COM303: Digital Signal Processing

Lecture 19: Quantization

overview

- quantization
- A/D and D/A converters
- oversampling

quantization

Overview:

- Quantization
- Uniform quantization and error analysis
- Clipping, saturation, companding

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$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
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The simplest quantizer:

- each sample is encoded individually (hence *scalar*)
- each sample is quantized independently (memoryless quantization)
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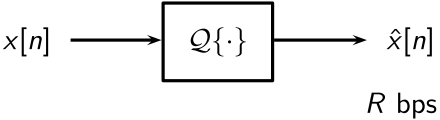

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Input signal: $A \leq x[n] \leq B$ (A, B can be ∞)

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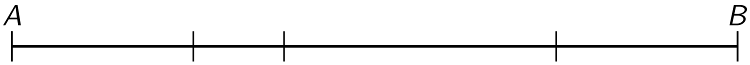

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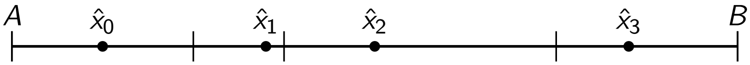
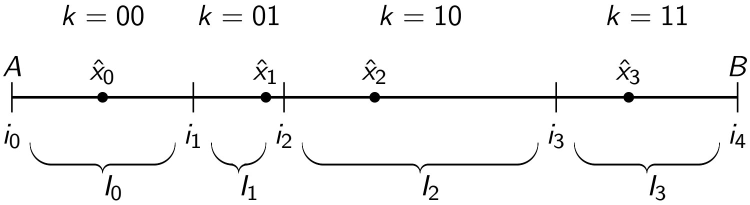


Scalar quantization

Example for $R = 2$:



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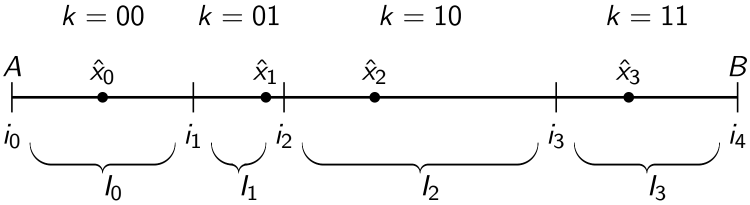

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Optimal Quantization

The optimal quantizer minimizes the energy of the quantization error:

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
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Quantization MSE

$$\sigma_e^2 = \int_{-\infty}^{\infty} (x - \mathcal{Q}\{x\})^2 \, f_x(x) \, dx$$

$$= \sum_{k=0}^{2^R-1} \int_{i_k}^{i_{k+1}} (x - \hat{x}_k)^2 \, f_x(x) \, dx$$

find global minimum wrt i_k, $\hat{x}_k$

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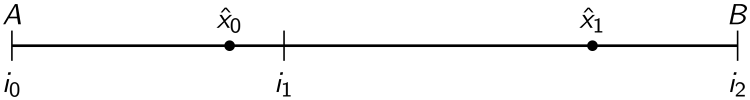
Simple example: optimal one-bit quantizer



3 free parameters: $i_1, \hat{x}_0, \hat{x}_1$

Simple example: optimal one-bit quantizer

$$\sigma_e^2 = \int_A^{i_1} (x - \hat{x}_0)^2 \, f_x(x) \, dx + \int_{i_1}^{B} (x - \hat{x}_1)^2 \, f_x(x) \, dx$$

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$$\frac{\partial \sigma_e^2}{\partial i_1} = \frac{\partial \sigma_e^2}{\hat{x}_0} = \frac{\partial \sigma_e^2}{\hat{x}_1} = 0$$

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little calculus reminder

$$\frac{\partial}{\partial t}\int_{\alpha}^{t} f(\tau)\,d\tau = \frac{\partial}{\partial t}\left[F(t) - F(\alpha)\right] = f(t)$$

Optimal one-bit quantizer: threshold

$$\frac{\partial \sigma_e^2}{\partial i_1} = \frac{\partial}{\partial i_1}\left[\int_A^{i_1} (x - \hat{x}_0)^2 f_x(x)\, dx + \int_{i_1}^{B} (x - \hat{x}_1)^2 f_x(x)\, dx\right]$$

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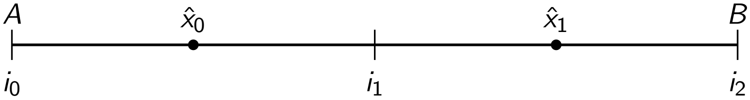

Uniform quantization of uniform input

- simple but very general case
- optimal subdivision: 2^R *equal* intervals of width $\Delta = (B - A)2^{-R}$
- optimal quantization values: interval's midpoint

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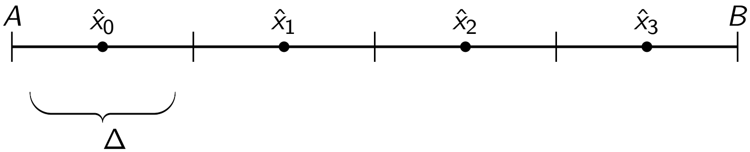
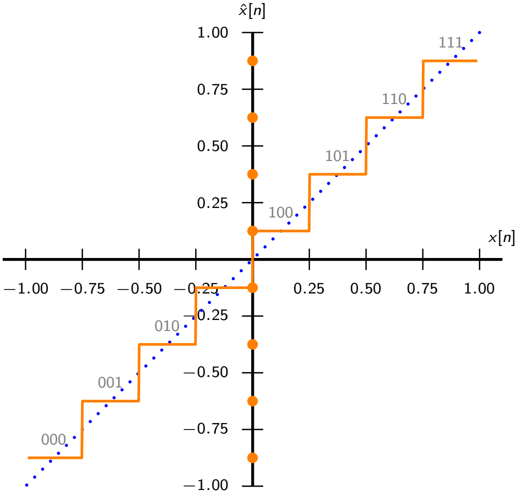


Uniform 3-Bit quantization function



Uniform quantization of uniform input: error analysis

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$$= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(x)(\hat{x}_k - x)^2 \, dx$$

$$f_x(s) = \frac{1}{B-A}$$

$$\Delta = \frac{B-A}{2^R}$$

$$I_k = [A + k\Delta, A + (k+1)\Delta]$$

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$$= 2^R \int_0^{\Delta} \frac{(\Delta/2 - x)^2}{B - A}\, dx$$

$$= \frac{\Delta^2}{12}$$

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Error analysis of the quantization error

fundamental assumptions:

- signal and quantization error are uncorrelated (ok-ish)
- quantization error process is white (stretch)

quantization noise acts as additive white noise

Error analysis

- error energy

$$\sigma_e^2 = \Delta^2/12, \qquad \Delta = (B - A)/2^R$$

- signal energy

$$\sigma_x^2 = (B - A)^2/12$$

- signal to noise ratio

$$\mathrm{SNR} = 2^{2R}$$

- in dB

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \log_{10} 2^{2R} \approx 6R \text{ dB}$$

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The "6dB/bit" rule of thumb

- a compact disk has 16 bits/sample:

$$\text{max SNR} = 96\text{dB}$$

- a DVD has 24 bits/sample:

