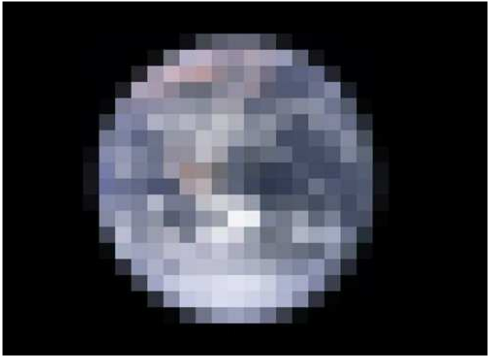
COM303: Digital Signal Processing

Lecture 16: Interpolation

overview

- the analog worldview
- interpolation of discrete-time signals
- bandlimited functions
- the sinc basis and sinc sampling

Two views of the world



Analog/continuous versus discrete/digital

Two views of the world

digital worldview:

- arithmetic
- combinatorics
- computer science
- DSP

analog worldview:

- calculus
- distributions
- system theory
- electronics

Two views of the world

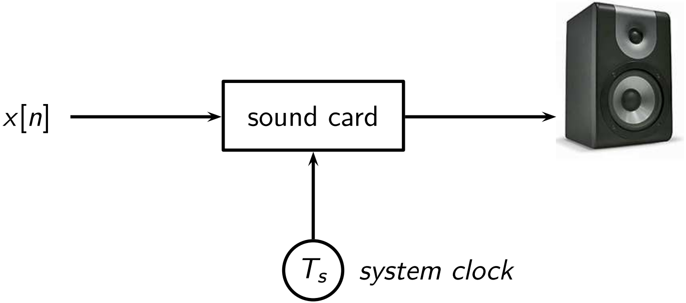
digital worldview:

- countable integer index n
- sequences $x[n] \in \ell_2(\mathbb{Z})$
- frequency $\omega \in [-\pi, \pi]$
- DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

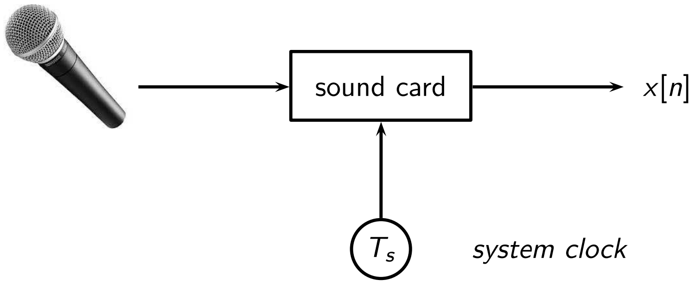
analog worldview:

- real-valued time t (sec)
- functions $x(t) \in L_2(\mathbb{R})$
- frequency $\Omega \in \mathbb{R}$ (rad/sec)
- FT: $L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$

Bridging the gap: interpolation



Bridging the gap: sampling



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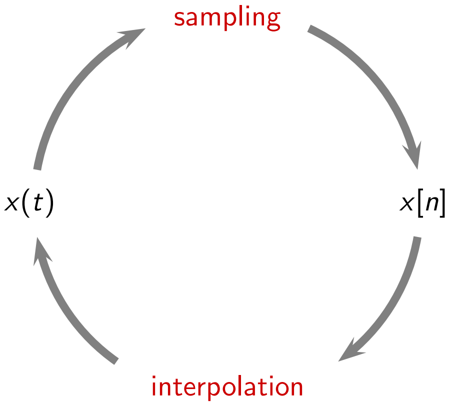


Today, processing is as digital as possible

- analog to digital to analog
- digital to analog
- analog to digital

Digital processing of signals from/to the analog world

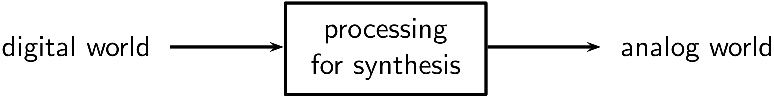
- input is continuous-time: $x(t)$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$

$$\text{analog world} \longrightarrow \boxed{\text{processing}} \longrightarrow \text{analog world}$$

examples: telephony, VOIP, sound effects, digital photography

Digital processing of signals to the analog world

- input is discrete-time: $x[n]$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$



examples: music synthesizers, computer graphics, video games

Digital processing of signals from the analog world

- input is continuous-time: $x(t)$
- output is discrete-time: $y[n]$
- processing is on sequences: $x[n], y[n]$

analog world $\longrightarrow$ processing for analysis $\longrightarrow$ digital world

examples: storage and compression (MP3, JPG), control systems, monitoring

continuous-time signal processing

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

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Analog LTI filters

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$
\begin{aligned}
y(t) &= (x * h)(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \langle h^*(t-\tau), x(\tau) \rangle
\end{aligned}
$$

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Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $\Omega \in \mathbb{R}$
- concept is the same: similarity to sinusoidal components

$$\begin{aligned} X(j\Omega) &= \langle e^{j\Omega t}, x(t) \rangle \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad \leftarrow \textit{not periodic!} \end{aligned}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

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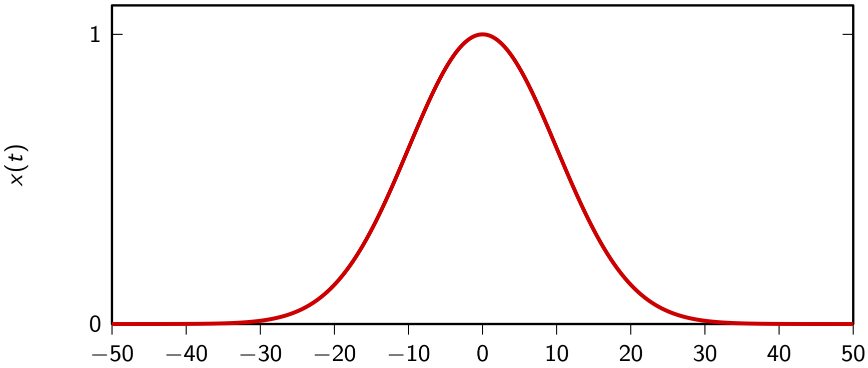
Real-world frequency

- Ω expressed in rad/s
- $F = \dfrac{\Omega}{2\pi}$, expressed in Hertz $(1/s)$
- period $T = \dfrac{1}{F} = \dfrac{2\pi}{\Omega}$

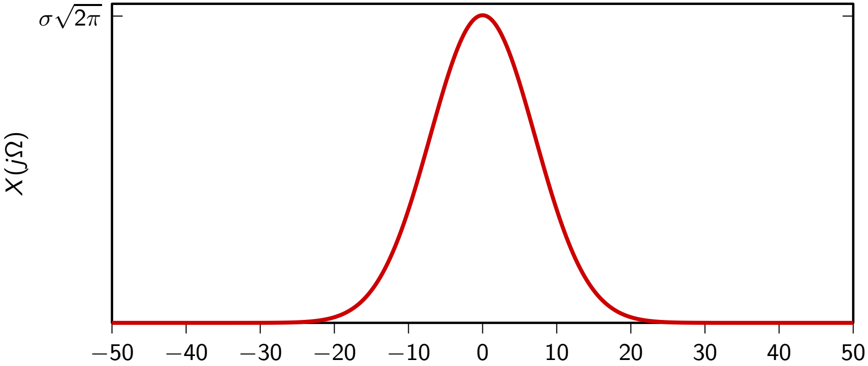
Example

$$x(t) = e^{-\frac{t^2}{2\sigma^2}}$$



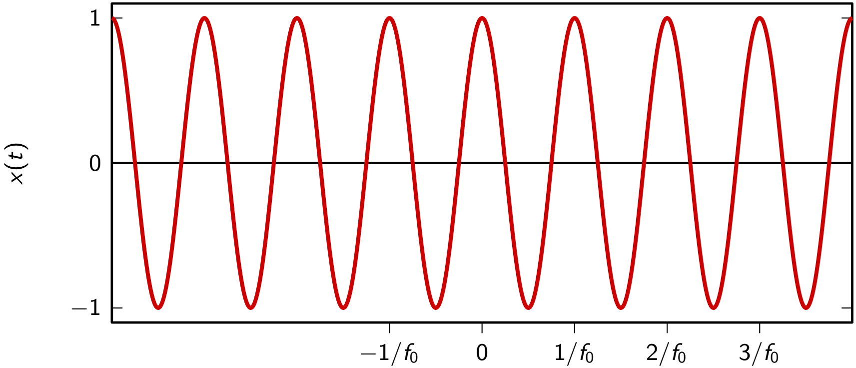
Example

$$X(j\Omega) = \sigma\sqrt{2\pi}e^{-\frac{\sigma^2}{2}\Omega^2}$$



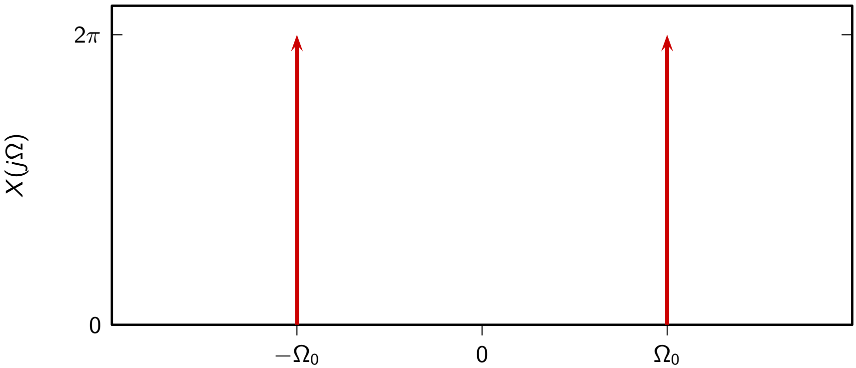
Example

$$x(t) = \cos(\Omega_0 t) = \cos(2\pi f_0 t)$$



Example

$$X(j\Omega) = 2\pi\delta(\Omega \pm \Omega_0)$$



Convolution theorem

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

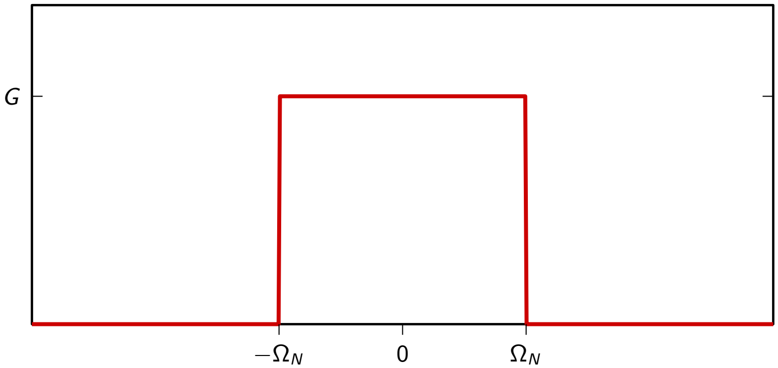
$$Y(j\Omega) = X(j\Omega)\,H(j\Omega)$$

A new concept: bandlimited functions

Ω_N-bandlimitedness:

$$X(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_N$$

Prototypical bandlimited function



The prototypical bandlimited function

$$\Phi(j\Omega) = G \ \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\begin{aligned}
\varphi(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(j\Omega) e^{j\Omega t} d\Omega \\
&= \ldots \\
&= G\,\frac{\Omega_N}{\pi}\,\mathrm{sinc}\left(\frac{\Omega_N}{\pi} t\right)
\end{aligned}$$

