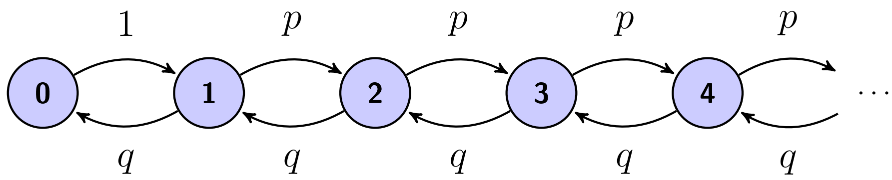
Solutions 10

Saliba, May 15 2019

Exercise 1. Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with transition matrix P. For a fixed state k, let us denote $\bar{P}^{(k)}$ the matrix obtained from P by suppressing the line k and colon k. Then the states (X_n) are recurrent if and only if

$$\begin{cases} \bar{P}^{(k)}\boldsymbol{x} = \boldsymbol{x} \\ 0 \leq x_i \leq 1 \ \forall i \end{cases}$$

has only the null vector as a solution.
Use this result to discuss the nature of the states of the reflected random walk represented by the graph $(p + q = 1)$:



Solution. As p and q are strictly positive, all states are of the same type. We see easily that the chain is irreducible and that all states are periodic with period 2. We know that the states are positive recurrent if and only if the system

$$\begin{cases} \pi P &= \pi, \\ \sum_{i=0}^{\infty} \pi_i &= 1, \end{cases}$$

has a unique solution, where P is the transition matrix associated to the Markov chain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & 0 & \cdots \\ 0 & 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The system is equivalent to

$$\begin{cases} q\pi_1 &= \pi_0, \\ \pi_0 + q\pi_2 &= \pi_1, \\ p\pi_{n-1} + q\pi_{n+1} &= \pi_n, \ \forall n \geq 2, \\ \sum_{i=0}^{\infty} \pi_i &= 1. \end{cases}$$

We get $\pi_n = \frac{p^{n-1}}{q^n}\pi_0$ for all $n \geq 1$.
The condition $\sum_{i=0}^{\infty} \pi_i = 1$ gives the value π_0:

$$\pi_0 + \sum_{n=1}^{\infty} \frac{p^{n-1}}{q^n}\pi_0 = 1 \Longleftrightarrow \pi_0(1 + \frac{1}{q}\sum_{n=0}^{\infty}\left(\frac{p}{q}\right)^n) = 1.$$

The system has a unique solution provided $\sum_{n=0}^{\infty}\left(\frac{p}{q}\right)^n$ converges. We deduce then that the states are positive recurrent if and only if $p<q$.
According to the result of the exercise, states are recurrent if and only if

$$\begin{cases} Q\overrightarrow{x}=\overrightarrow{x}, \\ 0\leq x_i\leq 1\ \forall i \end{cases} \tag{1}$$

has as unique solution $\overrightarrow{x}=\overrightarrow{0}$, where the matrix Q is obtained from P removing line k and colon k for a fixed k. Taking $k=0$, we obtain the system

$$\begin{cases} x_1 &= px_2, \\ x_n &= qx_{n-1}+px_{n+1}\ \forall n\geq 2. \end{cases}$$

Multiplying by $p+q(=1)$ on both sides, we have

$$\begin{cases} p(x_2-x_1) &= qx_1, \\ p(x_{n+1}-x_n) &= q(x_n-x_{n-1})\ \forall n\geq 2. \end{cases}$$

We finally get $x_{n+1}-x_n=\left(\frac{q}{p}\right)^n x_1$. For $n\geq 2$, we have then

$$x_n=\sum_{k=1}^{n-1}(x_{k+1}-x_k)+x_1=\sum_{k=1}^{n-1}\left(\frac{q}{p}\right)^k x_1+x_1=\sum_{k=0}^{n-1}\left(\frac{q}{p}\right)^k x_1.$$

If $p=q=\frac{1}{2}$, we get that $x_n=nx_1$. As $x_i\leq 1$ for all $i\geq 1$, we deduce that the only possibility is that $x_i=0$ for all i, implying that the states are recurrent.
If $p>q$, we have

$$x_n=\sum_{k=0}^{n-1}\left(\frac{q}{p}\right)^k x_1=\frac{1-\left(\frac{q}{p}\right)^n}{1-\frac{q}{p}}x_1.$$

In this case, taking for example $x_1=1-\frac{q}{p}\in]0,1[$, we get that $x_n=1-\left(\frac{q}{p}\right)^n\in]0,1[$ for all $n\geq 0$. We finally found a solution to the system (1), implying transience of the states in this case.

