COM303: Digital Signal Processing

Lecture 18: Multirate signal processing

overview

- ideal and practical sampling and interpolation
- bandpass sampling
- multirate signal processing

advanced topics in sampling

From Continuous to Discrete Time

$$x(t) \longrightarrow x[n]$$

From Continuous to Discrete Time

$$x(t) \longrightarrow x[n]$$

ideally

in practice

From Continuous to Discrete Time

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$$x[n] = \langle x(t), \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)\rangle$$

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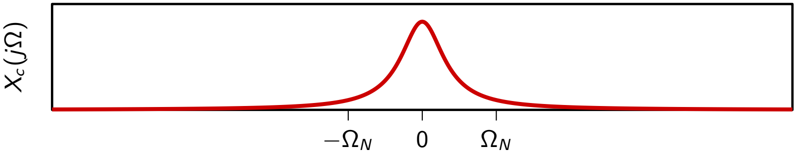
$$X(e^{j\omega}) = X\left(j\Omega_N \frac{\omega \mod 2\pi}{\pi}\right)$$

in practice

Ideal case



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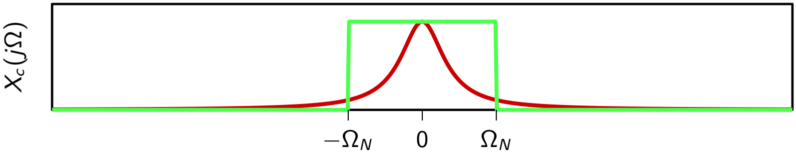

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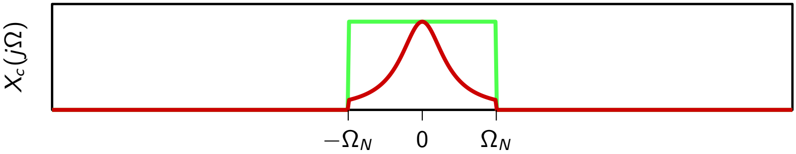

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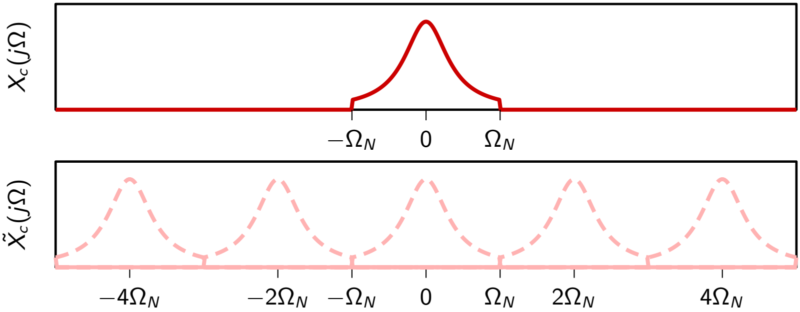

$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N
$\tilde{X}_c(j\Omega)$
$-4\Omega_N$
$-2\Omega_N$
$-\Omega_N$
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Ω_N
$2\Omega_N$
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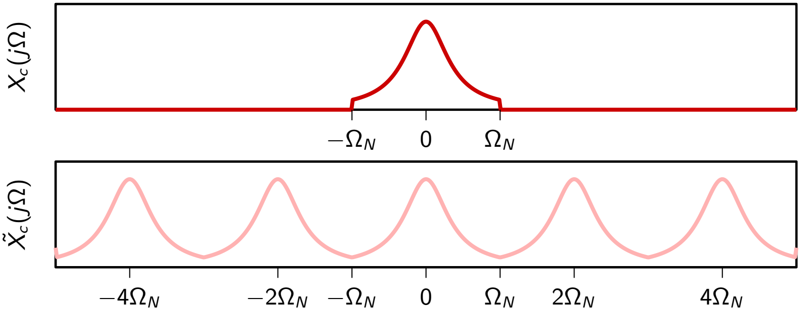

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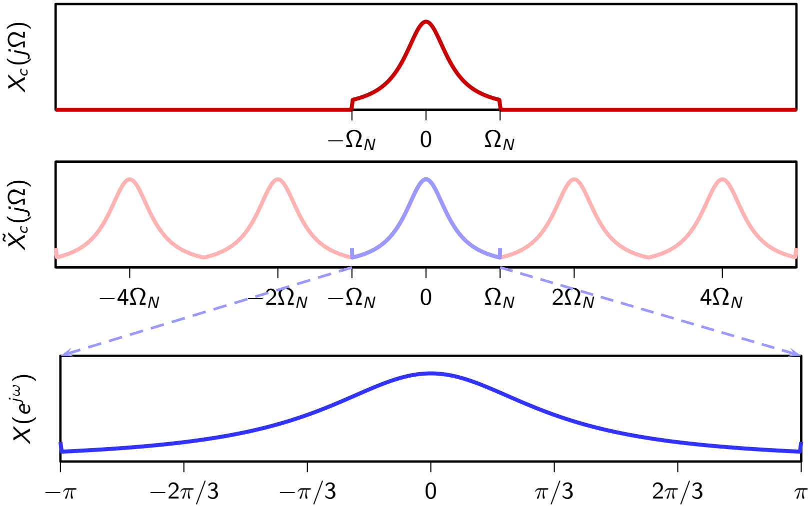

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From Continuous to Discrete Time

$$x(t) \longrightarrow x[n]$$

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in practice

$$x[n] = \langle x(t), \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)\rangle$$

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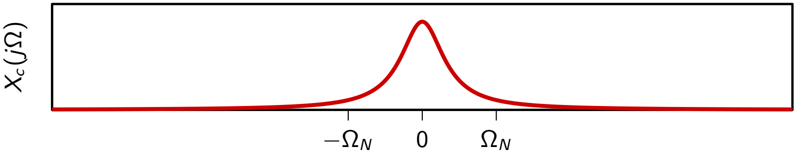
$$x[n] = x(nT_s)$$

$$X(e^{j\omega}) = \frac{\Omega_N}{\pi} \sum_{k=-\infty}^{\infty} X_c\left(j\Omega_N \frac{\omega}{\pi} - 2jk\Omega_N\right)$$

In practice



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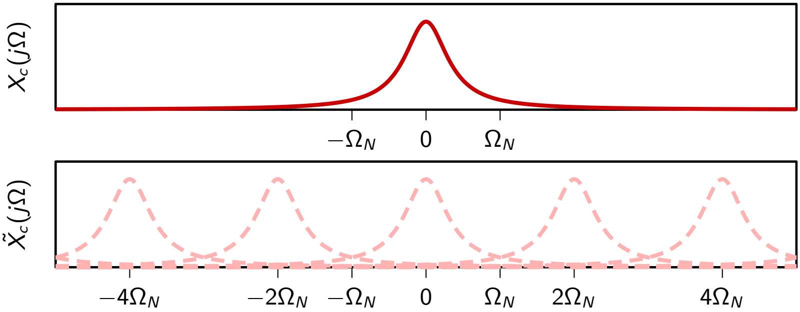

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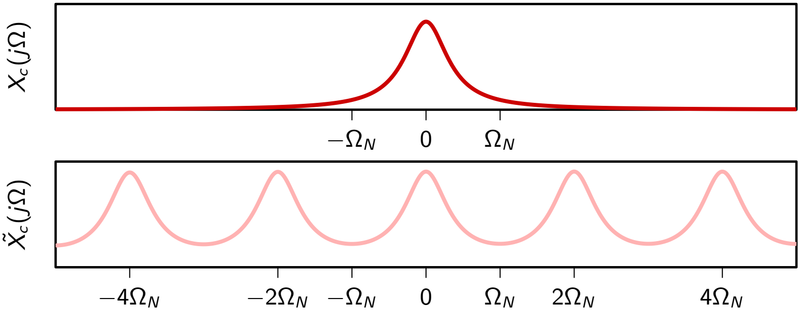

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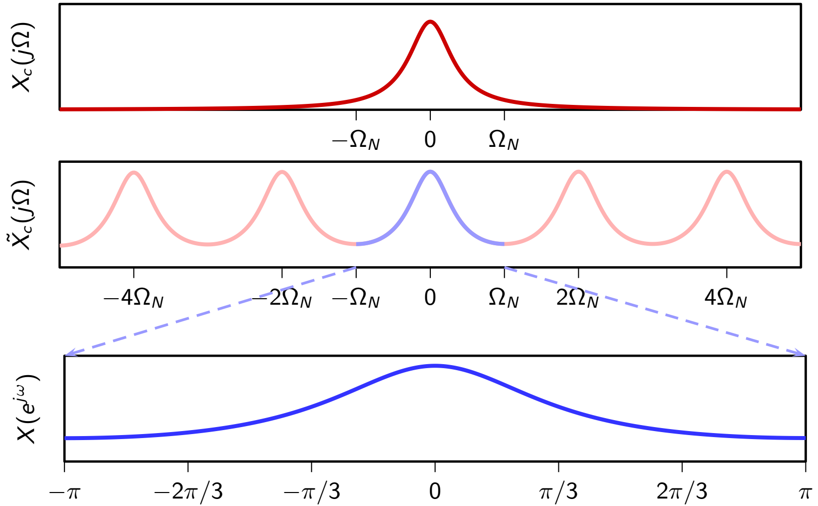

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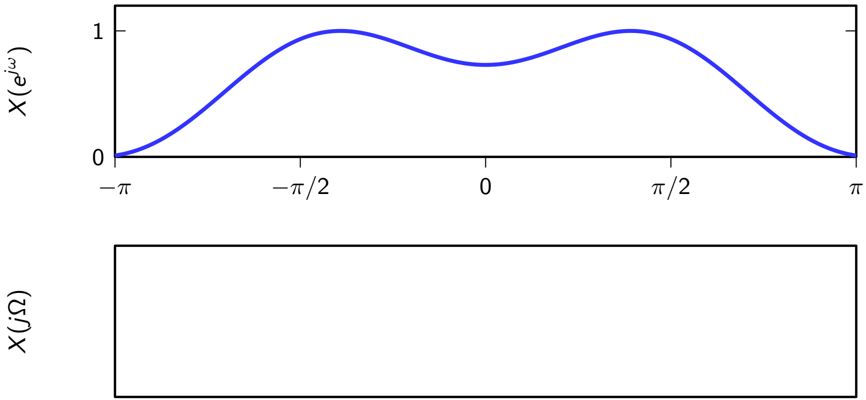
sinc interpolation



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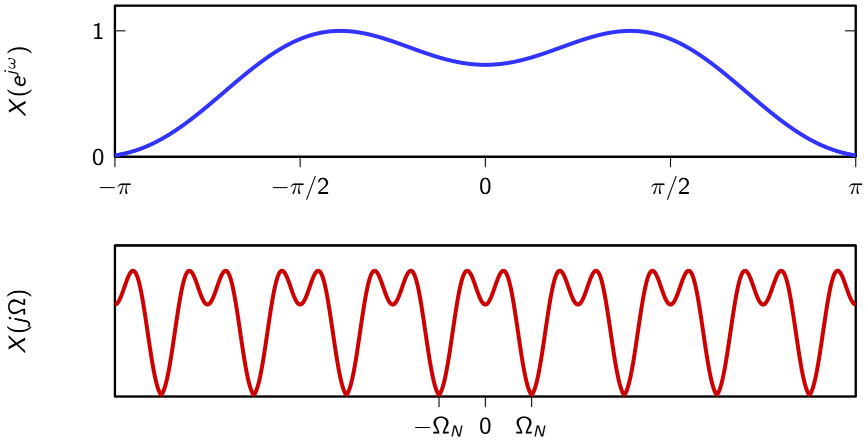

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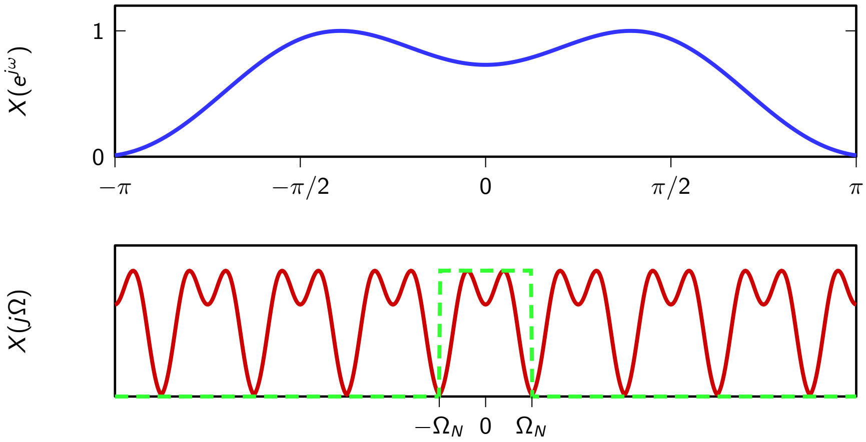

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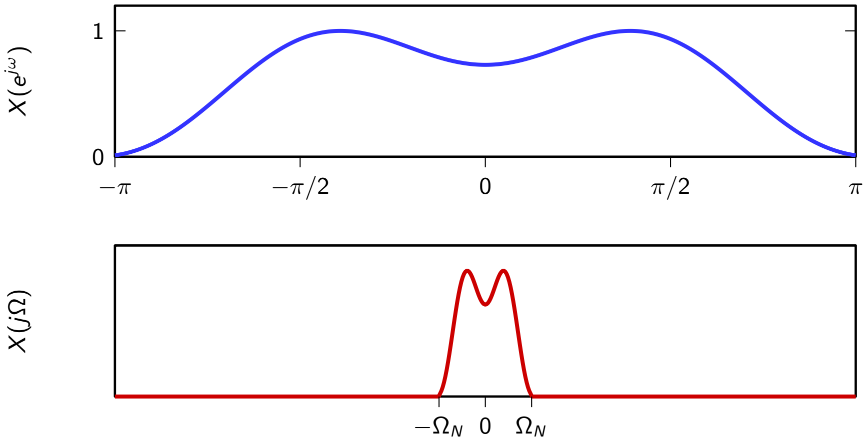

From Discrete to Continuous Time

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$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

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$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right)$$

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in practice

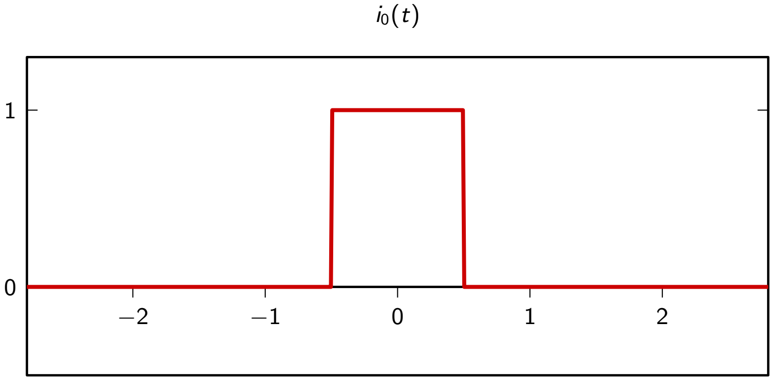
$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right)$$

$$X(j\Omega) = ?$$

Practical interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right)$$

Local interpolators



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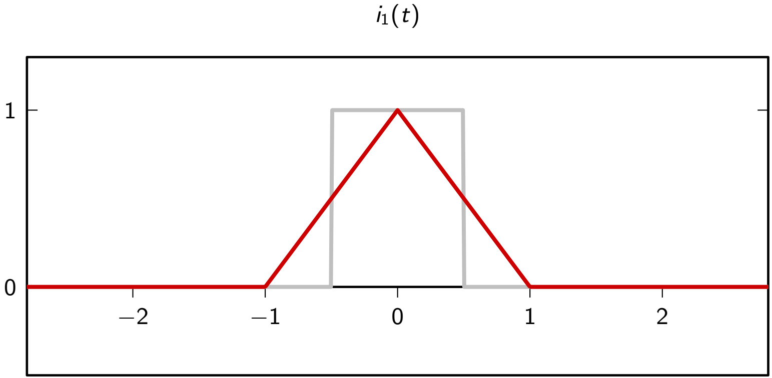

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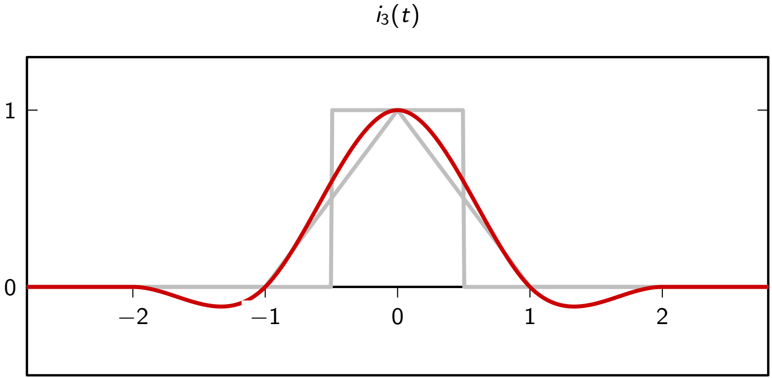

Spectral representation (I)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} i\left(\frac{t - nT_s}{T_s}\right) e^{-j\Omega t} dt$$

$$= T_s \sum_{n=-\infty}^{\infty} x[n] I\left(jT_s\Omega\right) e^{-jnT_s\Omega}$$

Spectral representation (I)