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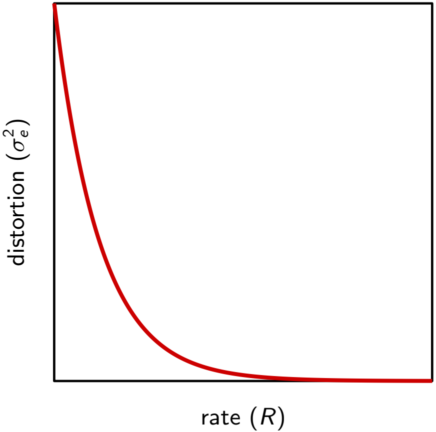
Rate/Distortion Curve



Other quantization errors

If input is not bounded to $[A, B]$:

- clip samples to $[A, B]$: linear distortion (can be put to good use in guitar effects!)
- smoothly saturate input: this simulates the saturation curves of analog electronics

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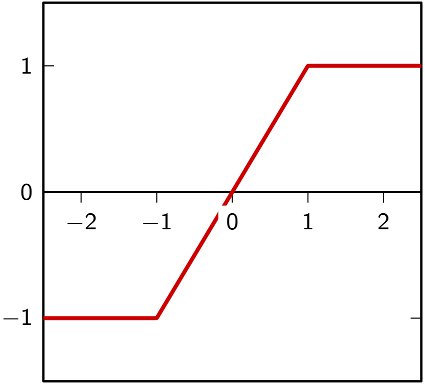
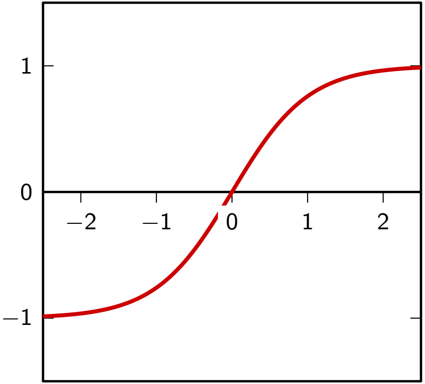
Clipping vs saturation





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- use uniform quantizer and accept increased error.
  For instance, if input is Gaussian:

$$\sigma_e^2 = \frac{\sqrt{3}\pi}{2}\,\sigma^2\,\Delta^2$$

- use "companders"
- design optimal quantizer for input distribution, if known (Lloyd-Max algorithm)

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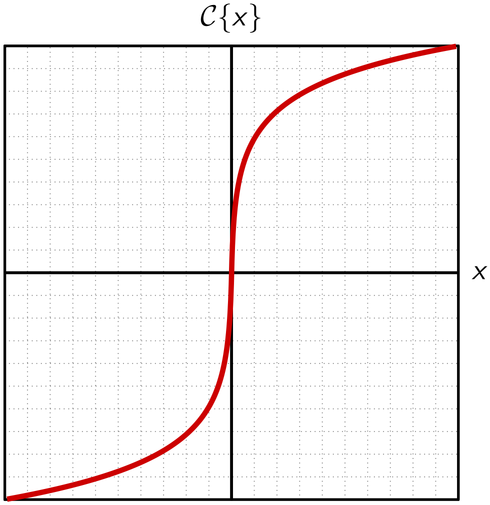
μ-law compander

$$\mathcal{C}\{x[n]\} = \operatorname{sgn}(x[n])\frac{\ln(1+\mu|x[n]|)}{\ln(1+\mu)}$$



Lloyd-Max Quantizer design

$$\sigma_e^2 = \sum_{k=0}^{2^R-1} \int_{i_k}^{i_{k+1}} (x - \hat{x}_k)^2 \, f_x(x) \, dx$$

$$\text{A)} \quad \frac{\partial \sigma_e^2}{\partial \hat{x}_k} = 0 \Rightarrow \hat{x}_k = \frac{\displaystyle\int_{i_{k-1}}^{i_k} x \, f_x(x) \, dx}{\displaystyle\int_{i_{k-1}}^{i_k} f_x(x) \, dx}$$

$$\text{B)} \quad \frac{\partial \sigma_e^2}{\partial i_k} = 0 \Rightarrow i_k = \frac{\hat{x}_{k-1} + \hat{x}_k}{2}$$

Lloyd-Max Quantizer design

- start with a guess for the i_k
- solve A and B iteratively until convergence

A/D and D/A converters

Overview:

- Analog-to-digital (A/D) conversion
- Digital-to-analog (D/A) conversion

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From analog to digital

- sampling discretizes time
- quantization discretized amplitude
- how is it done in practice?

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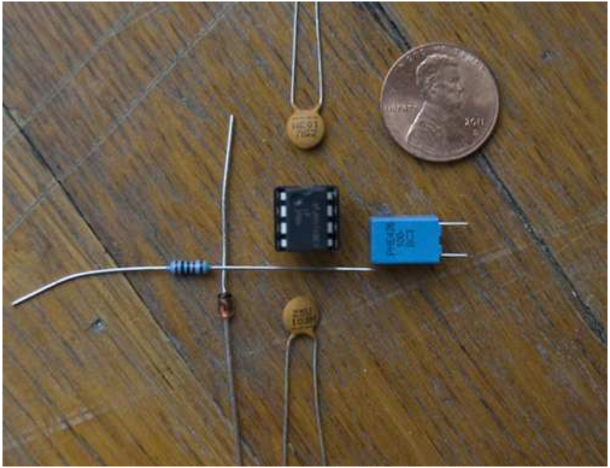
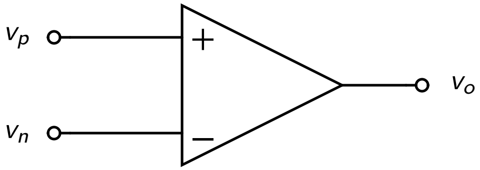
From analog to digital



A tiny bit of electronics: the op-amp



$$v_o = G(v_p - v_n)$$

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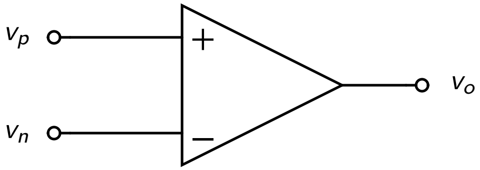

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The two key properties

- infinite input gain ($G \approx \infty$)
- zero input current

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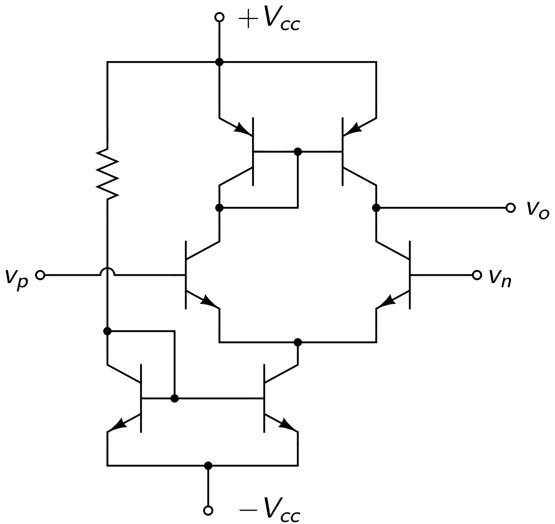
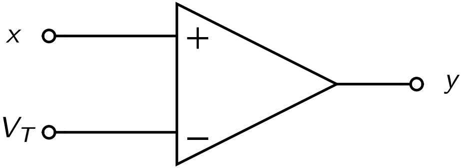
Inside the box



The op-amp in open loop: comparator



$$y = \begin{cases} +V_{cc} & \text{if } x > V_T \\ -V_{cc} & \text{if } x < V_T \end{cases}$$

