Solutions 3

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Exercise 1. Let $(X_n)_{n\geq 0}$ be a Markov chain given by the following transition matrix:

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Find a stationary distribution π for X, i.e. such that $\pi P = P$.

Proof. We look for a vector $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying $\pi P = \pi$ and such that the sum of their components is one. We obtain the following system:

$$\begin{cases} \pi_1 + \pi_2 + \pi_3 & = 1, \\ 0.5\pi_1 + 0.2\pi_2 + 0.1\pi_3 & = \pi_1, \\ 0.4\pi_1 + 0.5\pi_2 + 0.3\pi_3 & = \pi_2, \\ 0.1\pi_1 + 0.3\pi_2 + 0.6\pi_3 & = \pi_3. \end{cases}$$

We substitute π_3 by $1 - \pi_1 - \pi_2$ in the second and third equation. We have (by multiplying the equations by 10)

$$\begin{cases} 5\pi_1 & = 2\pi_2 + 1 - \pi_1 - \pi_2, \\ 5\pi_2 & = 4\pi_1 + 3 - 3\pi_1 - 3\pi_2. \end{cases} \iff \begin{cases} 6\pi_1 & = \pi_2 + 1, \\ 8\pi_2 & = \pi_1 + 3. \end{cases}$$

Substituing π_1 by $\frac{\pi_2+1}{6}$ in the second equation, we finally find $\pi = (\frac{11}{47}, \frac{19}{47}, \frac{17}{47})$. $\square$

Exercise 2. Let $(X_i)_{i\geq 0}$ be a Bernoulli process, which means that the X_i's are *iid* with a Bernoulli law of parameter p.

(a) Consider the process $(N_n)_{n\geq 0}$ of the number of successes: N_n is the number of successes of the Bernoulli process until time n included.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

(b) Consider the process $(T_n)_{n\geq 0}$ of the moment of successes: T_n is the time when the nth success happens in the Bernoulli process.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

Solution. (a) For any integer n, we can write $N_{n+1} = N_n + X$ where $X \sim Ber(p)$ is independent of $N_1, \cdots, N_n$. Thus, knowing $N_n = k$ for some integer k, $N_{n+1} = k + X$ is independent of $N_1, \cdots, N_{n-1}$. This shows that $(N_n)_{n\geq 0}$ is a Markov chain. For the homogeneity, it is easy to verify that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(N_{n+1} = j \mid N_n = i) = \begin{cases} p & \text{if } j = i + 1, \\ q & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The transition matrix corresponding to states $\{0, 1, 2, ...\}$ is then given by

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ 0 & q & p & 0 & 0 & \cdots \\ 0 & 0 & q & p & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The associated graph of this Markov chain is given by

$$\begin{tikzcd} 0 \arrow[loop above, "q"] \arrow[r, "p"] & 1 \arrow[loop above, "q"] \arrow[r, "p"] & 2 \arrow[loop above, "q"] \arrow[r, "p"] & 3 \arrow[loop above, "q"] \arrow[r, "p"] & \cdots \end{tikzcd}$$

All states are transient. Indeed, if we go from i to $i + 1$, we are sure that we are not returning to i. Thus, the probability, starting from i, to never return to i is strictly positive.

(b) Since the X_i's are i.i.d, we have for all integers n

$$T_{n+1} = T_n + S,$$

where S is the first time of a sucess of the Bernoulli process (S is independent of all the $T_i,\ i \leq n$). Hence, conditioning on T_n, T_{n+1} is independent of the T_i's for $i \leq n - 1$. It is easy to verify the homogeneity of the Markov chain:

$$\mathbb{P}(T_{n+1} = j \mid T_n = i) = \begin{cases} 0 & \text{if } j \leq i \\ q^{j-i-1}p & \text{otherwise.} \end{cases}$$

The associated transition matrix is then given by

$$Q = \begin{pmatrix} 0 & p & qp & q^2p & q^3p & \cdots \\ 0 & 0 & p & qp & q^2p & \cdots \\ 0 & 0 & 0 & p & qp & \cdots \\ 0 & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{tikzcd} 1 \arrow[r, "p"] \arrow[rr, bend left, "qp"] \arrow[rrr, bend right, "q^2p"'] & 2 \arrow[r, "p"] \arrow[rr, bend left, "qp"] & 3 \arrow[r, "p"] & 4 \cdots \end{tikzcd}$$

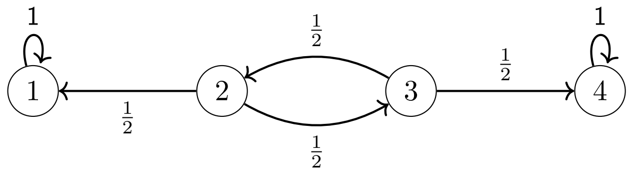
Similarly to the argument of the first part, it's easy to see that all states are transient.

