

Markov Chains and Algorithmic Applications: WEEK 8

1 Sampling

1.1 Introduction

In this lecture we are interested in finding good sampling techniques to obtain samples from a probability distribution. In other words, given a probability distribution π on S , how can we pick a random $i \in S$ such that $\mathbb{P}(i) = \pi_i$?

But why would we want to do this ?

Example 1.1 (Monte Carlo Integration). Suppose we want to compute $\mathbb{E}(f(X))$, with $X \sim \pi$ (i.e. $\mathbb{P}(X = i) = \pi_i, i \in S$). By the definition of expectation we have

$$\mathbb{E}(f(X)) = \sum_{i \in S} f(i) \pi_i \quad (1)$$

Depending on the set S , the above expression can be too expensive to compute exactly (i.e. computing it requires exponential time in $|S|$).

Instead of evaluating (1), we can compute the following approximation: take M i.i.d. samples X_1, \dots, X_M from distribution π and compute

$$\frac{1}{M} \sum_{k=1}^M f(X_k) \quad (2)$$

Given some conditions on $f(x)$, the law of large numbers guarantees

$$\frac{1}{M} \sum_{k=1}^M f(X_k) \xrightarrow{M \rightarrow \infty} \mathbb{E}(f(X)) \text{ almost surely}$$

But how big should M be for the approximation to be good ? The variance of (2) is given by

$$\text{Var} \left(\frac{1}{M} \sum_{k=1}^M f(X_k) \right) = \frac{1}{M} \text{Var}(f(X_1)) = \mathcal{O} \left(\frac{1}{M} \right)$$

so $\frac{1}{M} \sum_{k=1}^M f(X_k) \approx \mathbb{E}(f(X)) \pm \frac{C}{\sqrt{M}}$. We see that a good approximation requires taking M quite large.

A “simple” way to obtain samples is as follows:

Example 1.2 (“Simple” Sampling). Let X be a π -distributed random variable on $S = \mathbb{N}$. If we can generate a continuous $\mathcal{U}(0, 1)$ random variable U , then we decide

$$X = \begin{cases} 0 & 0 \leq U \leq \pi_0, \\ 1 & \pi_0 < U \leq \pi_0 + \pi_1, \\ \vdots & \\ i & \sum_{j=0}^{i-1} \pi_j < U \leq \sum_{j=0}^i \pi_j \\ \vdots & \end{cases}$$

Hence $\mathbb{P}(X = i) = \pi_i$.

As simple as the above sampling scheme seems, terms of the form $\sum_{j=0}^i \pi_j$ (cdf of X) can be difficult to compute because we need to know each term π_j exactly: for π_j of the form $\frac{h(j)}{Z}$, the normalization constant $Z = \sum_{j \in S} h(j)$ can be non-trivial to compute depending on S , as we will see below.

For the rest of the lecture, we will detail alternative sampling methods to try to side-step the issues above.

1.2 Importance Sampling

Consider again the Monte Carlo integration problem given above: our aim here is to find a better estimate of $\mathbb{E}(f(X))$.

For this purpose, take another distribution $\psi = (\psi_i, i \in S)$ from which we know how to sample and let us define the coefficients $w_i = \frac{\pi_i}{\psi_i}$. Then

$$\mathbb{E}(f(X)) = \sum_{i \in S} f(i)\pi_i = \sum_{i \in S} f(i)w_i\psi_i = \mathbb{E}(f(Y)w(Y))$$

with $Y \sim \psi$. Since we know how to sample from ψ , we can approximate $\mathbb{E}(f(Y)w(Y))$ by choosing M i.i.d. samples Y_1, \dots, Y_M from ψ and computing $\frac{1}{M} \sum_{k=1}^M f(Y_k)w(Y_k)$. We then have

$$\text{Var}\left(\frac{1}{M} \sum_{k=1}^M f(Y_k)w(Y_k)\right) = \frac{1}{M} \text{Var}(f(Y_1)w(Y_1))$$

As we did not assume anything in particular about the distribution ψ , we can choose it so as to *minimize* the variance of $f(Y_1)w(Y_1)$, which improves the approximation of the expectation (but note that the order in M remains the same).

Remark 1.3. Why is this method called *importance sampling* ? It turns out that the distribution ψ minimizing the above variance puts more weight than π itself on the states i with a large probability π_i , and less weight on those with a small probability π_i : only the “important” states are therefore sampled with this method.