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&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} i\left(\frac{t - nT_s}{T_s}\right) e^{-j\Omega t} dt \\
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Spectral representation (II)

$$X(j\Omega) = T_s \sum_{n=-\infty}^{\infty} x[n] I\left(jT_s\Omega\right) e^{-jnT_s\Omega}$$

$$= \frac{\pi}{\Omega_N} I\left(j\pi\frac{\Omega}{\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\pi\frac{\Omega}{\Omega_N}n}$$

$$= \frac{\pi}{\Omega_N} I\left(j\pi\frac{\Omega}{\Omega_N}\right) X(e^{j\pi\frac{\Omega}{\Omega_N}})$$

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\end{aligned}
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sinc interpolation

$$i(t) = \mathrm{sinc}(t)$$

$$I(j\Omega) = \mathrm{rect}\left(\frac{\Omega}{2\pi}\right)$$

$$\begin{aligned} X(j\Omega) &= \frac{\pi}{\Omega_N}\, I\left(j\pi\Omega/\Omega_N\right)\, X(e^{j\pi\Omega/\Omega_N}) \\ &= \frac{\pi}{\Omega_N}\, \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right) X(e^{j\pi\Omega/\Omega_N}) \end{aligned}$$

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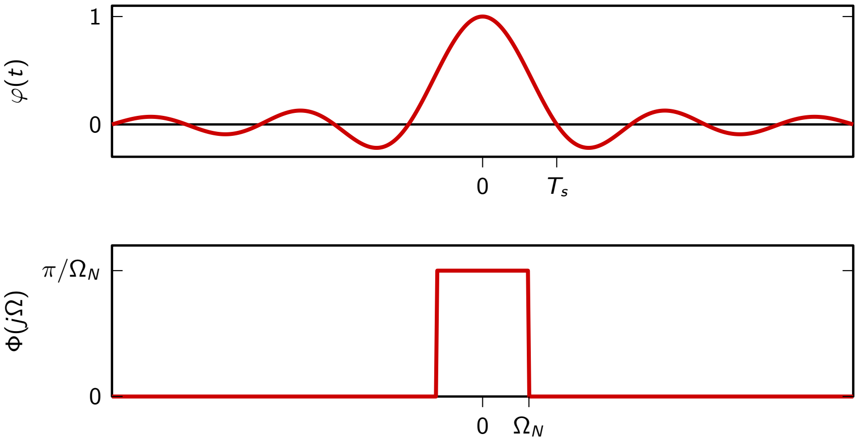

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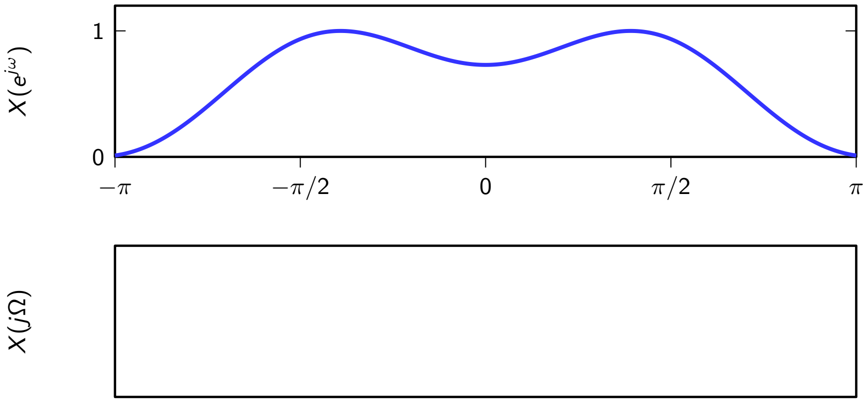

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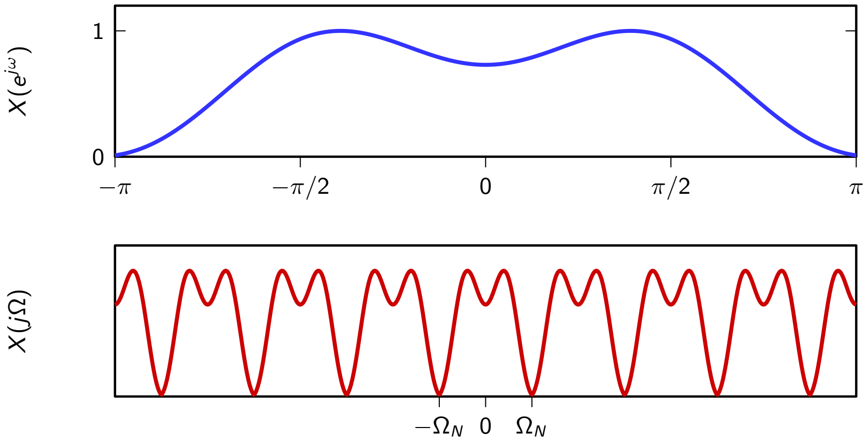

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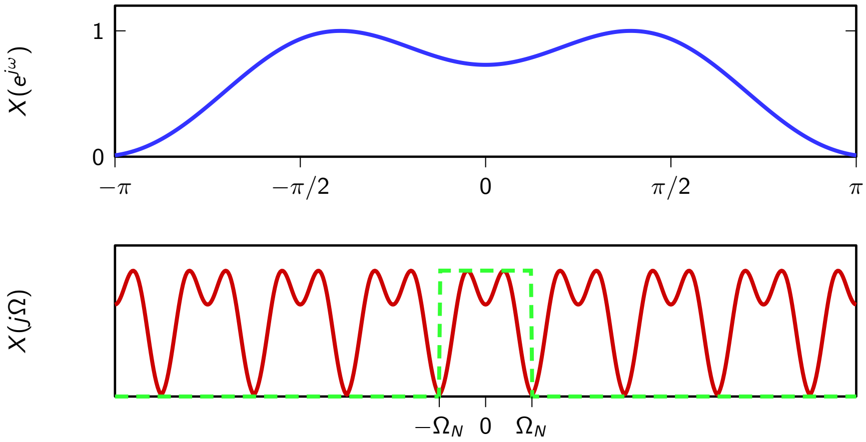

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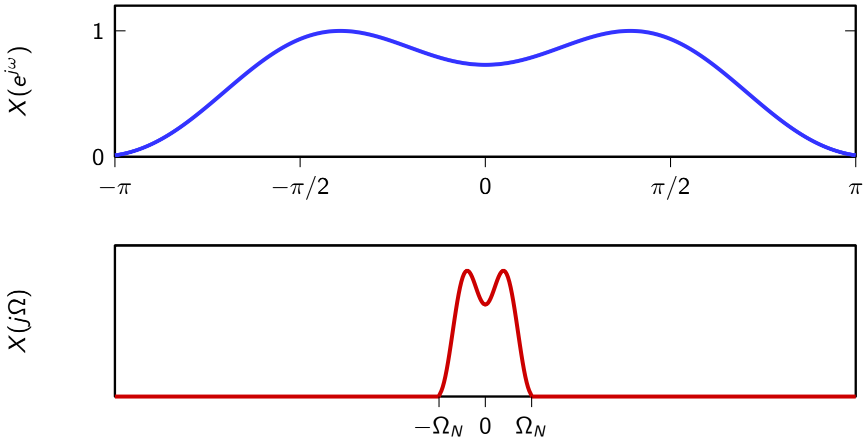

zero-order hold

$$i(t) = \mathrm{rect}(t)$$

$$I(j\Omega) = \mathrm{sinc}\left(\frac{\Omega}{2\pi}\right)$$

$$\begin{aligned}
X(j\Omega) &= \frac{\pi}{\Omega_N}\, I\left(j\pi\Omega/\Omega_N\right)\, X(e^{j\pi\Omega/\Omega_N}) \\
&= \frac{\pi}{\Omega_N}\, \mathrm{sinc}\left(\frac{\Omega}{2\Omega_N}\right)\, X(e^{j\pi\Omega/\Omega_N})
\end{aligned}$$

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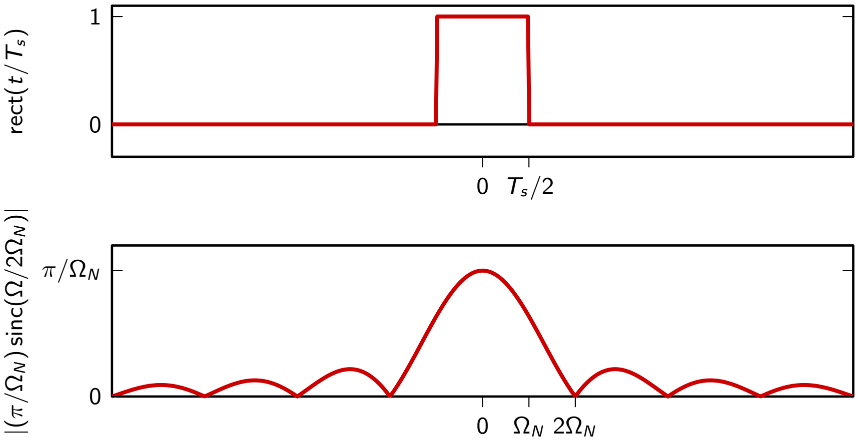

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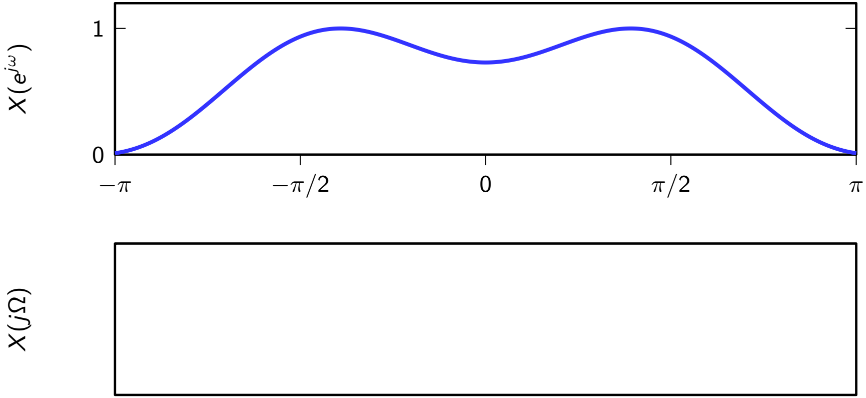

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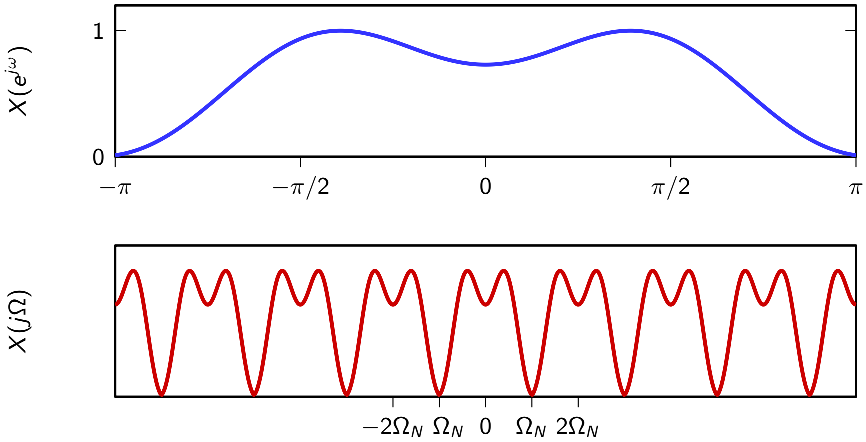

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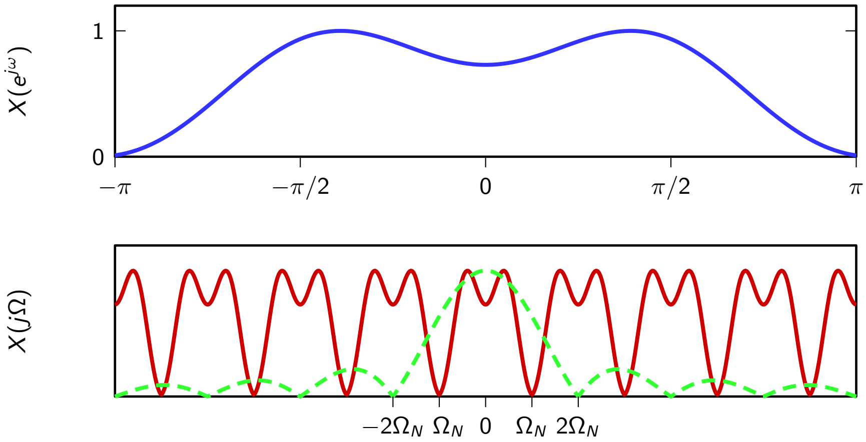

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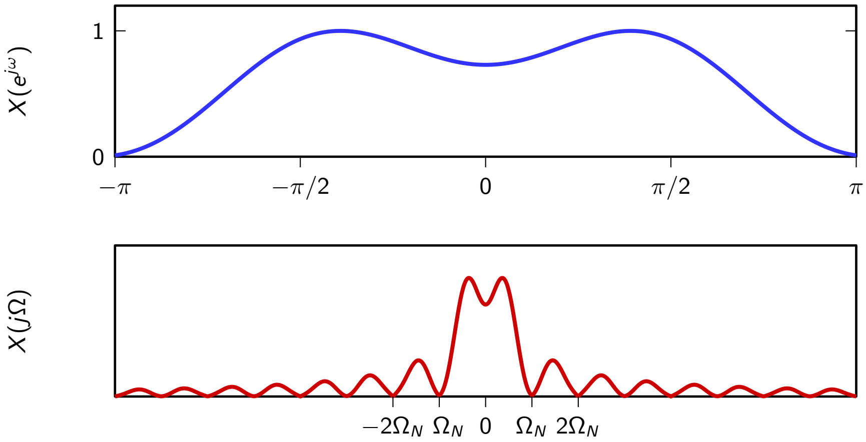
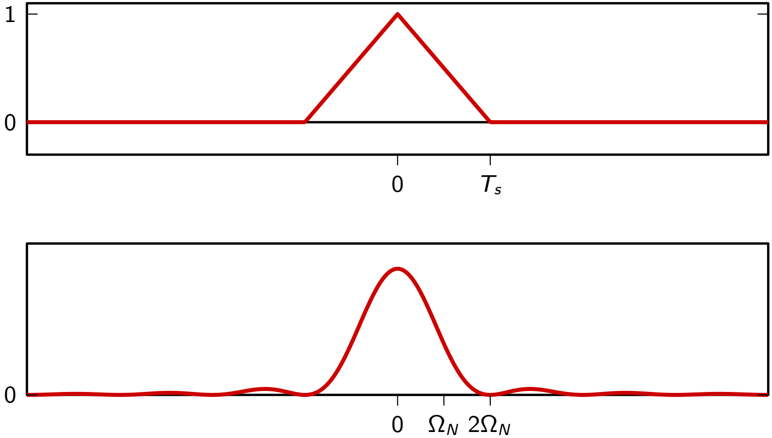


first-order interpolator



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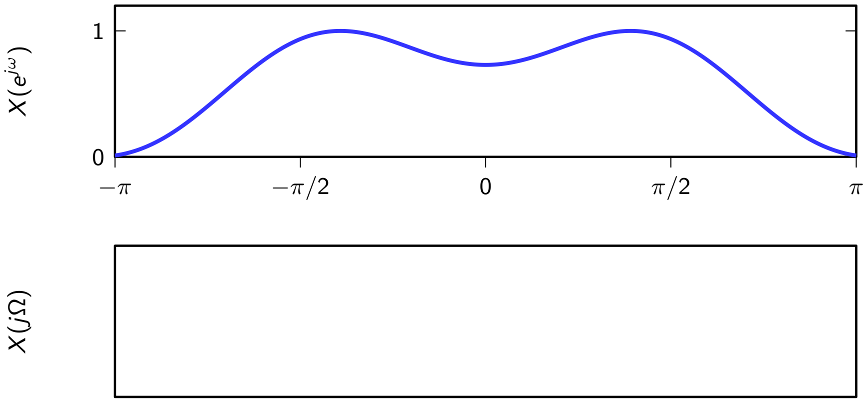

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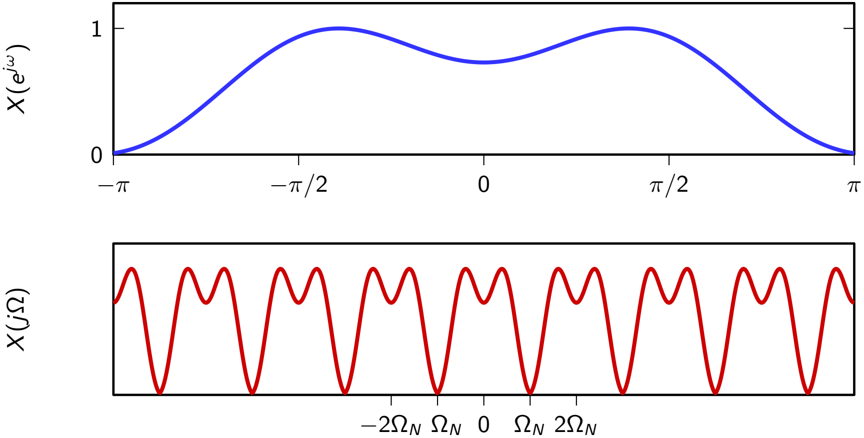

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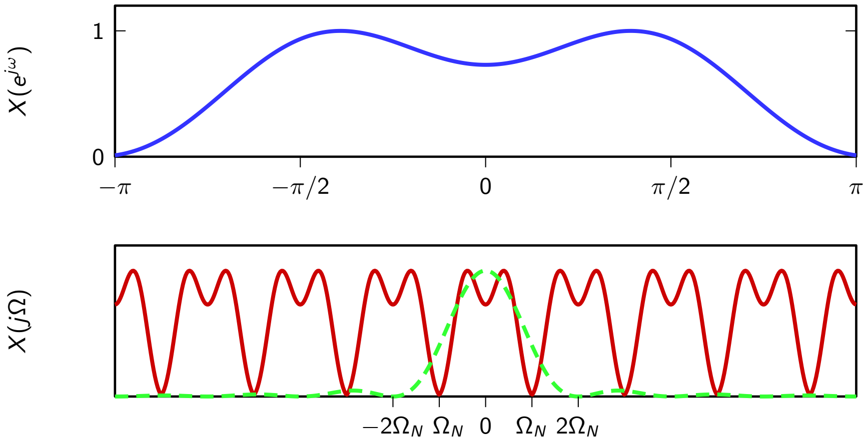

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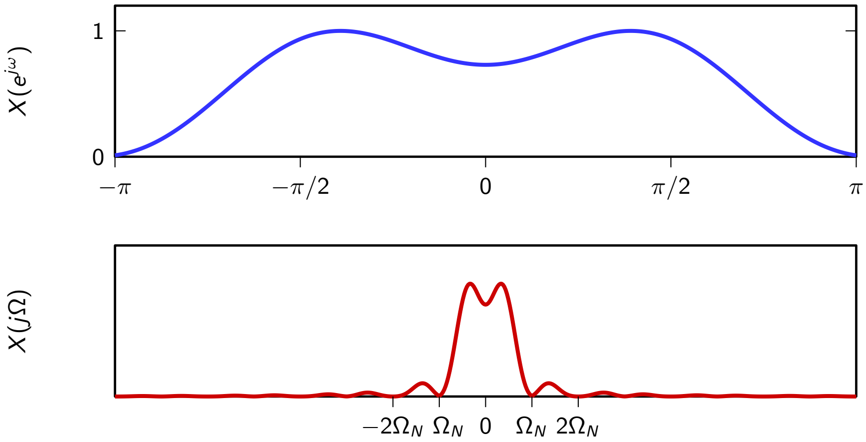

Bandpass Sampling

sampling theorem gives a *sufficient* condition

- in theory, $\Omega_N > \Omega_{\max}$
- what if signal is bandpass?

