# Solutions 6

Saliba, March 27, 2019

**Exercise 1.** Let $X$ be a Markov chain on $E$ (not necessarily irreducible). Suppose that state $j \in E$ is positive recurrent and aperiodic. Show that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i), \qquad \tau_j = \inf\{n \geq 0 : X_n = j\},$$

where $\pi$ is the stationary distribution of the chain restricted to the communicating class of $j$.

**Solution.** By the total probability formula, we have

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) p_{jj}^{(n-r)}.$$

By considering the sub-chain with valued in the communicating class of $j$, we see that this sub-chain is irreducible and $j$ is positive recurrent. We can thus apply a theorem from the class that gives that $p_{jj}^{(n-r)} \to \pi_j$. Since $\sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) < \infty$, we obtain, by the dominated convergence theorem

$$\begin{aligned} p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) p_{jj}^{(n-r)} &\to \pi_j \sum_{r=1}^{\infty} \mathbb{P}(\tau_j = r \mid X_0 = i) \\ &= \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i). \end{aligned}$$

**Exercise 2.** Let $X$ be a Markov chain with transition matrix $P$ on $E = \{1, 2, 3, 4, 5\}$ given by

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

(a) Find the communicating classes of $P$. For the recurrent classes, find the corresponding stationary distributions.

(b) Supposing that $X_0 \sim \alpha$ for a distribution $\alpha$ on $E$, find the limiting distribution of $X_n$ when $n \to \infty$.
*Hint:* Suppose that $X$ starts in a transcient state of $E$ and find the limiting distribution in this case.

**Solution.** (a) The communicating classes of $P$ are $\{1, 4\}$, $\{3, 5\}$ et $\{2\}$. Closed (and recurrent) classes are $\{1, 4\}$ et $\{3, 5\}$. The submatrix $P_1$ corresponding to $\{1, 4\}$ is given by:

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The stationary distribution of the system $\pi_1 = (a, b)$ verifies

$$\begin{cases} \frac{a}{3} + \frac{b}{2} &= a, \\ \frac{2a}{3} + \frac{b}{2} &= b, \\ a + b &= 1. \end{cases}$$

The solution of this system is given by $\pi_1 = (\frac{3}{7}, \frac{4}{7})$.
The submatrix $P_2$ corresponding to $\{3,5\}$ is given by:

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The stationary distribution of the system $\pi_2 = (c, d)$ verifies

$$\begin{cases} \frac{c}{2} + \frac{d}{4} & = c, \\ \frac{c}{2} + \frac{3d}{4} & = d, \\ c + d & = 1. \end{cases}$$

The solution of this system is given by $\pi_2 = (\frac{1}{3}, \frac{2}{3})$.

(b) State 2 is the only transient state of the system. Starting from 2, we know that $P_{2i}^n \overset{n\to\infty}{\longrightarrow} h(i)\pi(i)$ where

$$h(i) =: \mathbb{P}(T_i < \infty \mid X_0 = 2), \; i = 1, 3, 4, 5,$$

and $\pi(i)$ corresponds to the component of the stationary distribution relative to state $i$ $(\pi(1) = \pi_1(1), \pi(3) = \pi_2(1), \pi(4) = \pi_1(2), \pi(5) = \pi_2(2))$.

We need to compute $h(i)$ for $i = 1, 3, 4, 5$. Since $\{1, 4\}$ is a closed and recurrent class, we have that $\mathbb{P}(T_1 < \infty \mid X_0 = 4) = 1$. Since $\{3, 5\}$ is closed, we also have that $\mathbb{P}(T_1 < \infty \mid X_0 = 3) = 0$. Using this, we obtain:

$$h(1) = \frac{1}{4}h(1) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 1 \Longrightarrow h(1) = \frac{1}{3}.$$

Similarly, we find:

$$\begin{cases} h(3) = \frac{1}{4}h(3) + \frac{1}{2} & \Longrightarrow h(3) = \frac{2}{3}, \\ h(4) = \frac{1}{4}h(4) + \frac{1}{4} & \Longrightarrow h(4) = \frac{1}{3}, \\ h(5) = \frac{1}{4}h(5) + \frac{1}{2} & \Longrightarrow h(5) = \frac{2}{3}. \end{cases}$$