Exercise 2. $M/M/1/\infty$ queue
Let us suppose that the arrivals at the EPFL service desk follow a Poisson process with parameter λ. When a client arrives, its service starts immediately if the desk is free. Otherwise, he waits for his turn. A queue with infinite length is allowed.
We assume that the service time of one customer follow an exponential distribution with parameter μ, the service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.
Let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of customers in the system (while waiting or while being served) at time t.

1) Show that $(X(t))_{t\geq 0}$ is a Markov process (homogenous).

2) Compute the generator of the process.

3) Find the transition matrix of the jumps and deduce again the generator of the chain.

4) Determine the probability distribution of the time that the chain spends in each state.

5) Compute the asymptotic distribution of the number of customer in the queue.

6) Discuss the nature of the states with respect to the parameters μ and λ.

Solution. 1) Let $(N(t))_{t\geq 0}$ be a Poisson process with parameter λ that gives the arrival times in a system. For $s, t > 0$, we have

$$X(t+s) = X(t) + (N(t+s) - N(t)) - \#\text{ departures between } t \text{ and } t+s.$$

We known that $N(t+s)-N(t)$ is independent from the arrivals before t (as it is a Poisson process) and it is independent from $X(u),\ 0 \leq u \leq t$,by hypothesis.
On the other hand, the number of departures between t and $t+s$ is independent from what happens before t since the service times are exponentials.If a service starts between t and $t+s$, its length is independent from the past by hypothesis. If a service starts before t and is still alive between t and $t+s$, then it will be independent from the past (before t) by the memorylessness property of exponentials.
We deduce that the number of departures between t and $t+s$ should depend only on $X(t)$ and $N(t+s)-N(t)$. We then have

$$\mathbb{P}(X(t+s) = n \mid X(u),\ 0 \leq u \leq t) = \mathbb{P}(X(t+s) = n \mid X(t)),\ n \geq 0.$$

implying the Markov property of the process.
By homogeneity, note that $|\#$ departures $-$ $\#$ arrivals$|$ in a fixed interval of time does not dependent on the position of this interval. As the Poisson process $(N(t))_{t\geq 0}$ has stationary increments and that the service times are i.i.d. Write D_s^t and A_s^t for the number of departures/ arrivals between times s and t, respectively. We then have, for all $n, m \geq 0$

$$\begin{aligned}\mathbb{P}(X(t+s) = n \mid X(t) = m) &= \mathbb{P}(\#\ D_t^{t+s} - \#\ A_t^{t+s} = m - n) \\ &= \mathbb{P}(\#\ D_0^s - \#\ A_0^s = m - n) \\ &= \mathbb{P}(X(t) = n \mid X(0) = m).\end{aligned}$$

2) To compute the generator, we start by computing $P_{ij}(h)$ for h small and $i, j \geq 0$. In an interval of length $h(\to 0)$, we can have only one transition. For $i = 0$, we have then:

$$\begin{aligned}&\mathbb{P}(X(h) = 0 \mid X(0) = 0) = \mathbb{P}(N(h) = 0) = e^{-\lambda h} = 1 - \lambda h + o(h) \quad \text{(Taylor)}, \\ &\mathbb{P}(X(h) = 1 \mid X(0) = 0) = \mathbb{P}(N(h) = 1) = \lambda h + o(h).\end{aligned}$$

If $i > 0$, we have 3 possible transition in the interval $[0, h]$:

a) A new client arrives,

b) One departure,

c) No changes.

