COM-303 - Signal Processing for Communications

Solutions for Homework #4

Solution 1. Implementing the DFT

Set $W = e^{-j\pi/5}$; the DFT matrix for a 5-point DFT is

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W & W^3 \\ 1 & W^3 & W & W^4 & W^2 \\ 1 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

where we have exploited the fact that $W^n = W^{(n \mod 5)}$. The elements of the matrix that we need to compute are W to W^4. We have:

$$\begin{aligned} W &= C - jS \\ W^2 &= (C - jS)^2 = (C^2 - S^2) - 2jCS \\ W^3 &= e^{-j3\pi/5} = e^{j2\pi/5} = (W^2)^* = (C^2 - S^2) + 2jCS \\ W^4 &= e^{-j4\pi/5} = e^{j\pi/5} = W^* = C + jS \end{aligned}$$

With this, given a real-valued input vector $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, we have

$$\begin{aligned} X_0 &= x_0 + x_1 + x_2 + x_3 + x_4 \\ X_1 &= x_0 + (x_1 + x_3)(C^2 - S^2) + (x_1 + x_4)C - j[(x_1 - x_4)S + 2(x_2 - x_3)CS] \\ X_2 &= x_0 + (x_1 + x_4)(C^2 - S^2) + (x_2 + x_3)C - j[(x_3 - x_2)S + 2(x_1 - x_4)CS] \\ X_3 &= Y_2^* \\ X_4 &= Y_1^* \end{aligned}$$

Solution 2. DTFTs

(a) by linearity:

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \sum_{n=1}^{\infty} 4^{-n} e^{-j\omega n} \\
&= \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \sum_{n=0}^{\infty} 4^{-n} e^{-j\omega n} + 1 \\
&= \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{1-\frac{1}{4}e^{-j\omega}} + 1
\end{aligned}$$

(b) using the modulation theorem:

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{2}\frac{1}{1-ae^{-j(\omega-\omega_0)}} + \frac{1}{2}\frac{1}{1-ae^{-j(\omega+\omega_0)}} \\
&= \frac{1-a\cos\omega_0 e^{-j\omega}}{1-2a\cos\omega_0 e^{-j\omega}+a^2e^{-j2\omega}}
\end{aligned}$$

Solution 3. DTFT

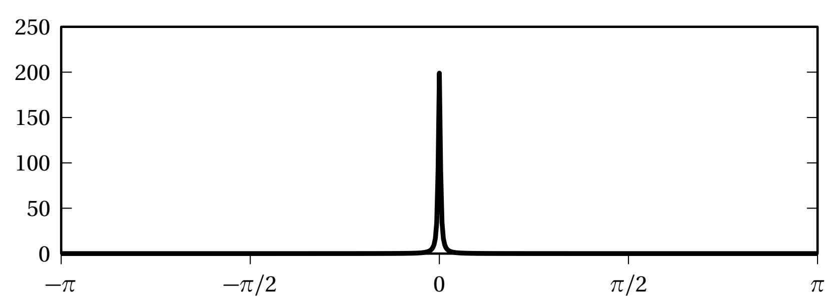
Set $t[n] = a^n u[n]$; then

$$x[n] = t[n] + t[-n] - \delta[n].$$

Since $T(e^{j\omega}) = (1-ae^{-j\omega})^{-1}$, it is

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}} - 1 \\
&= \frac{1-a^2}{1-2a\cos\omega+a^2}
\end{aligned}$$

Here is a plot for $a = 0.99$:



Solution 4. DTFT

The sequence $t[n]=(1+\cos\pi n)/2$ is equal to 1 for n even and to 0 for n odd. With this we can write

$$\begin{aligned}
y[n] &= x[n]\,t[n]+t[n+1]\\
&= \frac{1}{2}(x[n]+x[n]\cos\pi n+1+\cos\pi(n+1))\\
&= \frac{1}{2}(x[n]+x[n]\cos\pi n+1-\cos\pi n))
\end{aligned}$$

Now recall that the (generalized) DTFT of a cosine with frequency $\omega=\pi$ is

$$\begin{aligned}
\text{DTFT}\{\cos\pi n\} &= \frac{1}{2}\tilde{\delta}(\omega+\pi)+\frac{1}{2}\tilde{\delta}(\omega-\pi)\\
&= \tilde{\delta}(\omega-\pi)
\end{aligned}$$

since $\tilde{\delta}(\omega)$ is 2π-periodic. Therefore, by linearity,

$$Y(e^{j\omega})=\frac{1}{2}[X(e^{j\omega})+X(e^{j(\omega-\pi)})+\tilde{\delta}(\omega)+\tilde{\delta}(\omega-\pi)]$$

Solution 5. Modulation

(a) The modulated signal is

$$y[n]=\Re\{x[n]e^{j\omega_c n}\}=x[n]\cos(\omega_c n)$$

so that its DTFT is simply:

$$Y(e^{j\omega})=\frac{1}{2}\left[X\left(e^{j(\omega-\omega_c)}\right)+X\left(e^{j(\omega+\omega_c)}\right)\right].$$

(b) The baseband signal has spectral support $[\frac{-\omega_b}{2},\frac{\omega_b}{2}]$. First note that the maximum frequency of the modulated signal is $\omega_c+\frac{\omega_b}{2}$. To avoid overlap with the first repetition of the spectrum, we must guarantee that:

$$\omega_c+\frac{\omega_b}{2}<\pi,$$

which limit the maximum carrier frequency to: $\omega_c<\pi-\frac{\omega_b}{2}$

(c) The sampling frequency is $F_s=48000$ Hz and the signal bandwidth is 8 KHz and therefore

$$\omega_b=8000\text{ Hz}\frac{2\pi}{F_s}=\frac{\pi}{3}.$$

Applying the result of previous exercise:

$$\omega_c<\pi-\frac{\omega_b}{2}=\frac{5\pi}{6},$$

and converting back into Hz,

$$f_c = \omega_c \frac{F_s}{2\pi} < \frac{5\pi}{6} \frac{48000\text{Hz}}{2\pi} = 20\text{KHz}.$$