COM303: Digital Signal Processing

Lecture 7: The DTFT Formalism

Overview:

- the DTFT of non square-summable sequences
- relationships between transforms
- modulation

the DTFT formalism for non ℓ_2 sequences

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

Review: DFT

$$
\begin{tikzcd}
\mathbb{C}^N \arrow[rr, bend left, "X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n] \rangle"] & \text{basis: } \{e^{j\frac{2\pi}{N}nk}\}_k & \mathbb{C}^N \arrow[ll, bend left, "x[n] = (1/N) \sum X[k]\, e^{j\frac{2\pi}{N}nk}"]
\end{tikzcd}
$$

Review: DFS

$$\tilde{X}[k] = \langle e^{j\frac{2\pi}{N}nk}, \tilde{x}[n] \rangle$$

$$\tilde{\mathbb{C}}^N \qquad \text{basis: } \{e^{j\frac{2\pi}{N}nk}\}_k \qquad \mathbb{C}^N$$

$$\tilde{x}[n] = (1/N) \sum \tilde{X}[k]\, e^{j\frac{2\pi}{N}nk}$$

DTFT

$$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$$

$$\begin{tikzcd}
\ell_2(\mathbb{Z}) \arrow[rr, bend left, "X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle"] & \text{“basis”: } \{e^{j\omega n}\}_\omega & L_2([-\pi, \pi]) \arrow[ll, bend left, "x[n] = (1/2\pi) \int X(e^{j\omega}) e^{j\omega n} d\omega"]
\end{tikzcd}$$

$$x[n] = (1/2\pi) \int X(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT as basis expansion

Some things are OK:

- DFT $\{\delta[n]\} = 1$
- DTFT $\{\delta[n]\} = \langle e^{j\omega n}, \delta[n] \rangle = 1$

DTFT as basis expansion

Some things are OK:

- $\text{DFT}\{\delta[n]\} = 1$
- $\text{DTFT}\{\delta[n]\} = \langle e^{j\omega n}, \delta[n] \rangle = 1$

DTFT as basis expansion

Some things aren't:

- $\text{DFT}\{1\} = \delta[n]$
- $\text{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$

- problem: too many interesting sequences are *not* square summable!

The Dirac delta

The Dirac delta functional

the functional $\delta(t)$ is defined by its "sifting" property:

$$\int_{-\infty}^{\infty} \delta(t-s) f(t) dt = f(s)$$

for all $f(t), \quad t \in \mathbb{R}$.

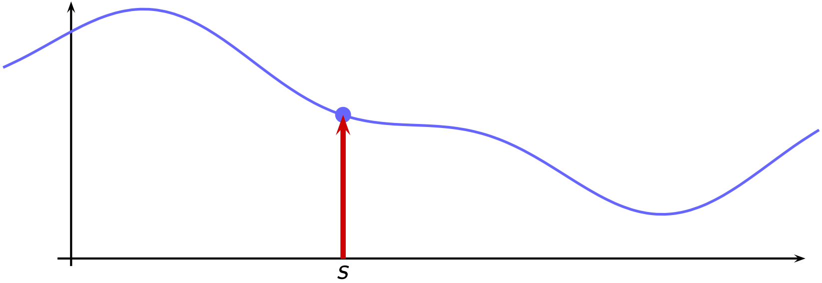
The Dirac delta functional

the functional $\delta(t)$ is defined by its "sifting" property:

$$\int_{s-\delta}^{s+\delta} \delta(t-s) f(t) dt = f(s) \qquad \forall \delta \in \mathbb{R}^+$$

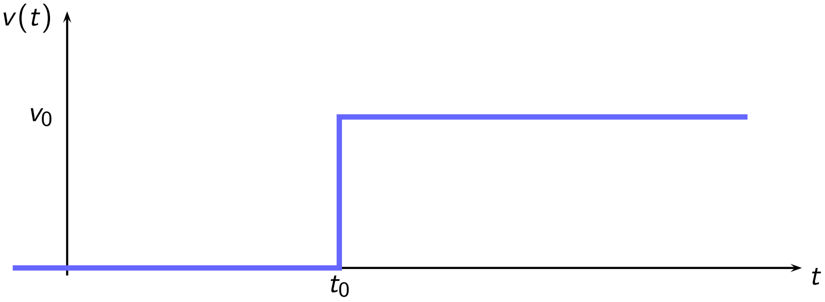
for all $f(t), \quad t \in \mathbb{R}$.

The Dirac delta functional



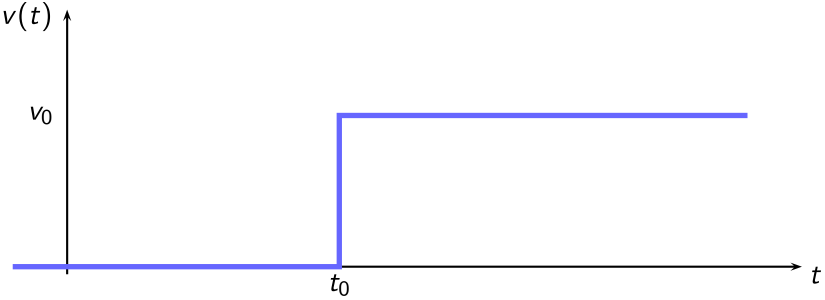
$$\int_{-\infty}^{\infty} \delta(t-s) f(t) dt = f(s)$$

The Dirac delta functional in physics



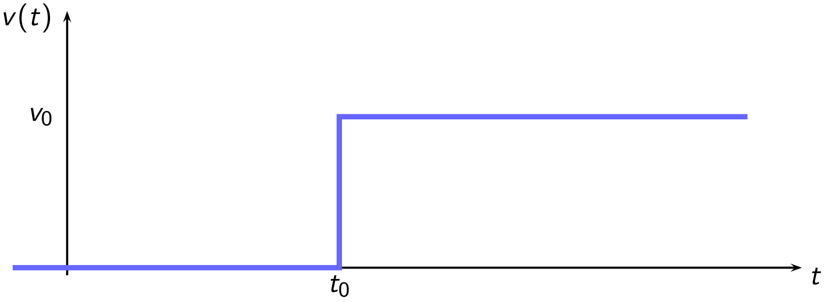
$$F(t) = ma(t) = m\frac{\partial v(t)}{\partial t}$$

The Dirac delta functional in physics



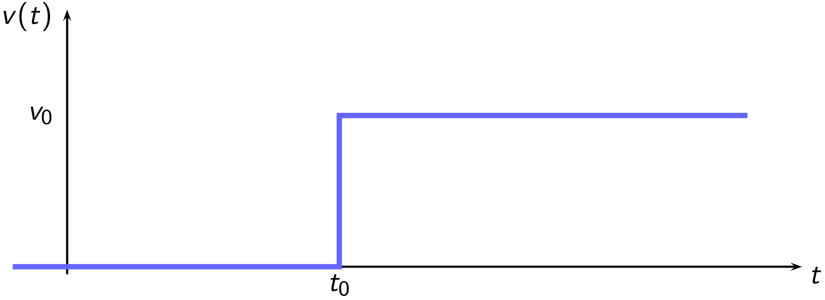
$$a(t_0) = \infty ?$$

The Dirac delta functional in physics



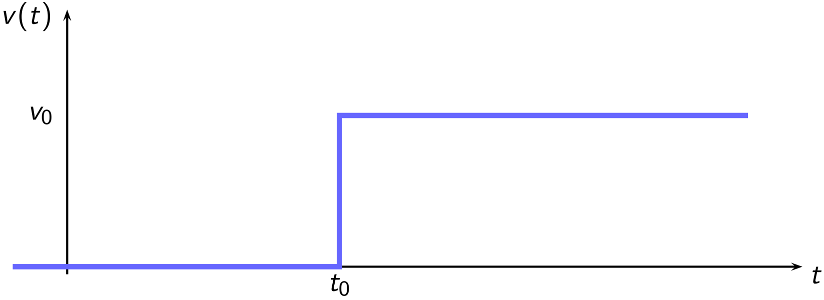
from the other side: $v(t) = \int_{-\infty}^{t} a(\tau) d\tau = \begin{cases} 0 & \text{for } t < t_0 \\ v_0 & \text{for } t > t_0 \end{cases}$