Similarly to the previous reasoning, we use the memorylessness property of the exponential random variable and the Taylor development to get

a)

$$\mathbb{P}(X(h) = i+1 \mid X(0) = i) = \mathbb{P}(N(h) = 1) = \lambda h + o(h).$$

b)

$$\mathbb{P}(X(h) = i-1 \mid X(0) = i) = \mathbb{P}(Exp(\mu) \leq h) = 1 - e^{\mu h} = \mu h + o(h).$$

c)

$$\begin{aligned}\mathbb{P}(X(h) = i \mid X(0) = i) &= 1 - \mathbb{P}(X(h) = i+1 \mid X(0) = i) - \mathbb{P}(X(h) = i-1 \mid X(0) = i) \\ &= 1 - (\lambda + \mu)h + o(h).\end{aligned}$$

The generator Q satisfy (for h small)

$$P(h) = I + Qh + o(h),$$

where I is the identity matrix. We have then, for all $i, j \geq 0$,

$$Q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h) - I_{ij}}{h}.$$

We get finally the following matrix Q (on the states $\{0, 1, 2, \cdots\}$)

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & \ddots \\ 0 & 0 & 0 & \mu & -(\lambda+\mu) & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3) Let $\hat{P}$ be the transition matrix of the Markov chain $(\hat{X}_n)_{n\geq 0}$. When we start from state 0, we move with probability one to the state 1 as soon as there is a jump. In other terms

$$\hat{P}_{0i} = \begin{cases} 1 & \text{si } i = 1 \\ 0 & \text{sinon.} \end{cases}$$

If we leave from $i > 0$, and if the system makes a jump, we move in $i+1$ or in $i-1$. Let us denote D the time of next leave and A the time of next arrival. By memorylessness, we have $D \sim Exp(\mu)$ and $A \sim Exp(\lambda)$ are independent. We get then

$$\begin{aligned}\hat{P}_{i,i+1} = \mathbb{P}(A < D) &= \int_0^\infty \mathbb{P}(A < D \mid D = x)\mu e^{-\mu x} dx \\ &= \int_0^\infty \mathbb{P}(A < x)\mu e^{-\mu x} dx \\ &= \int_0^\infty (1 - e^{-\lambda x})\mu e^{-\mu x} dx \\ &= \frac{\lambda}{\lambda + \mu}.\end{aligned}$$

We deduce then that $\hat{P}_{i,i-1} = \frac{\mu}{\lambda+\mu}$. Implying

$$\hat{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & 0 & \cdots \\ 0 & 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $p = \frac{\lambda}{\lambda+\mu}$ et $q = \frac{\mu}{\lambda+\mu}$. Knowing Q, we can then find again $\hat{P}$ directly since $\hat{P}_{ii} = 0$ for all i and for $i \neq j$:

$$\hat{P}_{ij} = \frac{Q_{ij}}{-Q_{ii}}.$$

4) If we are in state 0 at time t, the time while we are still at 0 is the time before the arrival of a new client and so follows an exponential distribution with parameter λ (by memorylessness).
If we are in the state $i > 0$, The time we spend in this state has the same law as $\min(X, Y)$ where $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ and both X and $Y$are independent. Then, this time follows an exponential distribution with parameter $\lambda + \mu$.

5) We have to solve the system

$$\begin{cases} \vec{\pi}Q & = \vec{0}, \\ \vec{\pi}\vec{1} & = 1. \end{cases} \tag{2}$$

We get easily that $\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0$ for all $n \geq 0$. as the sum of components of π should be equal to 1, π_0 should verify then

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \pi_0 = 1.$$

If $\mu > \lambda$, we have that $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{\mu}{\mu-\lambda}$ and so we get that $\pi_0 = 1 - \frac{\lambda}{\mu}$. The system (2) has a unique solution π given by $\pi_n = \left(\frac{\lambda}{\mu}\right)^n (1 - \frac{\lambda}{\mu})$ for all $n \geq 0$.
In the case that $\mu \leq \lambda$, the series $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$ diverges and so the system (2) has no solution (we could have deduce it thanks to the previous point, since the states are not positive recurrent in this case). We have then $\pi_n = 0$ for all $n \geq 0$.

