Solutions 11

Saliba, May 22, 2019

Exercise 1. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Find the stationary distribution of the process

(ii). Give an intuitive explanation of the result.

Solution. Starting in a given state i, we see that we have a cyclic path, we go deterministically to $i+1$ if $i \neq n$ and in 1 if $i = n$, spending a exponentially distributed time with parameter λ_i in each position.
Intuitively, the stationary distribution in i is the mean proportion of time spent in i during a full cycle, that is

$$\pi_i = \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^{n} \frac{1}{\lambda_k}}.$$

We check now that it is really a stationary distribution. Clearly,

$$\sum_{j=1}^{n} \pi_j = 1.$$

Note that $\pi Q = 0$, since

$$\begin{cases} \lambda_i \pi_i = \lambda_{i+1}\pi_{i+1}, si\ i \neq n \\ \lambda_n \pi_n = \lambda_1 \pi_1 \end{cases}$$

Note that unicity is obtained by a direct resolution of the system.

Exercise 2. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2\mu_2 & -\lambda_2 & \lambda_2(1-\mu_2) & 0 & \cdots & 0 & 0 \\ \lambda_3\mu_3 & 0 & -\lambda_3 & \lambda_3(1-\mu_3) & \cdots & 0 & 0 \\ \lambda_4\mu_4 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1}\mu_{n-1} & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1}(1-\mu_{n-1}) \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Guess the expression $\pi_1 = \lim_{t\to\infty} P[X(t) = 1]$.

(ii). Find the stationary distribution of the process.

Solution. At every state i except 1 and n, we jump to the next state with probability $1-\mu_i$, or we return to 1 with probability μ_i, we deterministically return to 1 in position n and we jump in 2 when we are in 1. As before, we infer that the stationary distribution in 1 is equal to the mean time spent in 1, divided by the duration of a return in 1—duration of a cycle. Let us denote C the duration of a cycle, then

$$\begin{aligned}\mathrm{E}[C] &= \sum_{k=2}^{n} \mathrm{E}\left[C | \textit{the last state visited before 1 is } k\right] \mathbb{P}\{\textit{the last state visited before 1 is } k\} \\ &= \sum_{k=2}^{n} \left(\tfrac{1}{\lambda_1} + \cdots + \tfrac{1}{\lambda_k}\right) \times (1-\mu_2)\cdots(1-\mu_{k-1})\mu_k,\end{aligned}$$

and so

$$\pi_1 = \frac{\frac{1}{\lambda_1}}{\mathrm{E}[C]}.$$

With the same methodology, we find π_i, $2 \le i \le n$. Let τ_i be the time spent in i during a cycle, then

$$\begin{aligned}\mathrm{E}[\tau_i] &= \mathrm{E}\left[\tau_i | \textit{i hit during the cycle}\right] \mathbb{P}\{\textit{i hit during the cycle}\} \\ &= \tfrac{1}{\lambda_i}(1-\mu_2)\cdots(1-\mu_{i-1}),\end{aligned}$$

we used that

$$\mathrm{E}\left[\tau_i | \textit{i is not hit during the cycle}\right] = 0.$$

Thus,

$$\pi_i = \frac{\mathrm{E}[\tau_i]}{\mathrm{E}[C]}.$$

Exercise 3. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be an irreducible Markov process on a finite space of n states, with generator Q. Let us take λ so that $\lambda > \max_i\{-Q_{ii}\}$ and define the matrix:

$$P = I + \frac{1}{\lambda} Q \qquad \text{(where } I \text{ is the identity matrix).}$$

(i). Show that P is a transition matrix, and that its stationary distribution is identical to the one of Q.

(ii). Let $\{N(t) \mid t \in \mathbb{R}^+\}$ be a Poisson process with parameter λ and let $\{Y_k \mid k \in \mathbb{N}\}$ be a Markov chain with matrix P, independent from $\{N(t)\}$. Let $T_0 = 0$, and $T_1, T_2, \cdots$ denote the arrival times in the Poisson process.

We define the process $\{Z(t) \mid t \in \mathbb{R}^+\}$ as follows:

$$Z(t) = Y_n \qquad \forall t \in [T_n, T_{n+1}[.$$

Show that $\{Z(t)\}$ is a Markov process with generator Q.

Solution. (i). Let π be the stationary distribution of Q, then

$$\pi P = \pi + \underbrace{\tfrac{1}{\lambda}\pi Q}_{=0} = \pi.$$

As there exists a unique solution (to a multiplicative constant) satisfying $xP = x$, π is the stationary distribution.
(ii). $\{Z(t)\}$ is a Markov process, since,