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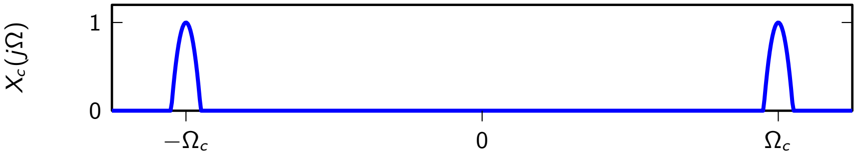
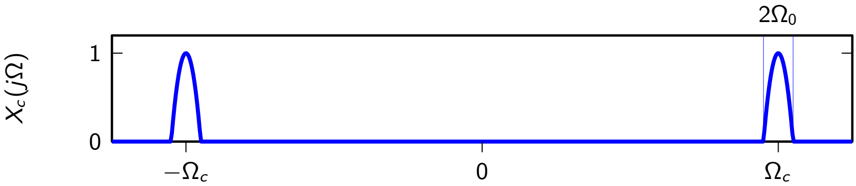
Bandpass sampling



Bandpass sampling



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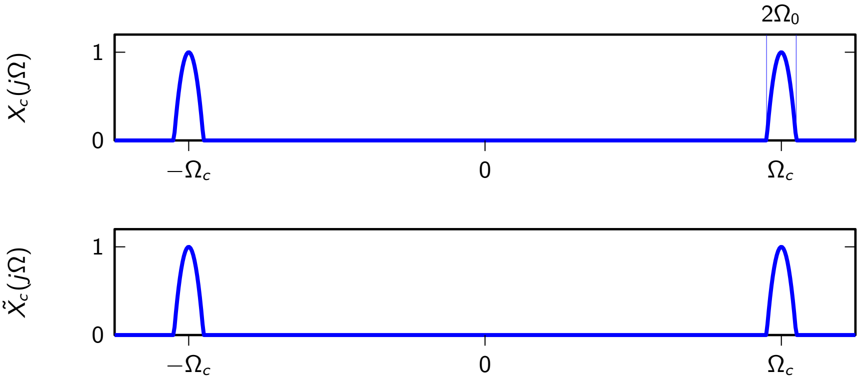

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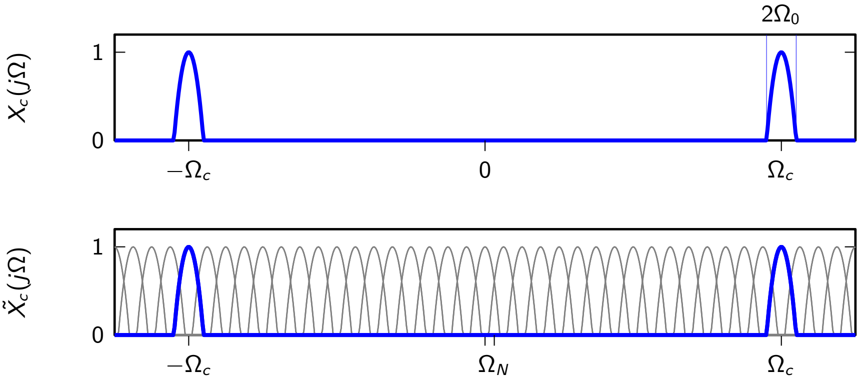

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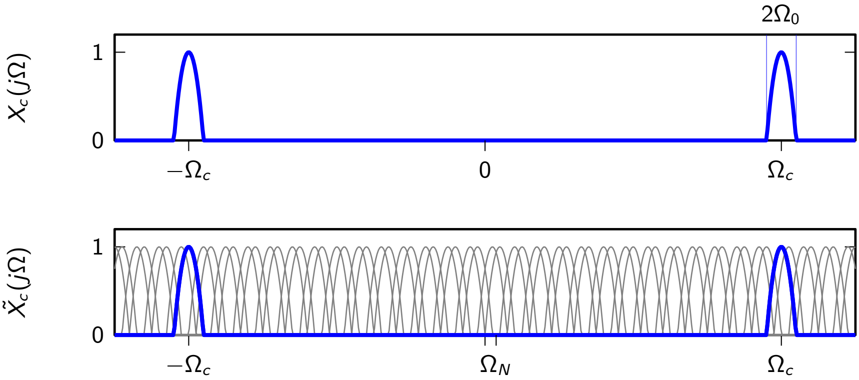

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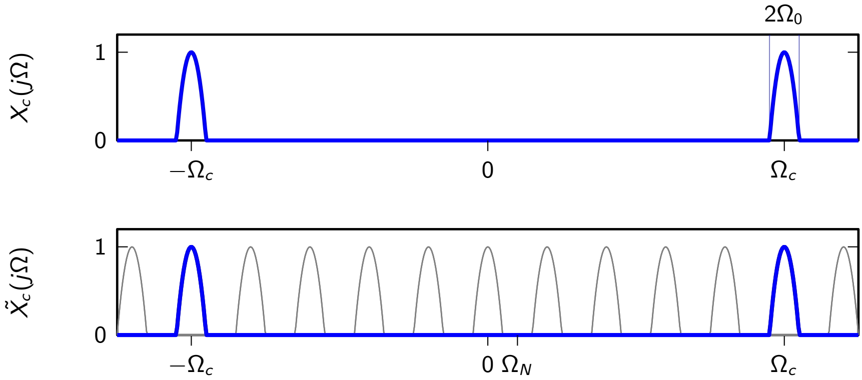

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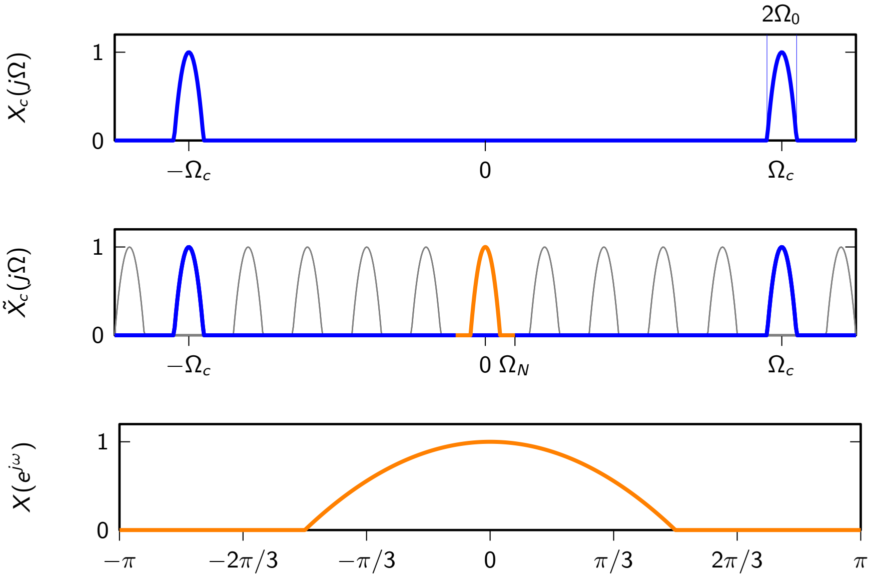

Bandpass sampling conditions

- bandpass signal: $X(j\Omega) = 0$ for $|\Omega - \Omega_c| > \Omega_0$
- no alias requires at least: $\Omega_N \geq \Omega_0$
- baseband condition: $\Omega_N = \Omega_c / k$ for some $k \in \mathbb{N}$

- for $T_s = \pi/\Omega_N$, $x[n] = x(nT_s)$ is an alias-free, baseband DT version of $x(t)$

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- bandpass signal: $X(j\Omega) = 0$ for $|\Omega - \Omega_c| > \Omega_0$
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Example: AM Channel

- AM radio band: 500kHz to 1.6MHz
- channel width is 9kHZ, i.e. $f_0 = 4.5$KHz
- take a channel at $f_c = 1.5$MHz
- in theory: $F_s \geq 2 * 1,504,500$Hz, $T_s < 10^{-6}$ seconds!

- antialias: $F_s \geq 2f_0 \Rightarrow F_s \geq 9$KHz
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multirate signal processing

Overview

- why multirate?
- upsampling
- downsampling
- applications

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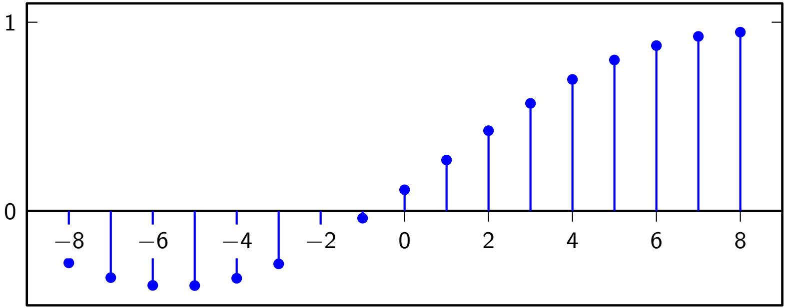
Upsampling

$$x_{NU}[n] = \begin{cases} x[k] & \text{for } n = kN, \quad k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

$$x[n] \longrightarrow N\uparrow \longrightarrow x_{NU}[n]$$

Example: upsampling by 4



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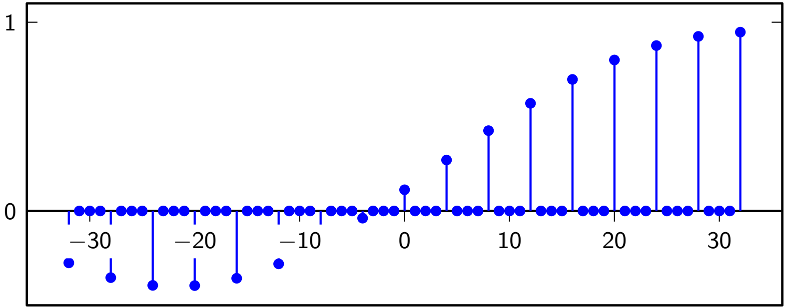

Spectral representation

$$X_{NU}(z) = \sum_{k=-\infty}^{\infty} x_{NU}[k]z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-Nk}$$

$$= X(z^N)$$

$$X_{NU}(e^{j\omega}) = X(e^{j\omega N})$$

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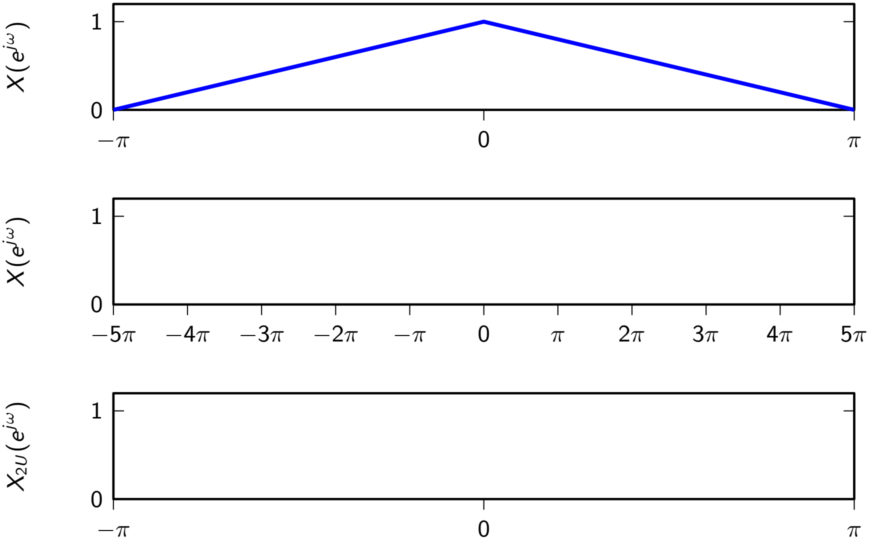
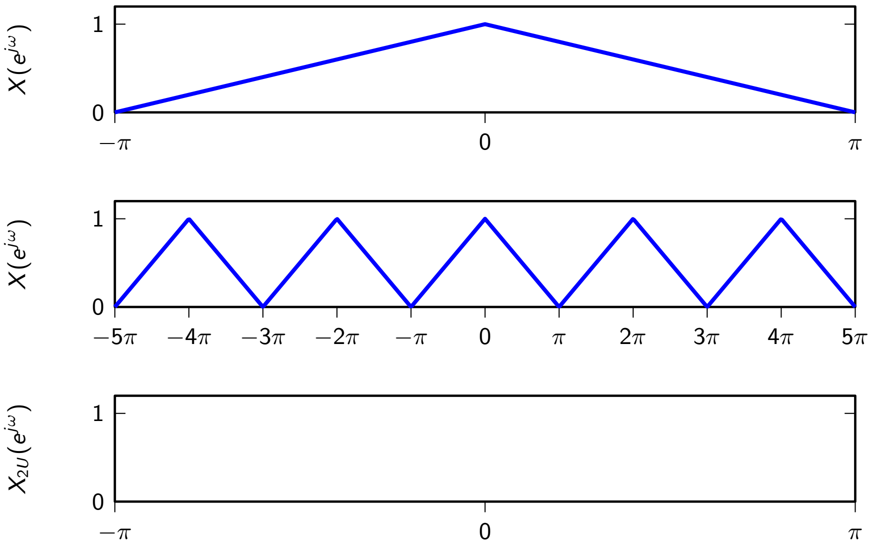
Upsampling by 2



Upsampling by 2



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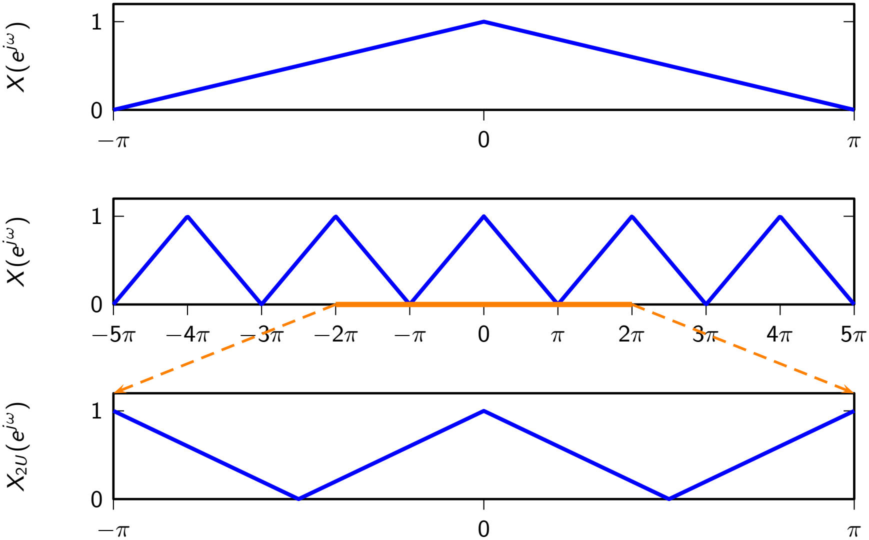
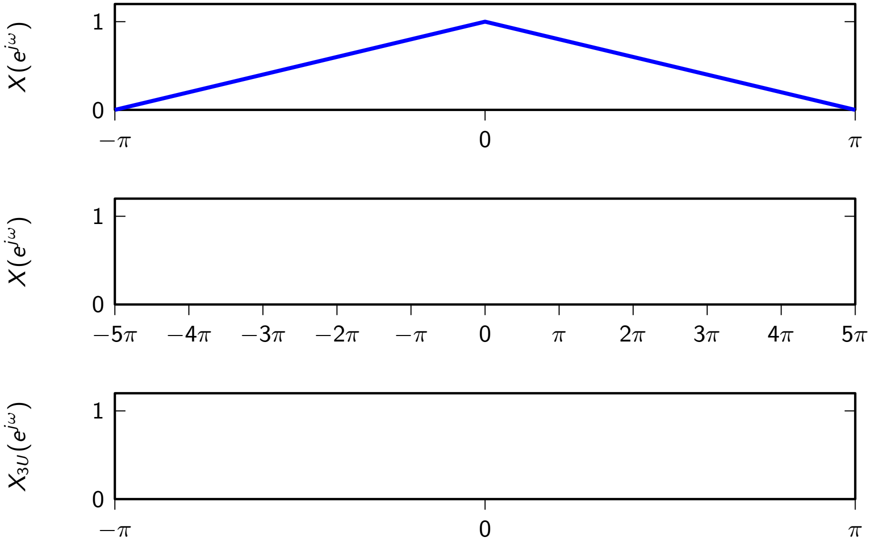


