Exercise Set 7

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Exercise 1 (**Countable exponential races**). Let I be a countable space and let $T_k, k \in I$, be independent exponential random variables with $T_k \sim Exp(q_k)$ with $0 < q := \sum_{k \in I} q_k < \infty$. Set $T = inf_k T_k$. Let K be the random variable with values in I that is equal to k whenever $T = T_k$ and $T_j > T_k$ for $j \neq k$. Show that T and K are independent with $T \sim Exp(q)$ and $\mathbb{P}(K = k) = q_k/q$. Deduce that $\mathbb{P}(K = k \text{ for some } k) = 1$.

Exercise 2 (**General construction of Markov processes**). Let us consider a countable state space E and an array of positive numbers $(\lambda_{i,j})_{i,j \in E; i \neq j}$ with $\sum_{j \in E; j \neq i} \lambda_{i,j} < \infty$ for all $i \in E$. We recursively define a continuous time stochastic process $(X(t))_{t \geq 0}$ on E starting at $i_0 \in E$ as follows:

(i). Define $T_0 = 0$ and set $X(T_0) = i_0 \in E$;

(ii). For $n \in \mathbb{N}$: suppose we know T_{n-1} and $X(T_{n-1}) = i_{n-1}$. Independently of the previous steps, generate independent exponential random variables $E_1, E_2, \ldots$ with $E_j \sim Exp(\lambda_{i_{n-1},j})$. Define $T_n = T_{n-1} + \inf_{j \in \mathbb{N}} E_j$ and $i_n = \text{argmin}_{j \in E} E_j$, that is, the (random) index of the exponential variable that is the smallest. Then put

$$X(t) = \begin{cases} i_{n-1} & \text{for } t \in [T_{n-1}, T_n) \\ i_n & \text{for } t = T_n. \end{cases}$$

a) What is the distribution of the time between the jumps of the process $(X(t))_{t \geq 0}$?

b) Let $\widehat{P}_{ij}$ be the probability

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1} = i)).$$

Find the matrix $\widehat{P} = (\widehat{P}_{ij})_{i,j \in E}$.

c) Show that $(X(t))_{t \geq 0}$ is a homogeneous Markov process.

Definition (**The Q-matrix**).
One way of thinking about the evolution of the Markov process $(X(t))_{t \geq 0}$ is in terms of its Q-matrix, which is known as the generator of the process. A matrix $Q = (q_{ij})_{i,j \in E}$ is a Q-matrix if it satisfies

(i). $-\infty < q_{ii} \leq 0$ for all $i \in E$;

(ii). $0 \leq q_{ij} < \infty$ for all $i \neq j$;

(iii). $\sum_{j \in E} q_{ij} = 0$ for all $i \in E$.

The Q-matrix of the Markov process $(X(t))_{t \geq 0}$ as constructed above is given by $q_{ii} = -\sum_{j \neq i} \lambda_{i,j}$ for $i \in E$, and $q_{ij} = \lambda_{ij}$ for $j \neq i$.

Exercise 3. In a population of size N, a rumor is begun by a single individual who tells it to everyone he meets; they in turn pass the rumor to everyone they meet, once a person has passed the rumor to somebody he exits the system. Assume that each individual meets another randomly with exponential rate $1/N$. Let $X(t)$, $t \geq 0$ be the number in $E = \{1, \ldots, N\}$ of people who know the rumor at time t.

a) Draw a graph to visualize the chain. Write down the Q-matrix of the chain.

b) How long does it take in average until everyone knows the rumor if $X(0) = 1$?

Exercise 4 (**Poisson process**). For $i \in \mathbb{N}$, let E_i be independent copies of an exponential random variable of parameter λ. We let $T_n := E_1 + \cdots + E_n$ and

$$N(t) := \sum_{i=1}^{\infty} \mathbb{1}_{\{T_n \le t\}},\ t \ge 0.$$

The process $(N(t))_{t\ge 0}$ is called a homogeneous Poisson process with intensity λ. Let $T_0 = 0$ and we say that $T_1, T_2, T_3, \ldots$ are the successive arrival times of the Poisson process, and E_n the intervals $T_n - T_{n-1}$.

(i). Show that T_n follows an Erlang law with parameters n and λ having density:

$$f_{T_n}(t) = \frac{\lambda^n}{(n-1)!}\ t^{n-1}\ e^{-\lambda t}\ \mathbb{1}_{\{t>0\}}.$$

(ii). Show that, $\forall t > 0$, $N(t)$ follows a Poisson law with parameter λt, i.e.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$