# COM-303 - Signal Processing for Communications

## Solutions for Homework #2

### Solution 1. Bases

Suppose by contradiction that the vector $\mathbf{z} \in S$ admits two distinct representations in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$. In other words, suppose that there exist two set of scalars $\alpha_0, \ldots, \alpha_{N-1}$ and $\beta_0, \ldots, \beta_{N-1}$, with $\alpha_i \neq \beta_i$ for all $i$, such that

$$\mathbf{z} = \sum_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)}$$

and

$$\mathbf{z} = \sum_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}.$$

In this case we can write

$$\sum_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)} = \sum_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}$$

or, equivalently,

$$\sum_{k=0}^{N-1} (\alpha_k - \beta_k) \mathbf{x}^{(k)} = 0.$$

The above expression is a linear combination of basis vectors that is equal to zero. Because of the linear independence of a set of basis vector, the only set of coefficients that satisfies the above equation is a set of null coefficients so that it must be $\alpha_i \neq \beta_i$ for all $i$, in contradiction with the hypothesis.

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### Solution 2. Plancherel-Parseval Equality

(a) We expand the sum of the multiplication of DFTs $X[k]$ and $Y[k]$, that is,

$$\begin{aligned}\sum_{k=0}^{N-1}(X[k]Y^*[k]) &= \sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1}x[n]e^{-j2\pi nk/N}\right)\left(\sum_{m=0}^{N-1}y[m]e^{-j2\pi mk/N}\right)^* \\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}x[n]y^*[m]\sum_{k=0}^{N-1}e^{-j2\pi(n-m)k/N}.\end{aligned}$$

Since

$$\sum_{k=0}^{N-1}e^{-j2\pi(n-m)k/N} = \begin{cases}0 & m \neq n \\ N & m = n,\end{cases}$$

the result is proved.

(b) Note that, if we consider signals as vectors in $\mathbb{C}^N$, the formula is just the inner product between the vectors. If $x[n] = y[n]$, then $\langle x[n], x[n]\rangle$ corresponds to the energy of the signal in the time domain while $\langle X[k], X[k]\rangle/N$ corresponds to the energy of the signal in the frequency domain. In this case, the Plancherel-Parseval equality illustrates the energy conservation property from the time domain to the frequency domain. This property is also known as the *Parseval theorem*. Note that, because we choose not to normalize the Fourier basis vectors, the energy is conserved up to a scaling factor $N$.

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## Solution 3. DFT of elementary functions

We have:

$$\begin{aligned}x[n] &= \frac{e^{j\phi}}{2}e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2}e^{-j(2\pi/N)Ln} \\ &= \frac{e^{j\phi}}{2}e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2}e^{-j(2\pi/N)Ln}e^{j(2\pi/N)Nn} \\ &= \frac{e^{j\phi}}{2}e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2}e^{j(2\pi/N)(N-L)n}.\end{aligned}$$

Therefore, we can write in vector notation:

$$\mathbf{x} = \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)},$$

and the result follows from the linearity of the expansion formula:

$$\begin{aligned}X[k] &= \langle \mathbf{w}^{(k)}, \mathbf{x}\rangle \\ &= \left\langle \mathbf{w}^{(k)}, \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)}\right\rangle = \frac{e^{j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(L)}\rangle + \frac{e^{-j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(N-L)}\rangle\end{aligned}$$

Now, if $L \neq N-L$, we have:

$$X[k] = \begin{cases}\frac{N}{2}e^{j\phi} & \text{if } k = L \\ \frac{N}{2}e^{-j\phi} & \text{if } k = N-L \\ 0 & \text{otherwise.}\end{cases}$$

Otherwise, if $L = N - L$, we have:

$$X[k] = \begin{cases} \frac{N}{2} e^{j\phi} + \frac{N}{2} e^{-j\phi} & \text{if } k = L = N - L \\ 0 & \text{otherwise.} \end{cases}$$