Upsampling by 3



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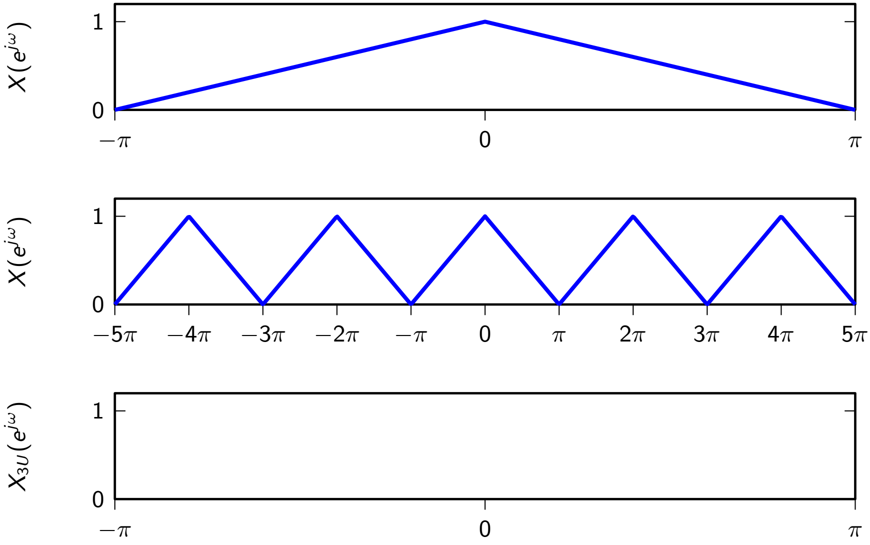

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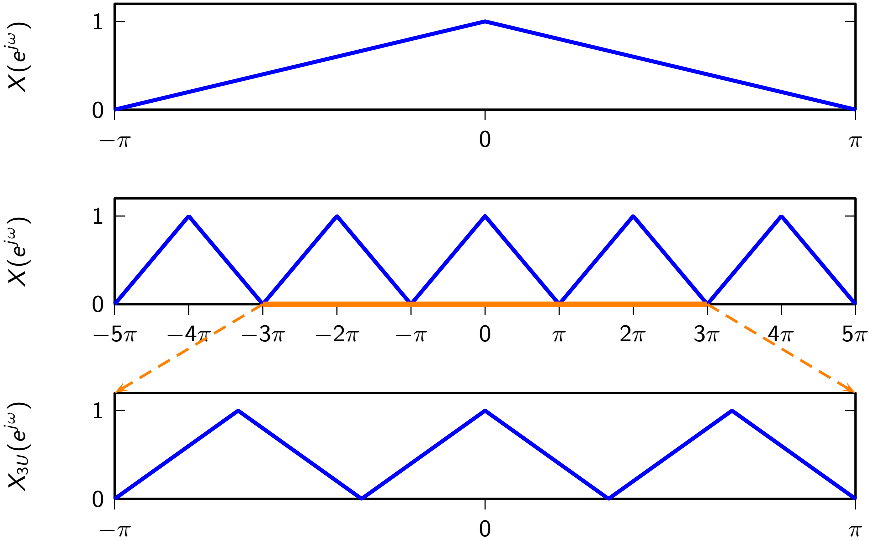
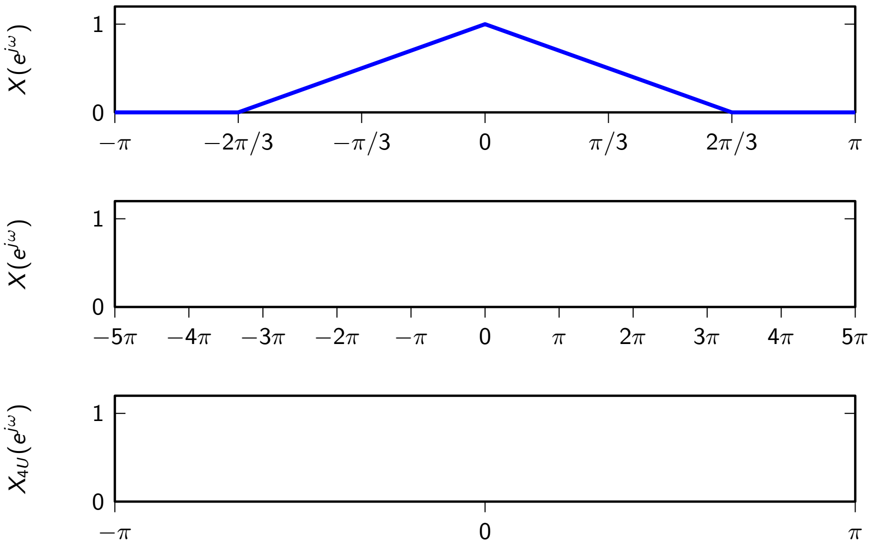
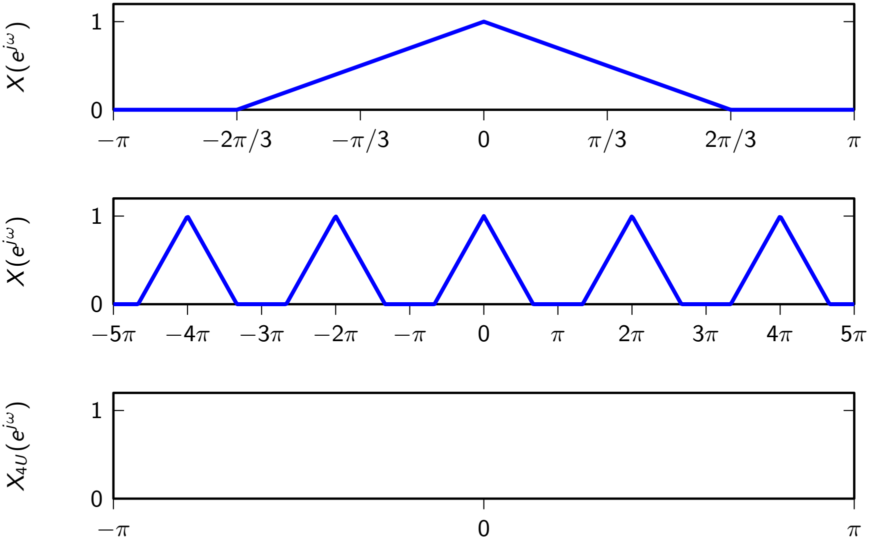


Upsampling by 4



Upsampling by 4



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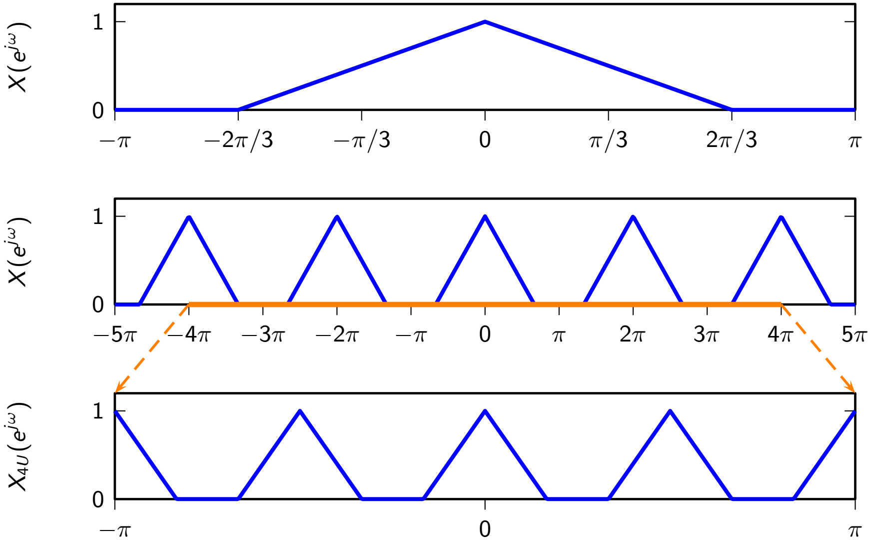

Upsampling: what we don't like

- in the time domain: zeros between nonzero samples are not "natural"
- in the frequency domain: extra replicas of the spectrum; can we get rid of them?

the two problems are the same!

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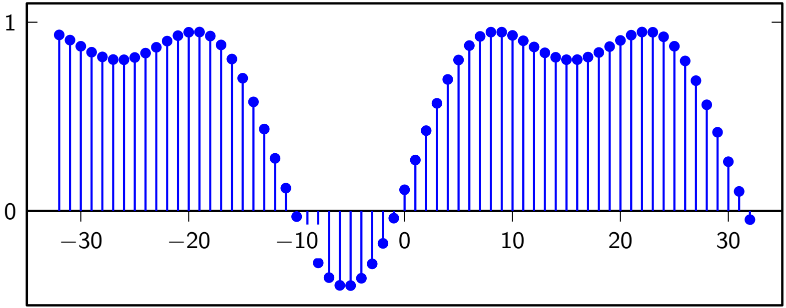
Upsampling in the time domain, revisited



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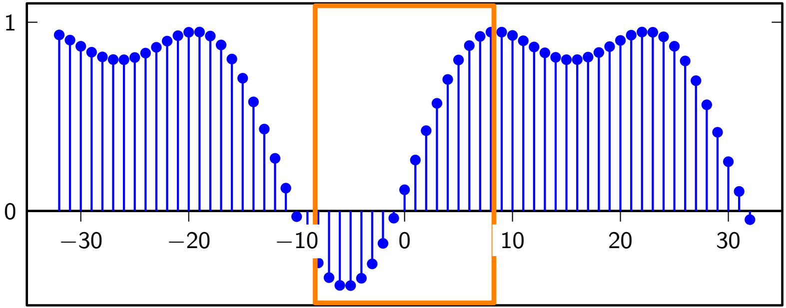

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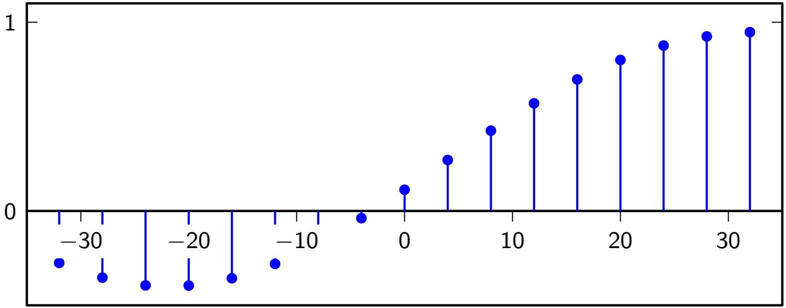

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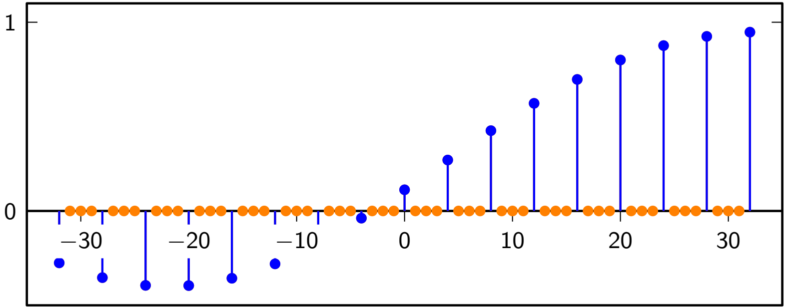

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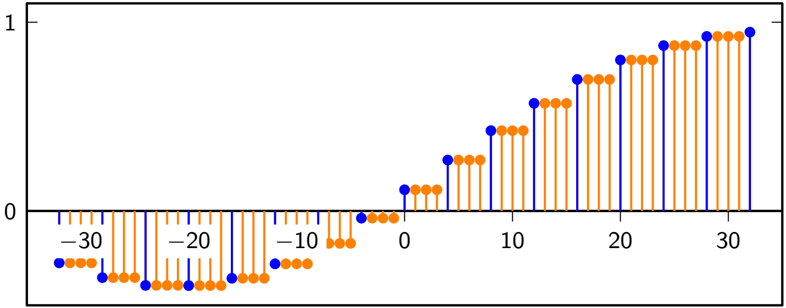
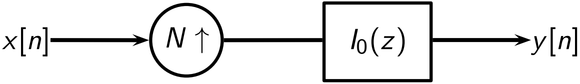
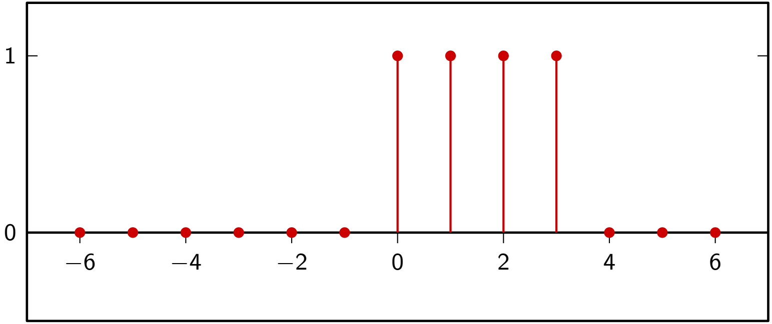


Zero-order interpolator



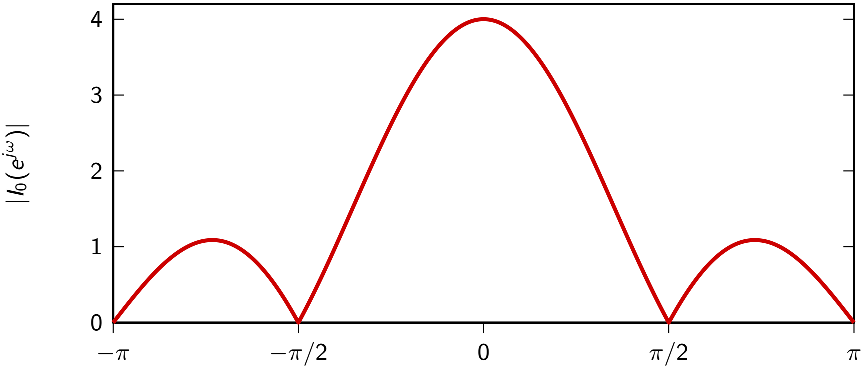
Zero-order interpolator for 4-upsampling

$$i_0[n] = u[n] - u[n-4]$$



Zero-order interpolator for 4-upsampling

$$|I_0(e^{j\omega})| = \left| \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} \right| \qquad N = 4$$



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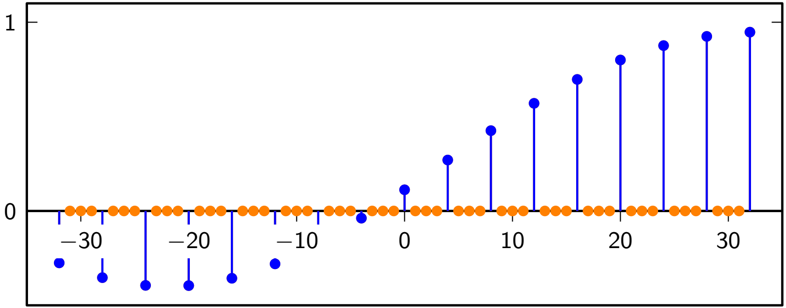