The Dirac delta functional in physics



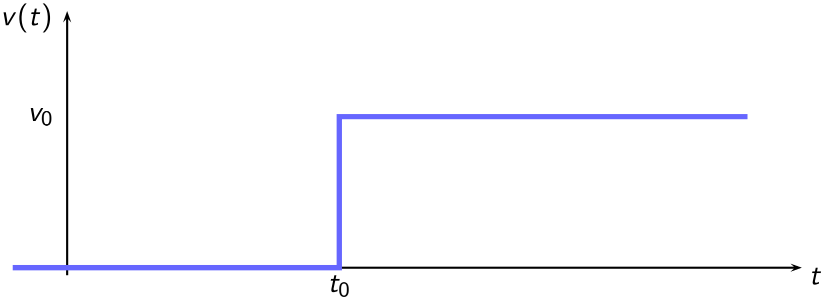
from the other side: $v(t) = \int_{-\infty}^{t} v_0 \delta(\tau - t_0) d\tau$

The Dirac delta functional in physics



$$a(t) = v_0 \delta(t - t_0)$$

The Dirac delta functional in physics



$$F(t) \propto \delta(t - t_0) \approx \begin{cases} \infty & \text{for } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

Intuition

consider a family of *localizing* functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$ where:

- support inversely proportional to k
- constant area

Intuition

consider a family of *localizing* functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$ where:

- support inversely proportional to k
- constant area

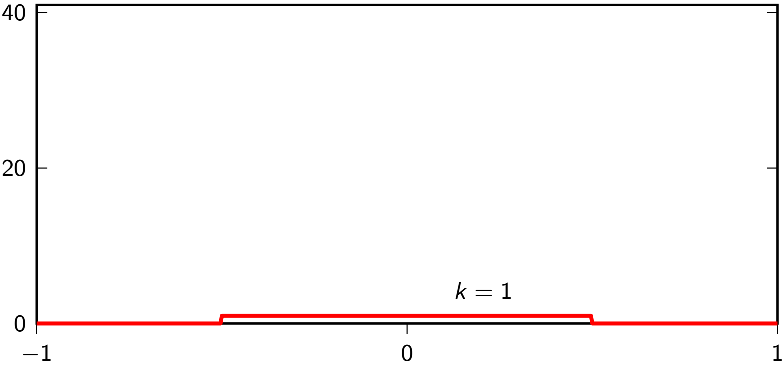
Intuition

$$\text{rect}(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

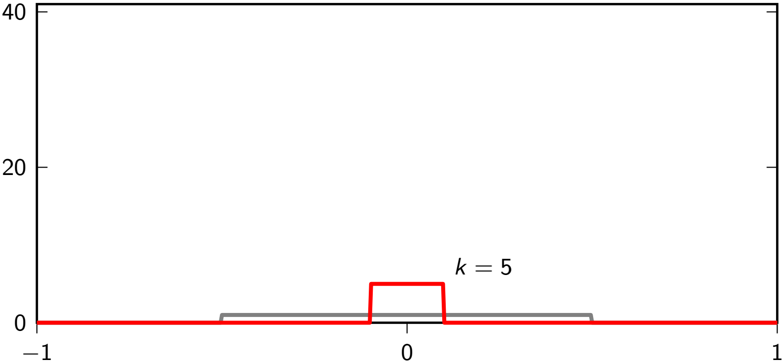
we can build a localizing family as $r_k(t) = k \,\text{rect}(kt)$:

- nonzero over $[-1/2k, 1/2k]$, i.e. support is $1/k$
- area is 1

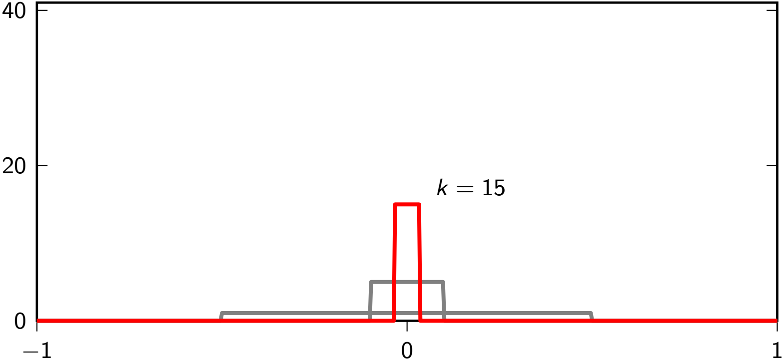
The family $r_k(t) = k\,\mathrm{rect}(kt)$



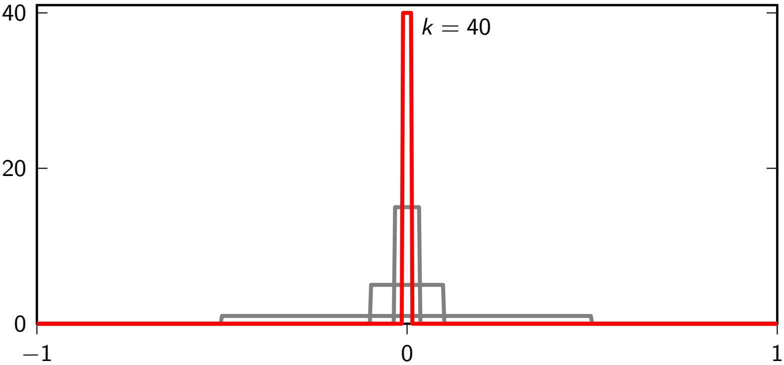
The family $r_k(t) = k\,\mathrm{rect}(kt)$



The family $r_k(t) = k\,\mathrm{rect}(kt)$



The family $r_k(t) = k\,\mathrm{rect}(kt)$

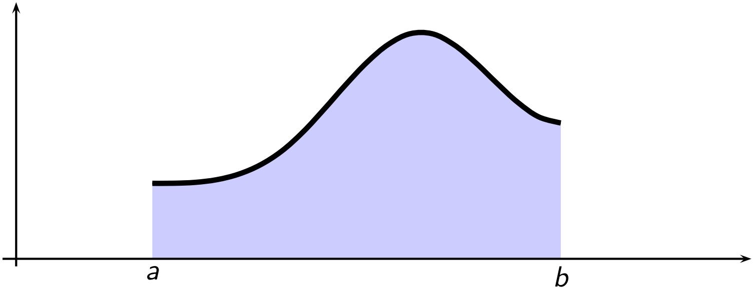


Remember the Mean Value Theorem?