## Solution 4. DFT Example

By simple visual inspection we can determine that

$$\begin{aligned} a[n] &= 2 \\ b[n] &= 3\cos(3(2\pi/64)n) \\ c[n] &= \sin(7(2\pi/64)n) = -\cos(7(2\pi/64)n + \pi/2). \end{aligned}$$

The DFT coefficients are $X[k] = A[k] + B[k] + C[k]$, with

$$\begin{aligned} A[k] &= 2N\delta[k] \\ B[k] &= (3N/2)\delta[k-3] + (3N/2)\delta[k-61] \\ C[k] &= -(jN/2)\delta[k-7] + (jN/2)\delta[k-57] \end{aligned}$$

and $N = 64$, so that in the end we have

$$\begin{aligned} X[0] &= 128 \\ X[3] &= 96 \\ X[7] &= -32j \\ X[57] &= 32j \\ X[61] &= 96 \end{aligned}$$

and $X[k] = 0$ for all the other values of $k$.

## Solution 5. DFT computation

There are many ways to solve this problem. A simple method is to observe that we can write $\mathbf{x} = \mathbf{a} + \mathbf{b}$ with

$$\mathbf{a} = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T$$
$$\mathbf{b} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}^T$$

which, in signal notation, corresponds to

$$a[n] = \sin((2\pi/8)2n - \pi/2)$$
$$b[n] = \cos((2\pi/8)2n + \pi/2)$$

Using the result from the previous exercise we have

$$A[k] = \begin{cases} -4je^{j\pi/2} = -4 & k = 2 \\ 4je^{-j\pi/2} = -4 & k = 6 \end{cases}$$

and

$$B[k] = \begin{cases} 4e^{j\pi/2} = 4j & k = 2 \\ 4e^{-j\pi/2} = -4j & k = 6 \end{cases}$$

so that

$$\mathbf{X} = \begin{bmatrix} 0 & 4(-1+j) & 0 & 0 & 0 & 0 & 4(-1-j) & 0 \end{bmatrix}^T$$

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### Solution 6. Structure of DFT formulas

Let $f[n] = \text{DFT}\{x[n]\}$. We have:

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{k=0}^{N-1} \left\{ \sum_{i=0}^{N-1} x[i] e^{-j\frac{2\pi}{N}ik} \right\} e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{i=0}^{N-1} x[i] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k}. \end{aligned}$$

Now,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} = \begin{cases} N & \text{for } (i+n) = 0, N, 2N, 3N, \ldots \\ 0 & \text{otherwise} \end{cases} = N\delta[(i+n) \mod N]$$

so that

$$\begin{aligned} y[n] &= \sum_{i=0}^{N-1} x[i] N\delta[(i+n) \mod N] \\ &= \begin{cases} Nx[0] & \text{for } n = 0 \\ Nx[N-n] & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, if $\mathbf{x} = [1\ 2\ 3\ 4\ 5]^T$ then $\text{DFT}\{\text{DFT}\{\mathbf{x}\}\} = 5[1\ 5\ 4\ 3\ 2]^T = [5\ 25\ 20\ 15\ 10]^T$.

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### Solution 7. Signal repetitions

Consider the auxiliary signal

$$\mathbf{y} = [x[0] \quad 0 \quad x[1] \quad 0 \quad x[2] \quad 0 \quad \ldots \quad x[N-1] \quad 0]^T$$

whose DFT coefficients are (for $k = 0, 1, \ldots, 2N-1$):

$$\begin{aligned} Y[k] &= \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}nk} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}2nk} \\ &= X[k] \end{aligned}$$

The signal $\mathbf{x_2}$ is the sum of $\mathbf{y}$ and of $\mathbf{y}$ circularly shifted by one. Since the circular shift in time corresponds to multiplication by a phase term $e^{-j\frac{2\pi}{2N}k}$ in frequency, we have

$$X_2[k] = (1 + e^{-j\frac{2\pi}{2N}k})X[k]$$