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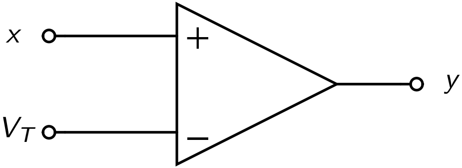

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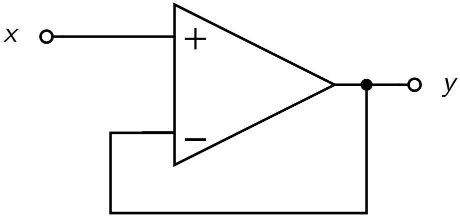
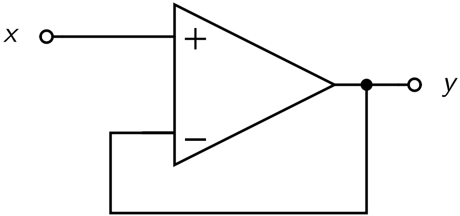
The op-amp in closed loop: buffer



$$y = x$$

The op-amp in closed loop: buffer



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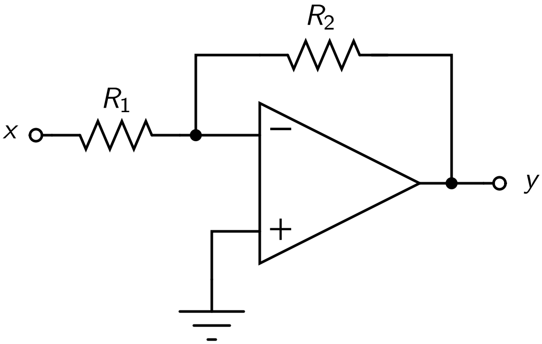
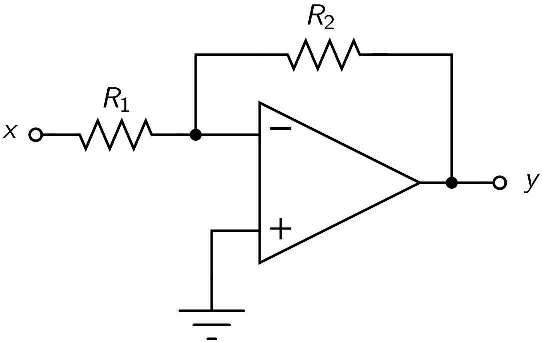
The op-amp in closed loop: inverting amplifier



$$y = -(R_2/R_1)x$$

The op-amp in closed loop: inverting amplifier



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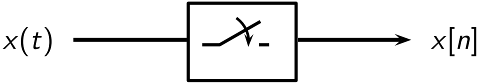
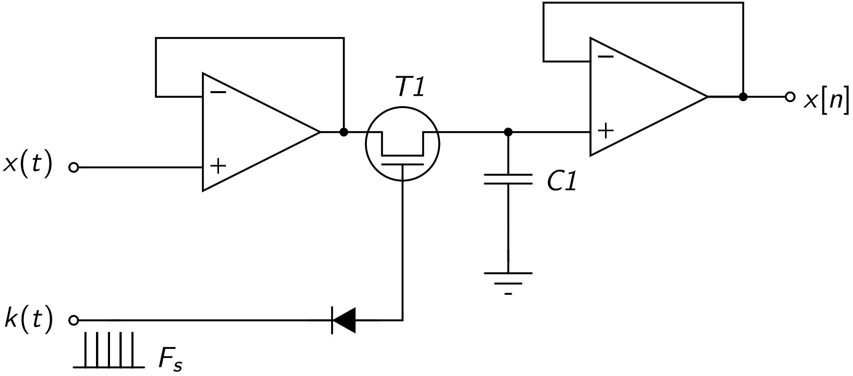
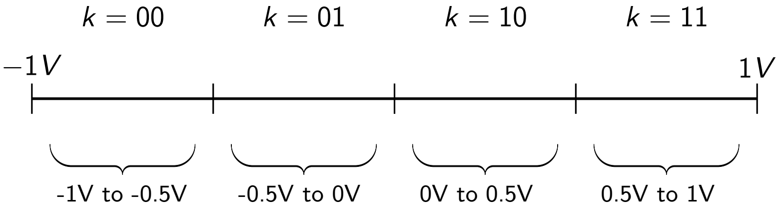
A/D Converter: Sampling



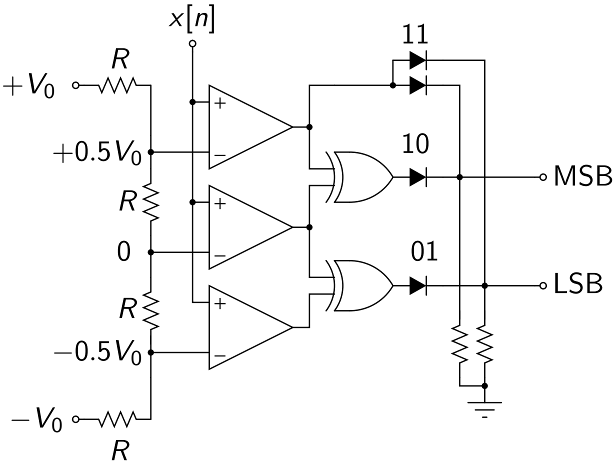
A/D Converter: Sample & Hold



A/D Converter: Quantization



A/D Converter: 2-Bit Quantizer

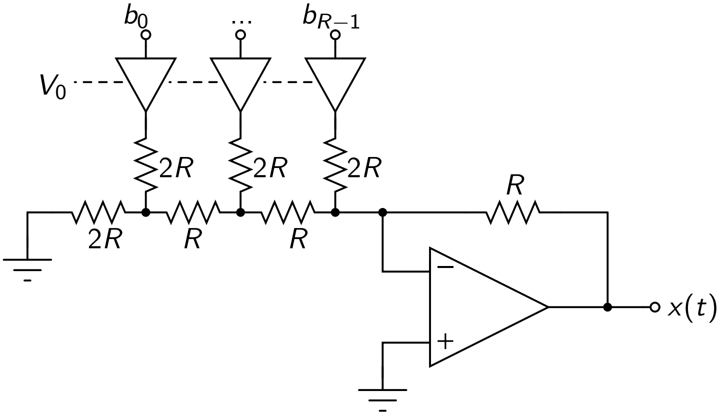


D/A Converter

$$x_B[n] = b_{R-1} b_{R-2} \ldots b_1 b_0$$

$$\hat{x}[n] = \sum_{k=0}^{R-1} \frac{V_0}{2^k} b_k$$

D/A Converter



oversampling

Oversampling

- oversampled A/D
  - reduce quantization error
- oversampled D/A
  - use cheaper hardware for interpolation

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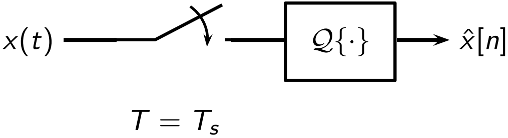
Oversampled A/D



$$\hat{x}[n] = x[n] + e[n]$$

Oversampled A/D

Key assumptions:

$e[n]$ i.i.d. process, independent of $x[n]$

$$P_e(e^{j\omega}) = \frac{\Delta^2}{12} \qquad \text{over } [-\pi, \pi] \text{ (no aliasing)}$$