Exercise 3. Let $(X_n)_{n \geq 0}$ be a Markov chain determined by the following diagram:
Compute for all $i = 1, 2, 3, 4$ the absorption probability

$$h_i = \mathbb{P}_i\{\exists n \geq 0 : X_n = 4\},$$

i.e. the probability that the chain is absorbed in state 4 knowing that the chain starts at $X_0 = i$. Then compute the mean absorption time knowing that the chain starts in state i

$$k_i = \mathbb{E}_i[\inf(n \geq 0 : X_n \in \{1, 4\})].$$



Proof. Note that the states 1 and 4 are absorbing. Clearly $h_4 = 1$, moreover as 1 is absorbing, we get $h_1 = 0$. Suppose that the chain is starting at 2 and consider the chain after a transition. The process jumps to 1 with probability 1/2 and to 3 with probability 1/2, then

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3.$$

By a similar argument, we obtain starting from 3

$$h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4.$$

The problem is equivalent to solving the following system of equations

$$\begin{cases} h_1 &= 0, \\ h_2 &= 0.5h_1 + 0.5h_3, \\ h_3 &= 0.5h_2 + 0.5h_4, \\ h_4 &= 1. \end{cases}$$

implying $h_2 = 1/3$ and $h_3 = 2/3$.
Let us compute now the mean times spent before absorption. Clearly, $k_1 = 0$ et $k_4 = 0$. By a similar argument as before, we have the equations

$$k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3, \;\; k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k_4,$$

where the term 1 is here since we count the first jump. We finally get $k_2 = 2$ and $k_3 = 2$. $\square$

Exercise 4 (Random walk). Let $(X_n)_{n\geq 0}$ be a one-dimensional random walk on the state space $\mathbb{Z}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p, & y = x+1, \\ q, & y = x-1, \end{cases}$$

(1) Prove that the random walk is recurrent if an only if $p = q$.
Hint: Note that $p_{00}^{2n+1} = 0$ for all $n \in \mathbb{N}$, and find the probability p_{00}^{2n}. You can then use Stirling's approximation to n!

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

(2) In the transient case $p \neq q$, find the limit $\lim_{n\to\infty} X_n$.

Proof. (1) This Markov chain is irreducible. Suppose we start at 0, then $p_{00}^{(2n+1)} = 0$ for all n. Any given sequence of $2n$ steps from 0 to 0 has probability $p^n q^n$ and the number of sequences is the number of ways of choosing n steps up from $2n$ steps is. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n.$$

We will use Stirling's approximation to $n!$

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

With this we obtain

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \sim \frac{C(4pq)^n}{\sqrt{n}}.$$

In the symmetric case $p = q = 1/2$, $4pq = 1$ and so

$$\sum_{n=0}^{\infty} p_{00}^{(2n)} \approx C \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

showing that the random walk is recurrent.

If $p \neq q$, then $4pq = r < 1$ and thus

$$\sum_{n=0}^{\infty} p_{00}^{(2n)} \approx C \sum_{n=0}^{\infty} \frac{r^n}{\sqrt{n}} < \infty,$$

where $C > 0$ is a constant. Thus the random walk is transient.

(2) The strong law of large numbers gives us that depending on the sign of $p - q$

$$\lim_{n\to\infty} X_n \overset{a.s.}{=} \operatorname{sgn}(p-q)\infty,$$

(please refer to the solution of the next exercice).

□

Exercise 5 (Birth and Death chain). Let us consider a Markov chain $(X_n)_{n\geq 0}$ on the state space $\mathbb{N}$ defined by the following transition probabilities:

$$p(x,y) = \begin{cases} p & \text{if } x > 0,\ y = x+1, \\ q & \text{if } x > 0,\ y = x-1, \\ 1 & \text{if } x = 0,\ y = 1. \end{cases}$$

Prove that:

(1) when $p \leq q$ the chain is recurrent.

Hint: study the probability $u(k) = P_k\left(X_n \neq 0, \forall n \in \mathbb{N}\right)$ by showing that

$$u(k+1) - u(k) = \frac{q}{p}\left(u(k) - u(k-1)\right).$$

(2) when $q < p$ the chain is transient.

Hint: consider writing the chain as $X_n = \sum_{i=1}^n Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0)$ where

$$Y_i \overset{\text{iid}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

and compare with the biased random walk $\sum_{i=1}^n Y_i$.

Proof. (1) in the case $p \leq q$, we define the probability $u(k) = \mathbb{P}_k(X_n \neq 0, \ \forall n \in \mathbb{N})$. The chain is recurrent if and only if $u(k) = 0$ for all $k \in \mathbb{N}$. Clearly $u(0) = 0$, and moreover by the Markov property

$$u(k) = qu(k-1) + pu(k+1),$$

which gives after rearranging

$$u(k+1) - u(k) = \frac{q}{p}(u(k) - u(k-1)) = \left(\frac{q}{p}\right)^k (u(1) - u(0)) = \left(\frac{q}{p}\right)^k u(1).$$

Consequently,

$$u(k+1) = (u(k+1) - u(k)) + (u(k) - u(k-1)) + \cdots + (u(1) - u(0)) = u(1) \sum_{j=0}^{k} \left(\frac{q}{p}\right)^j.$$

Thus, if the sum diverges, i.e., $q \geq p$, then $u(1) = 0 = u(k)$, for all $k \in \mathbb{N}$., since the $u(k)$ must be probabilities.

