Exercise Set 6

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Exercise 1. Let X be a Markov chain on E (not necessarily irreducible). Suppose that state $j \in E$ is positive recurrent and aperiodic. Show that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i), \qquad \tau_j = \inf\{n \geq 0 : X_n = j\},$$

where π is the stationary distribution of the chain restricted to the communicating class of j.

Exercise 2. Let X be a Markov chain with transition matrix P on $E = \{1, 2, 3, 4, 5\}$ given by

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

(a) Find the communicating classes of P. For the recurrent classes, find the corresponding stationary distributions.

(b) Supposing that $X_0 \sim \alpha$ for a distribution α on E, find the limiting distribution of X_n when $n \to \infty$.
Hint: Suppose that X starts in a transient state of E and find the limiting distribution in this case.

Exercise 3. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two independent Markov chains, aperiodic and irreducible, defined on the state spaces E and E', respectively. Show that $(X_n, Y_n)_{n\geq 0}$ is an aperiodic and irreducible Markov chain on $E \times E'$. Find an example of $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ independent and irreducible, but for which $(X_n, Y_n)_{n\geq 0}$ is not irreducible.

Exercise 4. (**Branching process with immigration**) For $n \in \mathbb{N}$, let $(N_k^n)_{k\geq 0}$ be a sequence of independent random variables on $\mathbb{Z}^+$ with a common generating function $\phi(t) = E(t^{N_k^n})$. The branching process with immigration is defined as

$$X_n = N_1^n + \ldots + N_{X_{n-1}}^n + I_n, \qquad n \geq 0,$$

where $(I_n)_{n\geqslant 0}$ is a sequence of independent random variables with values in $\mathbb{Z}^+$ with a common generating function $\psi(t) = E(t^{I_n})$. Show that if $X_0 = 1$ then

$$E(t^{X_n}) = \phi^{(n)}(t) \prod_{k=0}^{n-1} \psi(\phi^{(k)}(t)).$$

In the case where the number of immigrants in each generation is a Poisson random variable of parameter λ and $P(N_k^n = 0) = 1 - p$, $P(N_k^n = 1) = p$, find the proportion of time in the long run for which the population is 0.

Exercise 5. (**Metropolis–Hastings algorithm**) Suppose that we have a distribution p (called target distribution) on a countable space E. Then, for each $x \in E$, let q_x be a distribution on E (called the proposal distribution) with $q_x(y) > 0$ whenever $q_y(x) > 0$, for all $y \in E$. The Metropolis–Hastings algorithm constructs a Markov chain $(X_n)_{n\geq 0}$ as follows:

(i). Let $X_0 = x_0 \in E$ be random fixed state.

(ii). For $X_n = x$, choose a candidate y according to the proposal distribution q_x. Then let U be a uniform random variable on $[0, 1]$, the variable X_{n+1} is defined as

$$X_{n+1} = \begin{cases} y & \text{if } U \leq \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right) \\ x & \text{otherwise.} \end{cases}$$

Show that if $(X_n)_{n\geq 0}$ is irreducible and aperiodic, then it is a reversible chain with respect to its stationary distribution p.