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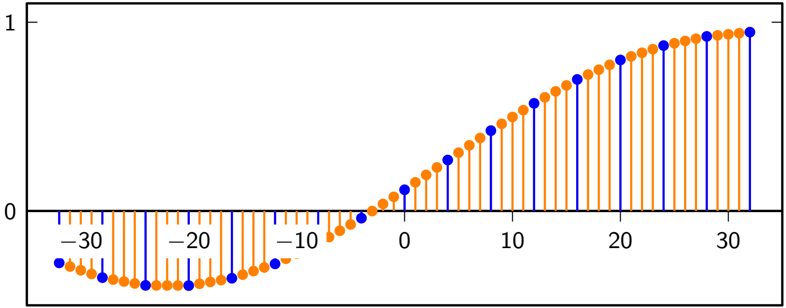
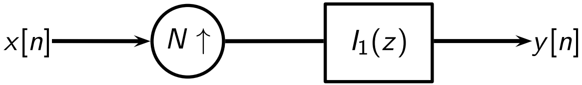
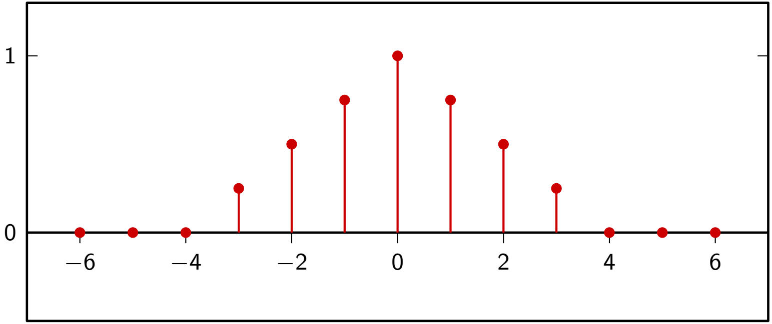


first-order interpolator



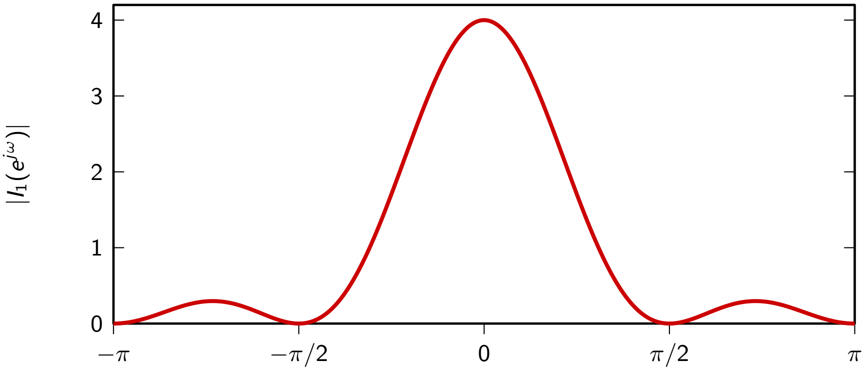
first-order interpolator for 4-upsampling

$$i_1[n] = (i_0[n] * i_0[n])/N$$

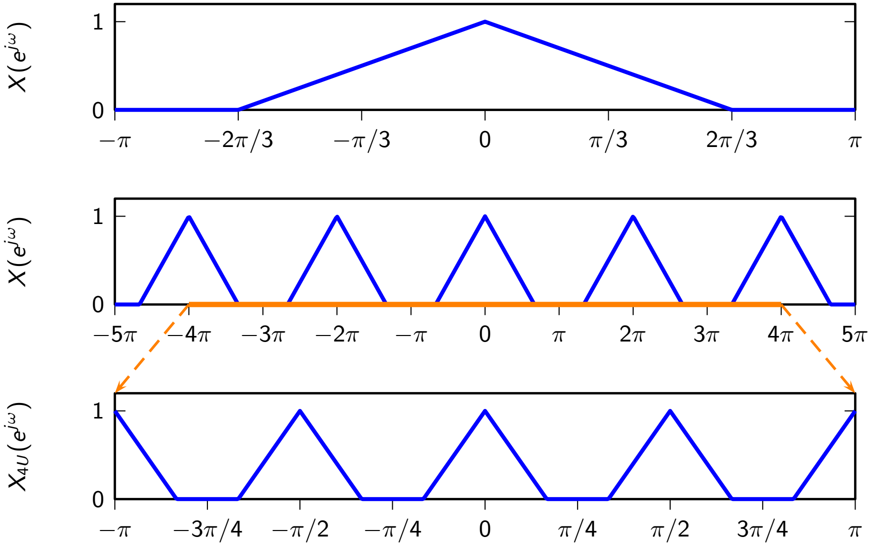


first-order interpolator for 4-upsampling

$$|I_1(e^{j\omega})| = |I_0(e^{j\omega})|^2/N \qquad N = 4$$



in the frequency domain...



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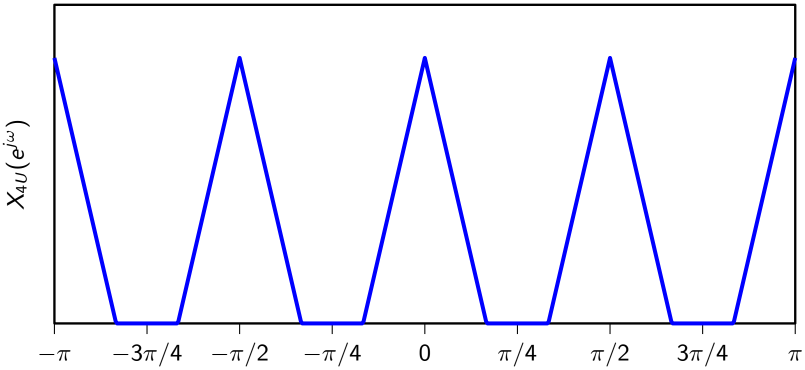

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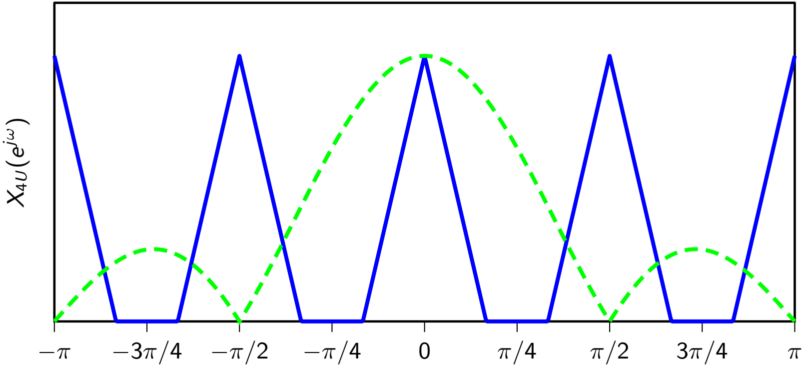

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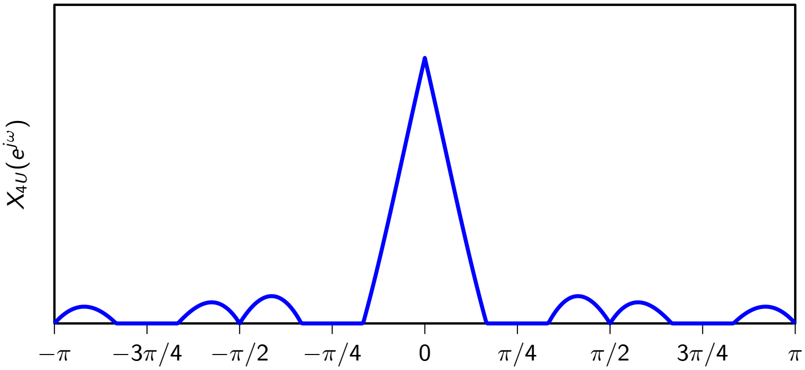

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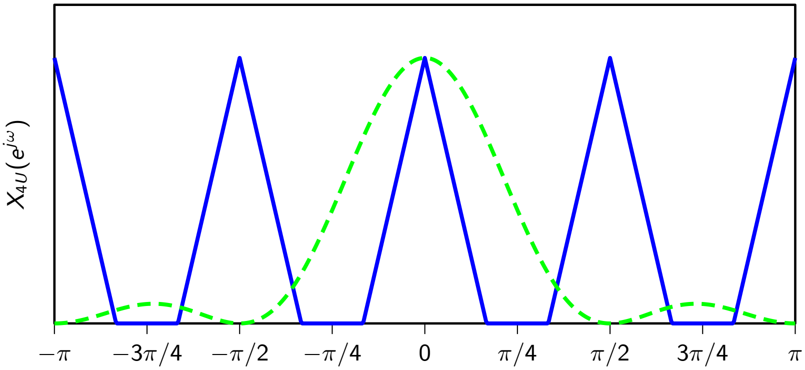

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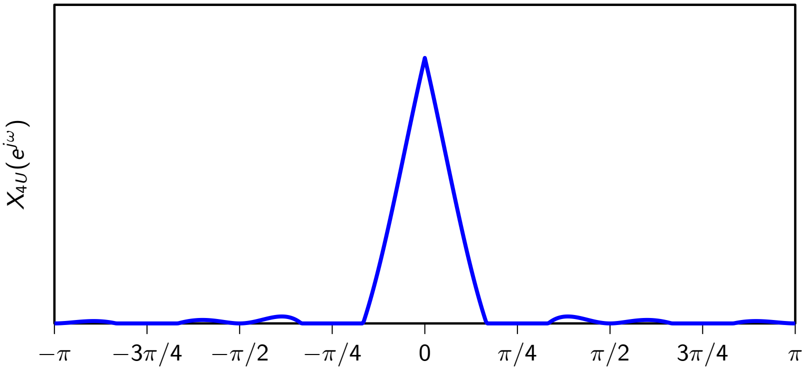

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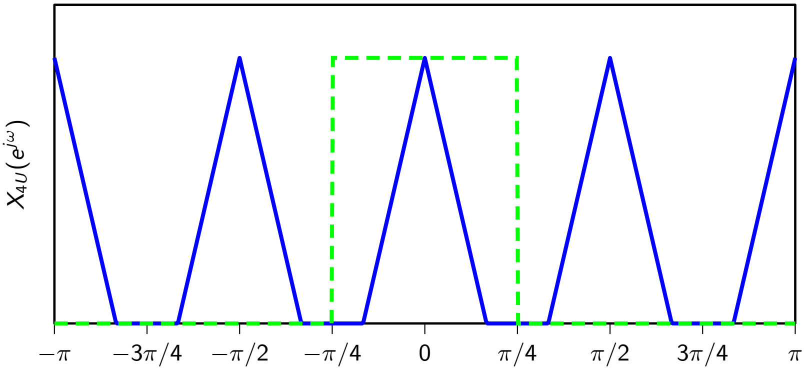

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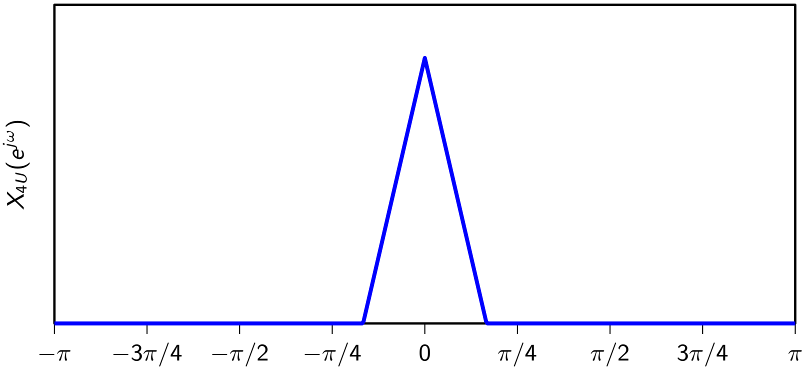


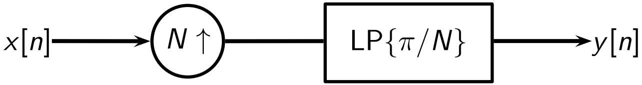
ideal digital interpolator

$$x[n] \longrightarrow \boxed{N\uparrow} \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$$H(e^{j\omega}) = \mathrm{rect}(\omega N/2\pi)$$

$$h[n] = (1/N)\,\mathrm{sinc}(n/N)$$

ideal digital interpolator

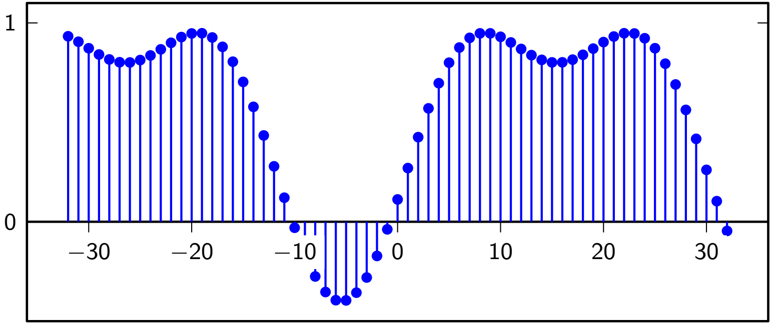


Downsampling

$$x_{ND}[n] = x[nN]$$

$$x[n] \longrightarrow \left(N \downarrow \right) \longrightarrow x_{ND}[n]$$

Example: downsampling by 4



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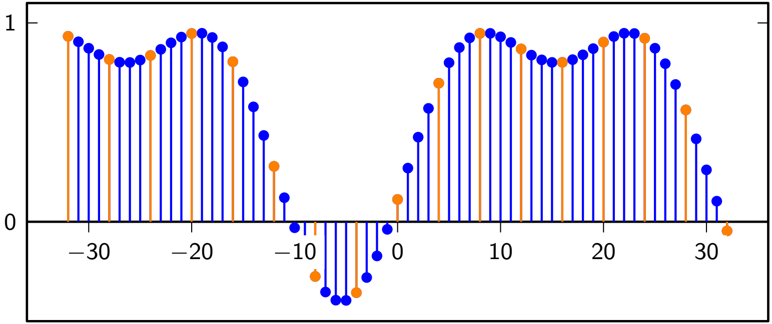

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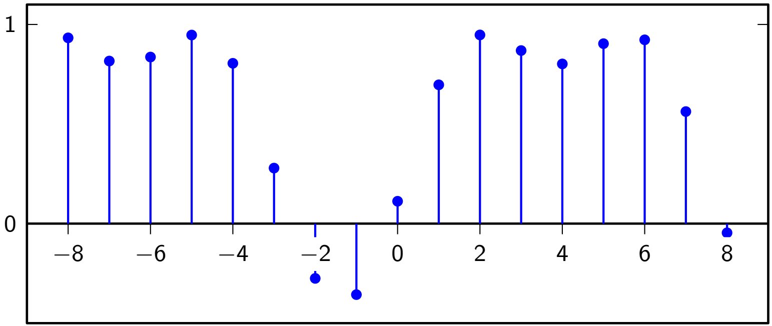

Spectral representation

$$X_{ND}(z) = \sum_{k=-\infty}^{\infty} x[kN]z^{-k} = \quad ?$$

if we can compute

$$A(z) = \sum_{k=-\infty}^{\infty} x[kN]z^{-kN}$$

then

$$X_{ND}(z) = A(z^{1/N})$$

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$$\xi[n] = \begin{cases} 1 & \text{for } n = kN \\ 0 & \text{otherwise} \end{cases}$$

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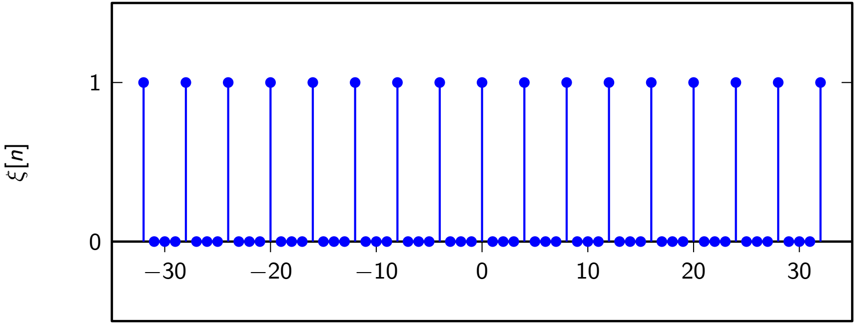
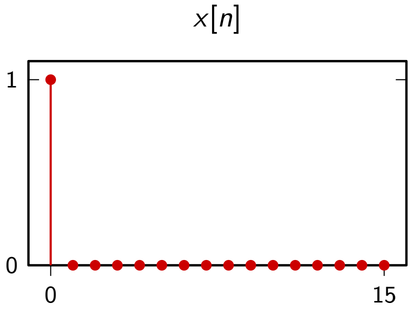
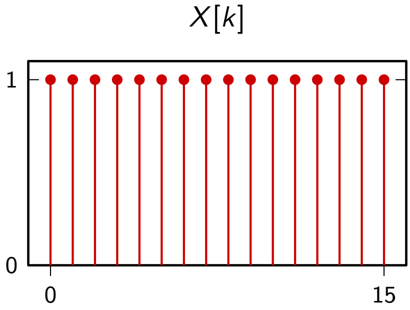
$\xi[n]$ for $N = 4$



Blast from the past: DFT of $x[n] = \delta[n], \quad x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}nk} = 1$$

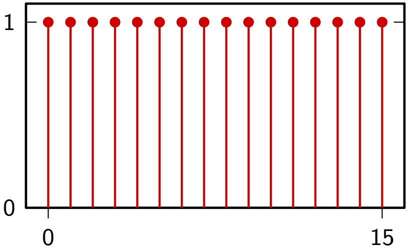




From the other side:

$$\mathrm{IDFT}\left\{1\right\} = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} = \delta[n]$$