Key observation:

$$X(e^{j\omega}) = \frac{1}{T_s} X\left(j\frac{\omega}{T_s}\right)$$

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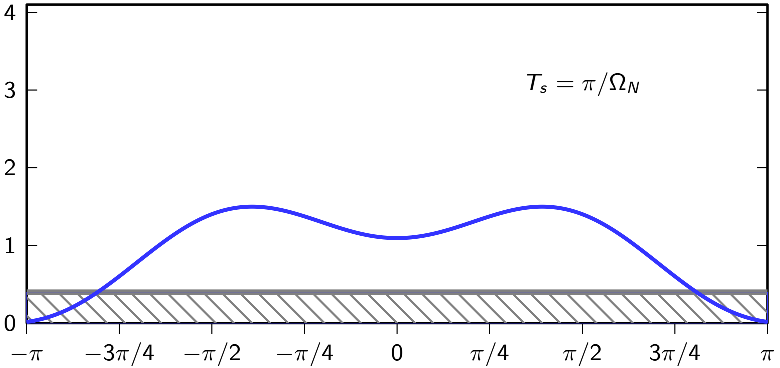
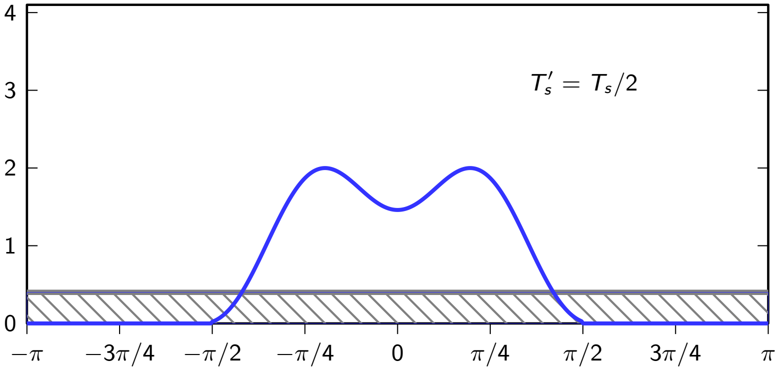
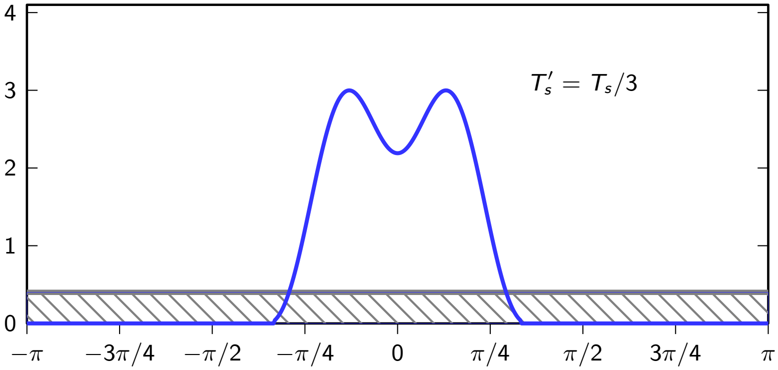


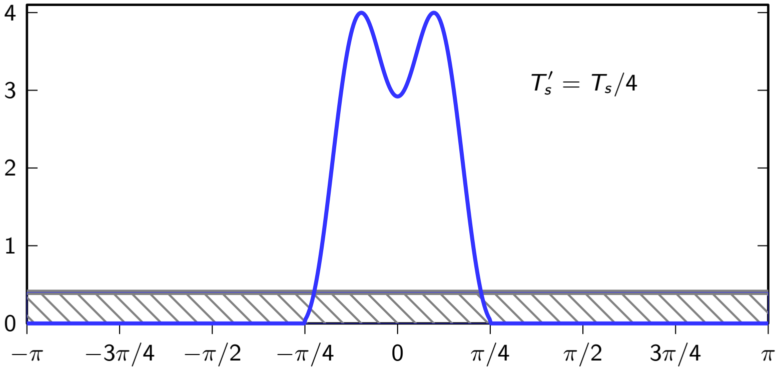
Oversampled A/D



Oversampled A/D



Oversampled A/D



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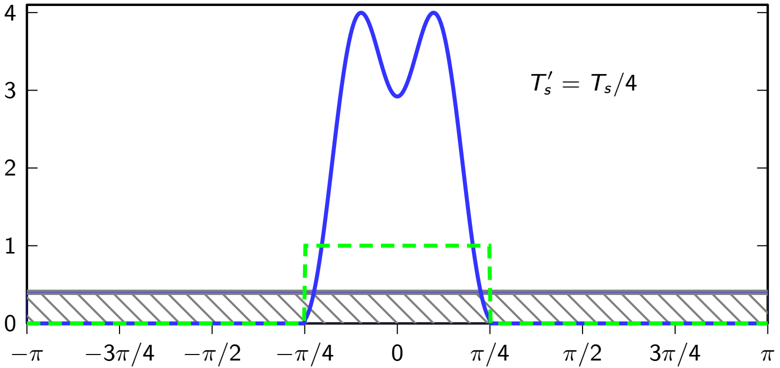
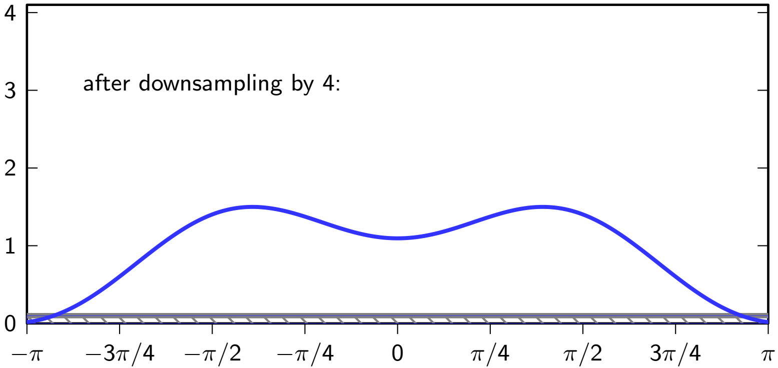


Oversampled A/D



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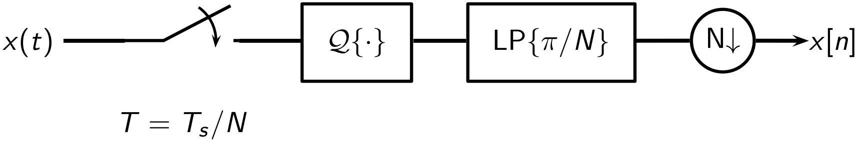


- $\mathrm{SNR}_O \approx N\,\mathrm{SNR}$
- 3dB per octave (doubling of F_s)
- but key assumption (independence) breaks down fast...