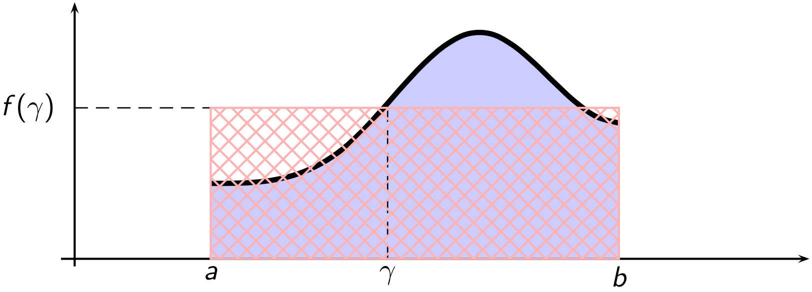
for any continuous function over the interval $[a, b]$ there exists $\gamma \in [a, b]$ s.t.

$$\int_a^b f(t)dt = (b - a)\, f(\gamma)$$

The Mean Value Theorem



The Mean Value Theorem



Extracting a point value

for our family of localizing functions:

$$\int_{-\infty}^{\infty} r_k(t) f(t) dt = k \int_{-1/2k}^{1/2k} f(t) dt$$
$$= f(\gamma)\big|_{\gamma \in [-1/2k, 1/2k]}$$

and so:

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} r_k(t) f(t) dt = f(0)$$

The Dirac delta functional

The delta functional is a shorthand. Instead of writing

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} r_k(t-s)f(t)dt$$

we write

$$\int_{-\infty}^{\infty}\delta(t-s)f(t)dt.$$

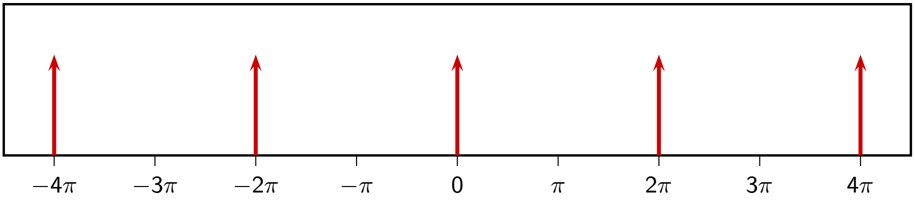
as if $\lim_{k\to\infty} r_k(t)=\delta(t)$,

The “pulse train”

$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

just a technicality to use the Dirac delta in the space of 2π-periodic functions

Graphical representation



Now let the show begin!

$$\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\delta}(\omega) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$$

$$= e^{j\omega n}\big|_{\omega=0}$$

$$= 1$$

Now let the show begin!

$$
\begin{aligned}
\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega \\
&= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega \\
&= e^{j\omega n}\big|_{\omega=0} \\
&= 1
\end{aligned}
$$

Now let the show begin!

$$
\begin{aligned}
\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega \\
&= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega \\
&= e^{j\omega n}\big|_{\omega=0} \\
&= 1
\end{aligned}
$$

Now let the show begin!

$$\begin{aligned}
\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega \\
&= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega \\
&= e^{j\omega n}\big|_{\omega=0} \\
&= 1
\end{aligned}$$

In other words

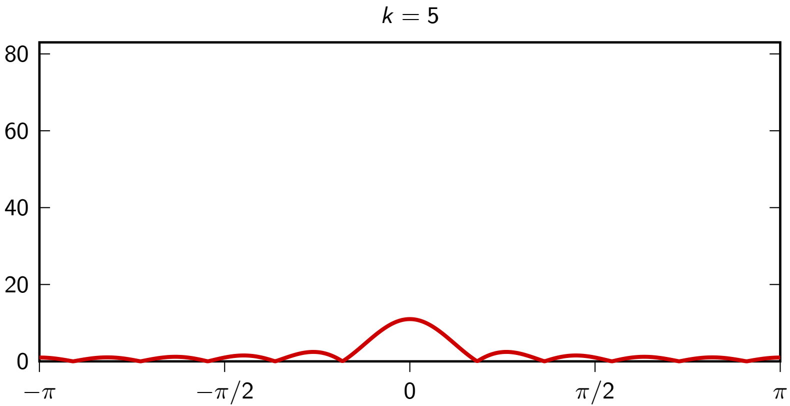
$$\mathsf{DTFT}\left\{1\right\} = \tilde{\delta}(\omega)$$

Does it make sense?

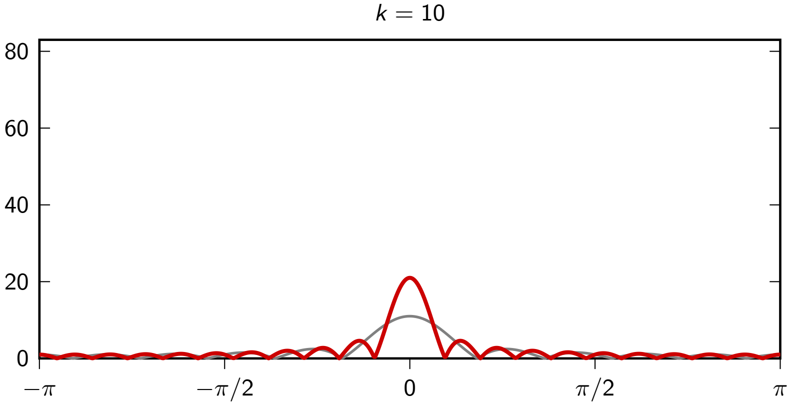
Partial DTFT sum:

$$S_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$$

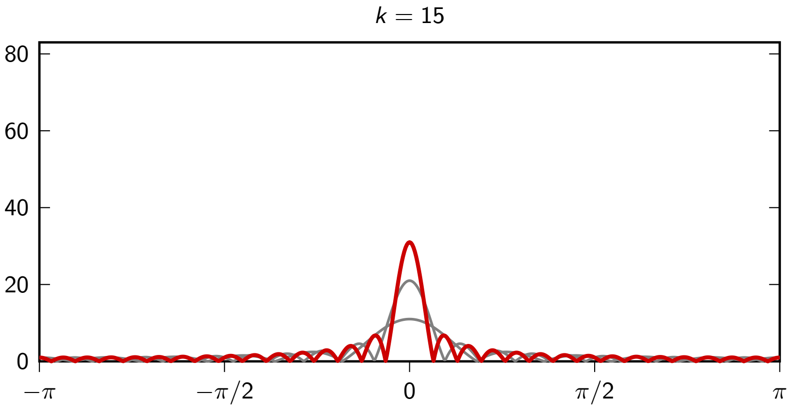
Plotting $|S_k(\omega)|$



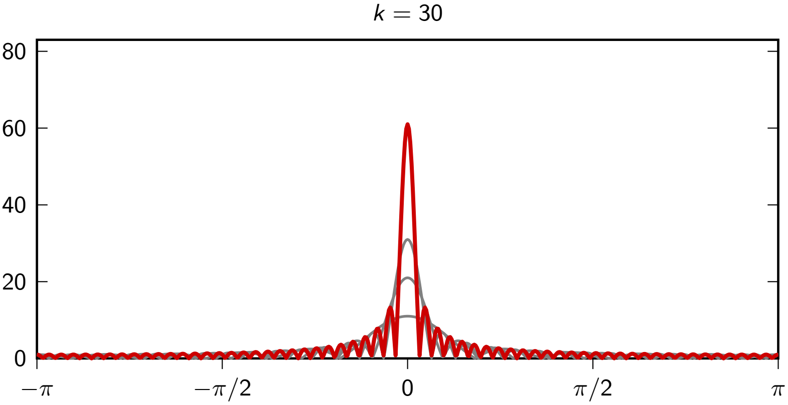
Plotting $|S_k(\omega)|$



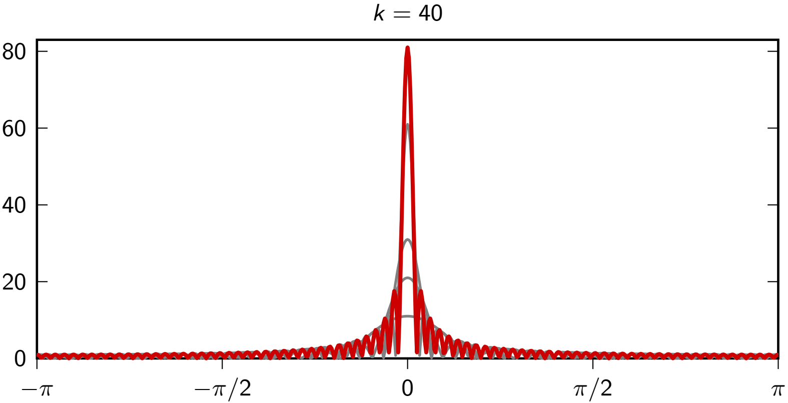
Plotting $|S_k(\omega)|$



Plotting $|S_k(\omega)|$



Plotting $|S_k(\omega)|$



Does it make sense?