$X[k]$



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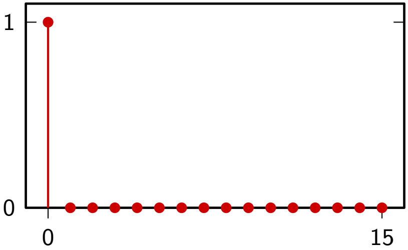
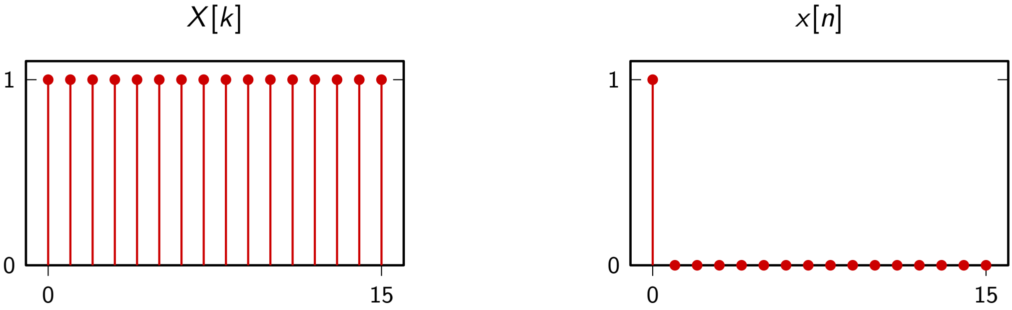


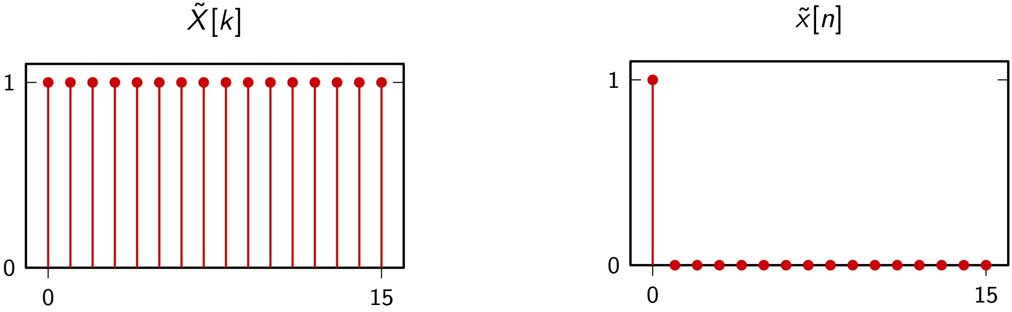
From the other side:

$$\mathrm{IDFT}\{1\} = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad n = 0, \ldots, N-1$$



From the other side:

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} = \begin{cases} 1 & \text{for } n \mod N = 0 \\ 0 & \text{otherwise} \end{cases}$$



Spectral representation

$$\xi[n] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn}$$

$$\begin{aligned}
A(z) &= \sum_{k=-\infty}^{\infty} \xi[k]x[k]z^{-k} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=-\infty}^{\infty} x[k]e^{j\frac{2\pi}{N}mk}z^{-k} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}z)
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Spectral representation

$$A(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{j(\omega - \frac{2\pi}{N}m)})$$

Spectral representation

$$X_{ND}(z) = A(z^{1/N}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}\, z^{\frac{1}{N}})$$

$$X_{ND}(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{j(\frac{\omega-2\pi m}{N})})$$

Spectral representation

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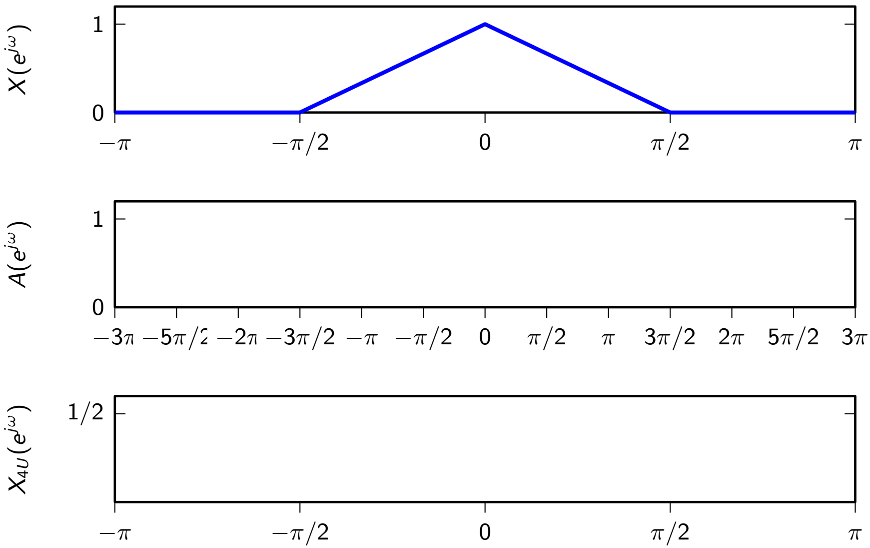
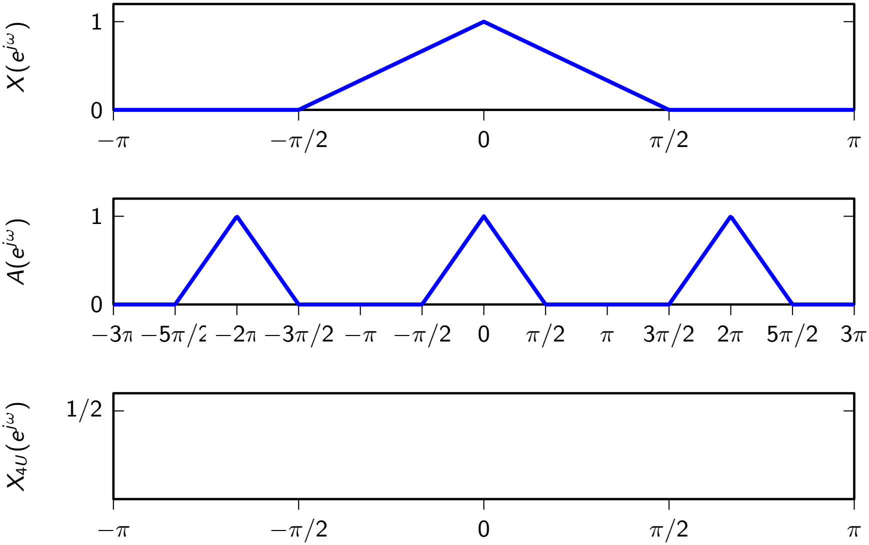
$$X_{ND}(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{j(\frac{\omega-2\pi m}{N})})$$

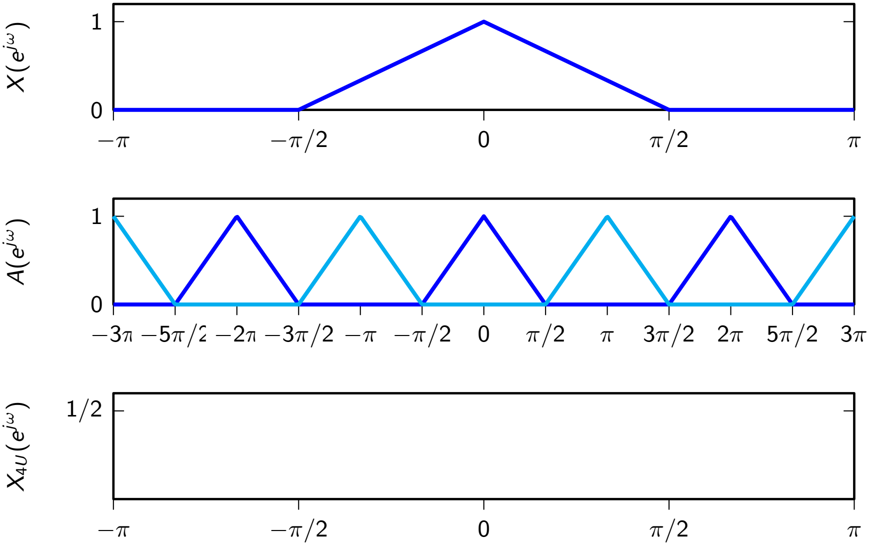
Downsampling by 2



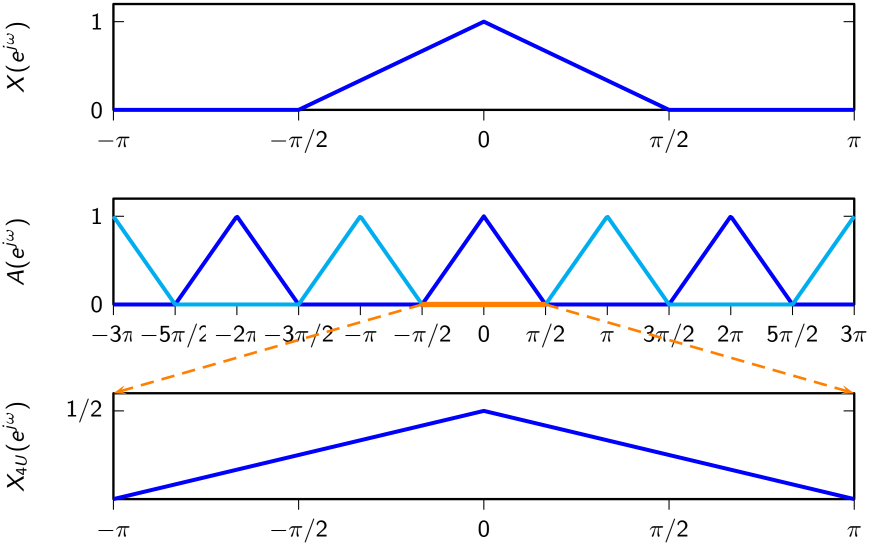
Downsampling by 2



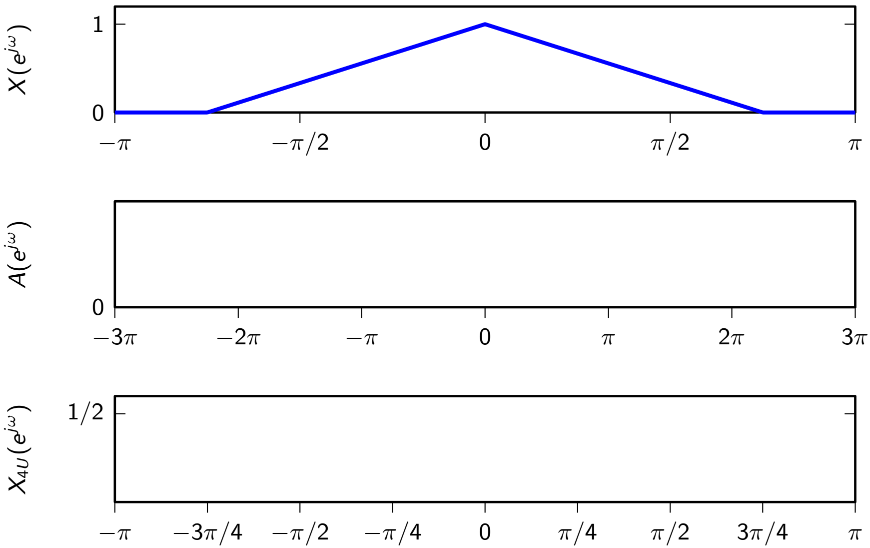
Downsampling by 2



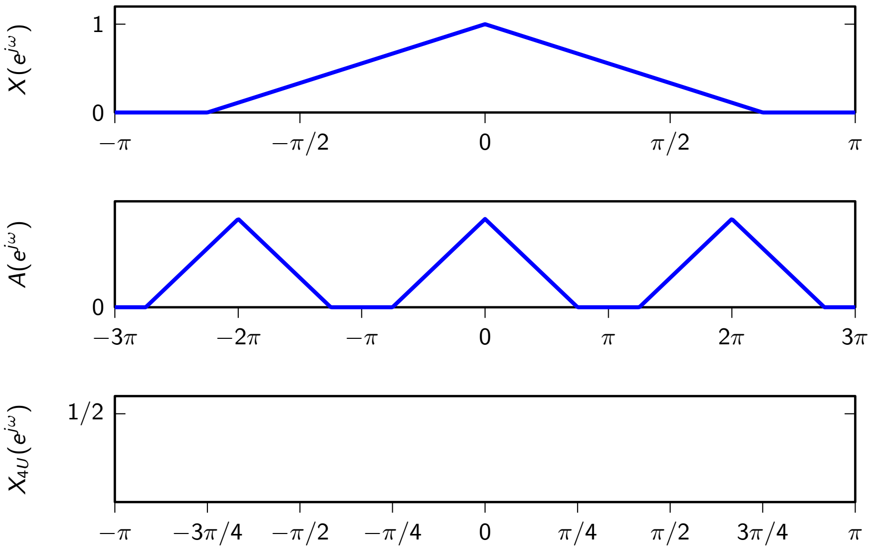
Downsampling by 2



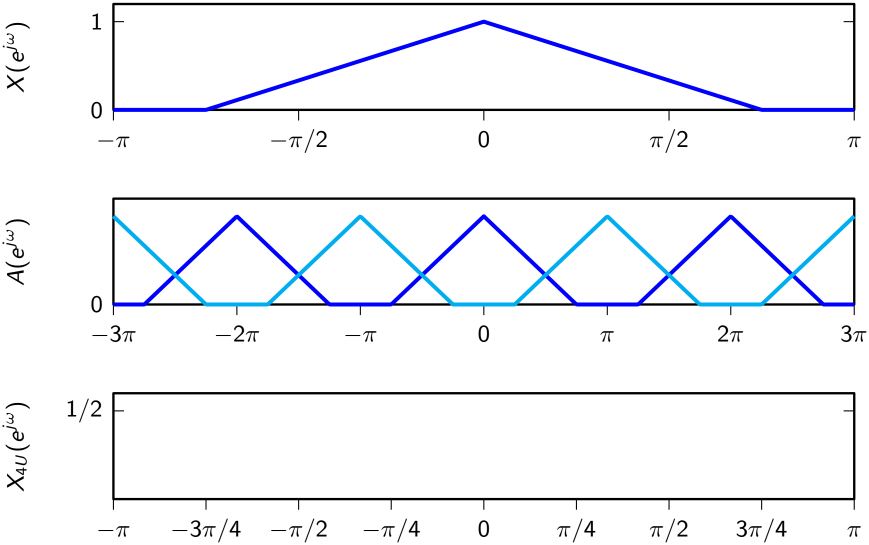
Downsampling by 2 with aliasing



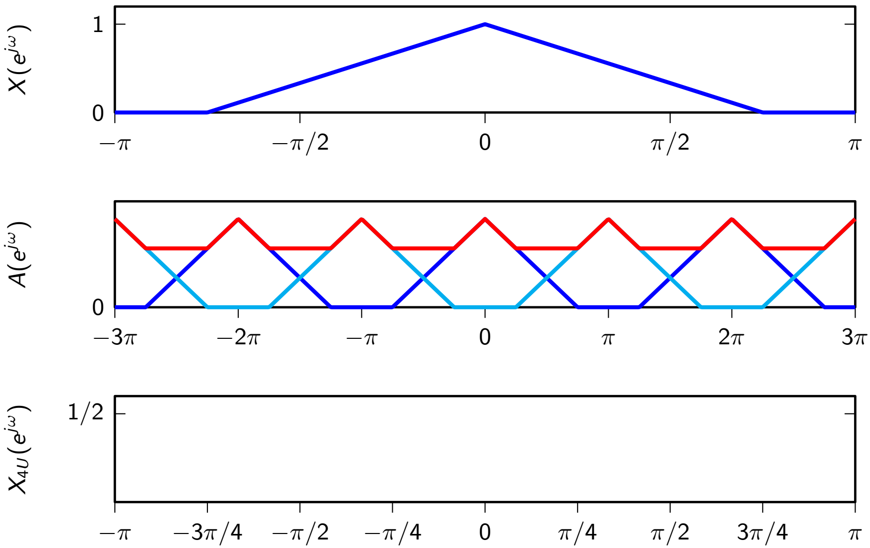
Downsampling by 2 with aliasing



Downsampling by 2 with aliasing



Downsampling by 2 with aliasing



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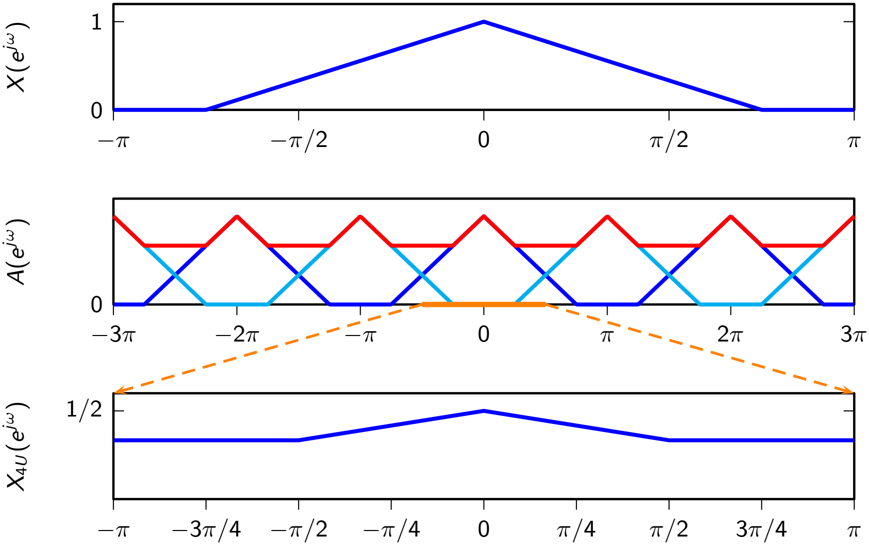
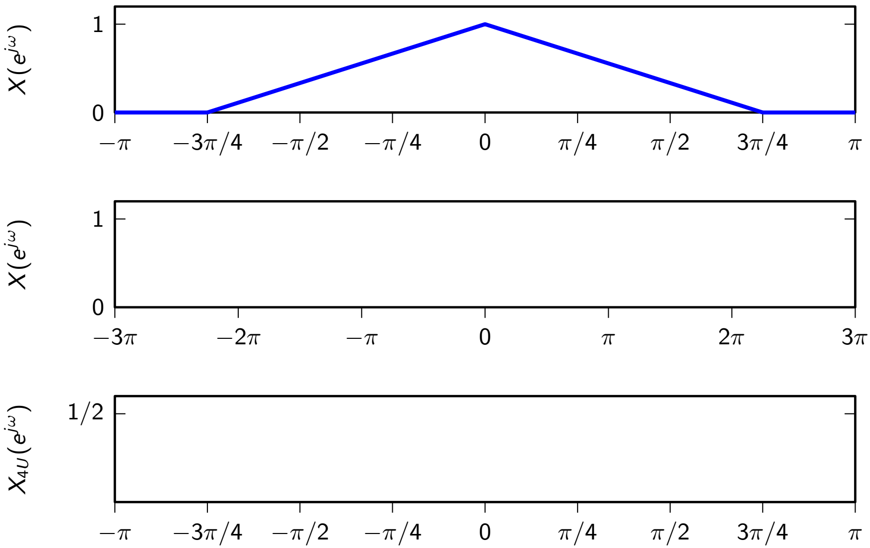


