1, \\ q, & y = x - 1, \end{cases}$$

(1) Prove that the random walk is recurrent if an only if $p = q$.

Hint: Note that $p_{00}^{2n+1} = 0$ for all $n \in \mathbb{N}$, and find the probability p_{00}^{2n}. You can then use Stirling's approximation to $n!$

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

(2) In the transient case $p \neq q$, find the limit $\lim_{n\to\infty} X_n$.

Exercise 5 (**Birth and Death chain**)**.** Let us consider a Markov chain $(X_n)_{n\geq 0}$ on the state space $\mathbb{N}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p & \text{if } x > 0,\ y = x+1, \\ q & \text{if } x > 0,\ y = x-1, \\ 1 & \text{if } x = 0,\ y = 1. \end{cases}$$

Prove that:

(1) If $p \leq q$ the chain is recurrent.

Hint: study the probability $u(k) = \mathbb{P}_k\left(X_n \neq 0, \forall n \in \mathbb{N}\right)$ by showing that

$$u(k+1) - u(k) = \frac{q}{p}\left(u(k) - u(k-1)\right).$$

(2) If $q < p$ the chain is transient.

Hint: consider writing the chain as $X_n = \sum_{i=1}^n Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0)$ where

$$Y_i \overset{\text{iid}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

and compare with the non-symmetric random walk $\sum_{i=1}^n Y_i$.

Exercise 6. Let X_0 be a random variable having values in a countable set I. Let $Y_1, Y_2, \ldots$ be a sequence of independent variables, uniformly distributed on $[0,1]$. Considering any function

$$G : I \times [0,1] \to I,$$

we define inductively

$$X_{n+1} = G(X_n, Y_{n+1}).$$

(1) Show that $(X_n)_{n\geqslant 0}$ is a Markov chain and write its transition matrix P as a function of G.

(2) Can all Markov chains be defined this way?

(3) How do you simulate a Markov chain on a computer?