6) We know that if i is transient/null recurrent for $\hat{P}$, then it is transient/null recurrent for the Markov process. This also is the case for the positive recurrence, whenever the jump rates of the Markov process are bounded away from 0 and ∞. By previous exercise, we have then that all states are positive recurrent for $\lambda < \mu$, null recurrent for $p = q = \frac{1}{2}$ and transients for $\lambda > \mu$.

Exercise 3. $M/M/1/m$ queue
We consider again the queuing system of last exercise, except that the waiting room has a maximum capacity of $m - 1$ customers. So that if the system starts with less than m clients, the number of clients in the system will never be bigger than m, because a client who arrives while m clients are already in the system goes away and never comes back. Nevertheless it is possible that the initial state is bigger than m.

1) Compute the generator and the transition matrix of the Markov chain for the jumps.

2) Determine the nature of the state for the associated Markov chain.

3) Compute the asymptotic distribution of the number of clients in the queue.

Solution. 1) The Q-matrix corresponding to the states $\{0, \cdots, m\}$is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \mu & -\mu \end{pmatrix}$$

The transition matrix of jumps is given by

$$\hat{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ q & 0 & p & 0 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & p & 0 & 0 & \cdots & 0 \\ 0 & 0 & q & 0 & p & 0 & \cdots & 0 \\ 0 & 0 & 0 & q & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

2) Starting with maximum m clients, we see that the chain is irreducible and finite, so all states are positive recurrent.

3) We look for the solution to the system

$$\begin{cases} \vec{\pi}Q & = \vec{0} \\ \sum_{i=0}^{m} \pi_i & = 1. \end{cases}$$

By a simple computation we get for $0 \leq n \leq m$

$$\pi_n = \begin{cases} \frac{(\frac{\lambda}{\mu})^n (1-\frac{\lambda}{\mu})}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}}, & \text{if } \lambda \neq \mu, \\ \frac{1}{m+1} & \text{si } \lambda = \mu. \end{cases}$$

Exercise 4. $M/M/\infty$ queue

Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from a countable collection of servers so that when a client arrives, his service starts immediately.

We assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.

Let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Compute the asymptotic distribution of the number of clients in the queue.

Solution. 1) The Q-matrix corresponding to the system is

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \cdots \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & \cdots \\ 0 & 0 & 0 & 4\mu & -(\lambda+4\mu) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

The transition matrix of jumps is given by

$$\hat{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & 0 & 0 & \cdots \\ 0 & q_2 & 0 & p_2 & 0 & 0 & \cdots \\ 0 & 0 & q_3 & 0 & p_3 & 0 & \cdots \\ 0 & 0 & 0 & q_4 & 0 & p_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $q_i = \frac{i\mu}{\lambda+i\mu}$ and $p_i = \frac{\lambda}{\lambda+i\mu}$ for all $i \geq 1$.

2) We look for the solution to the following system

$$\begin{cases} \vec{\pi}Q &= \vec{0} \\ \sum_{i=0}^{\infty} \pi_i &= 1. \end{cases}$$

A simple computation gives

$$\pi_n = \pi_0 \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}, \ n \in \mathbb{N}.$$

Comme $\sum_{i=0}^{\infty} \pi_i = 1$, on obtient que $\pi_0 = e^{-\frac{\lambda}{\mu}}$.

Exercise 5. $M/M/m/\infty$ queue
Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from m servers and a waiting room of infinite capacity (when a client arrives, his service starts immediately if one of the m desk is free, otherwise he goes in the waiting room).
The service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Compute the asymptotic distribution of the number of clients in the queue.

Solution. 1) If the number of clients k in the system is smaller than m, then all clients are being served, and so we will have to wait an amount of time distributed as the minimum of k exponential random variables μ to pass to $k-1$ clients. If $k > m$, we spend a time equivalent to the minimum of m exponential random variables with parameter μ to move to $k-1$ clients in the system. Coefficients of the Q-matrix are then given by

$$Q_{k,k+1} = \lambda,\ Q_{k,k-1} = \begin{cases} k\mu & \text{si } 0 \leq k \leq m, \\ m\mu & \text{si } k \geq m. \end{cases}$$