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The prototypical bandlimited function

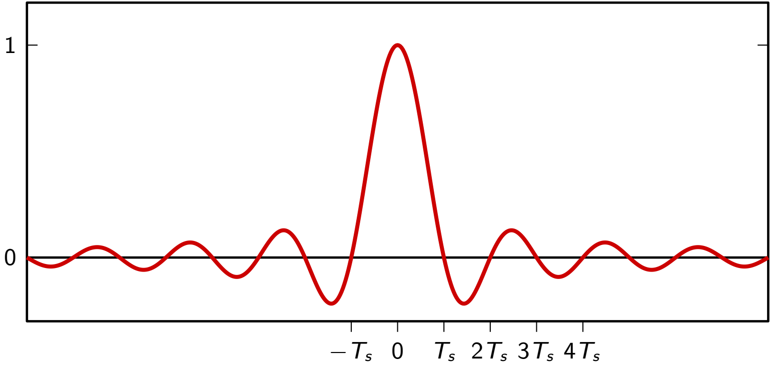
- normalization: $G = \dfrac{\pi}{\Omega_N}$
- total bandwidth: $\Omega_B = 2\Omega_N$
- define $T_s = \dfrac{2\pi}{\Omega_B} = \dfrac{\pi}{\Omega_N}$

The prototypical bandlimited function

$$\Phi(j\Omega) = \frac{\pi}{\Omega_N} \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\varphi(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$

The prototypical bandlimited function



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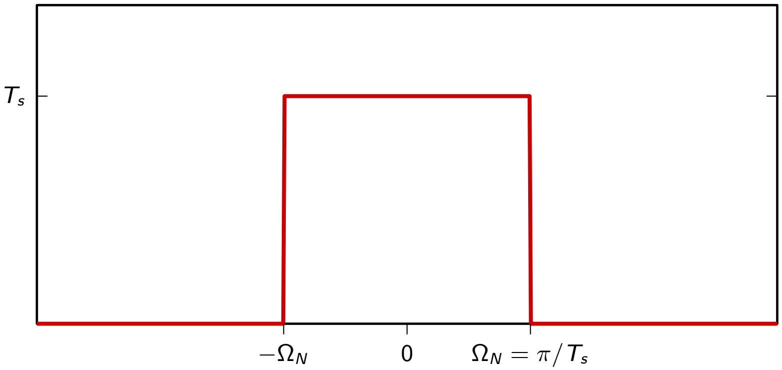
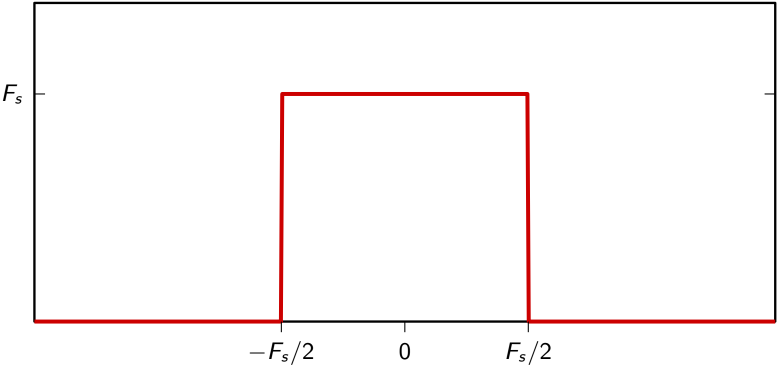


The prototypical bandlimited function (using Hz)

$$F_s = 1/T_s$$

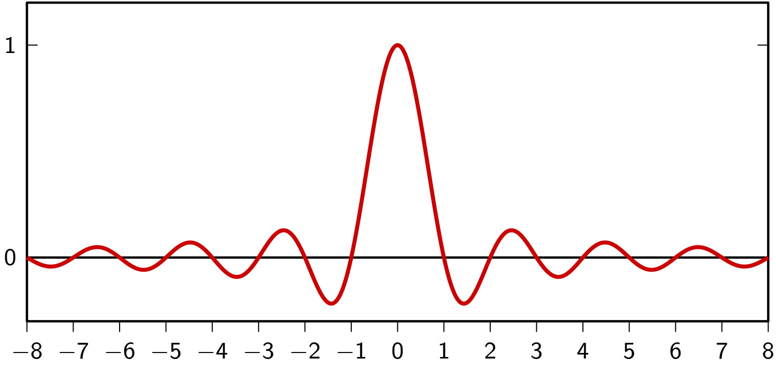


When $T_s = 1$

$$\Phi(j\Omega) = \mathrm{rect}\left(\frac{\Omega}{2\pi}\right)$$

$$\varphi(t) = \mathrm{sinc}(t)$$

The prototypical bandlimited function ($T_s = 1$)



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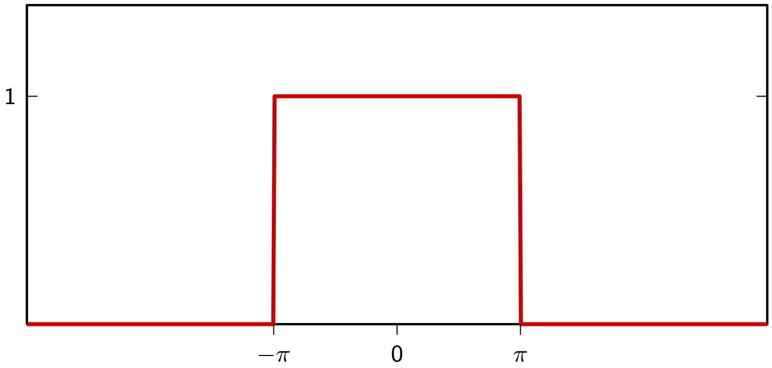
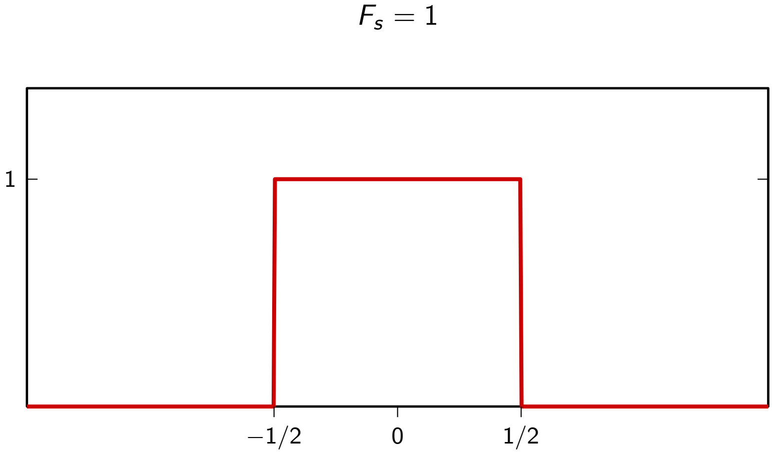


The prototypical bandlimited function ($T_s = 1$, using Hz)



interpolation

Overview:

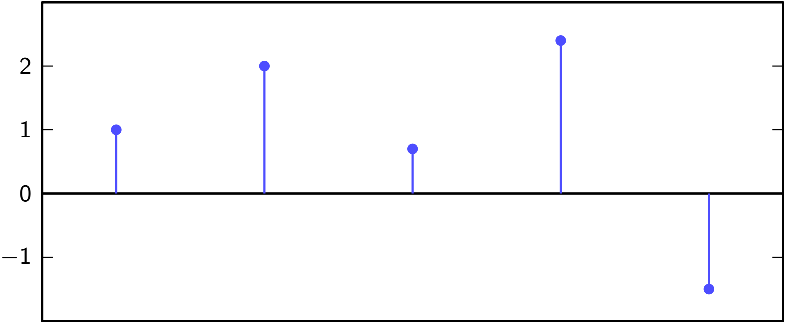
- Polynomial interpolation
- Local interpolation
- Sinc interpolation

Interpolation

$$x[n] \longrightarrow x(t)$$

"fill the gaps" between samples

Example



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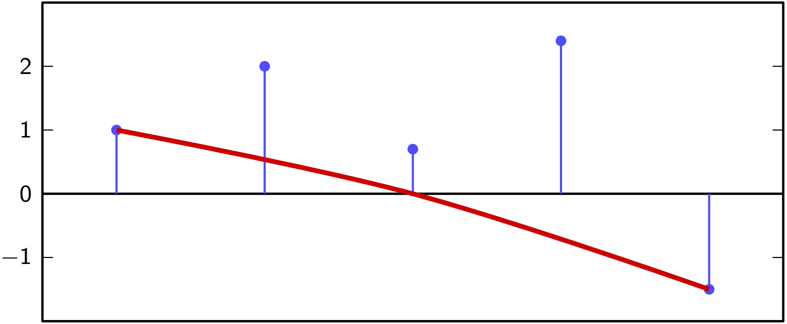

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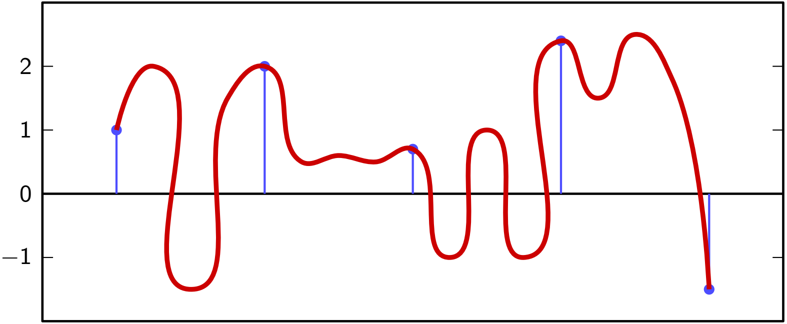

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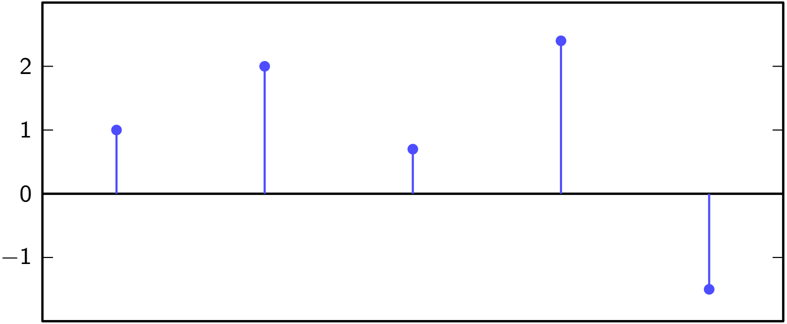

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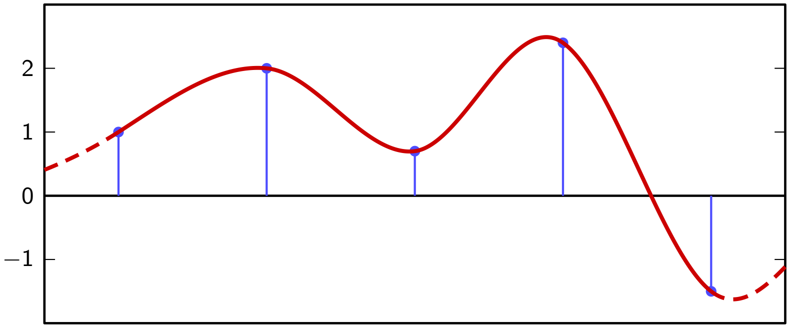

Interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

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Why smoothness?