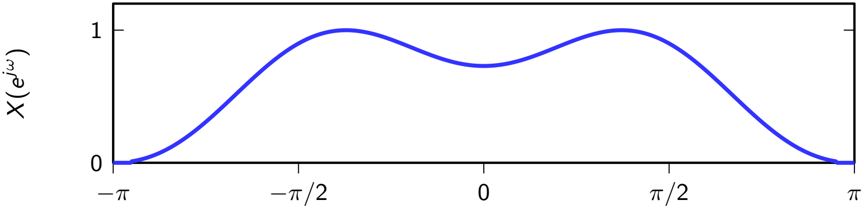
Oversampled D/A

Oversampled D/A

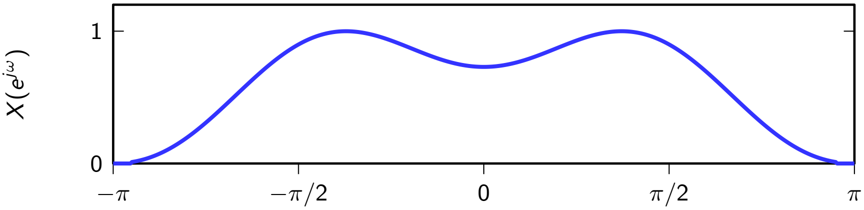
Sinc interpolation:

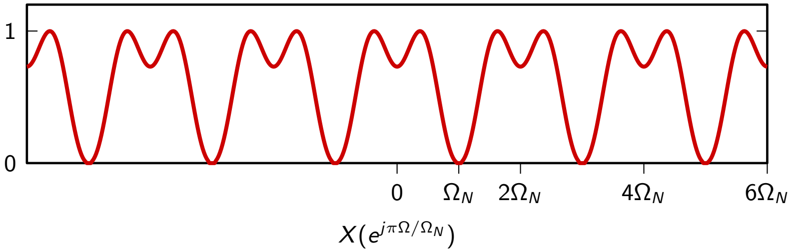
$$X_c(j\Omega) = \frac{\pi}{\Omega_N} X(e^{j\pi\Omega/\Omega_N}) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

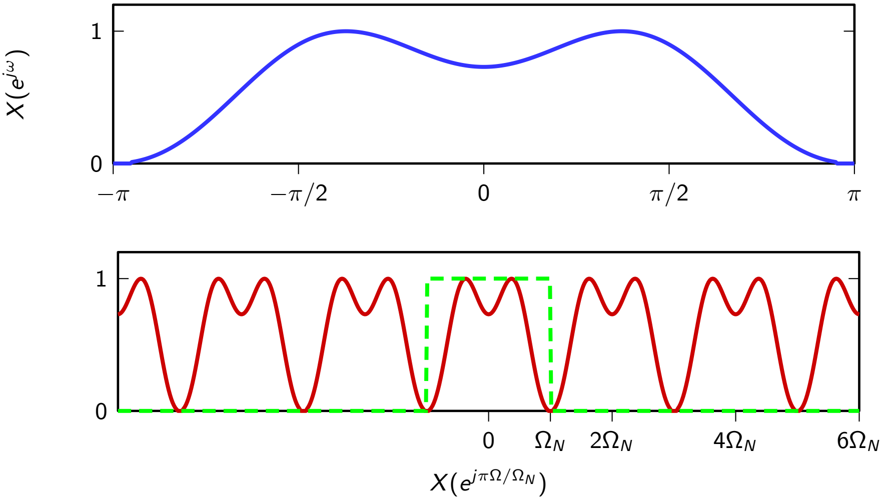
Sinc interpolation



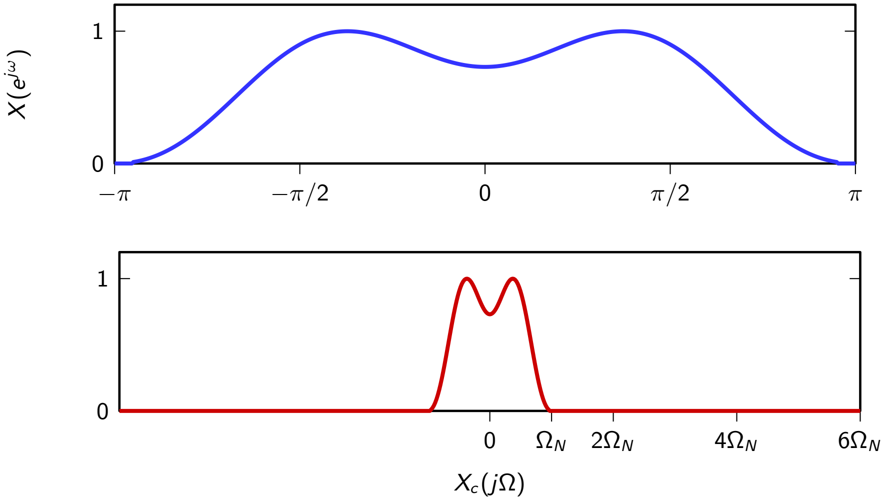
Sinc interpolation






1
0
$X(e^{j\omega})$
$-\pi$
$-\pi/2$
0
$\pi/2$
π
1
0
0
Ω_N
$2\Omega_N$
$4\Omega_N$
$6\Omega_N$
$X(e^{j\pi\Omega/\Omega_N})$

Sinc interpolation



Oversampled D/A

In general:

$$X_c(j\Omega) = \frac{\pi}{\Omega_N} X(e^{j\pi\Omega/\Omega_N})\, I\left(j\pi\frac{\Omega}{\Omega_N}\right)$$

The cheapest (hence most common) interpolator:

$$i(t) = \mathrm{rect}(t)$$

$$I(j\Omega) = \mathrm{sinc}\left(\frac{\Omega}{2\pi}\right)$$

Oversampled D/A

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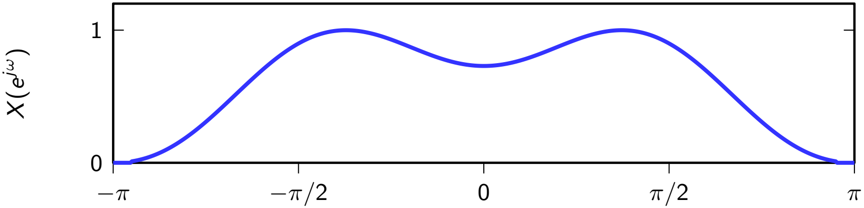
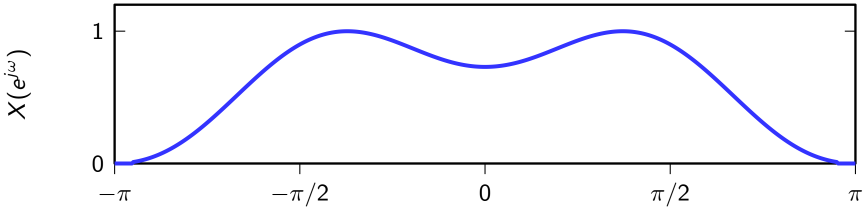
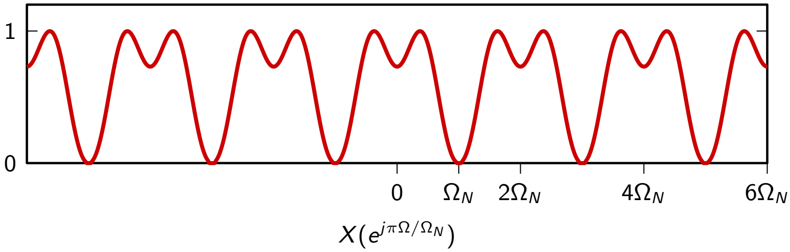
ZOH interpolation