Partial DTFT sums look like a family of localizing functions:

$$S_k(\omega) \to \tilde{\delta}(\omega)$$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega - \omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\left\{e^{j\omega_0 n}\right\} = \tilde{\delta}(\omega - \omega_0)$
- $\text{DTFT}\{\cos \omega_0 n\} = [\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]/2$
- $\text{DTFT}\{\sin \omega_0 n\} = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]/2$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega - \omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\{e^{j\omega_0 n}\} = \tilde{\delta}(\omega - \omega_0)$
- $\text{DTFT}\{\cos \omega_0 n\} = [\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]/2$
- $\text{DTFT}\{\sin \omega_0 n\} = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]/2$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega - \omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\left\{e^{j\omega_0 n}\right\} = \tilde{\delta}(\omega - \omega_0)$
- $\text{DTFT}\{\cos \omega_0 n\} = [\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]/2$
- $\text{DTFT}\{\sin \omega_0 n\} = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]/2$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega-\omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\left\{1\right\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\left\{e^{j\omega_0 n}\right\} = \tilde{\delta}(\omega-\omega_0)$
- $\text{DTFT}\left\{\cos\omega_0 n\right\} = [\tilde{\delta}(\omega-\omega_0)+\tilde{\delta}(\omega+\omega_0)]/2$
- $\text{DTFT}\left\{\sin\omega_0 n\right\} = -j[\tilde{\delta}(\omega-\omega_0)-\tilde{\delta}(\omega+\omega_0)]/2$

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\{x[n]\bar{w}[n]\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK
  - $h[n]$ stable filter $\Rightarrow \text{DTFT}\{x[n] * h[n]\} = X(e^{j\omega})H(e^{j\omega})$: Diracs still there

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\{x[n]\bar{w}[n]\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK
  - $h[n]$ stable filter $\Rightarrow \text{DTFT}\{x[n] * h[n]\} = X(e^{j\omega})H(e^{j\omega})$: Diracs still there

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\{x[n]\bar{w}[n]\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK
  - $h[n]$ stable filter $\Rightarrow \text{DTFT}\{x[n] * h[n]\} = X(e^{j\omega})H(e^{j\omega})$: Diracs still there

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\left\{x[n]\bar{w}[n]\right\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK
  - $h[n]$ stable filter $\Rightarrow \text{DTFT}\left\{x[n] * h[n]\right\} = X(e^{j\omega})H(e^{j\omega})$: Diracs still there

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\{x[n]\bar{w}[n]\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK
  - $h[n]$ stable filter $\Rightarrow \text{DTFT}\{x[n] * h[n]\} = X(e^{j\omega})H(e^{j\omega})$: Diracs still there

relationships between transforms

Overview:

- DFT, DFS, DTFT
- DTFT of periodic sequences
- DTFT of finite-support sequences
- Zero padding

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)$$

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)$$

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\begin{aligned}
\tilde{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)
\end{aligned}$$

We've seen this before

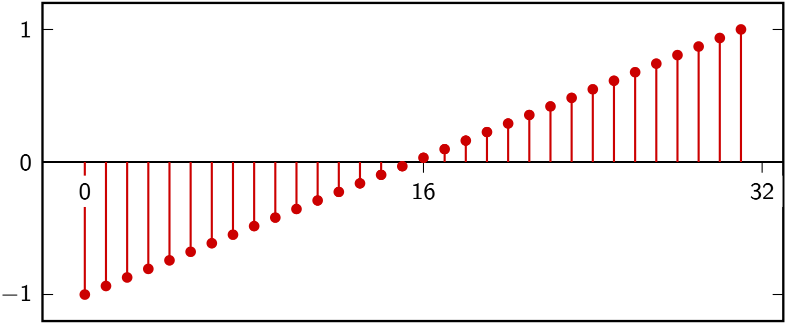
$$\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} = \text{DTFT}\left\{e^{j\frac{2\pi}{N}nk}\right\}$$

$$= \tilde{\delta}(\omega - \frac{2\pi}{N}k)$$

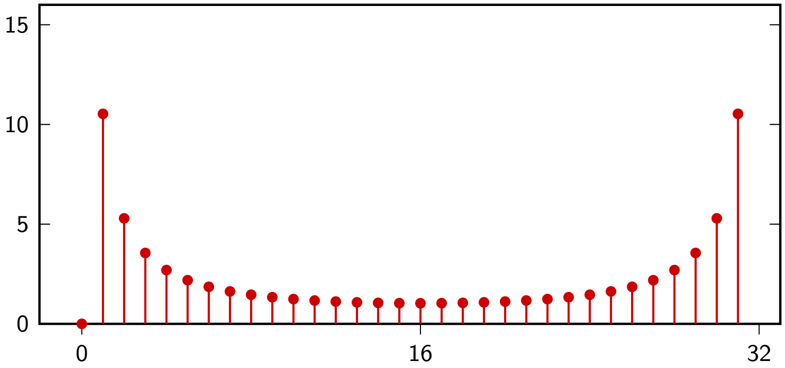
DTFT of periodic signals

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\tilde{\delta}(\omega - \frac{2\pi}{N}k)$$