Downsampling by 2 with antialiasing filter



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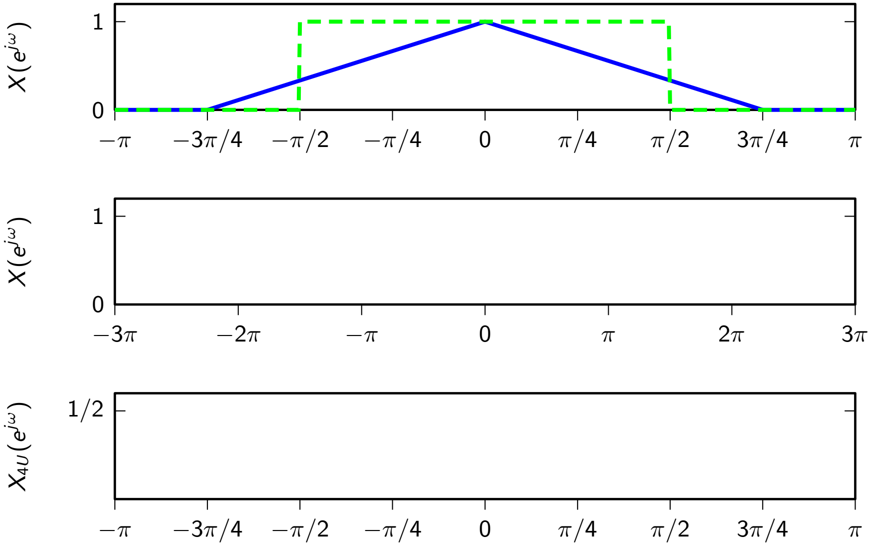
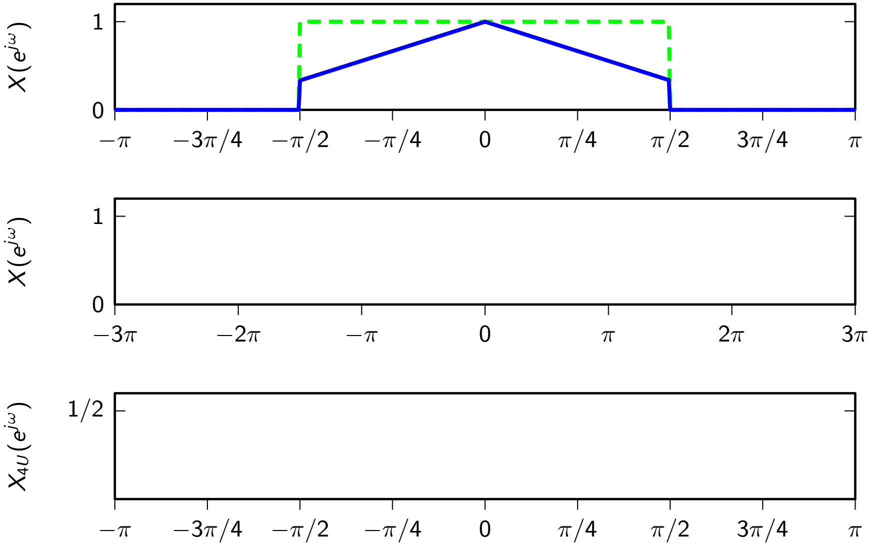


Downsampling by 2 with antialiasing filter



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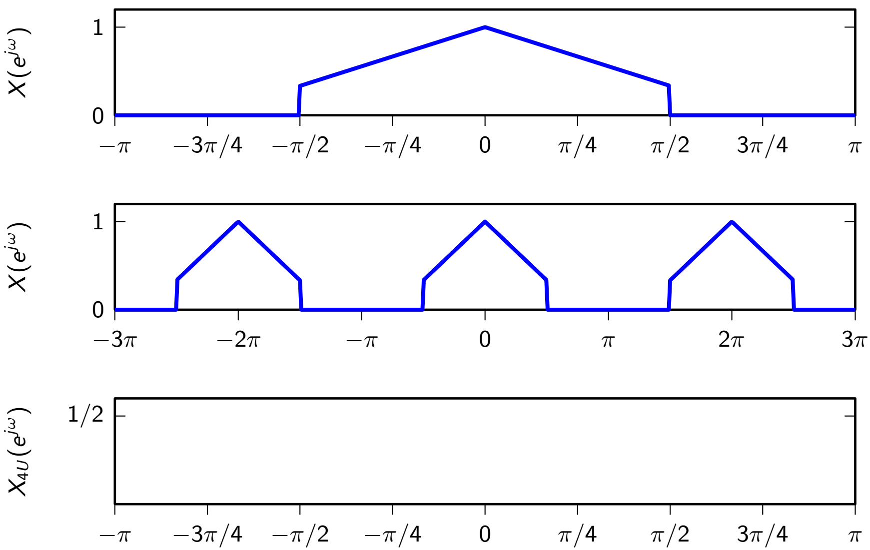
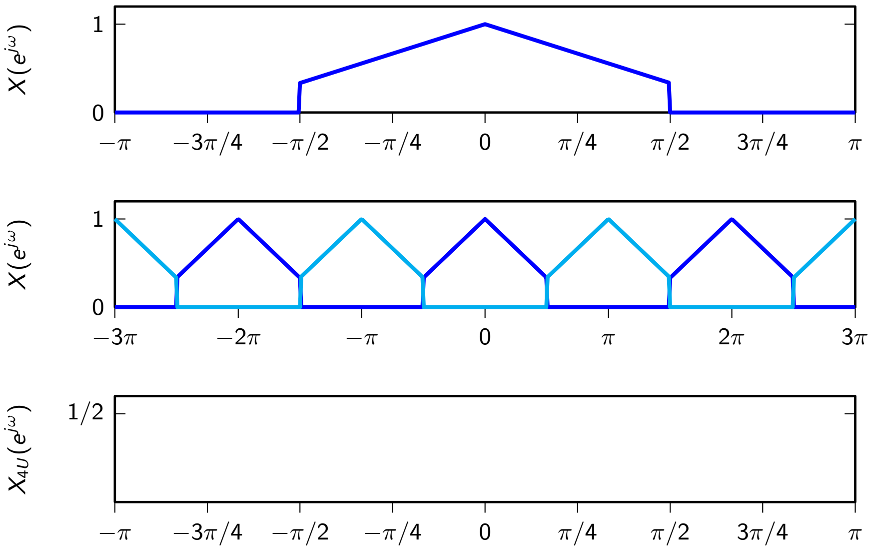
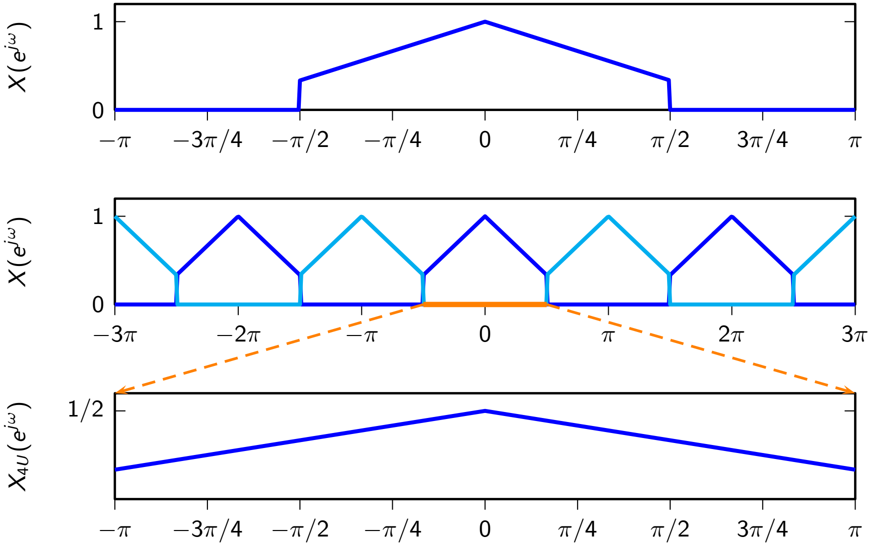


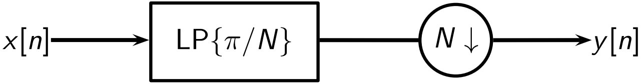
Downsampling by 2 with antialiasing filter



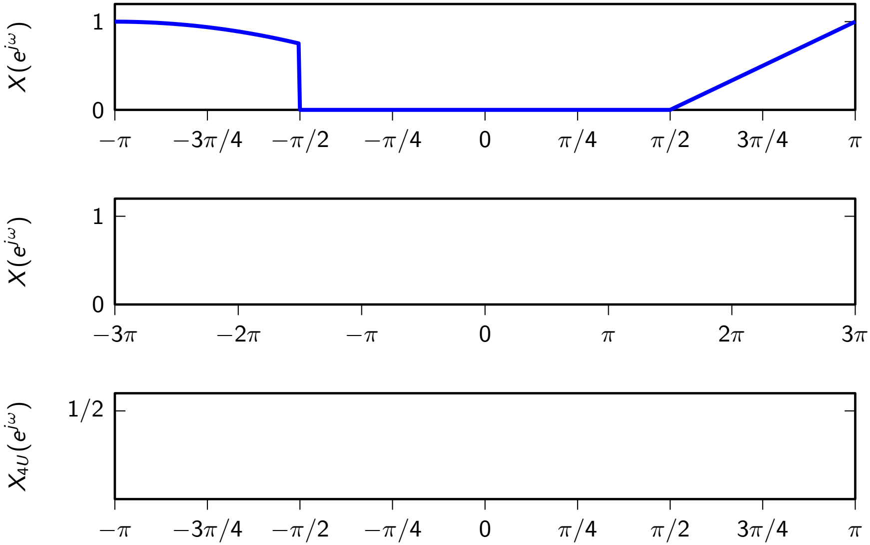
Downsampling by 2 with antialiasing filter



Downsampling



Downsampling by 2, careful with highpass signals



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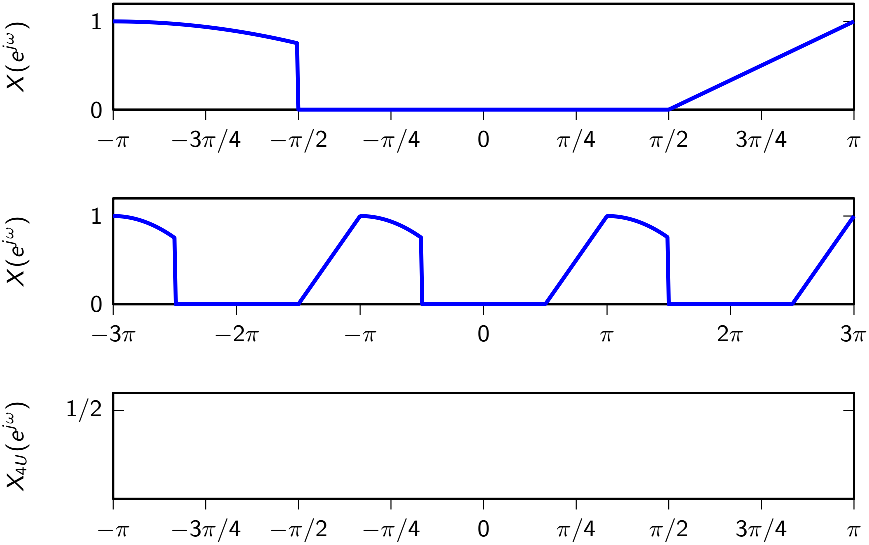

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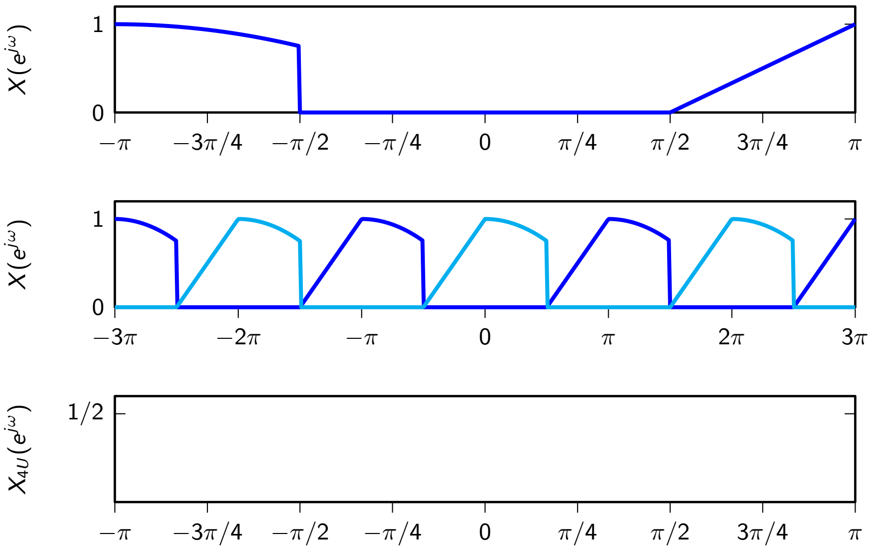

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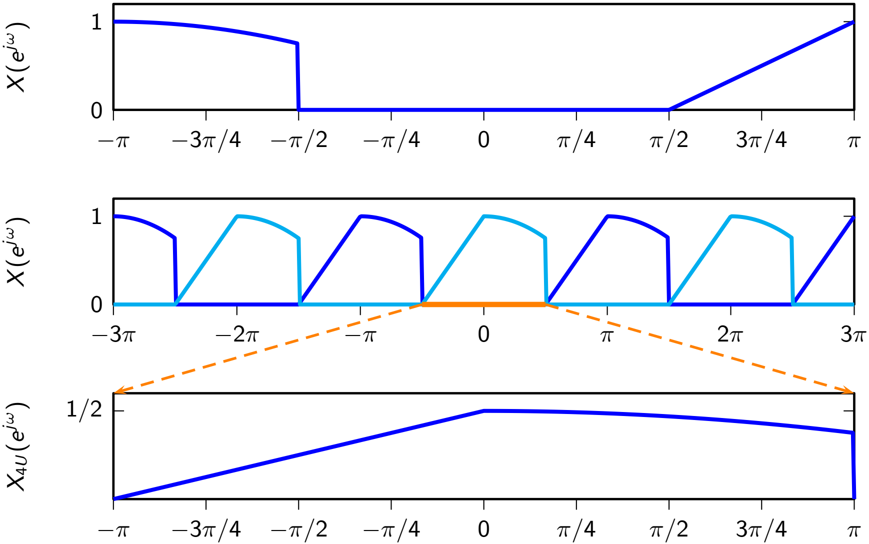
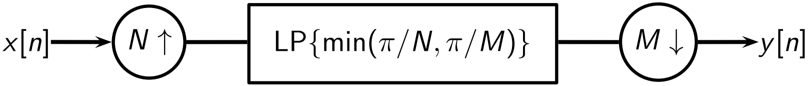


Rational Sampling Rate Change

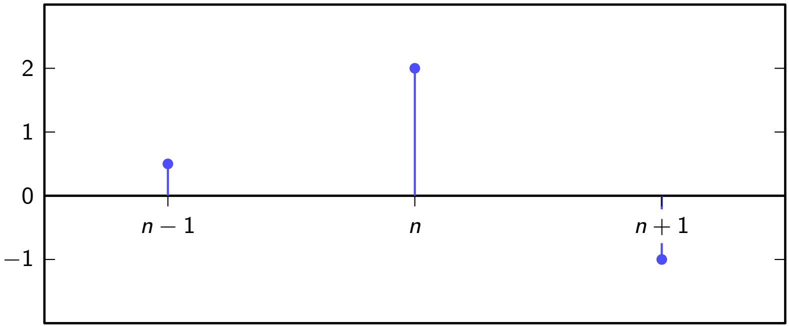


Rational Sampling Rate Change

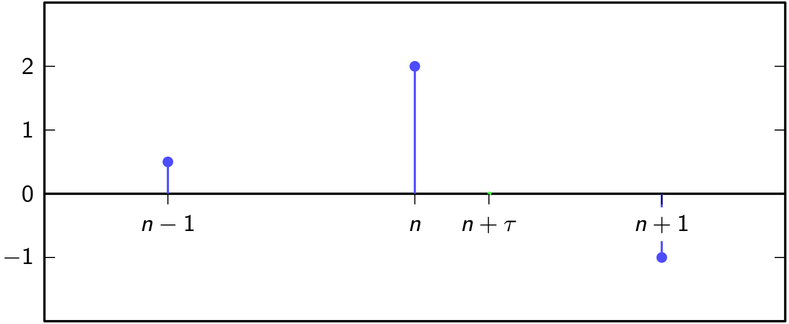
Example CD to DVD:

