Solutions 8

1. Because of the assumptions made, $a_{ij} > 0$ if $\psi_{ij} > 0$, so the chain with transition probabilities p_{ij} is also irreducible and aperiodic, therefore ergodic, as the state space S is finite. Let us check the detailed balance equation:

$$\pi_i \, p_{ij} = \pi_i \, \psi_{ij} \, a_{ij} = \frac{\pi_i \, \psi_{ij} \, \pi_j \, \psi_{ji}}{\pi_j \, \psi_{ji} + \pi_i \, \psi_{ij}}$$

which is clearly symmetric in i and j, and therefore equal to $\pi_j \, p_{ji}$.

2. First note that $Z = \frac{1-\theta^N}{1-\theta} \simeq \frac{1}{1-\theta}$ for large N.

a) The weights defined in class are given in this case by $w_i = \frac{\pi_i}{\psi_i} = \frac{N}{Z} \, \theta^{i-1}$, so that for $j \neq i$:

$$a_{ij} = \min\left(1, \frac{w_j}{w_i}\right) = \min\left(1, \theta^{j-i}\right) = \begin{cases} 1 & \text{if } j < i \\ \theta^{j-i} & \text{if } j > i \end{cases}$$

which leads to

$$p_{ij} = \begin{cases} \frac{1}{N} & \text{if } j < i \\ \frac{1}{N} \, \theta^{j-i} & \text{if } j > i \\ \frac{1}{N} + \frac{1}{N} \sum_{k=i+1}^{N} \left(1 - \theta^{k-i}\right) & \text{if } j = i \end{cases}$$

b) From the course, we know that

$$\|P_i^n - \pi\|^{\text{TV}} \leq \frac{\lambda_*^n}{2\sqrt{\pi_i}}$$

where

$$\lambda_* = 1 - \frac{1}{w_*} \quad \text{and} \quad w_* = \max_{i \in S} w_i = w_1 = \frac{N}{Z}$$

We conclude therefore that

$$\|P_i^n - \pi\|_{\text{TV}} \leq \frac{\sqrt{Z}}{2\sqrt{\theta^{i-1}}} \left(1 - \frac{Z}{N}\right)^n$$

For $i = 1$ and large N, this bound leads to:

$$\|P_1^n - \pi\|_{\text{TV}} \leq \frac{1}{2\sqrt{1-\theta}} \exp\left(-\frac{n}{N(1-\theta)}\right)$$

while for $i = N$ and large N, this bound leads to:

$$\|P_N^n - \pi\|_{\text{TV}} \leq \frac{1}{2\sqrt{(1-\theta)\,\theta^{N-1}}} \exp\left(-\frac{n}{N(1-\theta)}\right) = \frac{1}{2\sqrt{1-\theta}} \exp\left(\frac{N-1}{2} \log(1/\theta) - \frac{n}{N(1-\theta)}\right)$$

c) Because of the last estimate, in order for $\max_{i \in S} \|P_i^n - \pi\|_{\text{TV}}$ to be smaller than ε, we need that $n \gg N^2$, which gives the desired upper bound on the mixing time. What can actually be shown in this case (but this was not asked) is the following: using the more precise estimate

$$\|P_i^n - \pi\|_{\text{TV}} \leq \frac{1}{2} \sqrt{\sum_{k=1}^{N-1} \lambda_k^{2n} \, \left(\phi_i^{(k)}\right)^2}$$

we find that this quantity is small (uniformly in i) for $n \gg N$ already.

3. a) We have

$$\begin{aligned}\mathbb{P}(X \neq Y) = 1 - \mathbb{P}(X = Y) &= 1 - \sum_{i \in S} \xi_i = 1 - \sum_{i \in S} \min(\mu_i, \nu_i) \\ &= \sum_{i \in S \,:\, \mu_i > \nu_i} (\mu_i - \nu_i) = \sum_{i \in S \,:\, \nu_i > \mu_i} (\nu_i - \mu_i)\end{aligned}$$

where the last two equalities hold because both μ and ν are distributions. Summing these last two equalities, we obtain

$$2\,\mathbb{P}(X \neq Y) = \sum_{i \in S} |\mu_i - \nu_i| = 2\,\|\mu - \nu\|_{\mathrm{TV}}$$

b) Observe first that if $\sum_{i \in S} \xi_i = 1$, then $X = Y$ with probability one. When $\sum_{i \in S} \xi_i < 1$, we obtain for $i \in S$:

$$\begin{aligned}\mathbb{P}(X = i) = \mathbb{P}(X = Y = i) + \sum_{j \in S \setminus i} \mathbb{P}(X = i, Y = j) &= \xi_i + \sum_{j \in S \setminus i} \frac{(\mu_i - \xi_i)\,(\nu_j - \xi_j)}{1 - \sum_{k \in S} \xi_k} \\ &= \xi_i + \frac{\mu_i - \xi_i}{1 - \sum_{k \in S} \xi_k} \sum_{j \in S \setminus i} (\nu_j - \xi_j) = \xi_i + \frac{\mu_i - \xi_i}{1 - \sum_{k \in S} \xi_k} \left(1 - \nu_i - \sum_{j \in S} \xi_j + \xi_i\right) \\ &= \xi_i + (\mu_i - \xi_i) - \frac{(\mu_i - \xi_i)\,(\nu_i - \xi_i)}{1 - \sum_{k \in S} \xi_k} = \mu_i\end{aligned}$$

as $(\mu_i - \xi_i)\,(\nu_i - \xi_i) = 0$ necessarily. A similar reasoning shows that $\mathbb{P}(Y = j) = \nu_j$ for all $j \in S$.

c) Fix $i \in S$, let $X_0 = i$ and $Y_0 \sim \pi$ the stationary distribution, and fix also a time n. By parts a) and b), we can find a coupling of X_n and Y_n such that $d_i(n) = \|P_i^n - \pi\|_{\mathrm{TV}} = \mathbb{P}(X_n \neq Y_n)$. We can now define a new coupling for X_{n+1} and Y_{n+1} in the following way:

- If $X_n = Y_n$, then $X_{n+1} = Y_{n+1}$;
- Else, let X and Y evolve independently according to P.

Then

$$d_i(n+1) = \|P_i^{n+1} - \pi\|_{\mathrm{TV}} \leq \mathbb{P}(X_{n+1} \neq Y_{n+1}) \leq \mathbb{P}(X_n \neq Y_n) = d_i(n)$$

The first inequality holds by the coupling lemma, and the second inequality is by construction. Observe finally that $d(n) = \max_{i \in S} d_i(n)$ is also non-increasing in n (being the maximum of non-increasing functions).