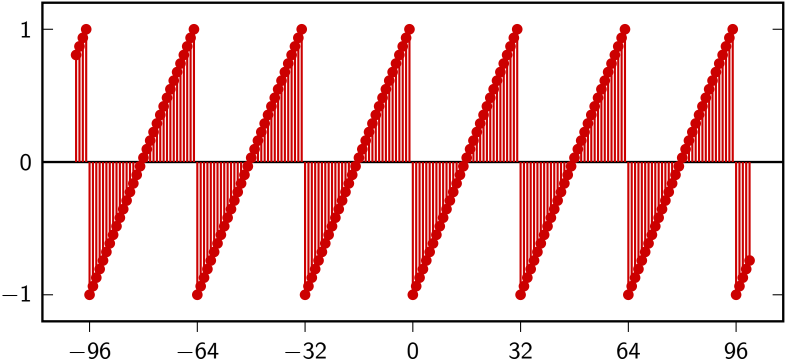
32-tap sawtooth



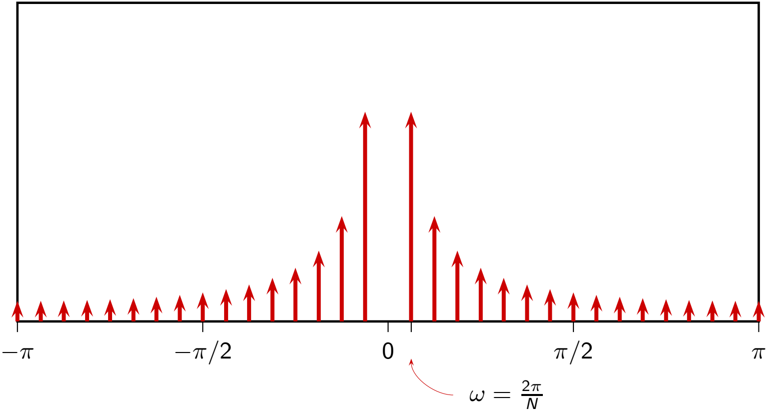
DFT of 32-tap sawtooth



32-periodic sawtooth



DTFT of periodic extension



DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)$$

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)$$

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)
\end{aligned}$$

DTFT of finite-support signals

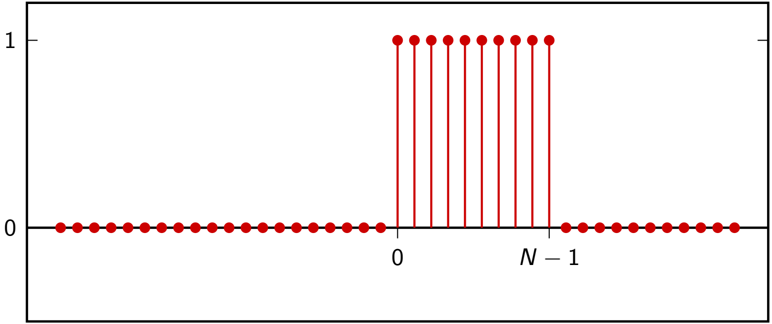
$$\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} = \bar{R}(e^{j(\omega - \frac{2\pi}{N}k)})$$

where $\bar{R}(e^{j\omega})$ is the DTFT of $\bar{r}[n]$, the rectangular signal:

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

Rectangular step signal

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$



DTFT of rectangular step signal

$$\bar{R}(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$= \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}$$

DTFT of rectangular step signal

$$
\begin{aligned}
\bar{R}(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} \\
&= \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

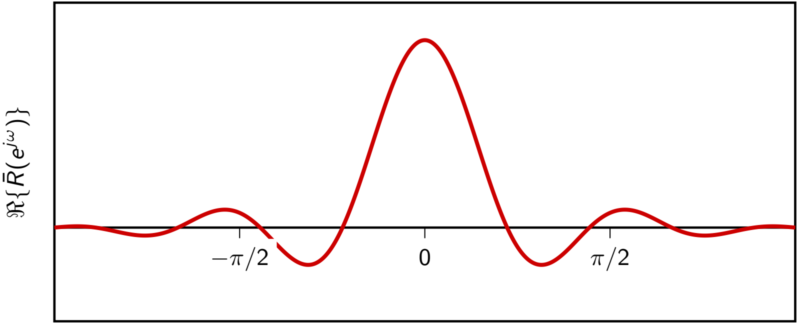
DTFT of rectangular step signal

$$
\begin{aligned}
\bar{R}(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{\omega N}{2}} \left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}} \right]}{e^{-j\frac{\omega}{2}} \left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right]} \\
&= \frac{\sin\left(\frac{\omega}{2} N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

DTFT of rectangular step signal

$$
\begin{aligned}
\bar{R}(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} \\
&= \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

DTFT of interval signal ($N = 9$)



note that $\bar{R}(e^{j\omega}) = 0$ for $\omega = 2\pi k/N$, $k \in \mathbb{Z}/\{0\}$

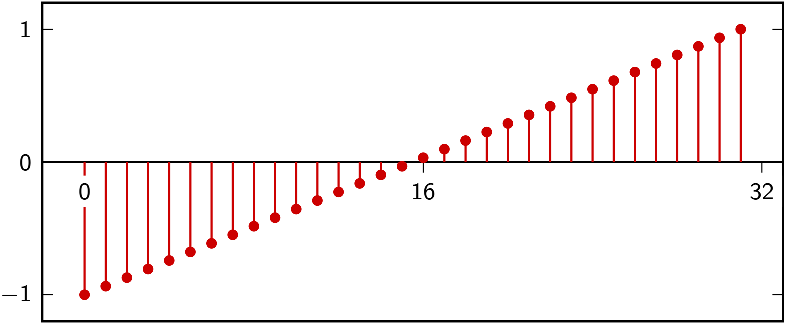
DTFT of finite-support signals

define $\Lambda(\omega) = \frac{1}{N}\bar{R}(e^{j\omega})$

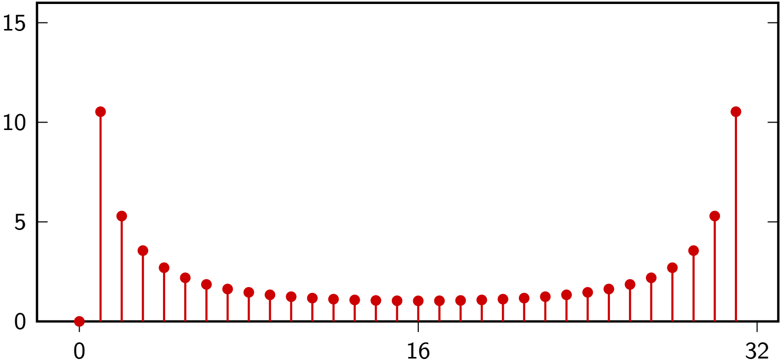
$$\bar{X}(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Lambda(\omega - \frac{2\pi}{N}k)$$