Since 2 is transient, $P_{22}^n \overset{n\to\infty}{\longrightarrow} 0$. We thus get:

$$P_{21}^n \to \frac{1}{3} \times \frac{3}{7} = \frac{1}{7}, P_{23}^n \to \frac{2}{9}, P_{24}^n \to \frac{4}{21}, P_{25}^n \to \frac{4}{9}.$$

Therefore, the transition matrix $P^n$ converges to $P_\infty$ given by:

$$P_\infty = \begin{pmatrix} \frac{3}{7} & 0 & 0 & \frac{4}{7} & 0 \\ \frac{1}{7} & 0 & \frac{2}{9} & \frac{4}{21} & \frac{4}{9} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{7} & 0 & 0 & \frac{4}{7} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}.$$

So, if $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is the initial distribution of $X$, the limiting distribution of $X_n$, denoted by $\alpha_\infty$, is given by

$$\alpha_\infty = \alpha P_\infty = (\frac{3\alpha_1}{7} + \frac{\alpha_2}{7} + \frac{3\alpha_4}{7}, 0, \frac{2\alpha_2}{9} + \frac{\alpha_3}{3} + \frac{\alpha_5}{3}, \frac{4\alpha_1}{7} + \frac{4\alpha_2}{21} + \frac{4\alpha_4}{7}, \frac{4\alpha_2}{9} + \frac{2\alpha_3}{3} + \frac{2\alpha_5}{3}).$$

**Exercise 3.** Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two independent Markov chains, aperiodic and irreducible, defined on the state spaces $E$ and $E'$, respectively. Show that $(X_n, Y_n)_{n\geq 0}$ is an aperiodic and irreducible Markov chain on $E \times E'$. Find an example of $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ independent and irreducible, but for which $(X_n, Y_n)_{n\geq 0}$ is not irreducible.

**Solution.** We write $(p_{ij})_{i,j\in E}$ and $(q_{ij})_{i,j\in E'}$ for the transition probabilities for $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ respectively. For all $i_0, i \in E$, $j_0, j \in E'$, there exist $r > 0$ and $s > 0$ with $p^{(r)}_{i_0 i} > 0$ and $q^{(s)}_{j_0 j} > 0$. If $X_n$ and $Y_n$ are aperiodic, then, by a theorem of the class, there exists $n_0$ such that for all $n \geq n_0$, we have $p^{(n)}_{ii} > 0$ and $q^{(n)}_{jj} > 0$. Thus, for all $m \geq r + s + n_0$, we have that $p^{(m)}_{i_0 i} > 0$ and that $q^{(m)}_{j_0 j} > 0$, and thus

$$P\Big\{(X_m, Y_m) = (i, j) | (X_0, Y_0) = (i_0, j_0)\Big\} = p^{(m)}_{i_0 i} q^{(m)}_{j_0 j} > 0.$$

Notice that this implies that the periodicity of $(X_n, Y_n)_{n\geq 0}$ is equal to 1.
For the counterexample, we consider $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ to be two independent Markov chain on $\mathbb{Z}$. They are independent. Notice that if we start at $(0, 0)$, we can only reach vertices for which the sum of its coordinates is even. Hence $(X_n, Y_n)$ is not irreducible.

**Exercise 4.** (**Branching process with immigration**) For $n \in \mathbb{N}$, let $(N^n_k)_{k\geq 0}$ be a sequence of independent random variables on $\mathbb{Z}^+$ with a common generating function $\phi(t) = E(t^{N^n_k})$. The branching process with immigration is defined as

$$X_n = N^n_1 + \ldots + N^n_{X_{n-1}} + I_n, \qquad n \geq 0,$$

where $(I_n)_{n\geqslant 0}$ is a sequence of independent random variables with values in $\mathbb{Z}^+$ with a common generating function $\psi(t) = E(t^{I_n})$. Show that if $X_0 = 1$ then

$$E(t^{X_n}) = \phi^{(n)}(t) \prod_{k=0}^{n-1} \psi(\phi^{(k)}(t)).$$

In the case where the number of immigrants in each generation is a Poisson random variable of parameter $\lambda$ and $P(N^n_k = 0) = 1 - p$, $P(N^n_k = 1) = p$, find the proportion of time in the long run for which the population is 0.

