Exercise 6: 9^{th} and 16^{th} of April 2019
CS-526 Learning Theory

Most of the following exercises are extracted from the books "Pattern Recognition and Machine Learning" by Bishop and "Bayesian Reasoning and Machine Learning" by Barber.

Problem 1

Suppose we have some data $(x_i, y_i), i = 1, \ldots, m$ where $x_i, y_i \in \mathbb{R}$ and we want to find a regression function $y = f(x)$. We use a fully Bayesian model:

$$y = \sum_{a=1}^{p} w_a x^a + \xi,$$

where the inputs $x \sim P_0$ are iid and generated according to a prior P_0 and $\xi \sim \mathcal{N}(0, \sigma^2)$ iid. The $w_a \in \mathbb{R}$ are regression parameters. We take for w_a the prior $\sim e^{-\alpha w_a^2}$ where α. The parameters α and σ^2 are supposed to be known. So our model for the data generating process is

$$\mathcal{D}(y \mid x, w)\mathcal{D}(x) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(y - \sum_{a=1}^{p} w_a x^a)^2} P_0(x)$$

1) Write down the joint distribution for $(y_1, \ldots, y_m, x_1, \ldots x_m, w_1, \ldots w_p)$.

2) Draw a Belief Network (directed acyclic graph) corresponding to this probabilistic model.

3) Remark 1: Show that the maximum likelihood principle (take $\alpha = 0$ or equivalently no prior on w_a's) is equivalent to empirical risk minimisation in the hypothesis class of functions $\mathcal{H} \ni f(x) = \sum_{a=1}^{p} w_a x^a$.

4) Remark 2: Consider the MAP principle for estimating w_a's and show that it is equivalent to an empirical risk minimization with additional penalty term proportional to $\alpha \sum_{a=1}^{p} w_a^2$ (this is called ridge regression).

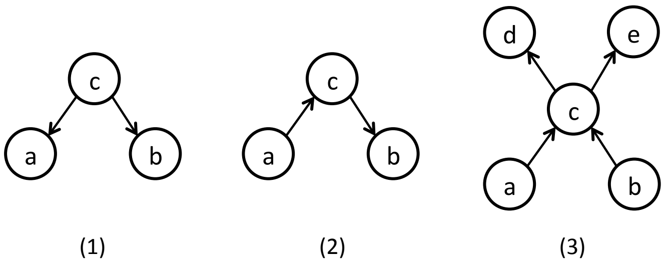
5) Remark 3: The ML or MAP estimates of w_a's are to viewed in general as summarized versions of a more detailled object, namely the complete posterior distribution $P(w_1, \ldots, w_p \mid (x_i, y_i)_{i=1}^{m})$. Show that the optimal regression function in a fully Bayesian approach is

$$f(x) = \sum_{a=1}^{p} \mathbb{E}_{w|data}[w_a] x^a$$

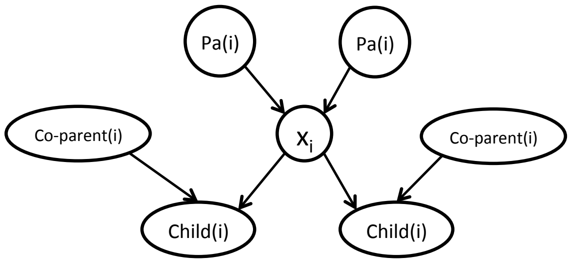
where $\mathbb{E}_{w|data}$ is the expectation with respect to the posterior distribution $P(w_1, \ldots, w_p \mid (x_i, y_i)_{i=1}^{m})$.

Problem 2

For each case below, is $a \perp\!\!\!\perp b$ true? And is $a \perp\!\!\!\perp b | c$ true? If yes, prove your answer.



