Solutions 12

Saliba, May 29, 2019

Exercise 1. (a) Prove Blackwell's theorem which states that, for a recurrent renewal process,

$$\lim_{t\to\infty}(R(t+h)-R(t)) = \frac{h}{E[W]}.$$

(b) We have seen already that, for a recurrent and irreducible Markov chain $(X_n)_{n\geq 0}$,

$$\lim_{n\to\infty}\mathbb{P}(X_n = j \mid X_0 = j) = \begin{cases} 0 & \text{if the states are null recurrent,} \\ \pi_j > 0 & \text{if the states are positive recurrent.} \end{cases}$$

Prove this result using Blackwell's theorem.

Solution. (a) It's sufficient to apply the Key Renewal Theorem with function $g(x) = \mathbb{1}_{\{[o,h]\}}(x)$. We then have

$$\begin{aligned}(R*g)(t) &= \int_0^t g(t-u)\,dR(u), \\ &= \int_{t-h}^t dR(u), \\ &= R(t)-R(t-h),\end{aligned}$$

and using the Key Renewal Theorem,

$$\begin{aligned}\lim_{t\to\infty}(R*g)(t) &= \frac{1}{\mathbb{E}[W]}\int_0^\infty g(u)\,du \\ &= \frac{1}{\mathbb{E}[W]}\int_0^h du, \\ &= \frac{h}{\mathbb{E}[W]}.\end{aligned}$$

(b) (Note: this was already done during the lectures). It's a consequence of Blackwell's theorem: for a Markov chain $\{X_n\}$, if we start at j, the time intervals between two transitions to j are i.i.d:

We then have a renewal process that is recurrent $\{S_n : n \geq 0\}$, where S_n is the time of the nth return to j,

$$\begin{cases} S_0 = 0, \\ S_{n+1} = S_n + W_{n+1}, \end{cases}$$

where W_i are i.i.d. and have the same distribution as the first return time T_j given that the chain starts at j ($\mathbb{E}[W] = \mathbb{E}[T_j \mid X_0 = j]$, see the chapter on Markov chains).

For $t \in \mathbb{N}$, we consider the renewal function corresponding to the renewal process:

$$\begin{aligned} R(t) - R(t-1) &= \mathbb{E}[N(t) - N(t-1)], \\ &= \mathbb{E}[\text{number of visits to } j \text{ between } t-1 \text{ and } t], \\ &= \mathbb{E}[\mathbb{1}_{\{X(t)=j\}}] \quad \text{(discrete time).} \end{aligned}$$

Hence, by Blackwell's theorem,

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(X_t = j \mid X_0 = j) &= \lim_{t\to\infty} (R(t) - R(t-1)), \\ &= \frac{1}{\mathbb{E}[W]} \\ &= \frac{1}{\mathbb{E}[T_j \mid X_0 = j]}, \end{aligned}$$

and therefore,

$$\begin{cases} \mathbb{E}[W] = +\infty \Rightarrow \lim = 0 \Rightarrow \text{null recurrent states,} \\ \mathbb{E}[W] < \infty \Rightarrow \lim > 0 \Rightarrow \text{ positive recurrent states.} \end{cases}$$

Exercise 2. Consider a machine installed at time $S_0 = 0$. When it breaks down, it is replaced by a new identical one and so on. Suppose that the lifetimes the machines are $U_1, U_2, \ldots$ and the replacement times are $V_1, V_2, \ldots$ unit of time. Hence, the machines start working successively at times $S_0 = 0$, $S_1 = U_1 + V_1$, $S_2 = U_2 + V_2$ and so on. It is reasonable to suppose that the U_i's are i.i.d as well as the V_i's. Moreover, we can suppose that U_i and V_i are independent. So $W_i = U_i + V_i$ are i.i.d and $\{S_n\}$ is a renewal process.
Write $\varphi(\cdot)$ for the distribution of the U_i's and $\psi(\cdot)$ for the distribution of the V_i's. Let Z_t be the indicator of the event that the machine is working at time t, and $f(t) = \mathbb{P}(Z_t = 1)$.

(a) Show that $f(t)$ satisfies a renewal equation and deduce the value of $\mathbb{P}(Z_t = 1)$.

(b) What is the asymptotic probability that the machine is working?