smooth interpolation of DFT values

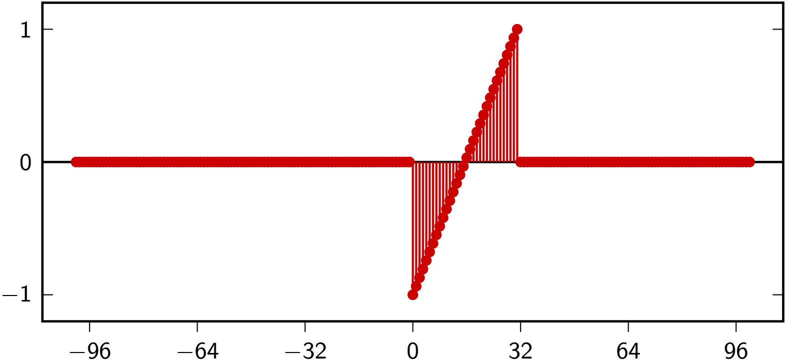
32-tap sawtooth



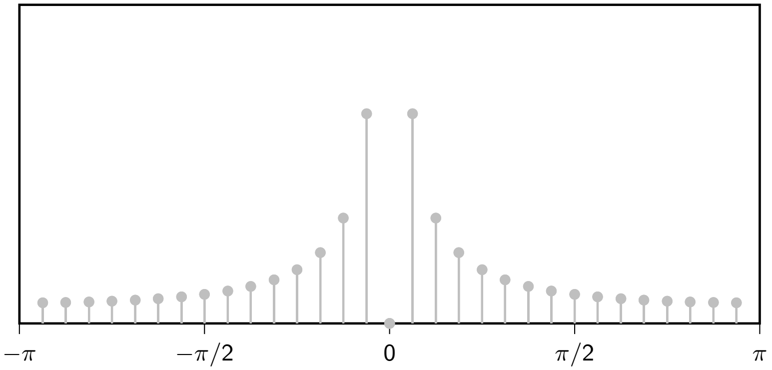
DFT of 32-tap sawtooth



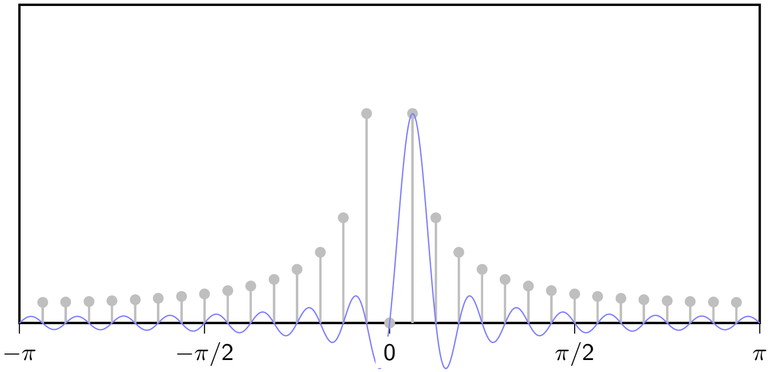
Sawtooth: finite support extension



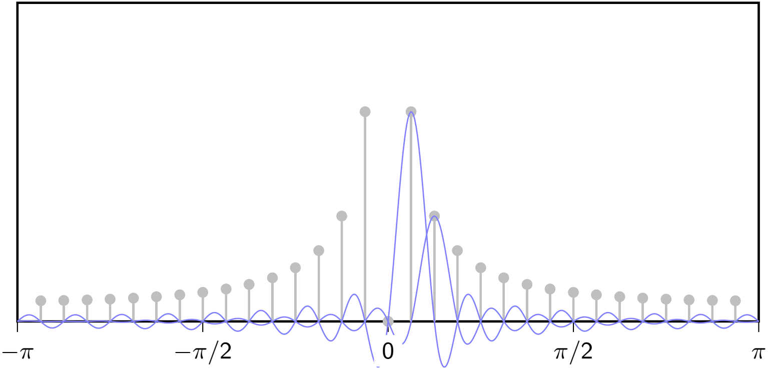
DTFT of finite support extension (sketch)



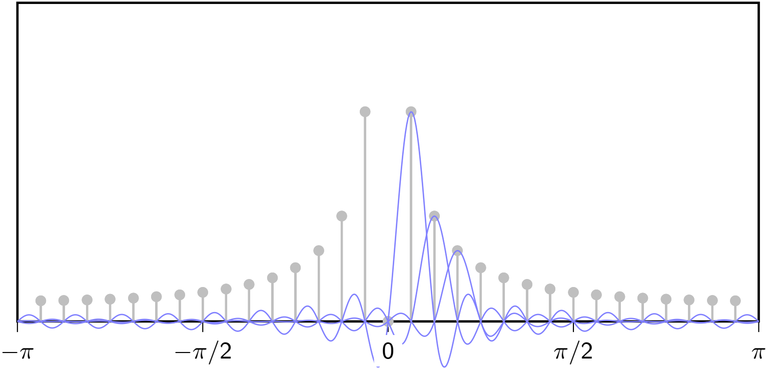
DTFT of finite support extension (sketch)



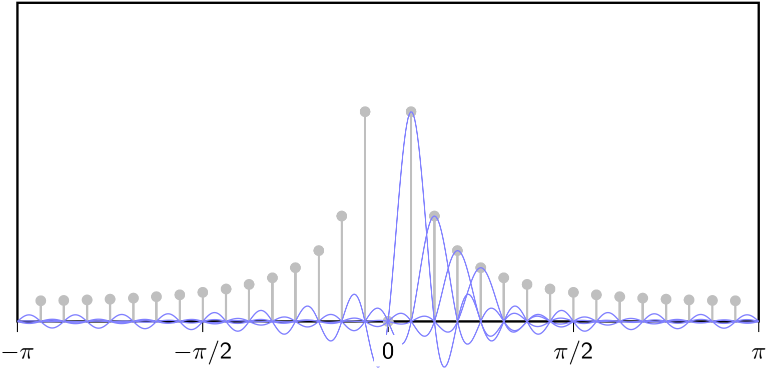
DTFT of finite support extension (sketch)



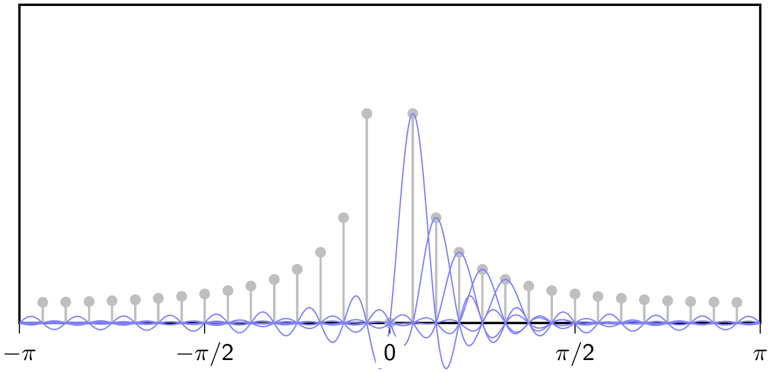
DTFT of finite support extension (sketch)



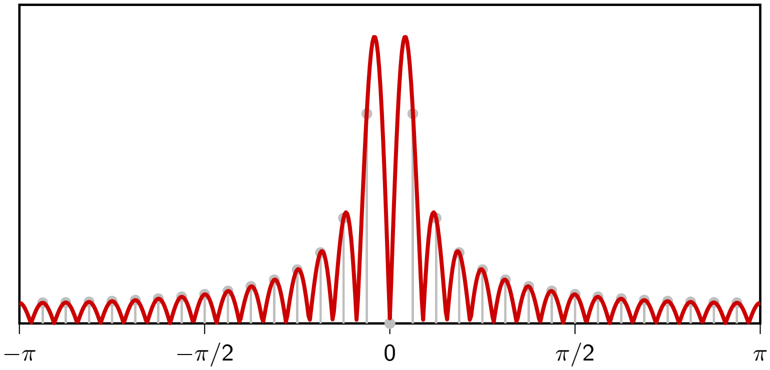
DTFT of finite support extension (sketch)



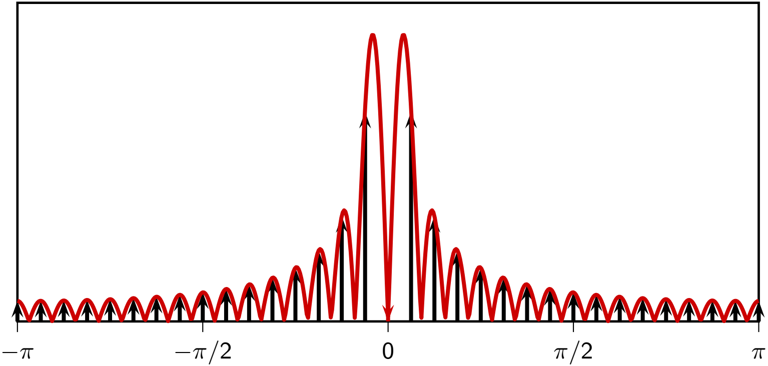
DTFT of finite support extension (sketch)



DTFT of finite support extension



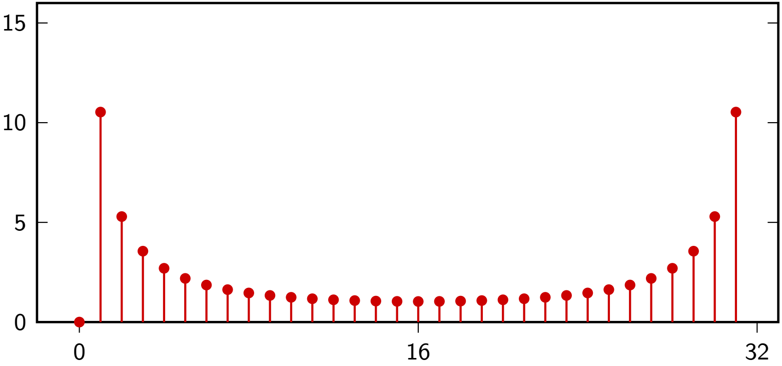
As a comparison...



