Solutions 9

1. a) There is no stationary distribution. By computing the stationary distribution, one would find an infinite normalization constant. The walk is irreducible but since there is no stationary distribution it cannot be positive-recurrent.

b) The acceptance probability for the move $i \to i + 1$ is

$$a_{i,i+1} = \min(1, e^{-a((i+1)^2 - i^2)}) = \begin{cases} 1, & i \le -1, \\ e^{-a(2i+1)}, & i \ge 0. \end{cases}$$

and for the move $i \to i - 1$ is

$$a_{i,i-1} = \min(1, e^{-a((i-1)^2 - i^2)}) = \begin{cases} 1, & i \ge 1, \\ e^{a(2i-1)}, & i \le 0. \end{cases}$$

In words: a move towards the origin $i = 0$ is always accepted and a move away from the origin is accepted with probability $e^{-a(1 \pm 2i)}$.

The transition probabilities are

$$p_{i,i+1} = \frac{1}{2} \min(1, e^{-a((i+1)^2 - i^2)}) = \begin{cases} \frac{1}{2}, & i \le -1, \\ \frac{1}{2} e^{-a(2i+1)}, & i \ge 0. \end{cases}$$

and

$$p_{i,i-1} = \frac{1}{2} \min(1, e^{-a((i-1)^2 - i^2)}) = \begin{cases} \frac{1}{2}, & i \ge 1, \\ \frac{1}{2} e^{a(2i-1)}, & i \le 0. \end{cases}$$

and for the self-loops

$$p_{ii} = 1 - p_{i,i-1} - p_{i,i+1} = \begin{cases} 1 - \frac{1}{2} e^{a(2i-1)} - \frac{1}{2} = \frac{1}{2}(1 - e^{a(2i-1)}), & i \le -1, \\ 1 - \frac{1}{2} e^{-a} - \frac{1}{2} e^{-a} = 1 - e^{-a}, & i = 0, \\ 1 - \frac{1}{2} - \frac{1}{2} e^{-a(2i+1)} = \frac{1}{2}(1 - e^{-a(2i+1)}), & i \ge 1. \end{cases}$$

c) The walk is obviously irreducible. A stationary distribution π_i^* exists by construction. Thus by the first fundamental theorem in class the walk is positive recurrent. Moreover because of the self loops it is aperiodic. An irreducible, positive recurrent, aperiodic walk satisfies the ergodic theorem and therefore

$$\lim_{n \to +\infty} (P^n)_{ij} = \pi_j^*$$

where P is the transition matrix elements $(P^n)_{ij} = \mathbb{P}(X_n = j \mid X_0 = i)$.

d) We start on the right of the origin because $z > 0$. The initial distance between the two walkers is d. When we propose a move towards the left $\xi_1 = -1$ both walkers accept the move with probability 1 so the distance stays equal to d. When we propose a move towards the right $\xi_1 = +1$ the walkers may accept-accept the move, accept-reject the move, reject-accept the move, or reject-reject the

move. In the first case and in the last case the distance stays equal to d, in the second case it decreases to $d-1$ and the third case it increases to $d+1$. Reasoning like this, we see that the minimum coalescence time corresponds to the events where we always propose a right move and the walkers accept-reject.

This minimum coalescence time is thus equal to d. We have

$$\mathbb{P}(T = d) = \frac{1}{2^d} \left\{ \prod_{i=z}^{z+d-1} e^{-a(2i+1)} \right\} (1 - e^{-a(2(z+d)+1)})^d$$