- $\mathbb{P}\{Z(t+s) = j|Z(u), 0 \le u \le s\} = \mathbb{P}\{Z(t+s) = j|Z(s)\}$, since the Poisson process N is Markovian, its arrivals between s and $s+t$ do not depend on the history of the process before s. Moreover, $\{Y_n\}$ satisfy the Markov property and thus the future transitions depend only the the current one s.
- $\mathbb{P}\{Z(t+s) = j|Z(s) = i\} = \mathbb{P}\{Z(t) = j|Z(0) = i\}$:

$$\begin{aligned}\mathbb{P}\{Z(t+s) = j|Z(s) = i\} = &\sum_k \mathbb{P}\{Z(t+s) = j|Z(s) = i,\ k \textit{ arrivals from } N \textit{ on } [s, s+t]\}\\ &\times \underbrace{\mathbb{P}\{N(t+s) - N(s) = k|Z(s) = i\}}_{=\mathbb{P}\{N(t)=k|Z(0)=i\}}\\ =&\sum_k (P^k)_{ij}\mathbb{P}\{N(t) = k|Z(0) = i\}\\ =&\mathbb{P}\{Z(t) = j|Z(0) = i\}.\end{aligned}$$

Let us show now that it is a Markov process with generator Q. We show that the transition matrix of $\{Z(t)\}$ can be expressed as the exponential of Q,

$$\begin{aligned}L_{ij}(t) :=&\mathbb{P}\{Z(t) = j|Z(0) = i\}\\ =&\sum_{k=0}^{\infty} \mathbb{P}\{Z(t) = j|Z(0) = i,\ k \textit{ arrivals of } N \textit{ on } [0, t]\}\\ &\times \mathbb{P}\{N(t) = k|Z(0) = i\}\\ =&\sum_{k=0}^{\infty} (P^k)_{ij} \exp\{-\lambda t\}\frac{(\lambda t)^k}{k!}.\end{aligned}$$

So that,

$$L(t) = \exp\{-\lambda I t\}\sum_{k=0}^{\infty}\frac{(\lambda t P)^k}{k!} = \exp\left\{(\lambda P - \lambda I)t\right\},$$

and $\lambda P - \lambda I = \lambda(I + \frac{1}{\lambda}Q) - \lambda I = Q$, thus

$$L(t) = \exp\{Qt\}.$$

Exercise 4. Show that the renewal function $R(t)$ satisfy a renewal equation, and specify the corresponding function $g(t)$.

Solution. We could get a renewal equation for $R(t)$ by conditioning on the first time S_1 after S_0. However, here, since $R(t) = \sum_{n\geq 0} F^{(n)}(t)$, we directly get that

$$\begin{aligned}(R * F)(t) &= \sum_{n\geq 0}(F^{(n)} * F)(t) = \sum_{n\geq 0} F^{(n+1)}(t),\\ &= \sum_{n\geq 1} F^{(n)}(t) = \sum_{n\geq 0} F^{(n)}(t) - F^{(0)}(t),\end{aligned}$$

and so

$$R(t) = F^{(0)}(t) + (R * F)(t).$$

with, $g(t) = F^{(0)}(t) = 1_{\{t\geq 0\}}$.
This also shows that $t \geq 0$, $(R * F)(t) = R(t) - 1$, which was used during the lecture in the proof of the theorem giving the distribution of the duration of life L of a transitive renewal process.

Exercise 5. Let $S_1, S_2, \ldots$ be the successive times at which cars cross a certain fixed position on the highway. We assume that the intervals of time $W_1, W_2, \ldots$ between each renewal are i.i.d. with cumulative distribution $F(\cdot)$. Suppose that at time $t = 0$, a pedestrian arrives at this fixed position, and wants to cross the road. Assume that he needs τ units of time to cross it. Let L be the time that the pedestrian has to wait before starting to cross the road.

(a) Find the distribution of L and its expectation.

(b) Same questions if we assume that the arrivals of cars follow a Poisson process with parameter λ.

Solution. (a). The pedestrian starts to cross the road at $L = S_n$ if and only if $W_1 \leq \tau, \ldots, W_n \leq \tau$ and $W_{n+1} > \tau$. So that, L is the duration of life of a renewal $\{\tilde{S}_n\}$ having its n^{th} interval of time given by

$$\tilde{W}_n = \begin{cases} W_n, & si\ W_n \leq \tau,\\ +\infty, & otherwise.\end{cases}$$

Thus, the distribution of $\tilde{W}_n$ is given by

$$\tilde{F}(t) = \begin{cases} F(t), & if\ t \leq \tau,\\ F(\tau), & if\ t > \tau.\end{cases}$$

We deduce that

$$\mathbb{P}\{L \leq t\} = \{1 - \tilde{F}(\infty)\}\tilde{R}(t) = \{1 - F(\tau)\}\tilde{R}(t),$$

where $\tilde{R}(t) = \sum_{n\geq 0} \tilde{F}^{(n)}(t)$.
(b). The mean waiting time of the pedestrian is

$$\mathrm{E}[L] = \frac{1}{1 - F(\tau)} \int_0^\tau \{F(\tau) - F(t)\}dt.$$

More precisely, if the arrivals of the cars follow a Poisson process with parameter λ, then

$$\mathrm{E}[L] = \frac{1}{\lambda}\big(\exp\{\lambda\tau\} - 1\big) - \tau.$$