Homework 2 (due Friday, October 5)

Exercise 1. Let $(X_n, n \geq 0)$ be an homogeneous Markov chain with transition probabilities

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i)$$

We define the probability of *first passage* as the probability that the chain passes from i to j in n steps without passing by j before the n^{th} step.

$$f_{ij}^{(n)} = \mathbb{P}(X_n = j,\, X_{n-1} \neq j, \ldots,\, X_1 \neq j | X_0 = i)$$

We also define the probability of *last exit* as the probability that the chain passes from i to j in n steps without revisiting i during these n steps.

$$l_{ij}^{(n)} = \mathbb{P}(X_n = j,\, X_{n-1} \neq i, \ldots,\, X_1 \neq i | X_0 = i)$$

Let

$$\begin{aligned} P_{ij}(s) &= \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \qquad p_{ij}(0) = \delta_{ij} \\ F_{ij}(s) &= \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n, \qquad f_{ij}(0) = 0 \\ L_{ij}(s) &= \sum_{n=0}^{\infty} l_{ij}^{(n)} s^n, \qquad l_{ij}(0) = 0 \end{aligned}$$

be the associated generating functions. Note that $L_{ii}(s) = F_{ii}(s)$. Recall that we proved in class that $P_{ii}(s) = 1 + P_{ii}(s)F_{ii}(s)$.

a) Prove that for $i \neq j$:

$$\begin{aligned} P_{ij}(s) &= F_{ij}(s)P_{jj}(s) \\ P_{ij}(s) &= P_{ii}(s)L_{ij}(s) \end{aligned}$$

b) Deduce the following statements:

1. If j is recurrent then $\sum_{n\geq 0} p_{ij}^{(n)} = \infty$ for all i such that $f_{ij} > 0$, where $f_{ij} = \sum_{n\geq 0} f_{ij}^{(n)}$.
2. If j is transient then $\sum_{n\geq 0} p_{ij}^{(n)} < \infty$ for all i.
3. If j is recurrent and i is transient then $\sum_{n\geq 0} l_{ij}^{(n)} = \infty$ as long as $f_{ij} > 0$.

c) Prove that if the Markov chain satisfies $P_{ii}(s) = P_{jj}(s)$ for all $i \neq j$, the probability distribution of last exit and first passage are equal.

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Exercise 2. Consider the symmetric random walk in 3 dimensions on $\mathbb{Z}^3$ defined during the first lecture:

$$S_0 = (0,0,0), \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1$$

where $(\xi_n, n \geq 1)$ are i.i.d. with

$$\mathbb{P}(\xi_n = e_i) = \mathbb{P}(\xi_n = -e_i) = 1/6$$

and $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

a) Argue that

$$\mathbb{P}(S_{2n} = (0,0,0) | S_0 = (0,0,0)) = \frac{1}{6^{2n}} \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2}$$

where i, j, k are ≥ 0.

b) We want to evaluate the asymptotic behaviour of this sum as $n \to \infty$ (we in fact want to derive a good upper bound). Derive the following inequality:

$$\mathbb{P}(S_{2n} = (0,0,0) | S_0 = (0,0,0)) \leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i+j+k=n} \frac{1}{3^n} \frac{n!}{i!j!k!}$$

where $M = \max\{\frac{n!}{3^n i!j!k!},\ i+j+k=n,\ i,j,k \geq 0\}$.

c) Next, assuming that the maximum is attained at $i,j,k \approx n/3$, deduce that

$$\mathbb{P}(S_{2n} = (0,0,0) | S_0 = (0,0,0)) \leq \frac{c}{n^{3/2}}$$

for some constant c.

d) Is the random walk in 3 dimensions recurrent or transient?