About zero-padding

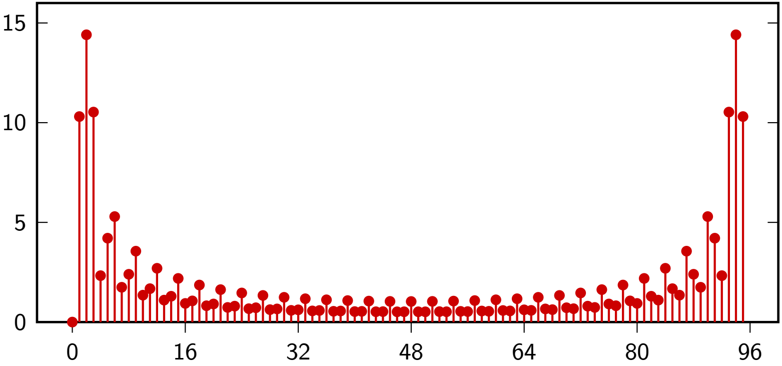
When computing the DFT numerically
one may “pad” the data vector with zeros to obtain “nicer” plots

DFT of 32-tap sawtooth



$$\mathbf{x} = [x_0\ x_1\ \dots\ x_{31}]$$

DFT of 32-tap sawtooth, zero-padded to 96 points



$$\mathbf{x} = [x_0 \; x_1 \; \ldots \; x_{31} \; 0 \; \ldots \; 0]$$

About zero-padding

$$x_M[n] = \begin{cases} x[n] & \text{for } 0 \leq n < N \\ 0 & \text{for } N \leq n < M \end{cases}$$

About zero-padding

$$X_M[h] = \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\frac{2\pi}{M}nh}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\frac{2\pi}{M}h - \frac{2\pi}{N}k)n} \right)$$

$$= \bar{X}(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}h}$$

About zero-padding

$$X_M[h] = \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\frac{2\pi}{M}nh}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\frac{2\pi}{M}h - \frac{2\pi}{N}k)n} \right)$$

$$= \bar{X}(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}h}$$

About zero-padding

$$
\begin{aligned}
X_M[h] &= \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\frac{2\pi}{M}nh} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\frac{2\pi}{M}h - \frac{2\pi}{N}k)n} \right) \\
&= \bar{X}(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}h}
\end{aligned}
$$

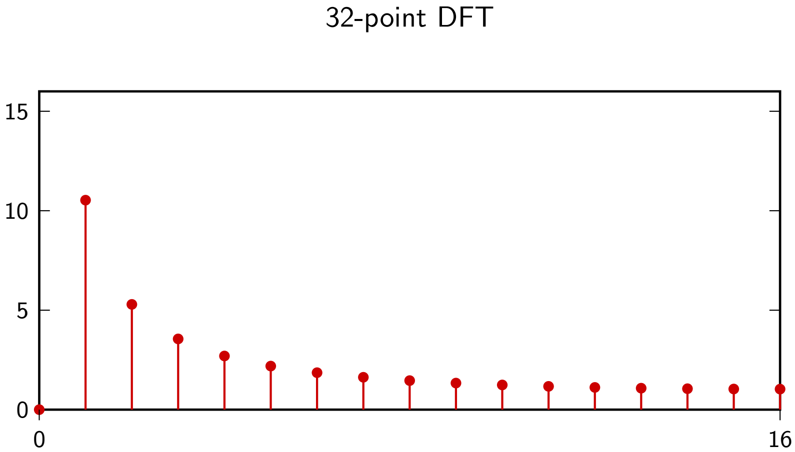
About zero-padding

$$
\begin{aligned}
X_M[h] &= \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\frac{2\pi}{M}nh} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\frac{2\pi}{M}h - \frac{2\pi}{N}k)n} \right) \\
&= \bar{X}(e^{j\omega})\big|_{\omega = \frac{2\pi}{M}h}
\end{aligned}
$$

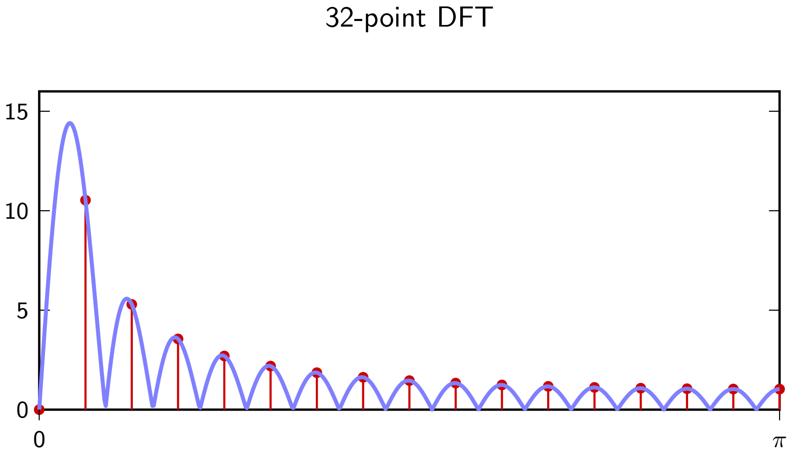
About zero-padding

- zero padding does not add information
- a zero-padded DFT is simply a sampled DTFT of the finite-support extension