(2) in the case $q < p$, we will use a coupling argument. Let us consider a sequence of independent and identically distributed random variables $(Y_n)_{n \geq 1}$ such that

$$Y_i \overset{\text{iid}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

This sequence will serve as a common source of randomness to couple the random walk on $\mathbb{Z}$ with the birth and death chain on $\mathbb{N}$. Indeed, if we consider the two processes:

$$\begin{aligned} X_n &= \sum_{i=1}^n Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0), \\ Z_n &= \sum_{i=1}^n Y_i, \end{aligned}$$

we remark that they both evolve according to the common sequence $(Y_n)_{n \geq 1}$ and we can check that X_n is exactly the birth and death chain on $\mathbb{N}$ and Z_n the biased random walk on $\mathbb{Z}$. We can now consider their asymptotic behaviour together. We have by construction the pathwise inequality

$$Z_n(\omega) \leq X_n(\omega), \text{ for all } n \text{ and } \omega \in E.$$

Then, since the expectation of the Y_i's is $p-q>0$, we can deduce by the law of large numbers that

$$\frac{1}{n} Z_i = \frac{1}{n}\sum_{i=1}^{n} Y_i \overset{a.s.}{\to} p-q>0.$$

implying that $Z_n \overset{a.s.}{\to} +\infty$ and finally $X_n \overset{a.s.}{\to} +\infty$. We thus conclude that the birth and death chain is transient in this setting.

□

Exercise 6. Let X_0 be a random variable having values in a countable set I. Let $Y_1, Y_2, \dots$ be a sequence of independent variables, uniformly distributed on $[0,1]$. Considering any function

$$G : I \times [0,1] \to I,$$

we define inductively

$$X_{n+1} = G(X_n, Y_{n+1}).$$

(1) Show that $(X_n)_{n\geqslant 0}$ is a Markov chain and write its transition matrix P as a function of G.

(2) Can all Markov chains be defined this way?

(3) How do you simulate a Markov chain on a computer?

Solution. (1) Writing $\overline{X} = (X_0, \dots, X_n)$, we have

$$\begin{aligned} & \mathbb{P}\left(X_{n+1} = j | X_n = i, \overline{X}\right) = \\ = \ & \mathbb{P}\left((X_n, Y_{n+1}) \in G^{-1}\left(\{j\}\right) | X_n = i, \overline{X}\right) = \\ = \ & \mathbb{P}\left((i, Y_{n+1}) \in \{i\} \times [0,1] \cap G^{-1}\left(\{j\}\right) | X_n, \overline{X}\right) = \\ = \ & \mathbb{P}\left(Y_{n+1} \in \pi\left(\{i\} \times [0,1] \cap G^{-1}\left(\{j\}\right)\right)\right), \end{aligned}$$

by the independence, where π is the projection operator defined by $\pi(x,y) = y$ from $I \times [0,1]$ to $[0,1]$.

(2) Yes. Given $(p_{i,j})_{i,j\in I}$, we choose an order (random) $j_1, j_2, \dots$ of the elements of I (makes sense, since I is countable) and we define:

$$G(i,t) = \begin{cases} j_1, & \text{if } \ 0 \leqslant t \leqslant p_{i,j_1}, \\ j_2, & \text{if } p_{i,j_1} \leqslant t \leqslant p_{i,j_1} + p_{i,j_2}, \\ \cdots \\ j_r, & \text{if } \sum_{n=1}^{r-1} p_{i,j_n} \leqslant t \leqslant \sum_{n=1}^{r} p_{i,j_n}, \\ \cdots \end{cases}$$

Hence

$$\mathbb{P}\left(X_{n+1} = j_r | X_n = i\right) = \sum_{n=1}^{r} p_{i,j_n} - \sum_{n=1}^{r-1} p_{i,j_n} = p_{i,j_r},$$

since $Y_1, Y_2, \cdots$ are uniform.

(3) To generate a Markov chain (λ, P), λ being a law on I, we take a sequence $Y_1, Y_2, \ldots$ of uniform random variables on $[0,1]$. We define:

$$X_0 = j_r \text{ if } \sum_{n=1}^{r-1} \lambda(j_n) \leqslant Y_1 \leqslant \sum_{n=1}^{r} \lambda(j_n),$$

then,

$$X_{n+1} = G\left(X_n, Y_{n+1}\right) \;\; n = 0, 1, \ldots.$$