- jumps (1st order discontinuities) would require the signal to move “faster than light”…
- 2nd order discontinuities would require infinite acceleration
- …
- the interpolation should be infinitely differentiable
- “natural” solution: polynomial interpolation

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- "natural" solution: polynomial interpolation

Polynomial interpolation

- N points $\rightarrow$ polynomial of degree $(N-1)$
- $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{N-1} t^{(N-1)}$
- "naive" approach:

$$\begin{cases} p(0) = x[0] \\ p(T_s) = x[1] \\ p(2T_s) = x[2] \\ \ldots \\ p((N-1)T_s) = x[N-1] \end{cases}$$

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Polynomial interpolation

Without loss of generality:

- consider a symmetric interval $I_N = [-N, \ldots, N]$
- set $T_s = 1$

$$
\begin{cases}
p(-N) = x[-N] \\
p(-N+1) = x[-N+1] \\
\ldots \\
p(0) = x[0] \\
\ldots \\
p(N-1) = x[N-1] \\
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Lagrange interpolation

- P_N: space of degree-$2N$ polynomials over I_N
- a basis for P_N is the family of $2N+1$ Lagrange polynomials

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k \neq n}}^{N} \frac{t-k}{n-k} \qquad\qquad n = -N, \ldots, N$$

Lagrange polynomials for l_2

$$L_{-2}^{(2)}(t) = \left(\frac{t+1}{-2+1}\right)\left(\frac{t}{-2}\right)\left(\frac{t-1}{-2-1}\right)\left(\frac{t-2}{-2-2}\right)$$

$$L_{-1}^{(2)}(t) = \left(\frac{t+2}{-1+2}\right)\left(\frac{t}{-1}\right)\left(\frac{t-1}{-1-1}\right)\left(\frac{t-2}{-1-2}\right)$$

$$L_{0}^{(2)}(t) = \left(\frac{t+2}{2}\right)\left(\frac{t+1}{1}\right)\left(\frac{t-1}{-1}\right)\left(\frac{t-2}{-2}\right)$$

$$L_{1}^{(2)}(t) = L_{-1}^{(2)}(-t)$$

$$L_{2}^{(2)}(t) = L_{-2}^{(2)}(-t)$$

Aside: N-degree polynomial bases on the interval

- naive basis: $1, t, t^2, \ldots, t^N$
- Legendre basis: orthonormal, increasing degree
- Chebyshev basis: orthonormal, increasing degree
- Lagrange: interpolation property, equal degree

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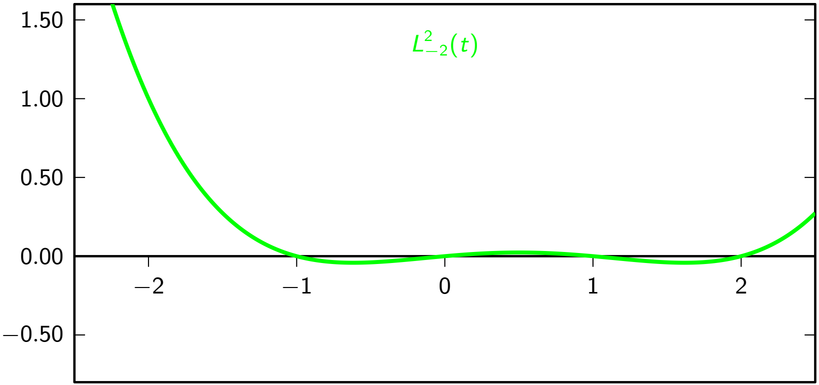
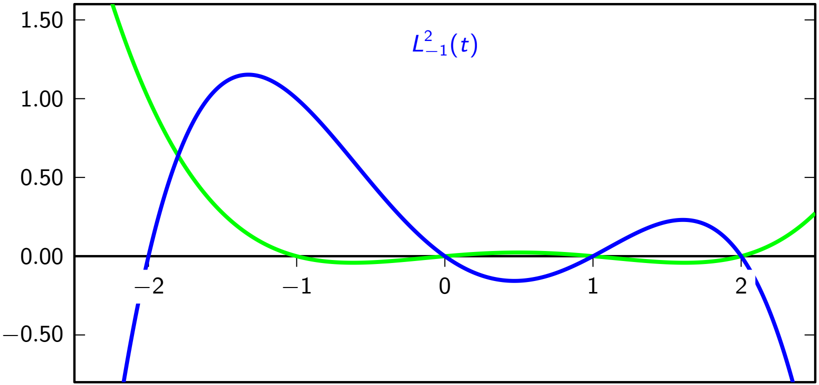
Lagrange interpolation polynomials



Lagrange interpolation polynomials



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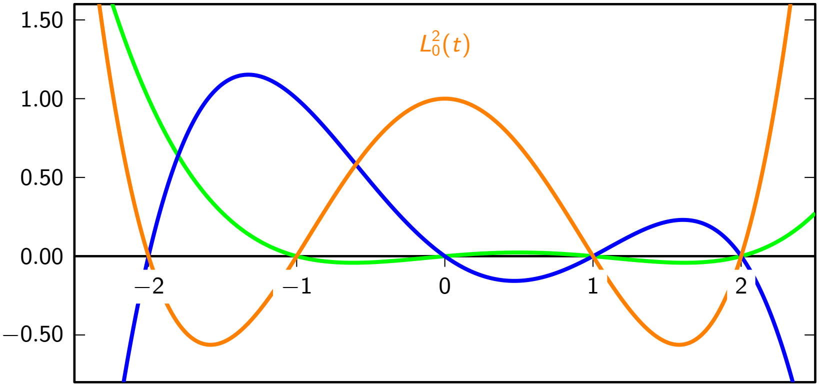

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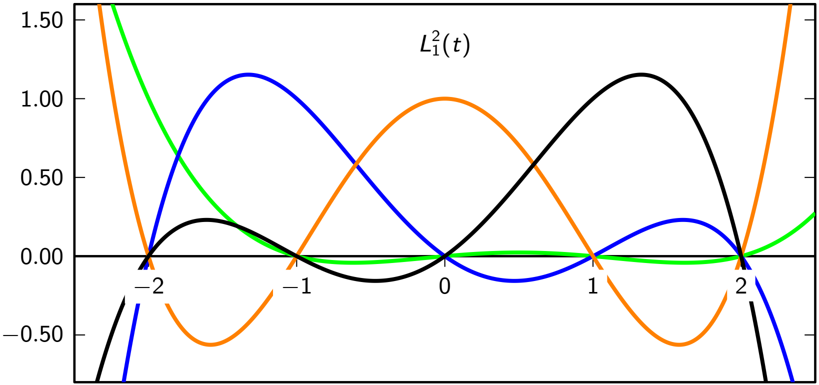

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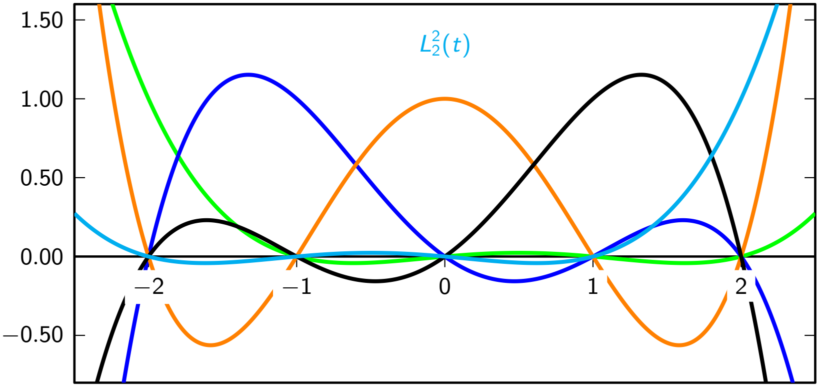


Lagrange interpolation

$$p(t) = \sum_{n=-N}^{N} x[n] L_n^{(N)}(t)$$

Lagrange interpolation

The Lagrange interpolation *is* the sought-after polynomial interpolation:

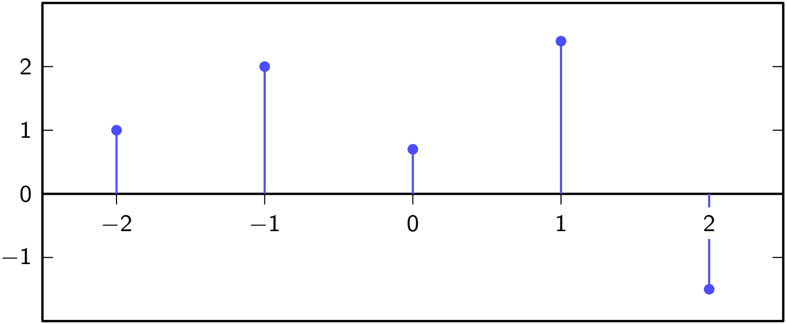
- polynomial of degree $2N$ through $2N+1$ points is unique
- the Lagrangian interpolator satisfies

$$p(n) = x[n] \qquad \text{for } -N \leq n \leq N$$

since

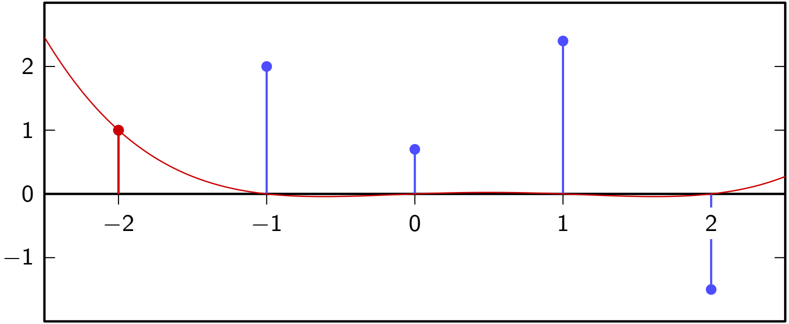
$$L_n^{(N)}(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad\qquad -N \leq n, m \leq N$$