ZOH interpolation





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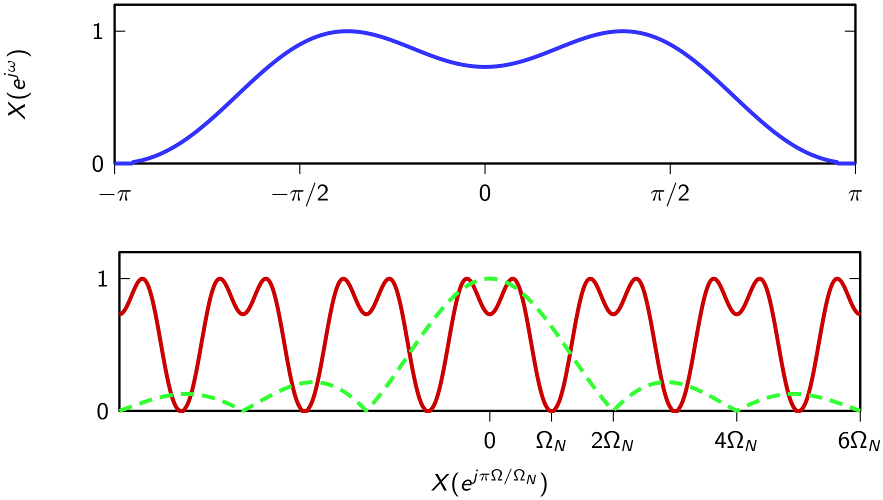

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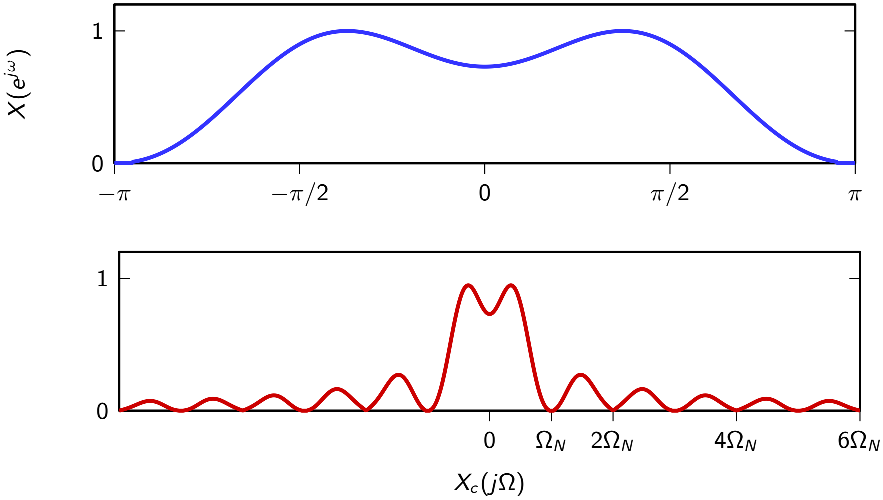
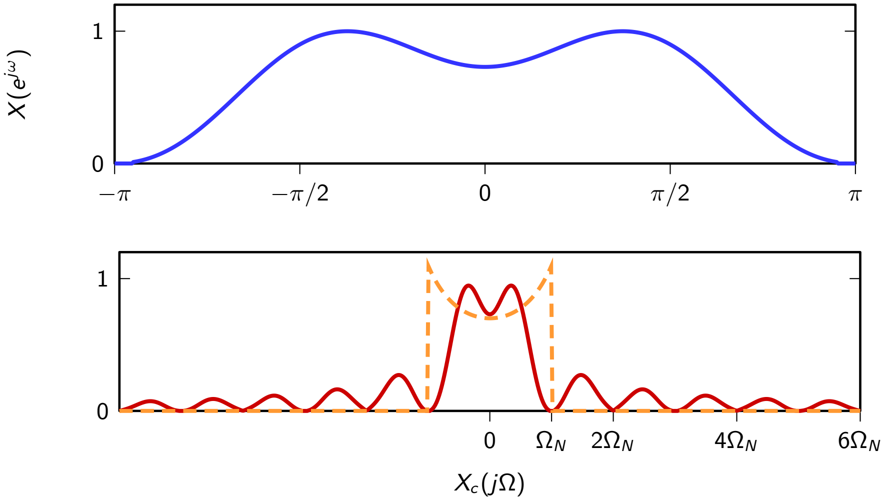


ZOH interpolation



Oversampled A/D

key problems:

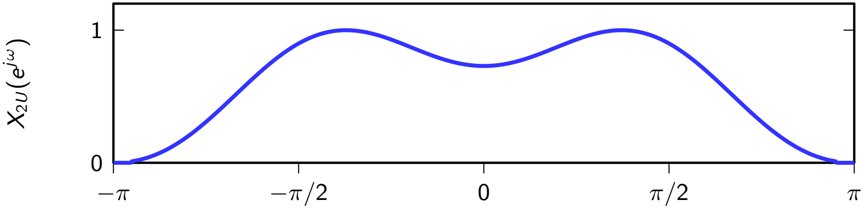
- we need to undo the in-band distortion in the analog domain
- we have a significant out-of-band distortion
- only advantage: minimal D/A rate

Oversampled D/A

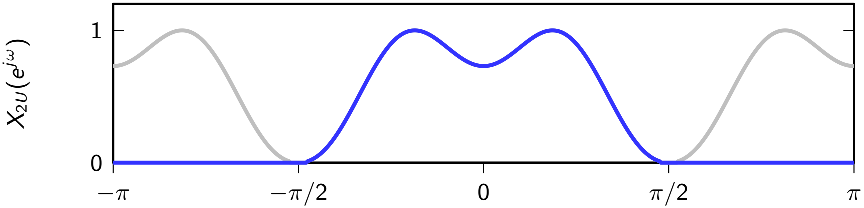
consider a K-upsampled and interpolated version of $x[n]$:

$$X_K(e^{j\omega}) = X(e^{j\omega K})\,\mathrm{rect}\left(\frac{\omega K}{2\pi}\right) \qquad 2\pi\text{-periodic}$$

Oversampled D/A



Oversampled D/A



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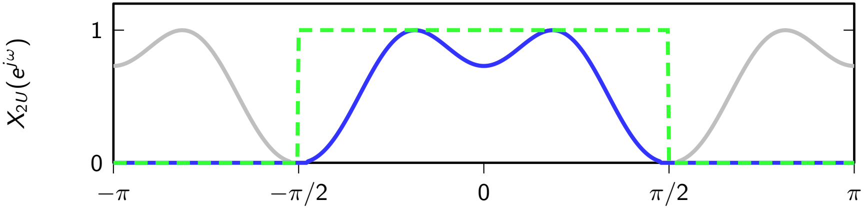
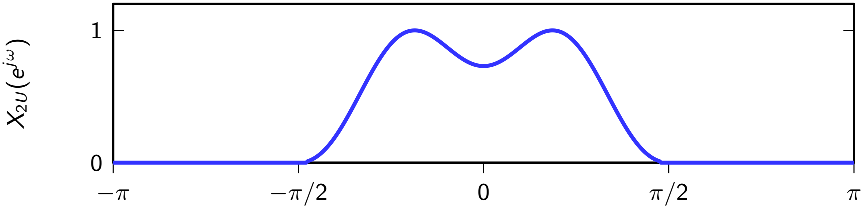