- CD: $F_s = 44100\text{Hz}$
- DVD: $F_S = 48000\text{Hz}$
- $\dfrac{N}{M} = \dfrac{160}{147}$
- in practice, we use time-varying local interpolation

Subsample Interpolation



Subsample Interpolation



Subsample Interpolation

- we want to compute $x(n+\tau)$, with $|\tau| < 1/2$
- local Lagrange approximation around n

$$x_L(n;t) = \sum_{k=-N}^{N} x[n-k] L_k^{(N)}(t)$$

$$L_k^{(N)}(t) = \prod_{\substack{i=-N \\ i\neq n}}^{N} \frac{t-i}{k-i} \qquad\qquad k = -N, \ldots, N$$

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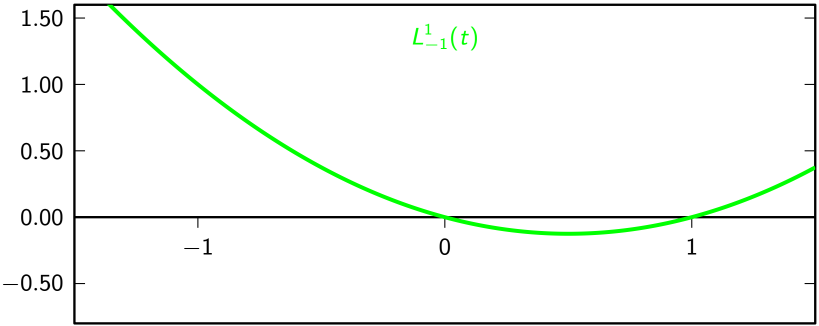
2nd-order Lagrange interpolation polynomials ($N = 1$)



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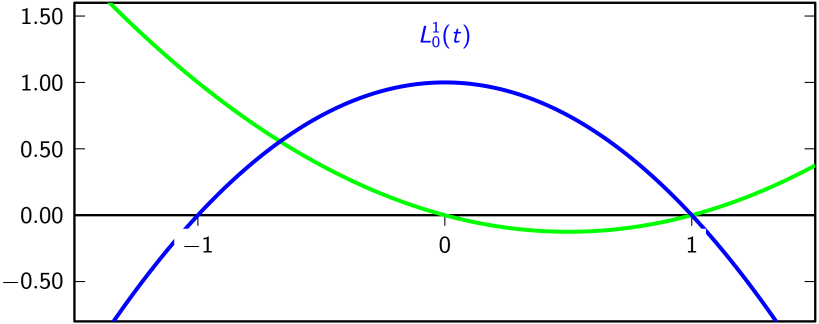

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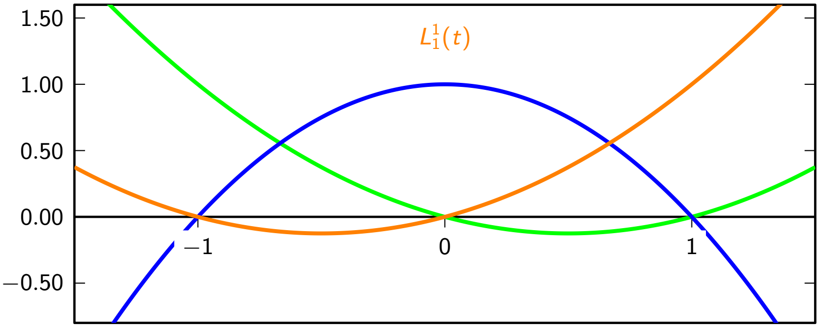

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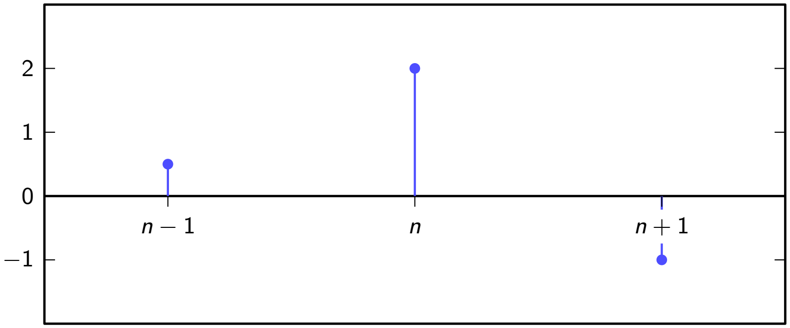

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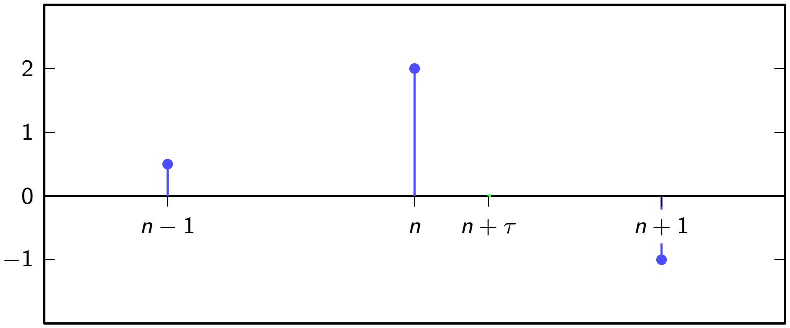

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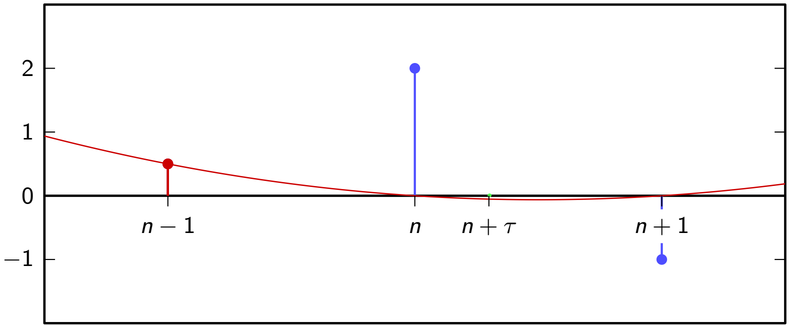

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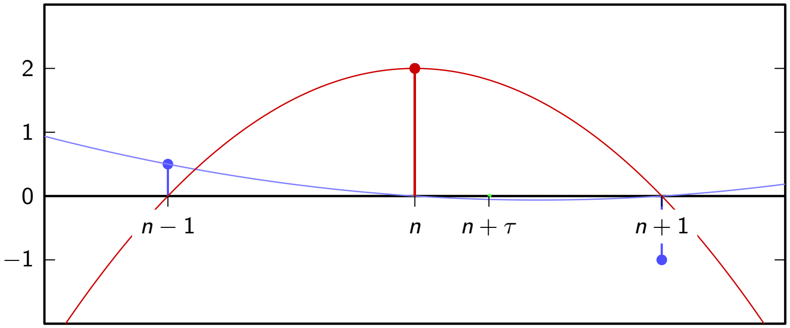

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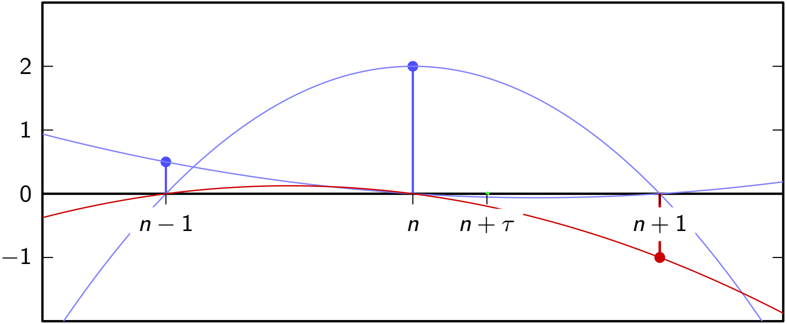
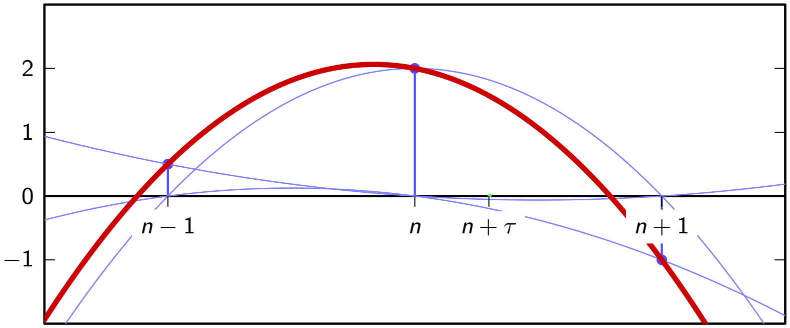


Lagrange interpolation ($N = 1$)



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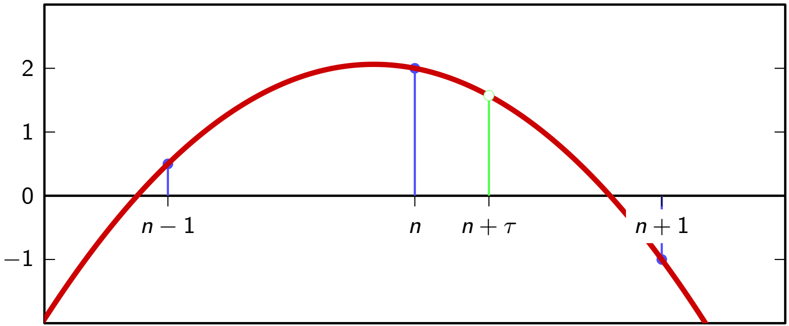

Lagrange interpolation as an FIR

- $x(n+\tau) \approx x_L(n;\tau)$
- $x_L(n;\tau) = \sum_{k=-N}^{N} x[n-k]L_k^{(N)}(\tau)$
- define $d_\tau[k] = L_k^{(N)}(\tau),\ k = -N, \ldots, N$
- $x_L(n;\tau) = \sum_{k=-N}^{N} x[n-k]d_\tau[k]$
- $x_L(n;\tau) = (x * d_\tau)[n]$
- $d_\tau[k]$ is a $(2N+1)$-tap FIR (dependent on τ)

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Example ($N = 1$, second order approximation)

$$L_{-1}^{(1)}(t) = t\frac{t-1}{2}$$
$$L_{0}^{(1)}(t) = (1-t)(1+t)$$
$$L_{1}^{(1)}(t) = t\frac{t+1}{2}$$

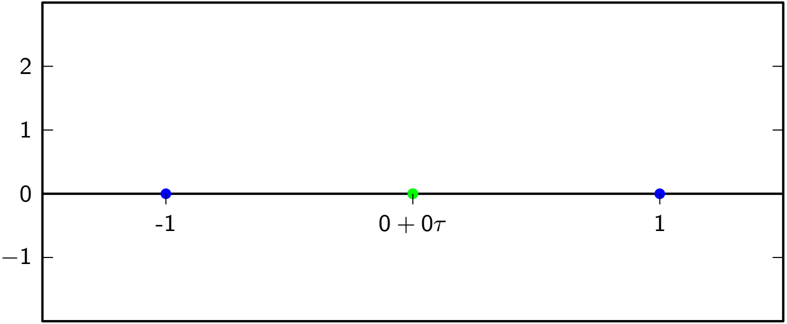
Example ($N = 1$, second order approximation)

$$d_{0.2}[n] = \begin{cases} -0.08 & n = -1 \\ 0.96 & n = 0 \\ 0.12 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

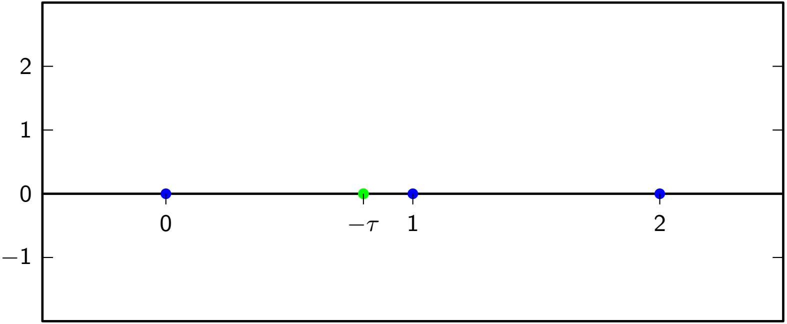
CD to DVD, revisited

for every 147 CD samples, generate 160 DVD samples

CD to DVD, revisited



CD to DVD, revisited, $\tau[1] = 0.06875$



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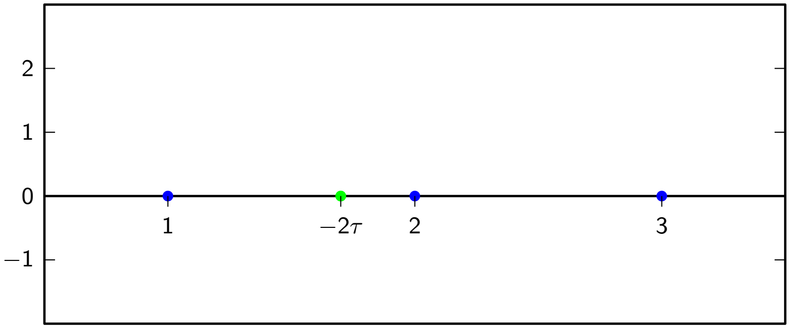

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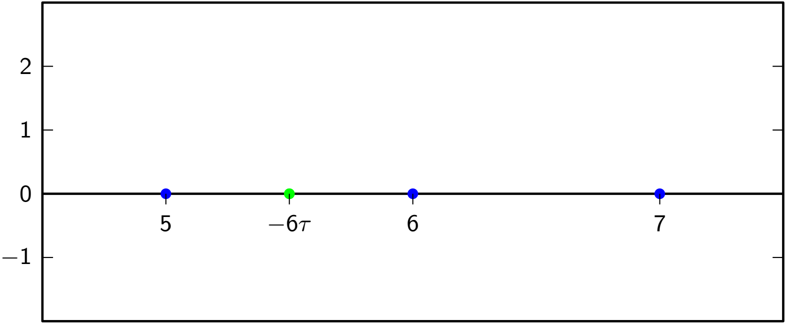

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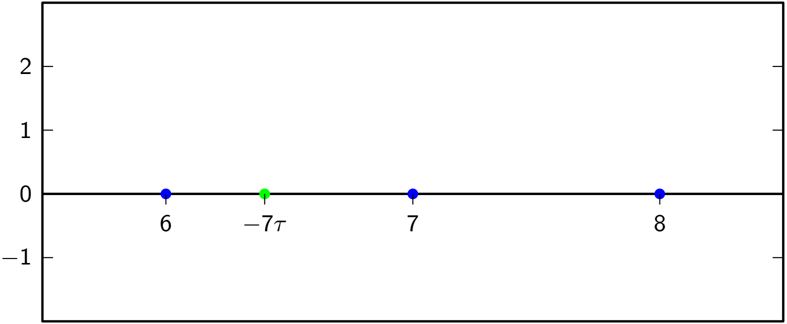
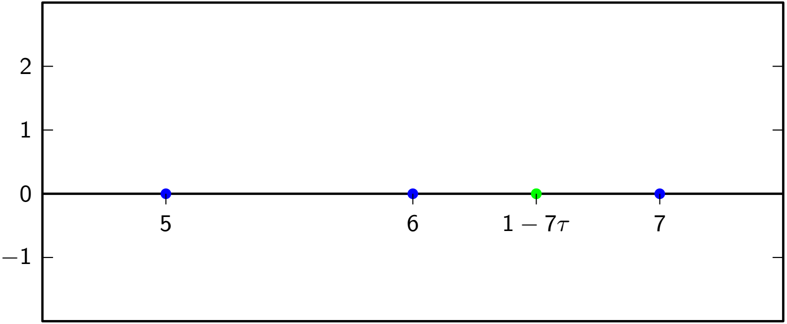
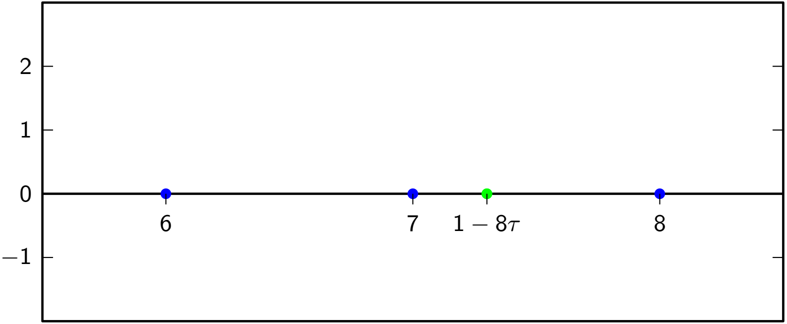


but $-7\tau < -0.5$

CD to DVD, revisited: repeat a sample



CD to DVD, revisited



CD to DVD, revisited

efficient local interpolation with 160 3-tap filters, used in sequence