The first m lines of the transition matrix corresponding to the Markov chain of jumps are the same as in the previous exericise. For $k > m$, we have

$$\hat{P}_{k,k-1} = \frac{m\mu}{-\lambda + m\mu},\ \hat{P}_{k,k+1} = \frac{\lambda}{\lambda + m\mu}.$$

2) conditions of detailed balance equations give

$$\begin{cases} \lambda\pi(k-1) = k\mu\pi(k) & \text{si } k \leq m, \\ \lambda\pi(k-1) = m\mu\pi(k) & \text{si } k \geq m. \end{cases}$$

A simple computation gives

$$\pi(k) = \begin{cases} \frac{c}{k!}\left(\frac{\lambda}{\mu}\right)^k & \text{si } k \leq m, \\ \frac{c}{m!m^{k-m}}\left(\frac{\lambda}{\mu}\right)^k & \text{si } k \geq m, \end{cases}$$

where $\pi(0) = c$ is such that $\sum_{i=0}^{\infty} \pi_i = 1$.

Exercise 6. Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from one server and a waiting room of infinite capacity (when a client arrives, his service starts immediately if the desk is free, otherwise he goes in the waiting room).
We assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
Clients are busy: a client that cannot be served immediately waits a random time exponentially distributed with parameter γ, then if his service has not already started he goes away and never comes back (there s however no restriction on the duration of service).
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Gives the stationary distribution for every state $i > 0$ of this process in term of the stationary probability that the system is void.

Solution. 1) The Q-matrix corresponding is

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu+\gamma & -(\lambda+\mu+\gamma) & \lambda & 0 & \cdots \\ 0 & 0 & \mu+2\gamma & -(\lambda+\mu+2\gamma) & \lambda & \ddots \\ 0 & 0 & 0 & \mu+3\gamma & -(\lambda+\mu+3\gamma) & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2) We are looking for the solution of the system

$$\begin{cases} \vec{\pi} Q & = \vec{0} \\ \sum_{i=0}^{\infty} \pi_i & = 1. \end{cases}$$

A simple computation gives

$$\pi_n = \pi_0 \frac{\lambda^n}{\prod_{i=0}^{n-1}(\mu + i\gamma)}, \ n \geq 1.$$

Exercise 7. The following theorem caracterizes the limiting distribution $\boldsymbol{\pi}$ of a postive recurrent Markov process as a solution of some linear equations systems.

Theorem. The following assertions are equivalent:

(i) The states of an irreducible Markov process are positive recurrent.

(ii) There exists $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} P(s) = \boldsymbol{\pi}$ for all s,

(iii) There exists $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} Q = \mathbf{0}$,

In (ii) and (iii), $\boldsymbol{\pi}$ is such that $\pi_j = \lim_{t\to\infty} \mathrm{P}(X(t) = j \mid X(0) = j)$, with $\boldsymbol{\pi} \geq \mathbf{0}$, and $\boldsymbol{\pi}\mathbf{1} = 1$. The goal of this exercise is to prove this theorem followig these steps:

1) Use the $h-$skeleton correspondant to $X(t)$ $(Z_n := X(nh), \ n \in \mathbb{N})$ to prove that $(i) \Leftrightarrow (ii)$.

2) Show that $(ii) \Leftrightarrow (iii)$ in the case where the number of states is finite using the Kolmogorov forward equation.

3) In the general case of a state space, show that $(iii) \Rightarrow (ii)$ using the expression of $P(s)$ as a function of Q.

4) It remains to show $(ii) \Rightarrow (iii)$ in the general case of a state space. For this,

   a) Using the relation between Q and $\widehat{P}$, show first that $\widehat{\boldsymbol{\pi}}$ is the stationary distribution of the jump chain if and only if $\widehat{\boldsymbol{\pi}}\Lambda^{-1}Q = \mathbf{0}$, where $\Lambda =$diag(δ_i).

   b) It remains to show that $\widehat{\boldsymbol{\pi}}\Lambda^{-1} = \boldsymbol{\pi}$ up to a multiplicative constant. To prove this, show first the following equality by conditionning on the last change of state before time t:

$$P_{ij}(t) = e^{-\delta_i t}\delta_{ij} + \int_0^t \sum_k P_{ik}(t-v)\delta_k \widehat{P}_{kj} e^{-\delta_j v} dv,$$

   or in a matrix form,

$$P(t) = e^{-\Lambda t} + \int_0^t P(t-v)\Lambda \widehat{P} e^{-\Lambda v} dv.$$