Oversampled D/A



Oversampled D/A

sinc-interpolate $x_K[n]$ with $T_s' = T_s/K$:

$$\Omega_N' = K\Omega_N$$

$$X_{c,K}(j\Omega) = \frac{\pi}{\Omega_N'} X_K(e^{j\omega})|_{\omega=\pi\Omega/\Omega_N'} \text{ rect}\left(\frac{\Omega}{2\Omega_N'}\right)$$

$$= \frac{\pi}{K\Omega_N} X(e^{j\pi\Omega/\Omega_N}) \text{ rect}\left(\frac{\Omega}{2\Omega_N}\right) \text{ rect}\left(\frac{\Omega}{2K\Omega_N}\right)$$

$$= \frac{1}{K} X_c(j\Omega) \quad \text{for } |\Omega| < \Omega_N$$

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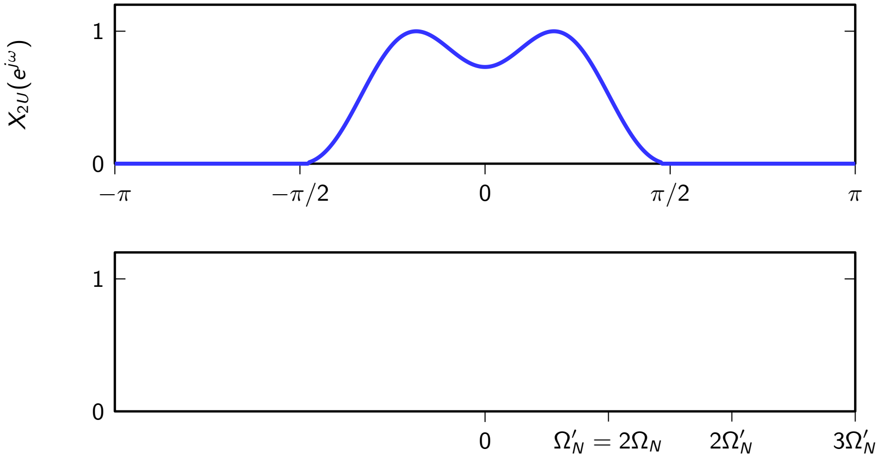
Oversampled D/A

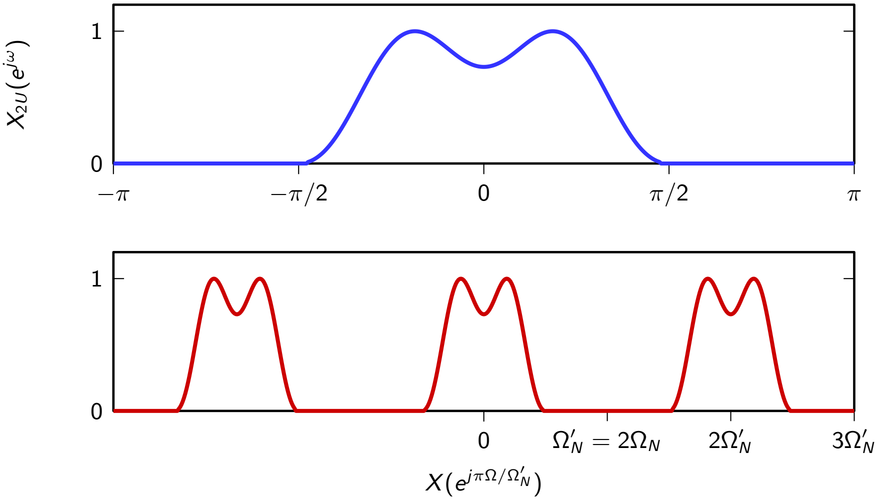
sinc-interpolate $x_K[n]$ with $T'_s = T_s/K$:

$$
\begin{aligned}
\Omega'_N &= K\Omega_N \\
X_{c,K}(j\Omega) &= \frac{\pi}{\Omega'_N} X_K(e^{j\omega})|_{\omega=\pi\Omega/\Omega'_N} \text{ rect}\left(\frac{\Omega}{2\Omega'_N}\right) \\
&= \frac{\pi}{K\Omega_N} X(e^{j\pi\Omega/\Omega_N}) \text{ rect}\left(\frac{\Omega}{2\Omega_N}\right) \text{ rect}\left(\frac{\Omega}{2K\Omega_N}\right) \\
&= \frac{1}{K} X_c(j\Omega) \quad \text{for } |\Omega| < \Omega_N
\end{aligned}
$$