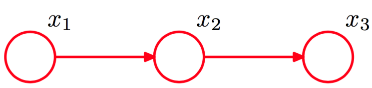
Problem 3



Consider a generic belief network (also called Bayesian network).
Let $\mathrm{MB}(i) = \{\mathrm{pa}(i), \mathrm{child}(i), \text{co-parent}(i)\}$ be the Markov blanket of x_i. Show that

$$p(x_i | \{x_j\}_{j \neq i}) = p(x_i | \{x_v\}_{v \in \mathrm{MB}(i)}).$$

Problem 4 (Bishop, p.371 & 419, Exercise 8.7)



The linear-Gaussian models for the above graph consists of three random variables x_1, x_2, x_3. The model has the structure equations

$$x_i = \sum_{j \in \mathrm{pa}(i)} w_{ij} x_j + b_i + \sqrt{v_i}\epsilon_i, \qquad i = 1, 2, 3$$

where $\mathrm{pa}(i)$ is the set of parent nodes of node i ($\mathrm{pa}(1) = \emptyset$, $\mathrm{pa}(2) = \{1\}, \mathrm{pa}(3) = \{2\}$).

Show that the mean and covariance of the joint distribution for the above graph are given by (hint: use a recursive calculation)

$$\mu = (b_1, b_2 + w_{21}b_1, b_3 + w_{32}b_2 + w_{32}w_{21}b_1)^\top$$
$$\mathbf{\Sigma} = \begin{pmatrix} v_1 & w_{21}v_1 & w_{32}w_{21}v_1 \\ w_{21}v_1 & v_2 + w_{21}^2 v_1 & w_{32}(v_2 + w_{21}^2 v_1) \\ w_{32}w_{21}v_1 & w_{32}(v_2 + w_{21}^2 v_1) & v_3 + w_{32}^2(v_2 + w_{21}^2 v_1) \end{pmatrix}$$

Problem 5 (Barber, p.75, Exercise 4.4)

The restricted Boltzmann machine (RBM) is a constrained Boltzmann machine on a bipartite graph, consisting of a layer of visible variables $\mathbf{v} = (v_1, \cdots, v_V)^\top$ and hidden variables $\mathbf{h} = (h_1, \cdots, h_H)^\top$:

$$p(\mathbf{v}, \mathbf{h}) = \frac{1}{Z(\mathbf{W}, \mathbf{a}, \mathbf{b})} \exp\left(\mathbf{v}^\top \mathbf{W} \mathbf{h} + \mathbf{a}^\top \mathbf{v} + \mathbf{b}^\top \mathbf{h}\right)$$

All variables are binary taking value 0 or 1. Here $\mathbf{W}$ is an $V \times H$ matrix of weight W_{ji}.

1) Show that the distribution of hidden units conditional on the visible unit is factorized as

$$p(\mathbf{h}|\mathbf{v}) = \prod_i p(h_i|\mathbf{v}), \qquad \text{with } p(h_i = 1|\mathbf{v}) = \sigma\Big(b_i + \sum_j W_{ji} v_j\Big)$$

   where $\sigma(x) = e^x/(1+e^x)$.

2) By symmetry arguments, write down the form of the conditional $p(\mathbf{v}|\mathbf{h})$.

3) Is $p(\mathbf{h}) = \prod_i p(h_i)$?

4) Can the partition function $Z(\mathbf{W}, \mathbf{a}, \mathbf{b})$ be computed efficiently for the RBM?

Problem 6 (Barber, p.77, Exercise 4.14)

Consider a pairwise binary Markov network defined on variables $x_i \in \{0, 1\}, i = 1, \ldots, N$, with $p(\mathbf{x}) = \frac{1}{Z} \prod_{ij \in \mathcal{E}} \phi_{ij}(x_i, x_j)$ where $\mathcal{E}$ is a given edge set and the factors ϕ_{ij} are arbitrary (here edges are non necessarily maximal cliques). Explain how to translate such a Markov network into a Boltzmann machine.

Problem 7

Let $G = (V, E)$ an undirected graph whose vertices $V = \{1, \ldots, n\}$ are associated to random variables, and edges are given by the set of pairs E. For simplicity the random variables are assumed to be discrete. Denote $\mathcal{C}$ the set of maximal cliques of G and consider a probability distribution $p(\mathbf{x})$ which factorizes as

$$p(\mathbf{x}) = \frac{1}{Z} \prod_{C \in \mathcal{C}} \psi_C(\mathbf{x}_C)\,,$$

where $Z = \sum_x \prod_{C \in \mathcal{C}} \psi_C(\mathbf{x}_C)$ and $\forall C \in \mathcal{C}, \forall \mathbf{x}_C : \psi_C(\mathbf{x}_C) > 0$. Remember that – for a Markov Random Field – the Markov blanket ∂S of a subset $S \subseteq V$ is the set of all vertices that are directly connected to a vertex in S and are not in S. Show that the following conditional independence properties is satisfied:

$$\forall S \subseteq V : p(\mathbf{x}_S | \mathbf{x}_{V \setminus S}) = p(\mathbf{x}_S | \mathbf{x}_{\partial S})\,.$$