**Solution.** The equation for $\mathbb{E}(t^{X_n})$ can be proved by induction by doing a one step decomposition. Indeed, for $n = 1$, the result is straightforward by independence

$$\mathbb{E}(t^{X_1}) = \mathbb{E}(t^{N^1 + I_1}) = \mathbb{E}(t^{N^1})\mathbb{E}(t^{I_1}) = \phi(t)\psi(t).$$

Suppose that this holds for $n$. We obtain

$$
\begin{aligned}
\mathbb{E}(t^{X_{n+1}}) = \mathbb{E}(\mathbb{E}(t^{X_{n+1}} | X_n)) = & \\
&= \sum_{k=0}^{\infty} \mathbb{E}(t^{X_{n+1}} | X_n = k)\mathbb{P}(X_n = k) = \\
&= \sum_{k=0}^{\infty} \mathbb{E}(t^{N_1^{n+1}+\ldots+N_k^{n+1}+I_{n+1}})\mathbb{P}(X_n = k) = \\
&= \psi(t)\sum_{k=0}^{\infty} \phi(t)^k \mathbb{P}(X_n = k) = \\
&= \psi(t)\mathbb{E}(\phi(t)^{X_n}).
\end{aligned}
$$

We finally get, by using the induction relation:

$$\mathbb{E}(t^{X_n}) = \phi^{(n)}(t)\prod_{k=0}^{n-1}\psi(\phi^{(k)}(t)).$$

In the special case of immigration that has a Poisson law, we get:

$$\mathbb{E}(t^{X_n}) = (1 + p^n(t-1))\exp\left(\lambda(t-1)\frac{1-p^n}{1-p}\right).$$

We conclude for $0 \leq p < 1$

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{P}(X_n = 0) &= \lim_{n\to\infty}\lim_{t\to 0}\mathbb{E}(t^{X_n}) \\
&= \lim_{n\to\infty}(1-p^n)\exp\left(-\lambda\frac{1-p^n}{1-p}\right) \\
&= \exp\left(-\frac{\lambda}{1-p}\right).
\end{aligned}
$$

That is the proportion of time for which the population is 0 since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}(X_k = 0) = \lim_{n\to\infty}\mathbb{P}(X_n = 0) = \exp\left(-\frac{\lambda}{1-p}\right).$$

**Exercise 5.** (**Metropolis–Hastings algorithm**) Suppose that we have a distribution $p$ (called target distribution) on a countable space $E$. Then, for each $x \in E$, let $q_x$ be a distribution on $E$ (called the proposal distribution) with $q_x(y) > 0$ whenever $q_y(x) > 0$, for all $y \in E$. The Metropolis–Hastings algorithm constructs a Markov chain $(X_n)_{n\geq 0}$ as follows:

(i). Let $X_0 = x_0 \in E$ be random fixed state.

(ii). For $X_n = x$, choose a candidate $y$ according to the proposal distribution $q_x$. Then let $U$ be a uniform random variable on $[0,1]$, the variable $X_{n+1}$ is defined as

$$X_{n+1} = \begin{cases} y & \text{if } U \leq \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right) \\ x & \text{otherwise.} \end{cases}$$

Show that if $(X_n)_{n\geq 0}$ is irreducible and aperiodic, then it is a reversible chain with respect to its stationary distribution $p$.

**Solution.** Note that in general, irreducibility and aperiodicity are simple to show given the proposal distribution. In particular, note that if there exist $x, y$ such that the ratio $\frac{p(y)q_y(x)}{p(x)q_x(y)}$ is not always equal to 1 (which is generally the case), then there is a positive probability to stay in state $x$, and the chain is therefore aperiodic.
It is easy to show that the transition probability from $x$ to $y$ with $x \neq y$ is

$$p_{xy} = q_x(y) \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right).$$

With this, we have by the detailed balance (suppose that $p(x)q_x(y) > p(y)q_y(x)$ wlog)

$$\pi_x \frac{q_x(y)\frac{p(y)q_y(x)}{p(x)q_x(y)}}{q_y(x)} = \pi_y$$

and thus $\pi_x = p(x)$ for all $x \in \mathbb{E}$. Moreover, by the previous exercise sheet, the chain is reversible with respect to its stationary distribution.