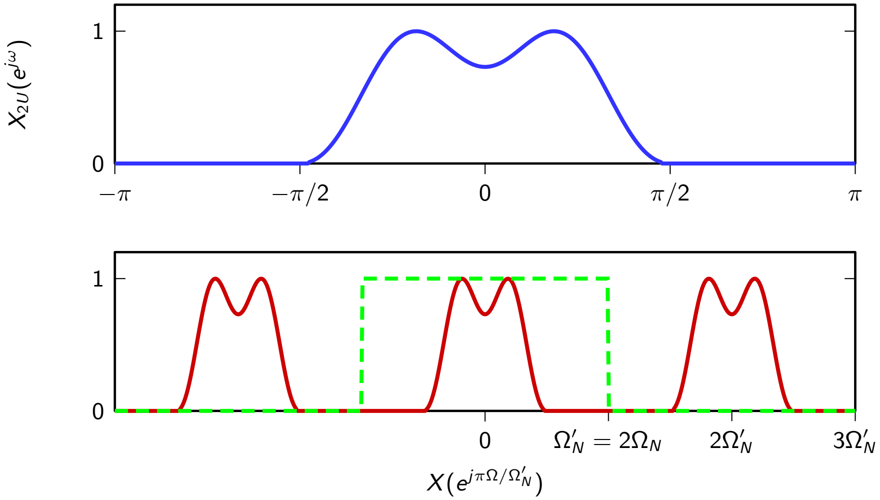
Oversampled D/A



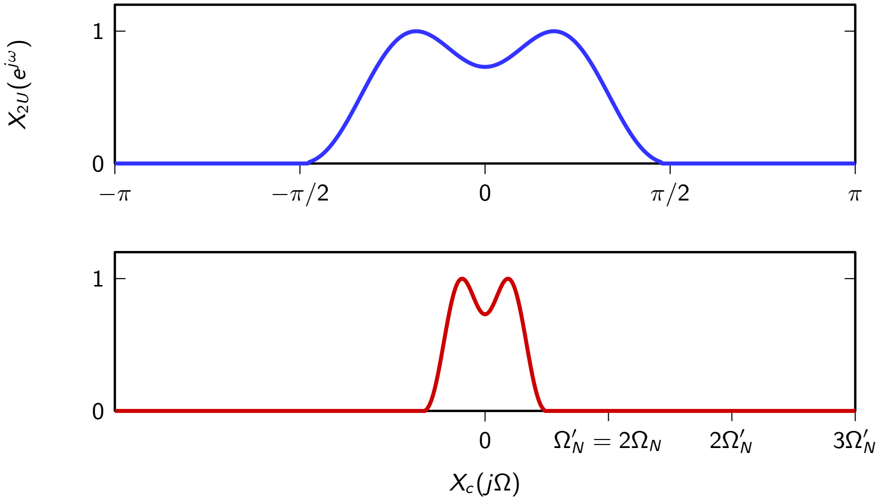
Oversampled D/A



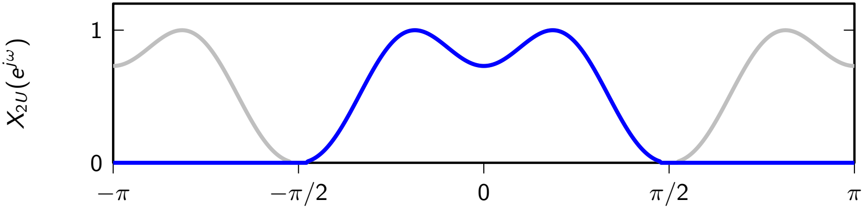
Oversampled D/A



Oversampled D/A



Oversampled D/A, using a ZOH



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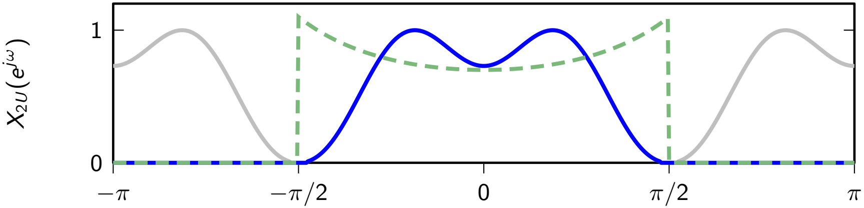
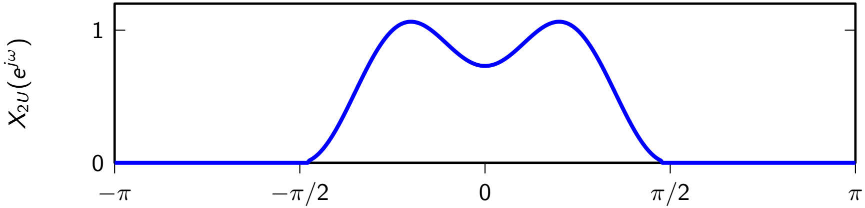
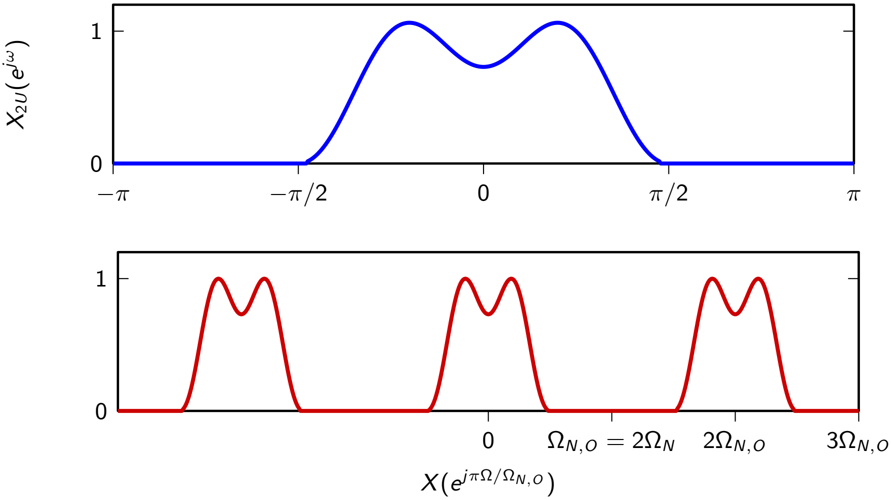


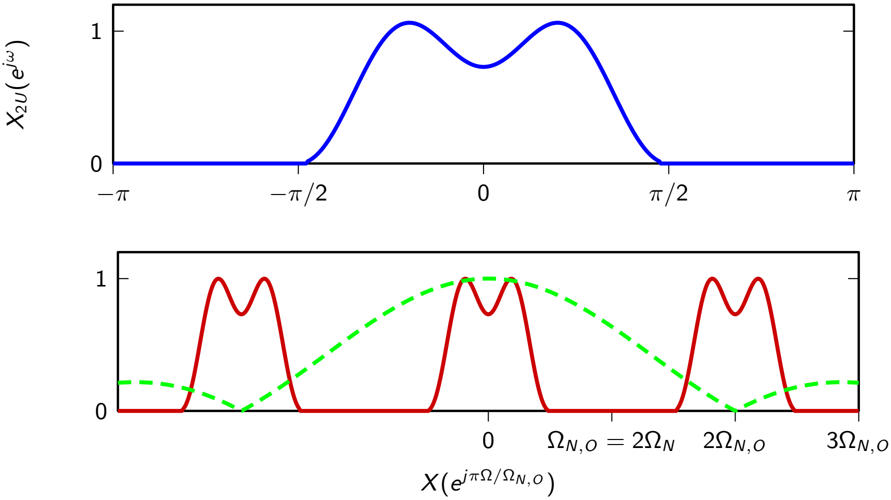
Oversampled D/A, using a ZOH



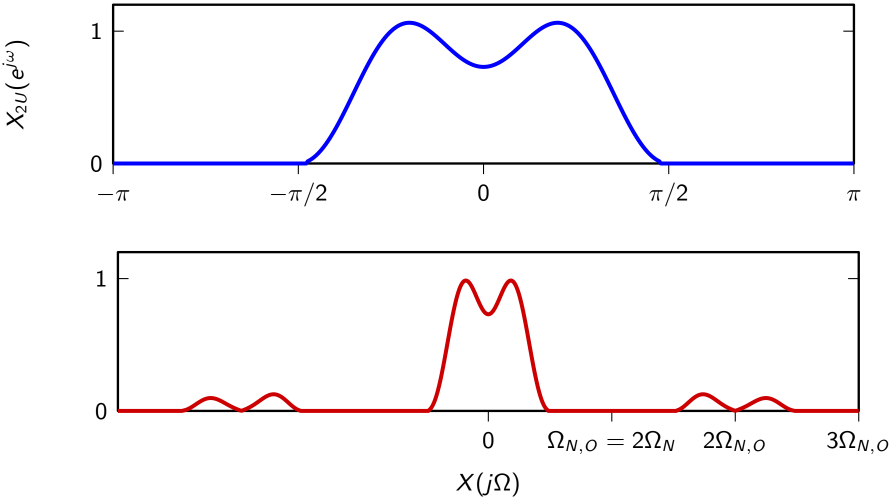
Oversampled D/A, using a ZOH



Oversampled D/A, using a ZOH



Oversampled D/A, using a ZOH



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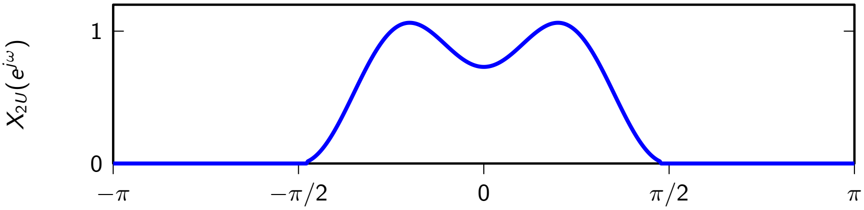

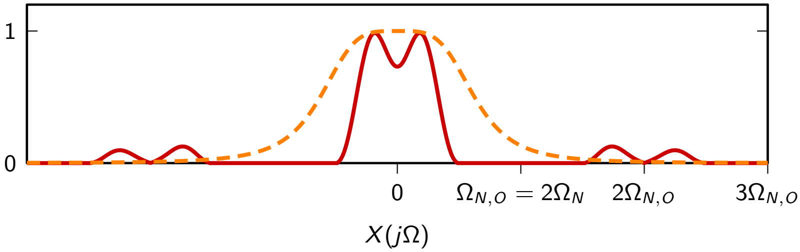

Oversampled A/D

key points:

- we can pre-compensate the in-band distortion in the digital domain
- we can interpolate with a cheap ZOH
- the higher the upsampling, the cheaper the analog lowpass needed to eliminate out-of-band distortion
- only price: higher D/A rate