Problem 8 (Barber, p.99, Exercise 5.4)

Consider the hidden Markov model (HMM)

$$p(\mathbf{v}, \mathbf{h}) = p(h_1)p(v_1|h_1)\prod_{t=2}^{T} p(v_t|h_t)p(h_t|h_{t-1})$$

in which $\mathrm{dom}(h_t) = \{1, \ldots, H\}$ and $\mathrm{dom}(h_t) = \{1, \ldots, V\}$ for all $t = 1, \ldots, T$.

1) Draw a belief network representation of the above distribution.

2) Show that the belief network for $p(h_1, \ldots, h_T)$ is a simple linear chain. Draw the belief network corresponding to $p(v_1, \ldots, v_T)$ (this is called a fully connected cascade belief network).

3) Draw a factor graph representation of the above distribution.

4) Use the factor graph to derive a Sum-Product algorithm to compute marginals $p(h_t|v_1, \ldots, v_T)$. Explain the sequence order of messages passed on your factor graph.

5) Explain how to compute $p(h_t, h_{t+1}|v_1, \ldots, v_T)$.

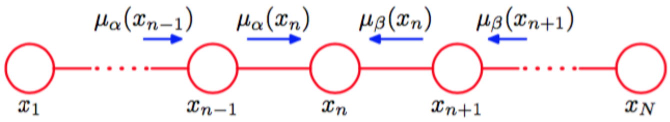
Problem 9 (Barber, p.98, Exercise 5.1)

Given a pairwise tree Markov network of the form

$$p(x) = \frac{1}{Z}\prod_{i\sim j} \phi(x_i, x_j),$$

explain how to efficiently compute the normalization factor (also called the partition function) Z as a function of the potentials ϕ.

Problem 10 (Bishop, p.397 & 421, Exercise 8.16 & 8.17)



The joint distribution for the above graph takes the form

$$p(\mathbf{x}) = \frac{1}{Z}\phi_{1,2}(x_1, x_2)\phi_{2,3}(x_2, x_3)\cdots\phi_{N-1,N}(x_{N-1}, x_N).$$

The marginal probability $p(x_n) = \sum_{i\in\{1,\ldots,N\}\backslash n} p(\mathbf{x})$ can be written as

$$p(x_n) = \frac{1}{Z_n}\mu_\alpha(x_n)\mu_\beta(x_n) \quad \text{with } Z_n = \sum_{x_n} \mu_\alpha(x_n)\mu_\beta(x_n)$$

where $\mu_\alpha(x_n)$ is the message passing forward from node $n-1$ to node n, and $\mu_\beta(x_n)$ is the message passing backward from node $n+1$ to node n. The computation of $\mu_\alpha(x_n)$ and $\mu_\beta(x_n)$ can be done recursively by the following message passing equations:

$$\begin{aligned}
\mu_\alpha(x_2) &= \sum_{x_1} \phi_{1,2}(x_1, x_2) \\
\mu_\beta(x_{N-1}) &= \sum_{x_N} \phi_{N-1,N}(x_{N-1}, x_N) \\
\mu_\alpha(x_n) &= \sum_{x_{n-1}} \phi_{n-1,n}(x_{n-1}, x_n)\mu_\alpha(x_{n-1}) \\
\mu_\beta(x_n) &= \sum_{x_{n+1}} \phi_{n,n+1}(x_n, x_{n+1})\mu_\beta(x_{n+1})
\end{aligned}$$

1) Discuss how to modify the above message passing algorithm in order to compute $p(x_n|x_N)$ efficiently.

2) Suppose $N = 5$, and nodes x_3, x_5 are observed. Show that if the message passing algorithm is applied to the evaluation of $p(x_2|x_3, x_5)$, the result will be independent of the value of x_5.