Lagrange interpolation

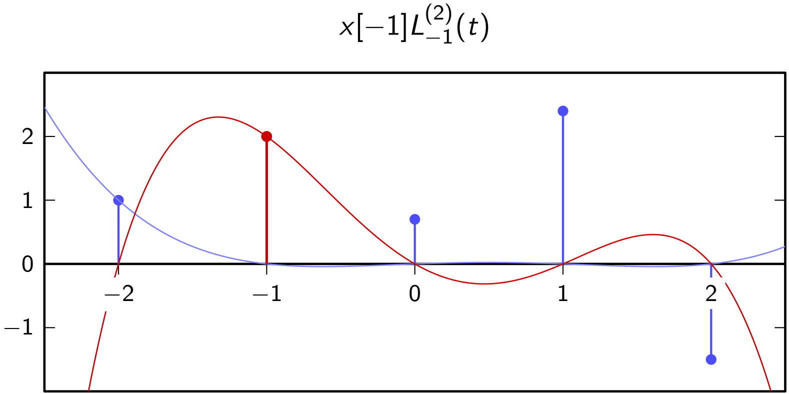


Lagrange interpolation

$$x[-2]L^{(2)}_{-2}(t)$$

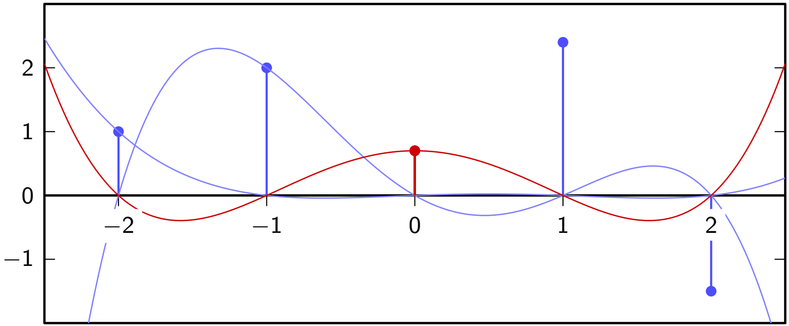


Lagrange interpolation



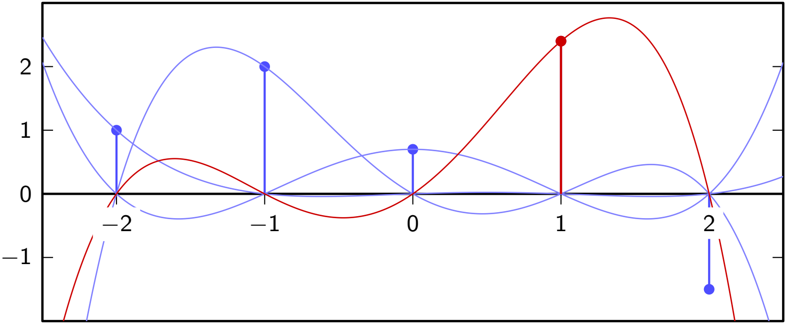
Lagrange interpolation

$$x[0]L_0^{(2)}(t)$$

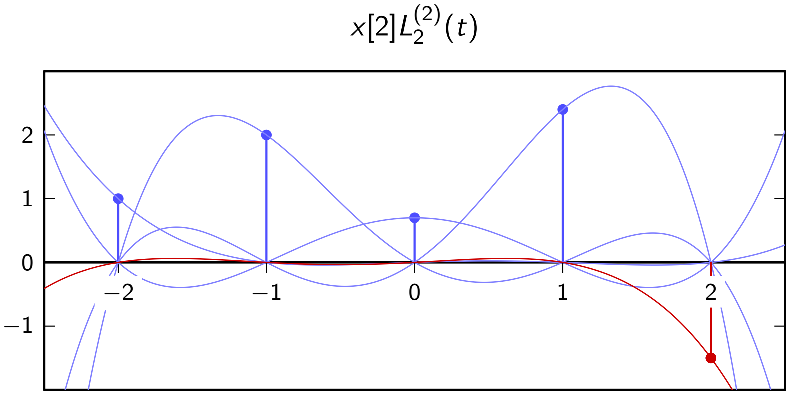


Lagrange interpolation

$$x[1]L_1^{(2)}(t)$$



Lagrange interpolation



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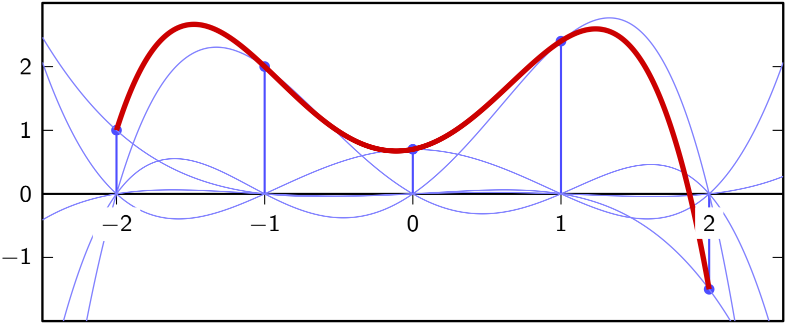

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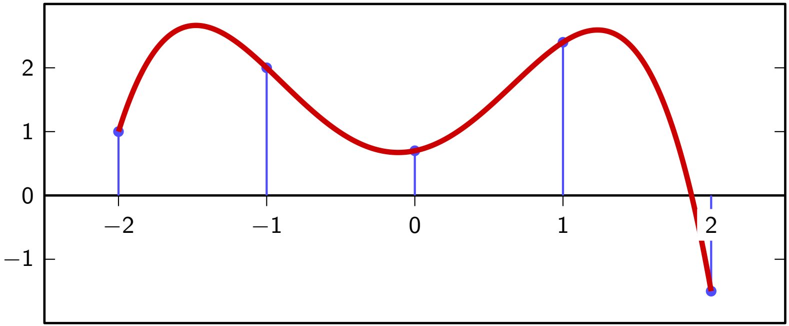

Polynomial interpolation

key property:

- maximally smooth (infinitely many continuous derivatives)

drawback:

- interpolation "machine" depend on N: we need to use a different set of polynomials if the length of the dataset changes

Relaxing the interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

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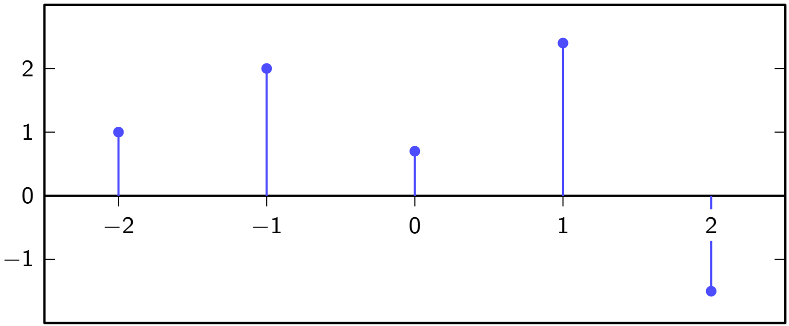

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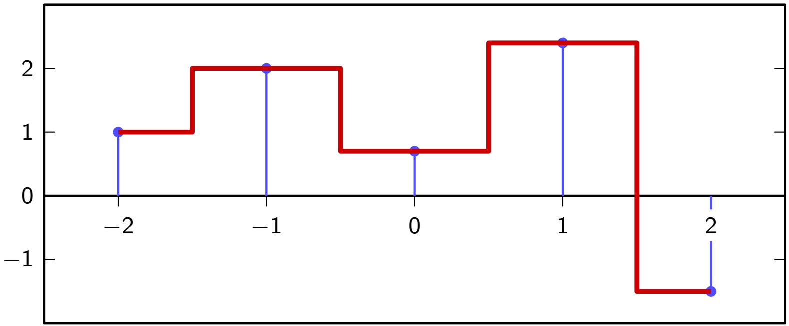

Zero-order interpolation

- $x(t) = x[\,\lfloor t + 0.5 \rfloor\,], \qquad -N \leq t \leq N$
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- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: “zero-order hold”
- interpolator’s support is 1
- interpolation is not even continuous

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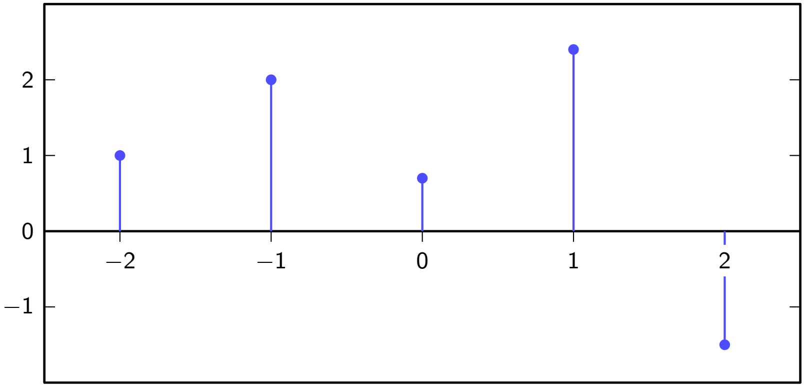

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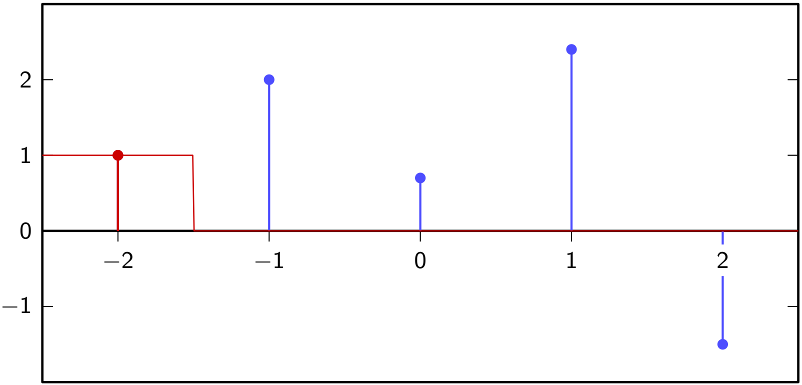

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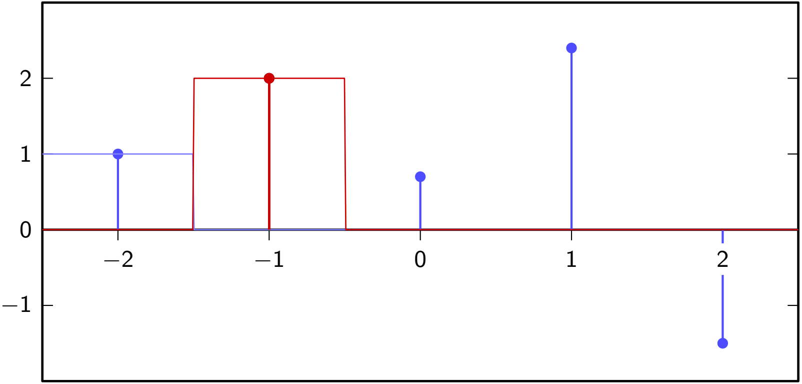