Solution. Let $Z_t = \mathbb{1}_{\{\text{the machine is working at time t}\}}$ and let S_1 be the first renewal time. We then have

$$\begin{aligned} \mathbb{P}(Z_t = 1) &= \mathbb{P}(Z_t = 1, S_1 > t) + \mathbb{P}(Z_t = 1, S_1 \le t) \\ &= \mathbb{P}(U_1 > t) + \int_0^t \mathbb{P}(Z_t = 1 | S_1 = u) dF(u). \end{aligned}$$

Notice that $\mathbb{P}(U_1 > t) = 1 - \varphi(t)$ and that when $u \le t$, $\mathbb{P}(Z_t = 1 | S_1 = u) = \mathbb{P}(Z_{t-u} = 1)$. Hence

$$\mathbb{P}(Z_t = 1) = 1 - \varphi(t) + \int_0^t \mathbb{P}(Z_{t-u} = 1) dF(u),$$

and $f(t) = \mathbb{P}(Z_t = 1)$ satisfies the renewal equation

$$f(t) = g(t) + (F * f)(t),$$

where

$$g(t) = 1 - \varphi(t).$$

Its solution is given by

$$f(t) = (R * g)(t) = \int_0^t (1 - \varphi(t-u))dR(u) = R(t) - (R * \varphi)(t).$$

Let's study two different cases:
Recurrent case If the mean lifetime is

$$a = \int_0^\infty (1 - \varphi(t))dt < \infty,$$

and the mean replacement time is b, then $E[W] = a + b$, and we have

$$\lim_{t\to\infty} \mathbb{P}(Z_t = 1) = \lim_{t\to\infty} f(t) = \frac{1}{a+b}\int_0^\infty (1-\varphi(t))dt = \frac{a}{a+b}.$$

Transient case:

- $\varphi(\infty) < 1$: $\psi(\infty) = 1$, then $F(\infty) = \varphi(\infty)\psi(\infty) = \varphi(\infty)$ this implies

$$\lim_{t\to\infty} f(t) = R(\infty)g(\infty) = \frac{1}{1-F(\infty)}(1-\varphi(\infty)) = 1.$$

- $\varphi(\infty) = 1, \psi(\infty) < 1$: then $F(\infty) = \psi(\infty)$ this implies

$$\lim_{t\to\infty} f(t) = R(\infty)g(\infty) = \frac{1}{1-\psi(\infty)}0 = 0.$$

- $\varphi(\infty) < 1, \psi(\infty) < 1$: then $F(\infty) = \varphi(\infty)\psi(\infty)$ this implies

$$\lim_{t\to\infty} f(t) = \frac{1-\varphi(\infty)}{1-\varphi(\infty)\psi(\infty)}.$$

Exercise 3. Let U_t be the time since the last renewal before t in a recurrent renewal process S with inter-renewal distribution F, that is, $U_t = t - S_{N_t - 1}$.

(a) Show that $f(t) = \mathbb{P}(U_t > x)$ satisfies, for a fixed x, the following renewal equation

$$f(t) = (1 - F(t))\mathbb{1}_{\{(x,\infty)\}} + \int_{[0,t]} f(t-s)dF(s).$$

(b) Show that for all $t > x$,

$$\mathbb{P}(U_t > x) = \int_{[0,t-x)} (1 - F(t-s))dR(s).$$

(c) Show that

$$\lim_{t\to\infty} \mathbb{P}(U_t > x) = \lim_{t\to\infty} \mathbb{P}(Z_t > x),$$

where $Z_t = S_{N_t} - t$ is the survival time of the process at time t.

Solution. (a) Conditioning on S_1, we have $f(t) = \int_0^\infty \mathbb{P}(U_t > x | S_1 = u)dF(u)$, with

(i) $u > t$: $\mathbb{P}(U_t > x | S_1 = u) = \mathbb{1}_{\{(x,\infty)\}}(t)$

(ii) $u < t$: $\mathbb{P}(U_t|S_1 = u) = \mathbb{P}(U_{t-u} > x)$,

and so

$$f(t) = \int_t^\infty \mathbb{1}_{\{(x,\infty)\}}(t)dF(u) + \int_0^t \mathbb{P}(U_{t-u} > x)dF(u) = (1-F(t))\mathbb{1}_{\{(x,\infty)\}}(t) + (F*f)(t).$$

(b) The solution of the renewal equation is

$$f(t) = (R*g)(t)$$

with $g(t) = (1-F(t))\mathbb{1}_{\{(x,\infty)\}}(t)$, which implies that for $t > x$, we have $\mathbb{1}_{\{(x,\infty)\}}(t) = 1$, and so

$$f(t) = \int_0^{t-x} (1 - F(t-u))dR(u).$$

(c) When $t \to \infty$, we have

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{t\to\infty} (R*g)(t) \\
&= \frac{1}{E[W]} \int_0^\infty g(u)du \\
&= \frac{1}{E[W]} \int_0^\infty (1-F(u))\mathbb{1}_{\{(x,\infty)\}}(u)du \\
&= \frac{1}{E[W]} \int_x^\infty (1-F(u))du,
\end{aligned}$$

which is equal to $\lim_{t\to\infty} \mathbb{P}(Z_t > x)$ where Z_t is the survival time at time t.

Exercise 4. Recall the context of Exercise 2. Suppose that the replacements of the machines are immediate (do not take any time), and instead of replacing only the damaged machines, suppose that the rule is to replace any damaged machine *or* any machine used for τ units of time. We now distinguish between these two cases.

(a) Show that the times S_n of successive replacements form a renewal process, and compute the distribution F of the time between two replacements.