   Use this equality, and the fact that $\boldsymbol{\pi} P(t) = \boldsymbol{\pi}$ for all t and $\lim_{t\to\infty} e^{-\Lambda t} = 0$, to show that $\widehat{\boldsymbol{\pi}}\Lambda^{-1} = \boldsymbol{\pi}$ up to a multiplicative constant.

Solution.
$\underline{(i) \Leftrightarrow (ii):}$ We have

$$\lim_{t\to\infty} \mathbb{P}(X(t)=j \mid X(0)=i)=\pi_j \text{ if } \lim_{n\to\infty} \mathbb{P}(X(nh)=j \mid X(0)=i)=\pi_j \ \forall h>0.$$

For the h-skeleton, the states are positive recurrent if and only if there exists $\boldsymbol{x}(h)$ such that

$$\begin{cases} \boldsymbol{x}(h)P(h) & = \boldsymbol{x}, \\ \boldsymbol{x}(h)\mathbf{1} & = 1. \end{cases}$$

In this case, $x_j(h)=\lim_{n\to\infty} \mathbb{P}(X(nh)=j \mid X(0)=i)=\pi_j$ (the limit does not depend on h).
$\underline{(ii) \Leftrightarrow (iii):}$ If the number of states is finite,

$$\begin{aligned} & \boldsymbol{\pi}P(s)=\boldsymbol{\pi} \ \forall s, \\ \Leftrightarrow\ & \boldsymbol{\pi}P'(s)=\mathbf{0} \quad \text{since the sum is finite}, \\ \Leftrightarrow\ & \boldsymbol{\pi}P(s)Q=\mathbf{0}, \\ \Leftrightarrow\ & \boldsymbol{\pi}Q=\mathbf{0}. \end{aligned}$$

For an arbitrary number of states, we notice first, for all $s\geq 0$,

$$\boldsymbol{\pi}Q=\mathbf{0} \Rightarrow \boldsymbol{\pi}P(s)=\boldsymbol{\pi}e^{Qs}=\boldsymbol{\pi}\sum_{n\geq 0}\frac{(Qs)^n}{n!}=\boldsymbol{\pi},$$

and therefore we have $(iii) \Rightarrow (ii)$.
To show $(ii) \Rightarrow (iii)$, we consider the corresponding jump chain, with its transition matrix $\hat{P}$.
We know that the states are positive recurrent if and only if there exists $\hat{\boldsymbol{\pi}}$ such that $\hat{\boldsymbol{\pi}}\hat{P}=\hat{\boldsymbol{\pi}}$.
And we have

$$\begin{cases} Q_{ij} & = \delta_i \hat{P}_{ij} \quad \text{if } i\neq j \\ Q_{ii} & = -\delta_i. \end{cases}$$

Writing $\Lambda=$diag(δ_i), we get

$$Q=\Lambda(\hat{P}-I) \Longleftrightarrow \hat{P}=I+\Lambda^{-1}Q$$

and $\hat{P}_{ii}=0$ for all i. Whence

$$\begin{aligned} & \hat{\boldsymbol{\pi}}\hat{P} && =\hat{\boldsymbol{\pi}}, \\ \Leftrightarrow\ & \hat{\boldsymbol{\pi}}+\hat{\boldsymbol{\pi}}\Lambda^{-1}Q && =\hat{\boldsymbol{\pi}}, \\ \Leftrightarrow\ & \hat{\boldsymbol{\pi}}\Lambda^{-1}Q && =\mathbf{0}. \end{aligned}$$