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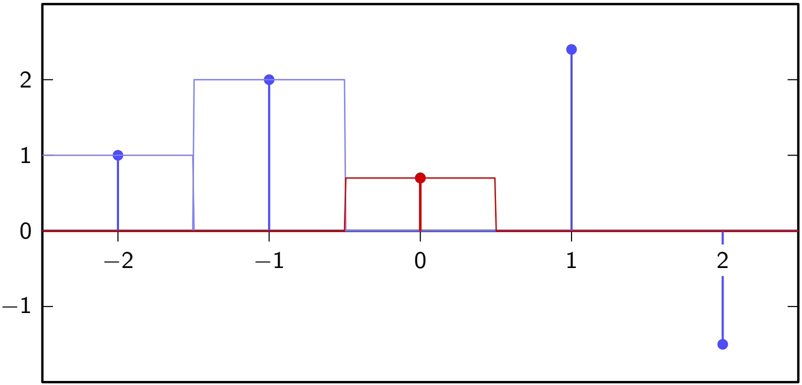

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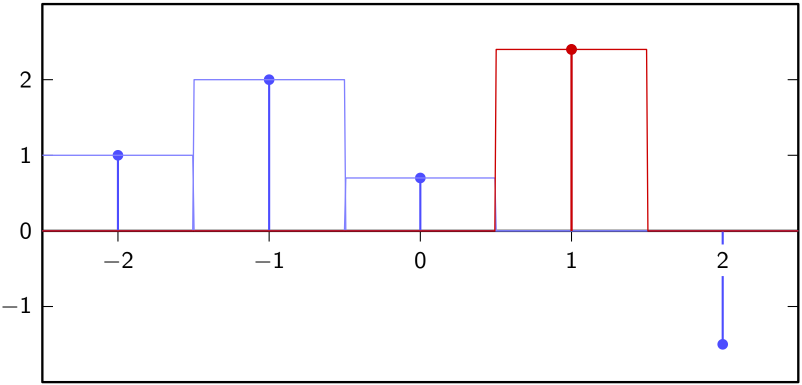

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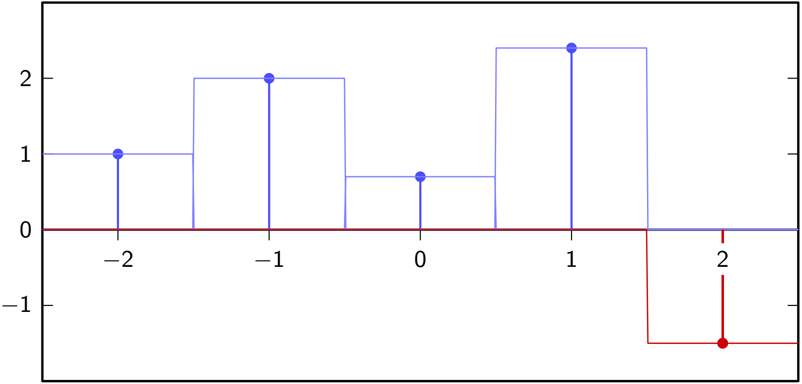

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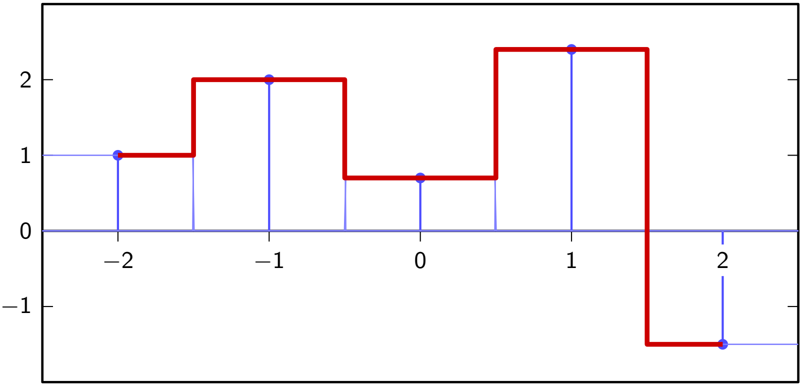
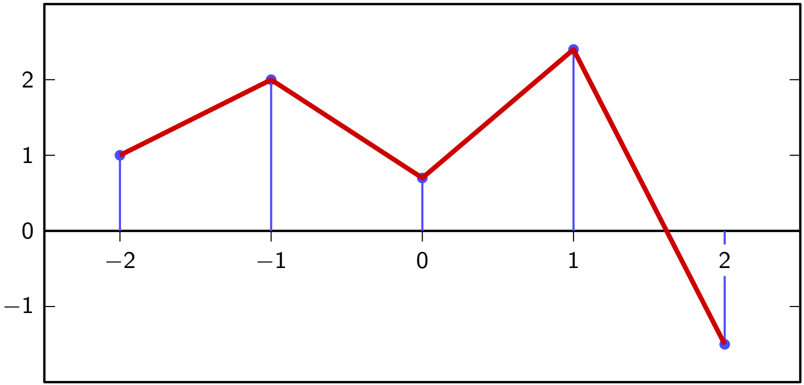


First-order interpolation



First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

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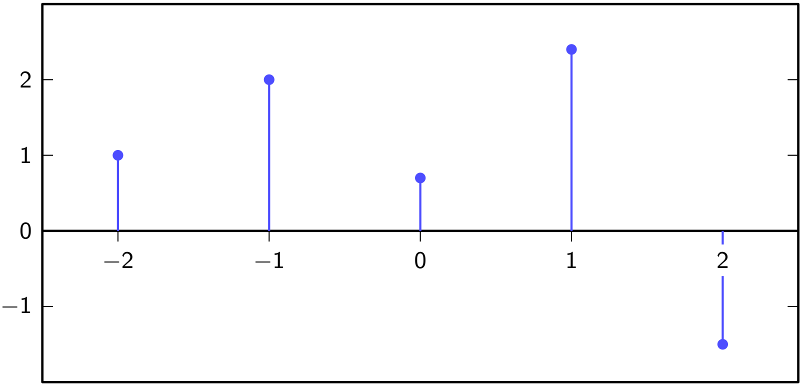

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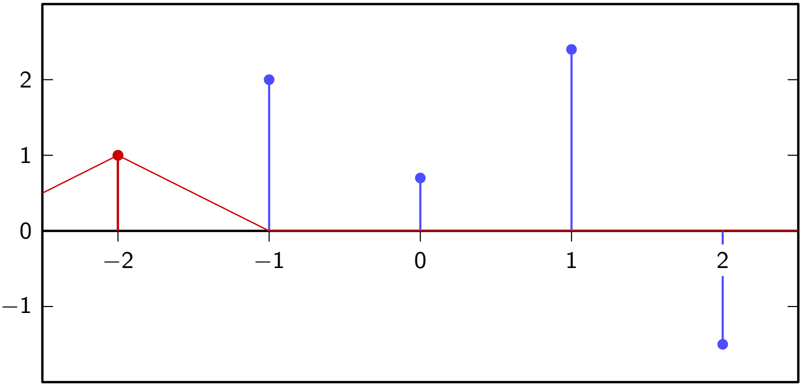

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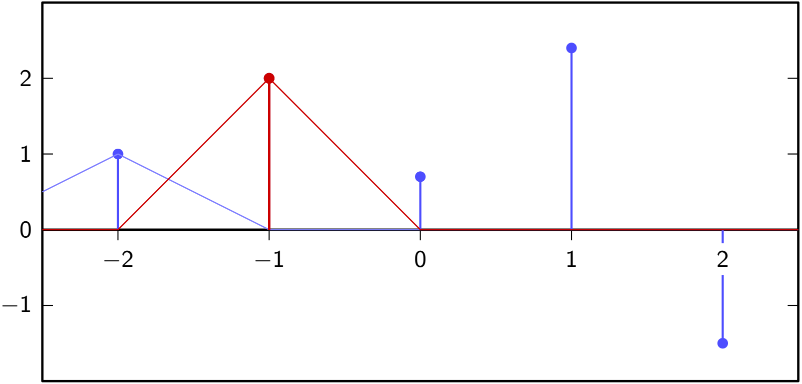

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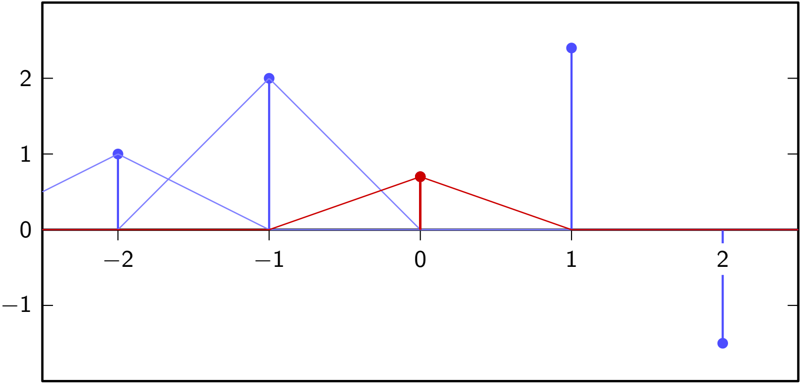

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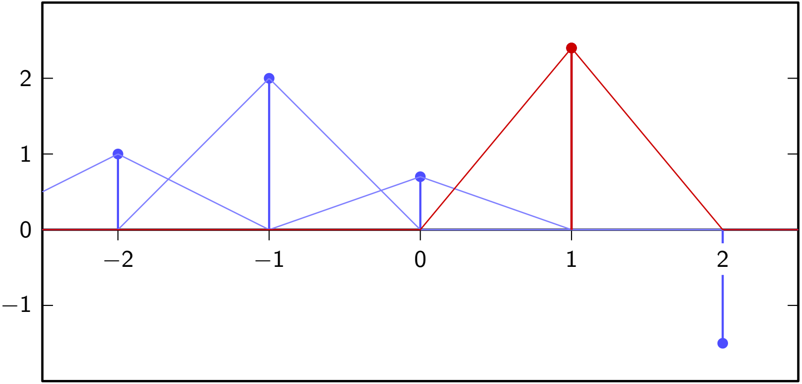

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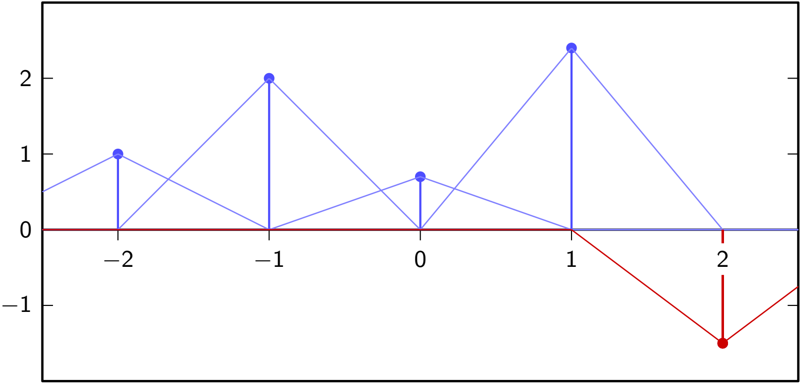