(b) Show that if $N(t)$ is the number of replacements in the time interval $[0,t]$, then

$$\lim_{t\to\infty} \frac{N(t)}{t} = 1 \Big/ \int_0^\tau [1-\varphi(u)]du.$$

(c) Show that the successive moments of breakdowns $T_0, T_1, \ldots$ form a renewal process with distribution G between two moments of breakdown given by

$$1 - G(t) = [1-\varphi(\tau)]^k[1-\varphi(t-k\tau)], \;\; k\tau \le t \le k\tau + \tau.$$

(d) Show that the mean number of breakdowns per unit of time, $M(t)/t$, has the following limit when $t \to \infty$,

$$\varphi(\tau) \Big/ \int_0^\tau [1-\varphi(u)]du.$$

Solution. (a) Let $\hat{W}_i = S_i - S_{i-1}$ be the time between the $(i-1)$th and the ith replacement, and $\hat{W}_i \sim F(\cdot)$, $W_i \sim \phi(\cdot)$. We have that

$$\hat{W}_i = \tau \mathbb{1}_{\{W_i > \tau\}} + W_i \mathbb{1}_{\{W_i \leq \tau\}},$$

and hence

$$\begin{aligned} \mathbb{P}(\hat{W}_i \leq t) &= \mathbb{P}(\hat{W}_i \leq t | W_i > \tau)\, \mathbb{P}(W_i > \tau) + \mathbb{P}(\hat{W}_i \leq t | W_i \leq \tau)\, \mathbb{P}(W_i \leq \tau) \\ &= \mathbb{1}_{(\tau,\infty)}(t)(1 - \varphi(\tau)) + \mathbb{1}_{(0,\tau)}(t)\varphi(t) + \mathbb{1}_{(\tau,\infty)}(t)\varphi(\tau) \end{aligned}$$

and so

$$F(t) = \mathbb{P}(\hat{W}_i \leq t) = \mathbb{1}_{(\tau,\infty)}(t) + \mathbb{1}_{(o,\tau)}(t)\varphi(t).$$

(b) We have

$$\begin{aligned} E[\hat{W}] &= \int_0^\infty (1 - F(u))du \\ &= \int_0^\infty ((1 - \mathbb{1}_{(\tau,\infty)})(u) - \mathbb{1}_{(0,\tau)}(u)\varphi(u))du \\ &= \int_0^\tau (1 - \varphi(u))du. \end{aligned}$$

Another possible method to compute it is the following:

$$\begin{aligned} E[\hat{W}] &= \tau\, \mathbb{P}(W_i > \tau) + E[W_i I_{\{W_i < \tau\}}] \\ &= \tau(1 - \varphi(\tau)) + \int_0^\tau (\varphi(\tau) - \varphi(u))du \\ &= \int_0^\tau (1 - \varphi(u))du. \end{aligned}$$

By the fundamental theorem, we get

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{E[W]} = \frac{1}{\int_0^\tau (1 - \varphi(u))du}$$

almost surely.

(c) Let X be the time between two breakdowns, and $G(t) = \mathbb{P}(X \leq t)$. For $k\tau \leq t < (k+1)\tau$, we have

$$1 - G(t) = \mathbb{P}(X > t) = \mathbb{P}(W > \tau)^k\, \mathbb{P}(W > t - k\tau)$$

. This implies

$$\begin{aligned} E[X] &= \int_0^\infty (1 - G(t))dt \\ &= \sum_{k=0}^\infty (\mathbb{P}(W > \tau))^k \int_{k\tau}^{(k+1)\tau} \mathbb{P}(W > t - k\tau)dt \\ &= \sum_{k=0}^\infty (\mathbb{P}(W > \tau))^k \int_0^\tau \mathbb{P}(W > u)du \\ &= \frac{\int_0^\tau \mathbb{P}(W > u)du}{1 - \mathbb{P}(W > \tau)} = \frac{\int_0^\tau (1 - \varphi(u))du}{\varphi(\tau).} \end{aligned}$$

(d) and by the fundamental theorem

$$\frac{M_t}{t} \to \frac{1}{E[X]}.$$

Exercise 5. Let S be a renewal process with inter-renewal distribution F, and let $T_1, T_2, \dots$ be the times of successes in a Bernouilli process with parameter p. As usual, we define $T_0 = 0$. Show that $\hat{S}_n = S_{T_n}$ form a renewal process with interval distribution between two renewals given by

$$\hat{F} = \sum_{n=1}^{\infty} pq^{n-1} F^{(n)}.$$

Solution. Let X be the time between two renewals associated to a success and $p =$ the probability that a renewal is successful. We then have

$$\begin{aligned} \mathbb{P}(X \le x) &= \mathbb{P}(W_1 \le x)p + \mathbb{P}(W_1 + W_2 \le x)qp + \mathbb{P}(W_1 + W_2 + W_3 \le x)q^2 p + \dots \\ &= \sum_{n \ge 1} pq^{n-1} F^{(n)}(x). \end{aligned}$$