To show (iii), we need to show that $\hat{\boldsymbol{\pi}}\Lambda^{-1}=\boldsymbol{\pi}$.
Let $t-v$ be the last time of change of state before time t. We have

$$P_{ij}(t)=e^{-\delta_i t}\delta_{ij}+\int_0^t \sum_k P_{ik}(t-v)\delta_k\hat{P}_{kj}e^{-\delta_j v}dv,$$

or in a matrix form

$$P(t)=e^{-\Lambda t}+\int_0^t P(t-v)\Lambda\hat{P}e^{-\Lambda v}dv. \tag{3}$$

The detailed proof of equation (3) is given in theorem 2.8.6 page 100 in Norris' book. Using that $\boldsymbol{\pi} P(t)=\boldsymbol{\pi}$ for all t, we obtain

$$\begin{aligned}
&\boldsymbol{\pi} e^{-\Lambda t}+\int_0^t \boldsymbol{\pi} P(t-v)\Lambda\hat{P}e^{-\Lambda v}dv &&=\boldsymbol{\pi},\\
\iff\ &\boldsymbol{\pi} e^{-\Lambda t}+\int_0^t \boldsymbol{\pi}\Lambda\hat{P}e^{-\Lambda v}dv &&=\boldsymbol{\pi}\ (\boldsymbol{\pi} P(t-v)=\boldsymbol{\pi}),\\
\iff\ &\boldsymbol{\pi} e^{-\Lambda t}+\boldsymbol{\pi}\Lambda\hat{P}\int_0^t e^{-\Lambda v}dv &&=\boldsymbol{\pi},\\
\iff\ &\boldsymbol{\pi} e^{-\Lambda t}-\boldsymbol{\pi}\Lambda\hat{P}\Lambda^{-1}[e^{-\Lambda v}]_0^t &&=\boldsymbol{\pi},\\
\iff\ &\boldsymbol{\pi} e^{-\Lambda t}-\boldsymbol{\pi}\Lambda\hat{P}\Lambda^{-1}e^{-\Lambda t}+\boldsymbol{\pi}\Lambda\hat{P}\Lambda^{-1} &&=\boldsymbol{\pi}\ \forall t.
\end{aligned}$$

Taking the limit when $t\to\infty$ of the last equivalence, and noticing that $\lim_{t\to\infty} e^{-\Lambda t}=0$, we obtain

$$\boldsymbol{\pi}\Lambda\hat{P}\Lambda^{-1}=\boldsymbol{\pi}\iff\boldsymbol{\pi}\Lambda\hat{P}=\boldsymbol{\pi}\Lambda.$$

Since $\hat{\boldsymbol{\pi}}$ is the unique solution of $\hat{\boldsymbol{\pi}}\hat{P}=\hat{\boldsymbol{\pi}}$ up to a multiplicative constant, we have

$$\hat{\boldsymbol{\pi}}=\boldsymbol{\pi}\Lambda$$

up to a multiplicative constant, and therefore

$$\hat{\boldsymbol{\pi}}\Lambda^{-1}Q=\mathbf{0}\Leftrightarrow\boldsymbol{\pi}\Lambda\Lambda^{-1}Q=\mathbf{0}\Leftrightarrow\boldsymbol{\pi} Q=\mathbf{0}.$$

This shows $(ii)\Rightarrow(iii)$.

Exercise 8 (Birth and Death process)**.** We let $X(t)$ represents a population size at time t, where the death rate of one individual is μ (i.e. the individuals have an exponential lifetime of parameter μ), and the reproduction rate of an individual is λ (i.e. individuals give birth to one new individual at a time, according to the arrival moments of a Poisson process of parameter λ). We suppose that the individuals behave independently from each other. Let V_t be the remaining time, starting from t, until the next birth of an individual, and W_t the remaining time, starting from t, until the next death of an individual. We add the following hypothesis: If $X(t)=i$, there exist positive numbers p_i and q_i such that:

$$P(V_t>u;W_t>u|X(s),0\le s\le t)=e^{-q_i u},$$

$$P(V_t\le W_t|X(s),0\le s\le t)=p_i.$$

These hypothesis imply that $X:=\{X(t):t\in\mathbb{R}\}$ is a Markov process.

(i). Find the constants p_i and q_i, and compute the generator and the transition matrix of the jump chain corresponding to X.

(ii). Write in details the Kolmogorov backward and forward equations, supposing that $X(0)=1$.