1.3 Rejection Sampling

Consider yet again the Monte Carlo integration problem (i.e. for $X \sim \pi$, compute $\mathbb{E}(f(X))$), but assume now that we are unable to sample directly from π (essentially because of the computation cost of this operation).

The idea behind rejection sampling is the following:

1. Take a distribution ψ on S from which samples can be easily produced (e.g. take ψ uniform).
2. Take a sample X from ψ .
3. Accept X with some probability, or reject it with the complement probability.

Formally, let $\psi = (\psi_i, i \in S)$ be a distribution from which we can sample and define weights $\tilde{w}_i = \frac{1}{c} \frac{\pi_i}{\psi_i}$ with $c = \max_{i \in S} \frac{\pi_i}{\psi_i} (\geq 1)$. The weights \tilde{w}_i play the role here of acceptance probabilities. Then

$$\begin{aligned} \mathbb{P}(X = i) &= \psi_i \tilde{w}_i = \frac{\pi_i}{c} \\ \mathbb{P}(X \text{ is rejected}) &= 1 - \sum_{i \in S} \mathbb{P}(X = i) = 1 - \sum_{i \in S} \frac{\pi_i}{c} = 1 - \frac{1}{c} \end{aligned}$$

We therefore have

$$\mathbb{E}(f(X)) \approx \frac{1}{M'} \sum_{k=1: X_k \text{ accepted}}^M f(X_k)$$

where M' is the number of accepted samples among the X_1, \dots, X_M .

The disadvantage of rejection sampling is that it may end up requiring much more samples than needed due to the sample rejection process (especially when the distance between π and ψ is large, i.e. when c is large).

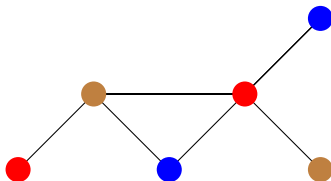
1.4 Markov Chain Monte Carlo (MCMC) Sampling

The idea behind the MCMC method to obtain samples of a distribution π on S is to construct a Markov chain on S with transition matrix P having π as its stationary distribution. The samples of π are then obtained by iterating P long enough to reach the stationary distribution π , then sampling among the states of the Markov chain. The advantage here is that a) we do not have to sample directly from π , and b) we do not even need to know everything about π , as we will see below.

For practical reasons, we want P to have certain properties:

1. π should be the unique limiting distribution of P .
2. Convergence to the stationary distribution π should be fast, so as to obtain samples within a reasonable amount of time.

Example 1.4 (Graph Coloring). Let $G = (V, E)$ be a graph with vertex set V and edge set E . We want to color each vertex of the graph with one of the q colors at our disposal such that a vertex's color differs from that of all its neighbors, as seen below:



More formally, let $x = (x_v, v \in V)$ be a particular color configuration of the vertex set V . A *proper q -coloring* of G is any configuration x such that $\forall v, w \in V$, if $(v, w) \in E$ then $x_v \neq x_w$.

If S represents the set of all possible color configurations, then the uniform distribution π over all proper q -colorings is given by

$$\pi(x) = \frac{1}{Z} \mathbb{1}\{x \text{ is a proper } q\text{-coloring}\}$$

where Z is the total number of proper q -colorings in G .

Computing Z would require enumerating all possible proper q -colorings which is non-trivial depending on G . Still, we would like to sample from π without computing Z explicitly.

1.4.1 Metropolis-Hastings Algorithm

The Metropolis-Hastings algorithm is a procedure to construct a Markov chain on S having as limiting distribution π (for convenience, we assume that $\pi_i > 0$ for all $i \in S$). Here is the algorithm:

1. Select an easy-to-simulate irreducible and aperiodic Markov chain ψ on S with the constraint that $\psi_{ij} > 0$ if and only if $\psi_{ji} > 0$.¹ We call ψ the *base chain*.
2. Design acceptance probabilities $a_{ij} = \mathbb{P}(\text{transition from } i \text{ to } j \text{ is accepted})$ such that the matrix P given below has limiting distribution π .
3. Construct the matrix P as such:

$$\begin{cases} p_{ij} &= \psi_{ij} a_{ij}, & j \neq i \\ p_{ii} &= \psi_{ii} + \sum_{k \neq i} \psi_{ik} (1 - a_{ik}) = 1 - \sum_{k \neq i} \psi_{ik} a_{ik} \end{cases}$$

In other words, we are adding self-loops of different weights to each state.

¹If S is finite, then these conditions imply positive-recurrence, hence ψ is ergodic and has a unique limiting distribution, but this limiting distribution is of no interest to the algorithm.