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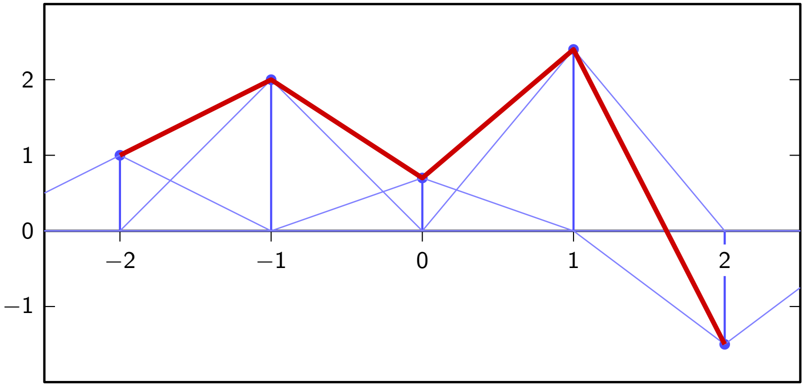

Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
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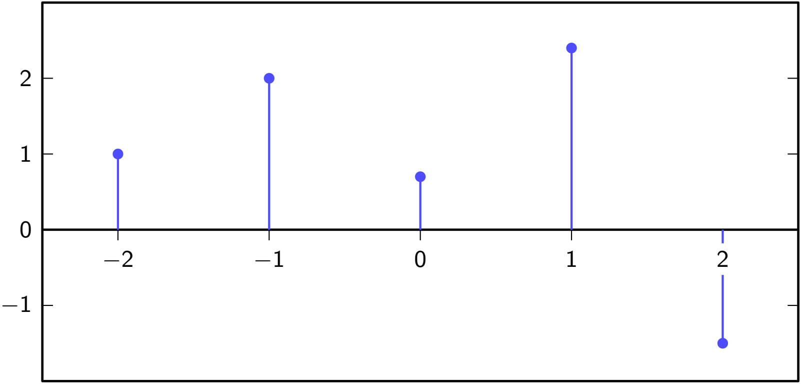

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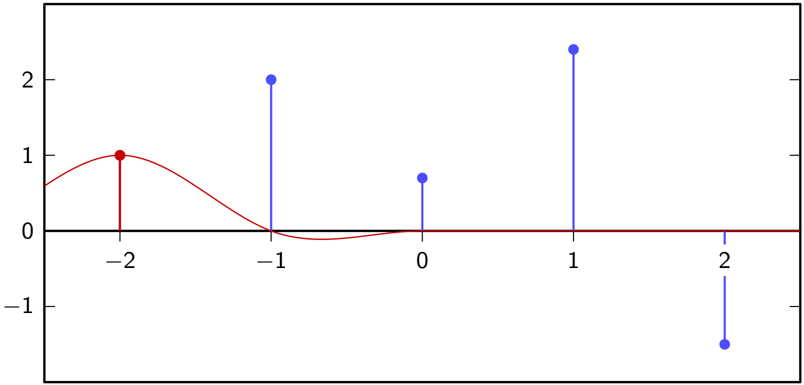

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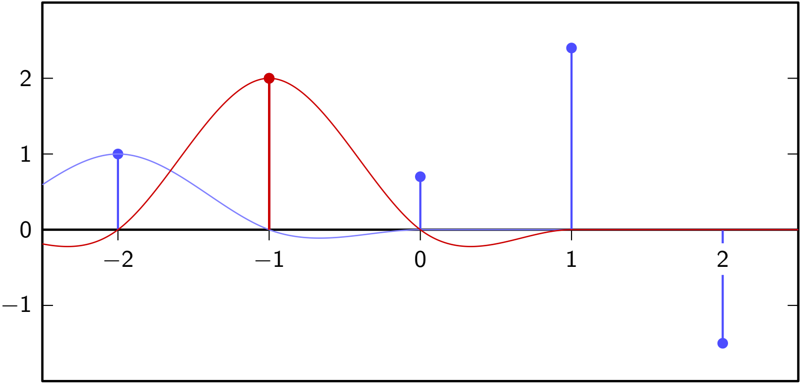

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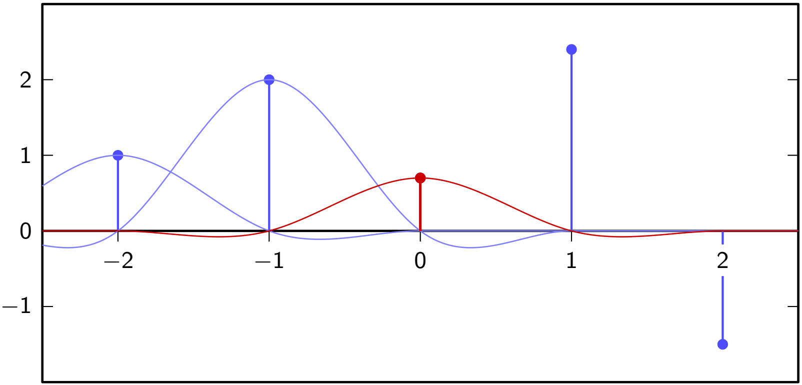

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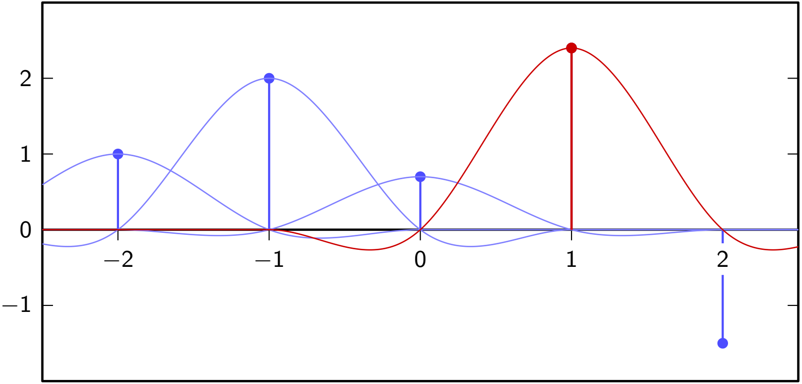

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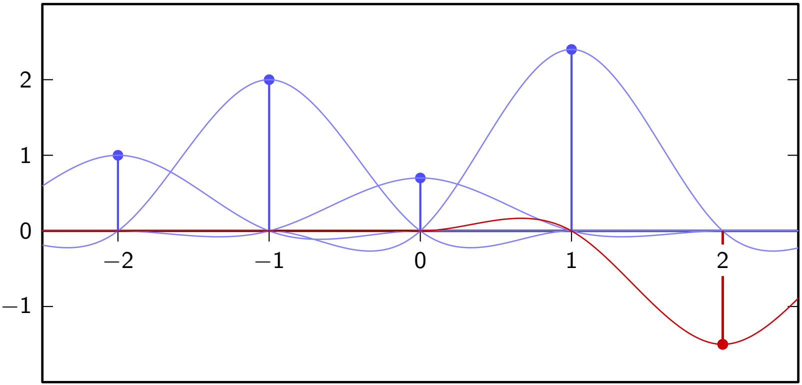

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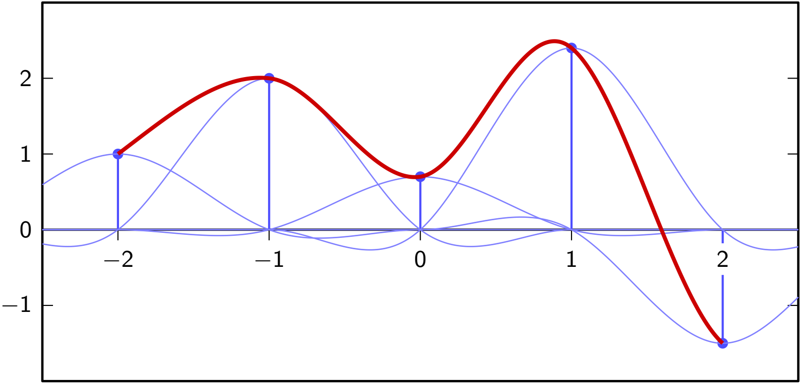

Local interpolation schemes

$$x(t) = \sum_{n=-N}^{N} x[n]\, i_c(t-n)$$

Interpolator's requirements:

- $i_c(0) = 1$
- $i_c(t) = 0$ for t a nonzero integer.

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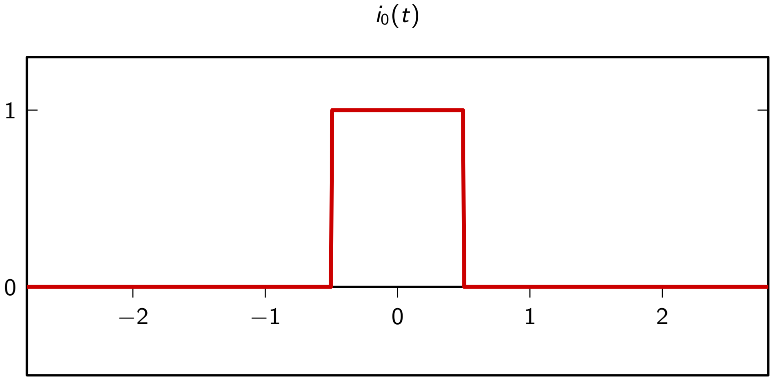

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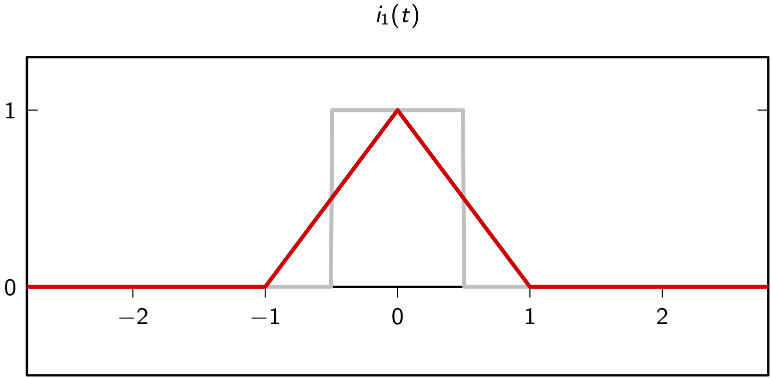

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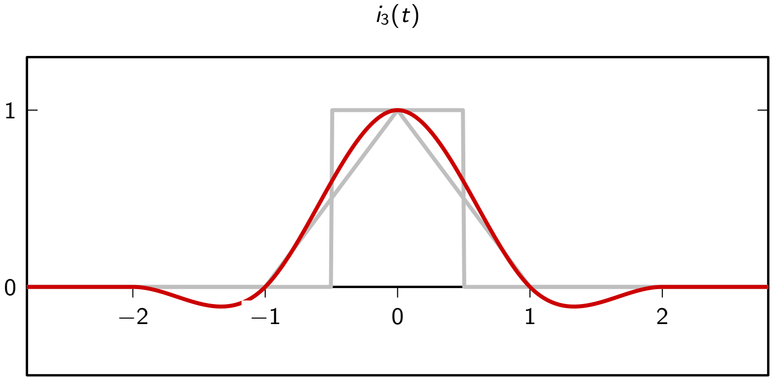


Local interpolation

key property:

- same interpolating function independently of N

drawback:

- lack of smoothness

Polynomial interpolation

key property:

- maximally smooth (infinitely many continuous derivatives)

drawback:

- interpolation kernels depend on N

A remarkable result:

$$\lim_{N\to\infty} L_n^{(N)}(t) = f(t-n)$$

in the limit, local and global interpolation are the same!