(iii). Write the resulting differential equations that you get (ordinary for the backward and with partial derivatives for the forward) for the generating function of the population size at time t, defined as

$$F(t,s)=\sum_{k\ge 0}\mathbb{P}(X(t)=k|X(0)=1)s^k.$$

Hint: for the backward equation, notice that $F^{(2)}(t,s) := \sum_{k\geq 0} \mathbb{P}(X(t) = k|X(0) = 2)s^k = F(t,s)^2$ by independence.

(iv). Verify that these equations have the following solution

$$F(t,s) = \begin{cases} 1 + \dfrac{(\lambda-\mu)(s-1)}{(\lambda s - \mu)\, e^{(\mu-\lambda)t} - \lambda(s-1)}, & \text{if } \lambda \neq \mu \\ 1 + \dfrac{(s-1)}{1-\lambda t(s-1)}, & \text{if } \lambda = \mu. \end{cases}$$

Solution. (i). We have $p_i = i\lambda/(i\lambda + i\mu) = \lambda/(\lambda+\mu)$, and $q_i = i(\lambda+\mu)$. The generator is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & 2\mu & -2(\mu+\lambda) & 2\lambda & \dots \\ 0 & 0 & 3\mu & -3(\mu+\lambda) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We see that the state 0 is absorbing: if the population is empty, it remains empty forever.

(ii)–(iv). Backward equation: $P'(t) = QP(t)$, that is, if we suppose that $X(0) = 1$,

$$\begin{aligned} \frac{d}{dt}P(X(t)=0|X(0)=1) &= \mu P(X(t)=0|X(0)=0) \\ &\quad -(\lambda+\mu)P(X(t)=0|X(0)=1) \\ &\quad +\lambda P(X(t)=0|X(0)=2) \\ &= \mu - (\lambda+\mu)P(X(t)=0|X(0)=1) + \lambda P(X(t)=0|X(0)=2) \end{aligned}$$

et pour $j \geq 1$,

$$\begin{aligned} \frac{d}{dt}P(X(t)=j|X(0)=1) &= \mu P(X(t)=j|X(0)=0) \\ &\quad -(\lambda+\mu)P(X(t)=j|X(0)=1) \\ &\quad +\lambda P(X(t)=j|X(0)=2) \\ &= -(\lambda+\mu)P(X(t)=j|X(0)=1) + \lambda P(X(t)=j|X(0)=2) \end{aligned}$$

Multiplying the jth equation by s^j and taking the sum of all the equations for $j \leq 0$, we get

$$\frac{d}{dt}F(t,s) = \mu - (\lambda+\mu)F(t,s) + \lambda F(t,s)^2,$$

with initial condition $F(0,s) = s$ (since we supposed that $X(0) = 1$). The solution of this Riccati differential equation is

$$F(t,s) = \begin{cases} 1 + \dfrac{(\lambda-\mu)(s-1)}{(\lambda s - \mu)\, e^{(\mu-\lambda)t} - \lambda(s-1)}, & \text{if } \lambda \neq \mu \\ 1 + \dfrac{(s-1)}{1-\lambda t(s-1)}, & \text{if } \lambda = \mu. \end{cases}$$

Forward equation: $P'(t) = P(t)Q$, that is,

$$\frac{d}{dt}P(X(t)=0|X(0)=1) \quad = \quad \mu P(X(t)=1|X(0)=1),$$

and for $j \geq 1$,

$$\begin{aligned}\frac{d}{dt}P(X(t)=j|X(0)=1) \quad = \quad & (j-1)\lambda P(X(t)=j-1|X(0)=1) \\ & -j(\lambda+\mu)P(X(t)=j|X(0)=1) \\ & +(j+1)\mu P(X(t)=j+1|X(0)=1)\end{aligned}$$

Multiplying the jth equation by s^j and taking the sum of all the equations for $j \leq 0$, and using the definition of $F(t,s)$, we get

$$\frac{\partial}{\partial t}F(t,s) - \frac{\partial}{\partial s}F(t,s)(\mu - (\lambda+\mu)s + \lambda s^2) = 0$$

with the same initial condition (and the same solution) as for the backward equation.