We must now choose the weights a_{ij} so that $p_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$. Moreover, we were able to upper-bound the mixing time of chains satisfying detailed balance in the previous lectures, so we would like P to satisfy this condition too: $\pi_i p_{ij} = \pi_j p_{ji}$

Theorem 1.5 (Metropolis-Hastings). If $a_{ij} = \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right)$, then the matrix P constructed above is ergodic with stationary distribution π . Moreover, P satisfies detailed balance.

Proof. By assumption, ψ is irreducible and aperiodic, and $\forall i, j \in S, \psi_{ij} > 0$ iff $\psi_{ji} > 0$. So if $\psi_{ij} > 0$, then $a_{ij} > 0$ and $p_{ij} > 0$ also. Therefore, P is also irreducible and aperiodic. We then have

$$\pi_i p_{ij} = \pi_i \psi_{ij} a_{ij} = \pi_i \psi_{ij} \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right) = \min(\pi_i \psi_{ij}, \pi_j \psi_{ji})$$

whose expression is symmetric in i, j . It is therefore also equal to $\pi_j p_{ji}$: detailed balance holds and P has π as stationary distribution.

Finally, since P is irreducible and has a stationary distribution π , then by a previously seen theorem, P must be positive-recurrent and p_i must be unique. therefore P is ergodic and π is also a limiting distribution. \square

Remark 1.6. If $\psi_{ij} = \psi_{ji}$, then the expression for a_{ij} simplifies to $a_{ij} = \min\left(1, \frac{\pi_j}{\pi_i}\right)$.

The intuition behind choosing a_{ij} as such is the following: if $\pi_j > \pi_i$ the transition $i \rightarrow j$ should be taken with probability 1 since the chain is heading towards the more probable state j . However if $\pi_j < \pi_i$, then the move $i \rightarrow j$ should be taken with probability $\frac{\pi_j}{\pi_i} < 1$. In other words, the chain should tend towards the states having high probability, but it should be able to return to less probable states in order not to get stuck in a state that locally maximizes π .

Remark 1.7. The advantage of the Metropolis-Hastings algorithm is that the acceptance probabilities a_{ij} depend on π only through the ratios $\frac{\pi_j}{\pi_i}$, which can be significantly easier to compute than π_i and π_j separately! In the graph coloring example given previously, $\frac{\pi_j}{\pi_i} = \frac{\mathbb{1}\{j \text{ is a proper } q\text{-coloring}\}}{\mathbb{1}\{i \text{ is a proper } q\text{-coloring}\}}$, so we can avoid computing the expensive normalization constant Z entirely.

Example 1.8 (Metropolized Independent Sampling). To obtain samples of distribution π on S , we choose the base chain ψ such that $\psi_{ij} = \psi_j > 0 \forall i, j \in S$ (i.e. the process realizations are just sequences of i.i.d. random variables).

The acceptance probabilities are $a_{ij} = \min\left(1, \frac{w_j}{w_i}\right)$ with $w_i = \frac{\pi_i}{\psi_i}$, so the transition probabilities of P are given by

$$\begin{cases} p_{ij} &= \psi_{ij} a_{ij} = \psi_j \min\left(1, \frac{w_j}{w_i}\right), \quad j \neq i \\ p_{ii} &= 1 - \sum_{k \neq i} \psi_{ik} a_{ik} = 1 - \sum_{k \neq i} \psi_k \min\left(1, \frac{w_k}{w_i}\right) \end{cases}$$

In this particular example, one can show the following (no proof given here):

Theorem 1.9 (Liu). Let $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{N-1}$ be the eigenvalues of P , and $\lambda_* = \max(\lambda_1, -\lambda_{N-1})$. Then

$$\lambda_* = 1 - \frac{1}{w_*}, \quad \text{where } w_* = \max_{i \in S} \frac{\pi_i}{\psi_i} > 1$$

Correspondingly, the spectral gap $\gamma = \frac{1}{w_*}$.

From the above and the previous lectures, we find that

$$\|P_i^n - \pi\|_{\text{TV}} \leq \frac{\lambda_*^n}{2\sqrt{\pi_i}} \leq \frac{1}{2\sqrt{\pi_i}} e^{-\gamma n} = \frac{1}{2\sqrt{\pi_i}} e^{-\frac{n}{w_*}}$$

Therefore, if w_* is large (i.e. if the distance between π and ψ is large), then convergence to the stationary distribution π is slow (this resembles the situation we already encountered with rejection sampling).