A remarkable result:

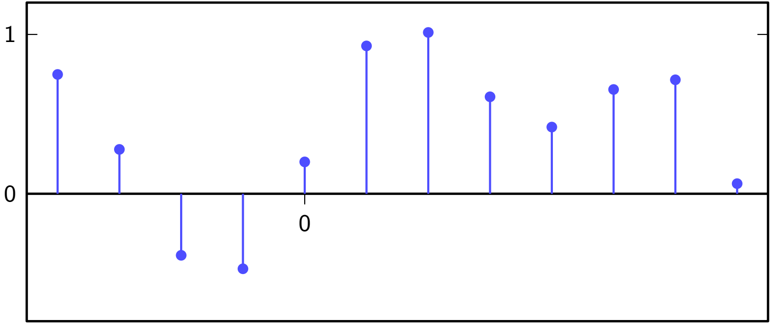
$$\lim_{N\to\infty} L_n^{(N)}(t) = \operatorname{sinc}(t-n)$$

in the limit, local and global interpolation are the same!

Sinc interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Sinc interpolation



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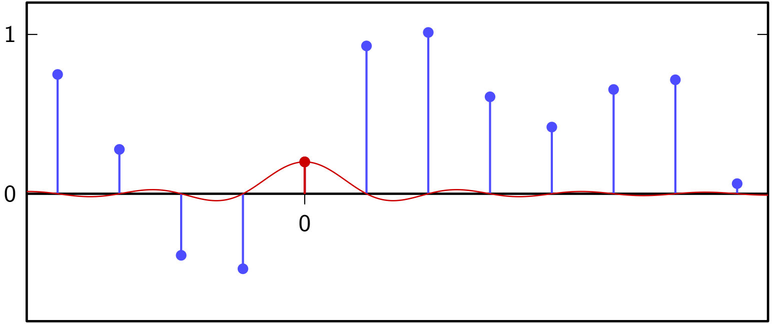

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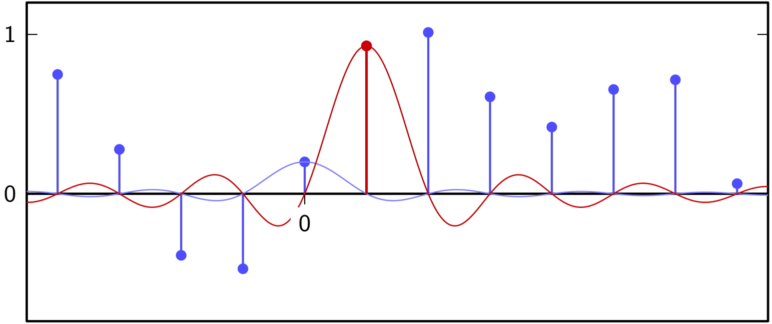

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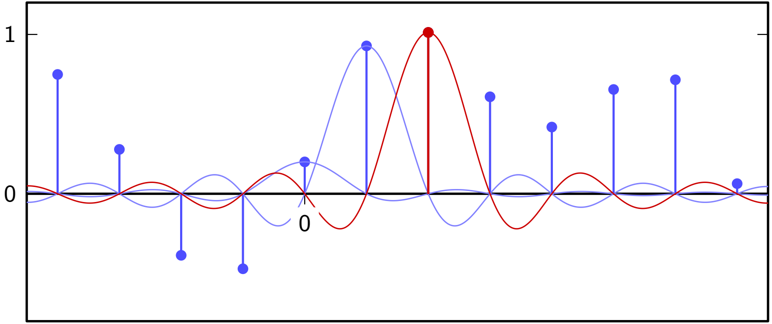

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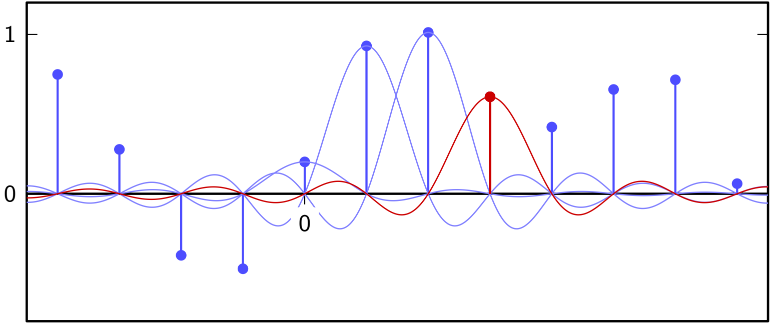

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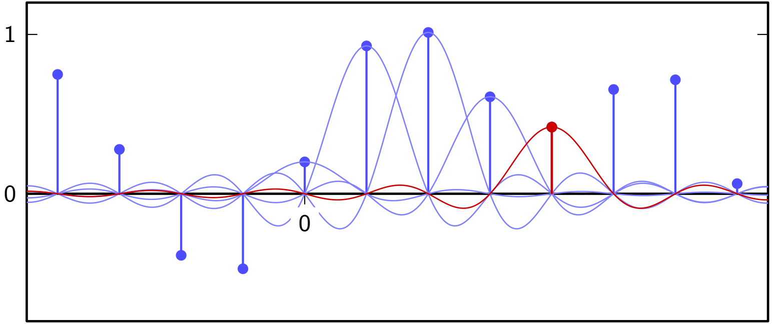

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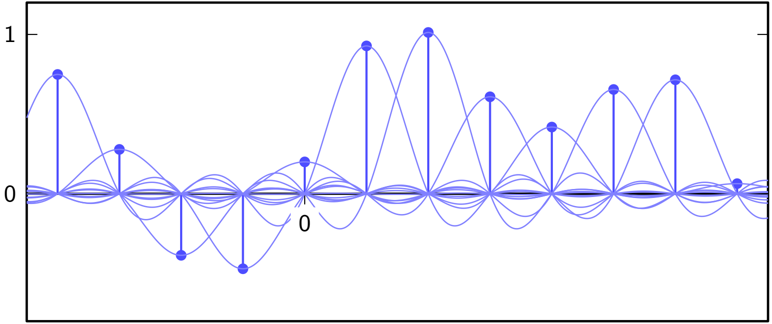

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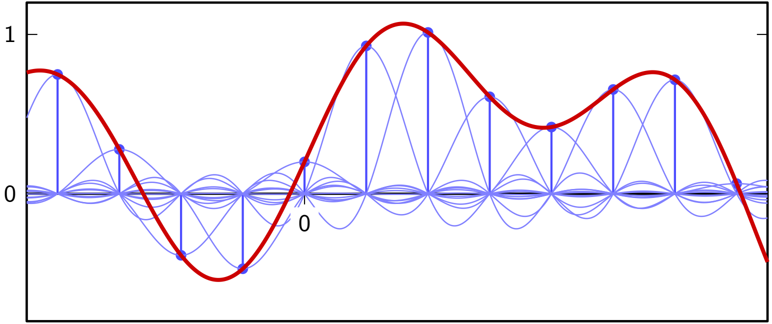

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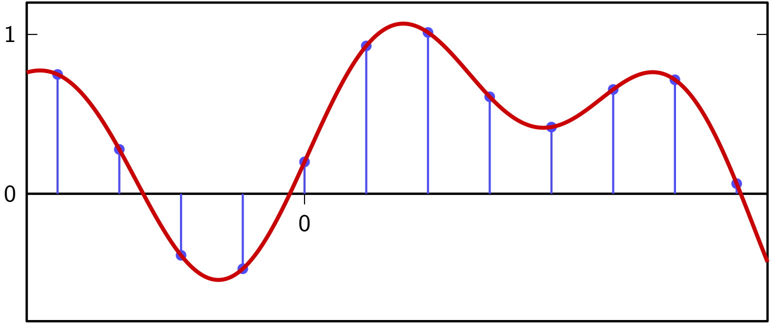

Convergence: graphical “proof”

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{t-k}{n-k}$$

$$L_0^N(t) = \prod_{\substack{k=-N \\ k\neq 0}}^{N} \frac{t-k}{-k} = \prod_{k=-N}^{-1} \frac{t-k}{-k} \prod_{k=1}^{N} \frac{t-k}{-k}$$

$$= \prod_{k=1}^{N} \frac{t+k}{k} \prod_{k=1}^{N} \frac{t-k}{-k}$$

$$= \prod_{k=1}^{N} \left(1 - \frac{t^2}{k^2}\right)$$

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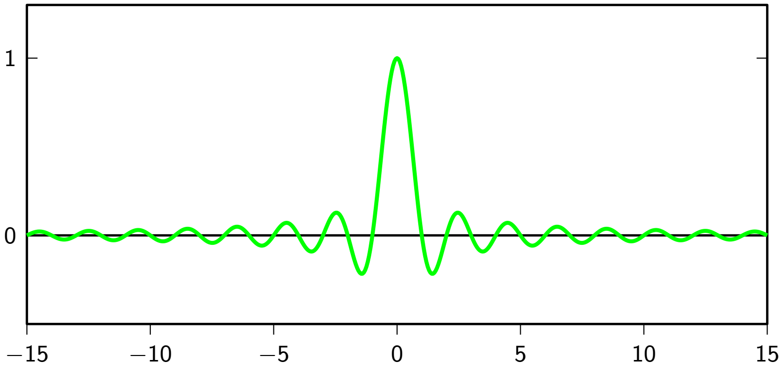

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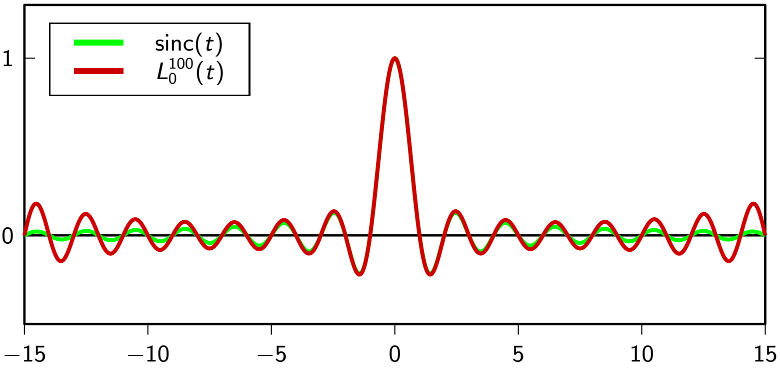

