Exercise Set 5

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Exercise 1. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

$0 < a, b < 1$. Use the Markov property to show that

$$\mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\{\mathbb{P}(X_n = 1) - \frac{b}{a+b}\},$$

and conclude that

$$\mathbb{P}(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n\{\mathbb{P}(X_0 = 1) - \frac{b}{a+b}\}.$$

Further show that $\mathbb{P}(X_n = 1)$ converges exponentially fast to its limit distribution $b/(a+b)$.

Exercise 2. (**Reversible Processes**)

a) Let P be an irreducible matrix with stationary distribution π. We assume that $(X_n)_{0\le n\le N}$ is $Markov(\pi, P)$. The process $Y_n = X_{N-n}$, $0 \le n \le N$ is called the *reverse process* of $(X_n)_{0\le n\le N}$. Show that $(Y_n)_{0\le n\le N}$ is $Markov(\pi, \hat{P})$, where $\hat{P} = (\hat{p}_{ij})$ is given by

$$\pi_j \hat{p}_{ji} = \pi_i p_{ij}, \text{ pour tout } i, j,$$

and $\hat{P}$ is also irreducible with stationary distribution π.

b) A transition matrix P is said to be *doubly stochastic* if its columns sum also to 1, that is $\sum_i p_{ij} = 1$ for all j.
Show that the stationary distribution of an irreducible Markov chain on N states is the uniform distribution ($\pi(i) = \frac{1}{N}$, $1 \le i \le N$) if and only if its transition matrix is doubly stochastic.

c) We say that an irreducible Markov chain $X \sim Markov(\lambda, P)$ is *reversible* if $\hat{P} = P$ (in that case λ should be stationary). Find an irreducible chain on $E = \{1, 2, 3\}$ with a stationary distribution but not reversible.

Exercise 3. Consider two boxes filled with gas molecules and joined by a small gap allowing them to pass from one box to the other. Assume that in total N molecules are in this configuration. We model the system so that at each time only one (randomly chosen) molecule is able to move from one box to the other.

(1) Show that the number of molecules in a box evolves according to a Markov process.

(2) Give the transition probabilities.

(3) What is the stationary distribution (detailed balance equations)?

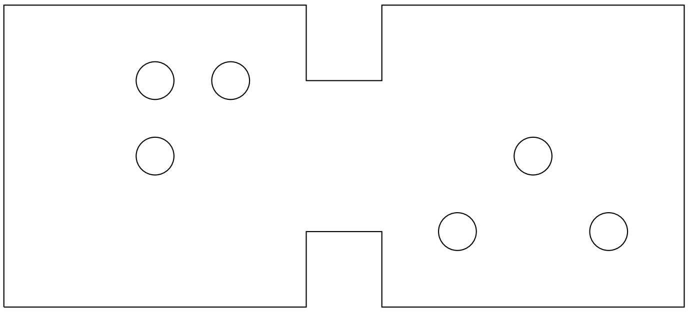

Figure 1: Configuration of the problem.

Exercise 4. Consider the aging chain on $\{0, 1, 2, \cdots\}$ in which for any $n \geq 0$ the individual gets one day older from n to $n+1$ with probability p_n but dies and returns to age 0 with probability $1 - p_n$. Find conditions that guarantee that

(a) 0 is recurrent,

(b) 0 is positive recurrent.

(c) Find the stationary distribution of the chain.

Exercise 5. (**Random walk on a graph**)
An undirected graph $\mathcal{G}$ is a countable collection of states (that we call vertices) along with some edges connecting them. The degree d_i of a vertex i is the number of edges incident to i. We suppose the graph to be locally finite (i.e., each edge is incident to a finite number of edges). We say that a Markov chain on the state space $E = \mathcal{G}$ is a random walk on the graph if the transition probabilities are given by

$$p_{i,j} = \begin{cases} 1/d_i & \text{if } (i,j) \text{ is an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \mathcal{G}$.

a) We assume that $\mathcal{G}$ is connected (implying that P is irreducible) and that $\sum_i d_i < \infty$. Find the stationary distribution of the random walk on $\mathcal{G}$.
Hint: Assume that the random walk is reversible and find a stationary distribution verifying the detailed balance equations. Explain why P is reversible.

b) We assume now that the graph is a chessboard, i.e., the vertices are $\mathcal{G} = \{1, \ldots, 8\}^2$ and the edges are the possible moves of a King. We assume that the King starts its random walk in one of the four corners of the chessboard $c \in \mathcal{G}$. Compute the mean return time to the initial state $\mathbb{E}_c(T_c)$ of the King. Compute the same quantity for a Knight instead of a King.