Solutions 7

1. a) Let us write by convention that $y \sim x$ if there exists a unique $j \in 1, \dots, d$ such that $y_j \neq x_j$. Observing that the described process is a random walk on the graph described by the relation $\sim$, we deduce that the transition matrix of the chain is given by

$$p_{xy} = \begin{cases} \frac{1}{(m-1)d}, & \text{if } y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

The chain is clearly irreducible, aperiodic and positive-recurrent, therefore ergodic. Its stationary distribution π is uniform (i.e. $\pi_x = m^{-d}\ \forall x \in S$), and the detailed balance equation is satisfied.

b) Assume that $|z| = k$ and denote by A the set of indices $j \in \{1, \dots, d\}$ such that $z_j \neq 0$ (so that $|A| = k$). Then

$$\begin{aligned} \left(P\phi^{(z)}\right)_x &= \sum_{y \in S} p_{xy}\, \phi_y^{(z)} = \frac{1}{(m-1)d} \sum_{y \sim x} \exp\left(2\pi i y \cdot z/m\right) \\ &= \frac{1}{(m-1)d} \sum_{j=1}^{d} \sum_{t=0\,:\,t \neq x_j}^{m-1} \exp\left(2\pi i \left(\sum_{l=1\,:\,l \neq j}^{d} x_l z_l + t z_j\right)/m\right) \\ &= \frac{1}{(m-1)d} \sum_{j=1}^{d} \exp\left(2\pi i \left(\sum_{l=1\,:\,l \neq j}^{d} x_l z_l\right)/m\right) \times \sum_{u=0\,:\,u \neq x_j}^{m-1} \exp(2\pi i u z_j/m). \end{aligned}$$

Observe now that if $z_j = 0$, then

$$\sum_{u=0\,:\,u \neq x_j}^{m-1} \exp(2\pi i u z_j/m) = \sum_{u=0\,:\,u \neq x_j}^{m-1} 1 = m-1 = (m-1)\exp(2\pi i x_j z_j/m)$$

while if $z_j \neq 0$, then

$$\sum_{u=0\,:\,u \neq x_j}^{m-1} \exp(2\pi i u z_j/m) = \sum_{u=0}^{m-1} \exp(2\pi i u z_j/m) - \exp(2\pi i x_j z_j/m) = 0 - \exp(2\pi i x_j z_j/m).$$

This finally gives

$$\begin{aligned} \left(P\phi^{(z)}\right)_x &= \frac{1}{(m-1)d} \left(\sum_{j \in A} (-1)\exp\left(2\pi i x \cdot z/m\right) + \sum_{j \in A^c} (m-1)\exp\left(2\pi i x \cdot z/m\right)\right) \\ &= \frac{1}{(m-1)d} \left(-d\exp\left(2\pi i x \cdot z/m\right) + (d-k)m\exp\left(2\pi i x \cdot z/m\right)\right) = \frac{(d-k)m - d}{(m-1)d}\, \phi_x^{(z)} \\ &= \left(1 - \frac{km}{(m-1)d}\right) \phi_x^{(z)}. \end{aligned}$$

The eigenvalue λ_z corresponding to $\phi^{(z)}$ is therefore given by

$$\lambda_z = 1 - \frac{|z|m}{(m-1)d}.$$

c) The second largest eigenvalue is equal to $1 - \frac{m}{(m-1)d}$, while the least eigenvalue is equal to $1 - \frac{m}{m-1} = -\frac{1}{m-1}$. When $d > 2$ (remember also that by assumption, $m > 2$), the spectral gap is therefore determined by the second largest eigenvalue and equal to $\gamma = \frac{m}{(m-1)d}$. This leads to the following upper bound on the total variation distance:

$$\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2\sqrt{\pi_0}} \exp(-\gamma n) = \frac{m^{d/2}}{2} \exp\left(-\frac{nm}{(m-1)d}\right),$$

which becomes small only when $n \geq c\, d^2 \log m$ for some constant $c > 0$.

d) The lower bound obtained in class applies here, as $\phi_x^{(z)}|^2 = 1$ for all z and x. It reads

$$\|P_0^n - \pi\|_{\mathrm{TV}} \geq \frac{1}{2}\lambda_*^n \simeq \frac{1}{2} \exp(-\gamma n) = \frac{1}{2} \exp\left(-\frac{nm}{(m-1)d}\right),$$

which is small for $n \geq cd$ already, so the two bounds do not match.

e) [NOT REQUIRED] A tighter upper bound on the total variation distance can be found via the following analysis:

$$\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2} \sqrt{\sum_{z \in S \backslash \{0\}} \lambda_z^{2n}} = \frac{1}{2} \sqrt{\sum_{t=1}^{d} \sum_{z \in S\,:\,|z|=t} \left(1 - \frac{tm}{(m-1)d}\right)^{2n}}$$

As

$$\sum_{z \in S\,:\,|z|=t} = \binom{d}{t} (m-1)^t \leq \frac{((m-1)d)^t}{t!} \quad \text{and} \quad \left(1 - \frac{tm}{(m-1)d}\right)^{2n} \leq \exp\left(-\frac{2tmn}{(m-1)d}\right) \leq \exp\left(-\frac{2tn}{d}\right)$$

we finally obtain

$$\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{d} \frac{1}{t!} \exp\left(-t\left(\frac{2n}{d} - \log((m-1)d)\right)\right)}$$

Taking now $n = \frac{d}{2}\left(\log((m-1)d) + c\right)$, we obtain

$$\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{\infty} \frac{1}{t!} \exp(-tc)} = \frac{1}{2}\sqrt{\exp\left(e^{-c}\right) - 1}$$

which can be made arbirarily small by taking c large. So finally, the upper bound on the mixing time is $O(d\, \max(\log m, \log d))$.

2. Following what has been done in class, we obtain first

$$
\begin{aligned}
\|P_0^n - \pi\|_2 &= \left(\sum_{y \in S} \left(\frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y} \right)^2 \right)^{1/2} = \left(\sum_{z \in S \,:\, z \neq 0} \lambda_z^{2n} \, (\phi_0^{(z)})^2 \right)^{1/2} \\
&\geq \left(\sum_{k=1}^{\lfloor d/2 \rfloor} \binom{d}{k} \left(1 - \frac{2k}{d+1} \right)^{2n} \right)^{1/2} \geq \sqrt{d} \left(1 - \frac{2}{d+1} \right)^n
\end{aligned}
$$

by retaining only the term $k = 1$ in the above sum. Using now the fact that $e^{-x} \simeq 1 - x$ for x small, we obtain further

$$
\|P_0^n - \pi\|_2 \geq \exp\left(\frac{1}{2} \log d - \frac{2n}{d+1} \right) = \exp(c/2)
$$

for $n = \frac{d+1}{4} (\log d - c)$. The above expression can therefore be made arbitrarily large by taking $c > 0$ arbitrarily large.