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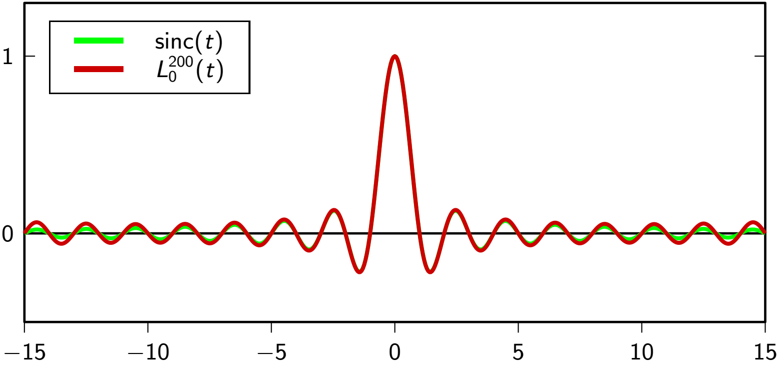

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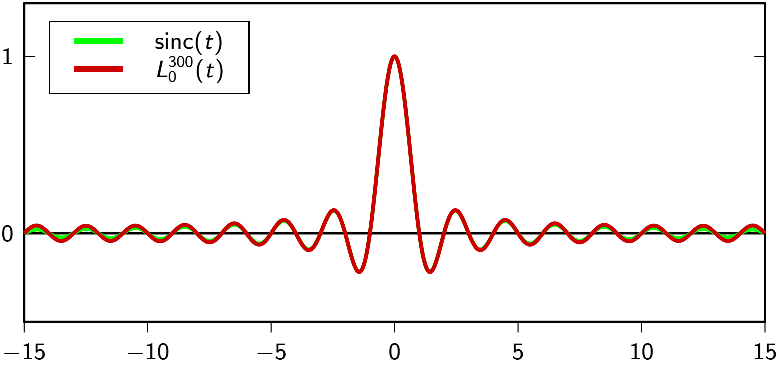

Convergence: mathematical intuition

- $\mathrm{sinc}(t-n)$ and $L_n^{(\infty)}(t)$ share an infinite number of zeros:

$$\mathrm{sinc}(m-n) = \delta[m-n] \qquad m, n \in \mathbb{Z}$$

$$L_n^{(N)}(m) = \delta[m-n] \qquad m, n \in \mathbb{Z}, \quad -N \leq n, m \leq N$$

Convergence: Euler's "proof" (1748)

very cute (if non-rigorous) proof – see handout or book for details

Convergence: rigorous proof

uses the properties of Fourier series expansions – see handout or book for details

bandlimited functions and sampling

Overview:

- Spectrum of interpolated signals
- Space of bandlimited functions
- Sinc sampling
- The sampling theorem

Sinc interpolation

the ingredients:

- discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- interpolation interval T_s
- the sinc function

the result:

- a smooth, continuous-time signal $x(t)$, $t \in \mathbb{R}$

what does the spectrum of $x(t)$ look like?

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Key facts about the sinc

$$\varphi(t) = \text{sinc}\left(\frac{t}{T_s}\right) \quad \longleftrightarrow \quad \Phi(j\Omega) = \frac{\pi}{\Omega_N}\,\text{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$T_s = \frac{\pi}{\Omega_N} \qquad\qquad \Omega_N = \frac{\pi}{T_s}$$

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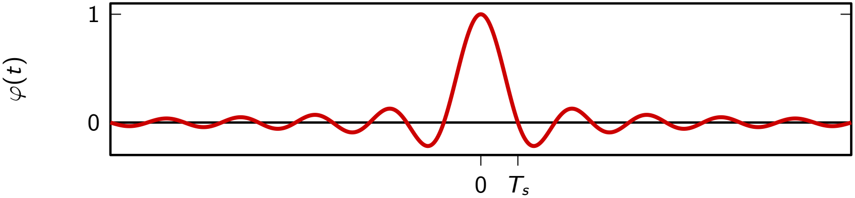

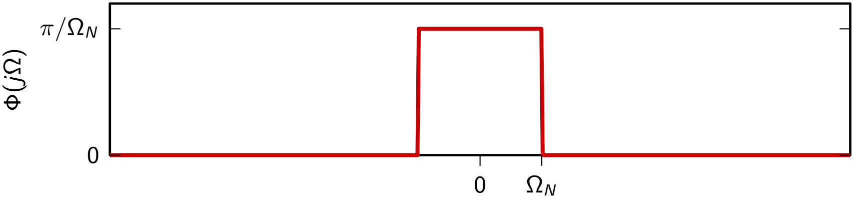

Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Spectral representation (I)

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X(j\Omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt \\
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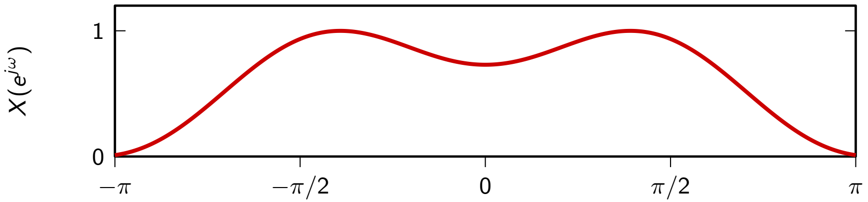
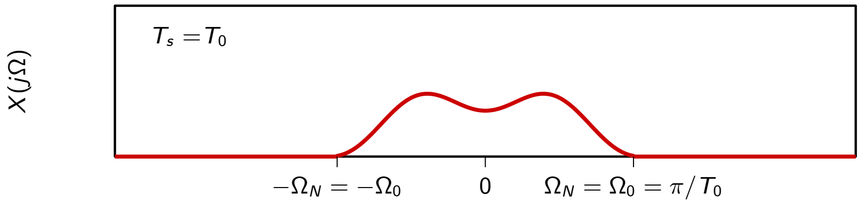
Spectrum of interpolated signals





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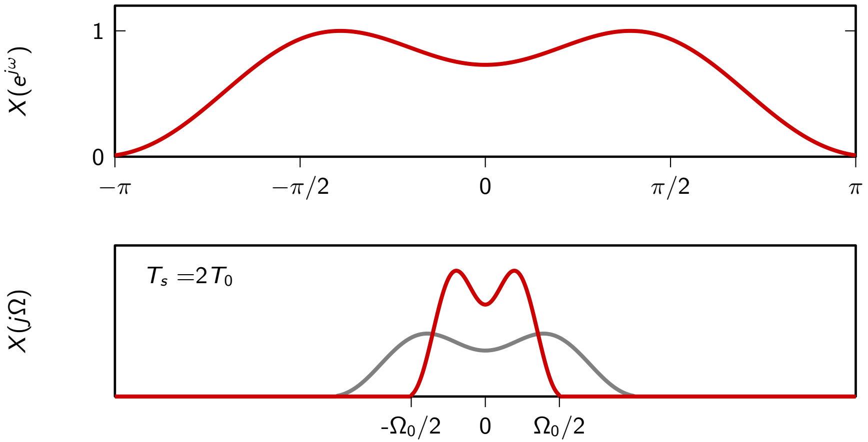

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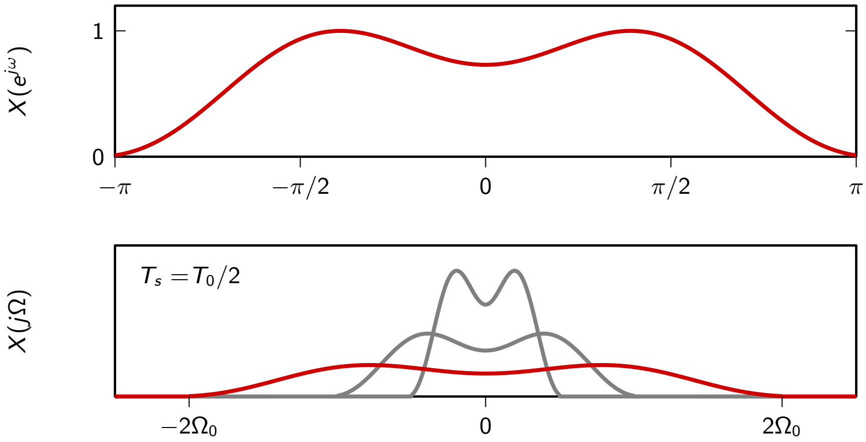

Spectrum of interpolated signals

pick interpolation period T_s:

- $X(j\Omega)$ is Ω_N-bandlimited, with $\Omega_N = \pi/T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

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Space of bandlimited functions

$$x[n] \in \ell_2(\mathbb{Z}) \xrightarrow{\quad T_s \quad} \begin{array}{c} x(t) \in L_2(\mathbb{R}) \\ \Omega_N\text{-BL} \end{array}$$

Space of bandlimited functions

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Let's lighten the notation

for a while we will proceed with

- $T_s = 1$
- $\Omega_N = \pi$

(derivations in the general case are in the book)

The road to the sampling theorem

claims:

- the space of π-bandlimited functions is a Hilbert space
- the functions $\varphi^{(n)}(t) = \mathrm{sinc}(t - n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if $x(t)$ is π-BL, the sequence $x[n] = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct $x(t)$ from $x[n]$)

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The space π-BL

- clearly a vector space because π-BL $\subset L_2(\mathbb{R})$ (and linear combinations of π-BL functions are π-BL functions)
- inner product is standard inner product in $L_2(\mathbb{R})$
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The space of π-BL functions

recap:

- inner product:

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- convolution:

$$(x * y)(t) = \langle x^*(\tau), y(t - \tau) \rangle$$

A basis for the π-BL space

$$\varphi^{(n)}(t) = \text{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
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now use the convolution theorem knowing that:

$$\mathrm{FT}\{\mathrm{sinc}(t)\} = \mathrm{rect}\left(\frac{\Omega}{2\pi}\right)$$

$$(\mathrm{sinc} * \mathrm{sinc})(m-n) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\mathrm{rect}\left(\frac{\Omega}{2\pi}\right)\right]^2 e^{j\Omega(m-n)}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(m-n)}\,d\Omega$$

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Sampling as a basis expansion

for any $x(t) \in \pi$-BL:

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Sampling as a basis expansion, π-BL

Analysis formula:

$$x[n] = \langle \mathrm{sinc}(t-n), x(t) \rangle$$

Synthesis formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}(t-n)$$

Sampling as a basis expansion, Ω_N-BL

Analysis formula:

$$x[n] = \left\langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle = T_s\, x(nT_s)$$

Synthesis formula:

$$x(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

The sampling theorem

- the space of Ω_N-bandlimited functions is a Hilbert space
- set $T_s = \pi/\Omega_N$
- the functions $\varphi^{(n)}(t) = \mathrm{sinc}((t - nT_s)/T_s)$ form a basis for the space
- for any $x(t) \in \Omega_N$-BL the coefficients in the sinc basis are the (scaled) samples $T_s\, x(nT_s)$

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The sampling theorem, corollary

- Ω_N-BL $\subseteq$ Ω-BL for any $\Omega \geq \Omega_N$

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The sampling theorem, in hertz

any signal $x(t)$ bandlimited to F_N Hz can be sampled with no loss of information using a sampling frequency $F_s \geq 2F_N$ (i.e. a sampling period $T_s \leq 1/2F_N$)