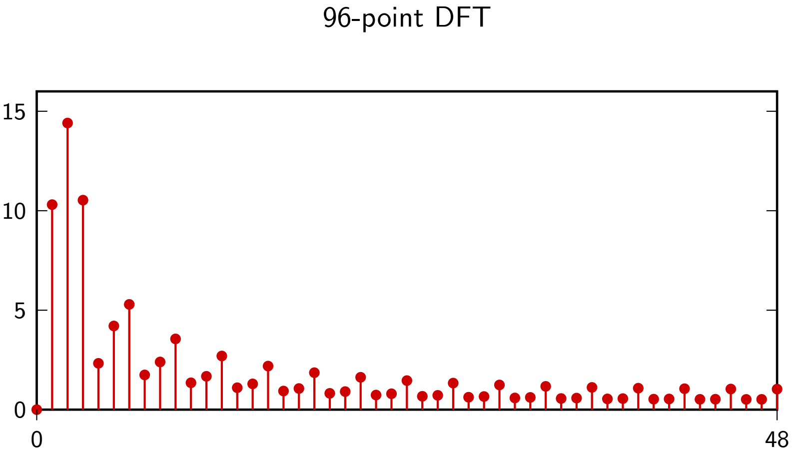
DFT of 32-tap sawtooth, zero-padded



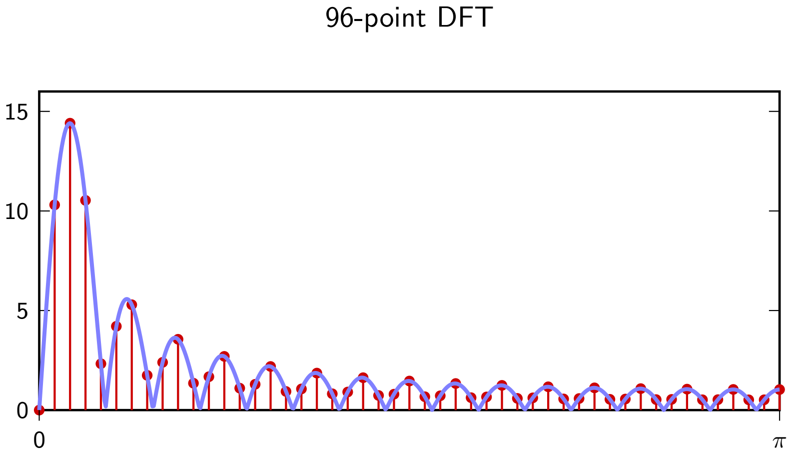
DFT of 32-tap sawtooth, zero-padded



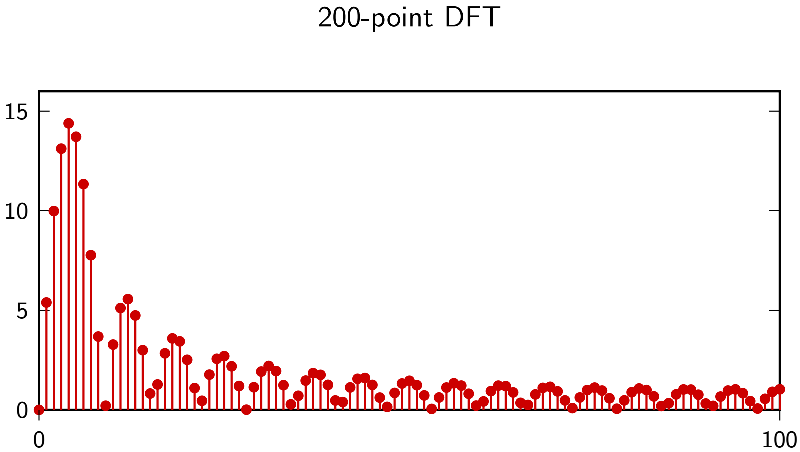
DFT of 32-tap sawtooth, zero-padded



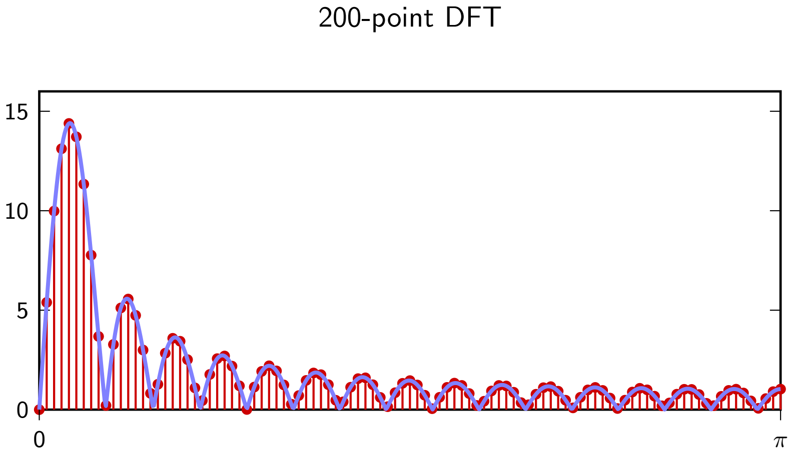
DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



modulation

Overview:

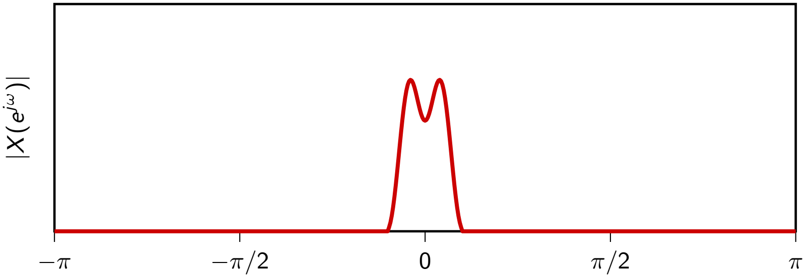
- Lowpass, highpass and bandpass signals
- Sinusoidal modulation
- Tuning a guitar

Classifying signals in frequency

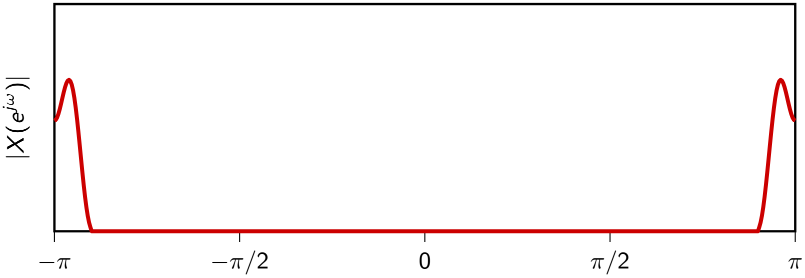
Three broad categories according to where most of the spectral energy resides:

- lowpass signals (also known as “baseband” signals)
- highpass signals
- bandpass signals

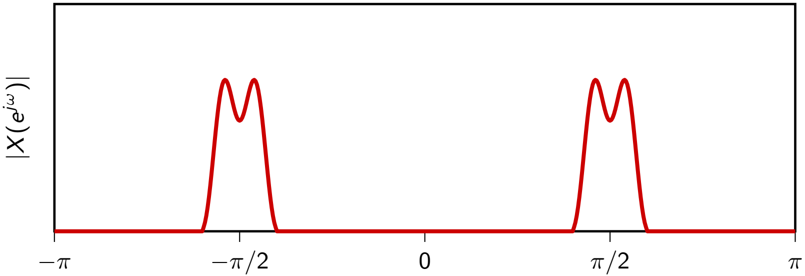
Lowpass example



Highpass example



Bandpass example



Sinusoidal modulation

$$\text{DTFT}\left\{x[n]\cos(\omega_c n)\right\} = \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\}$$

$$= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

- usually $x[n]$ baseband
- ω_c is the *carrier* frequency

Sinusoidal modulation

$$\text{DTFT}\{x[n]\cos(\omega_c n)\} = \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\}$$

$$= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

- usually $x[n]$ baseband
- ω_c is the *carrier* frequency

Sinusoidal modulation

$$
\begin{aligned}
\text{DTFT}\left\{x[n]\cos(\omega_c n)\right\} &= \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\} \\
&= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]
\end{aligned}
$$

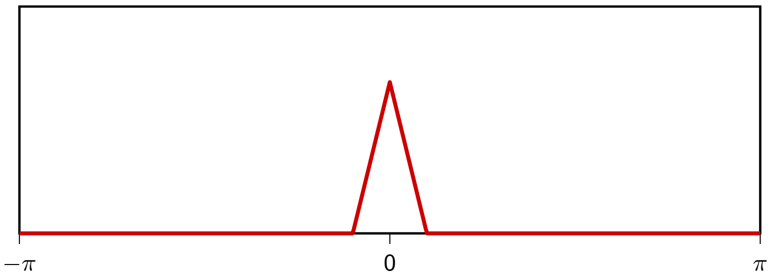
- usually $x[n]$ baseband
- ω_c is the *carrier* frequency

Sinusoidal modulation

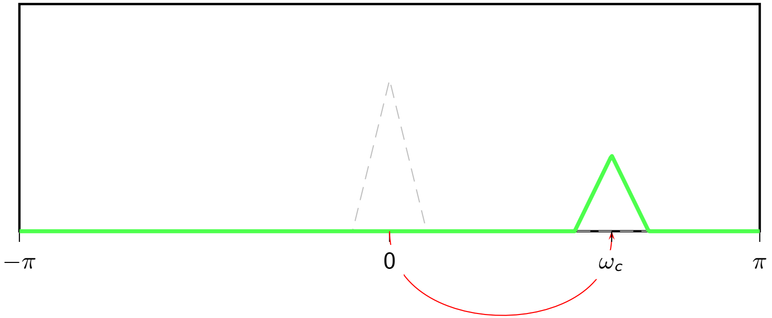
$$\begin{aligned}\text{DTFT}\,\{x[n]\cos(\omega_c n)\} &= \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\} \\ &= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]\end{aligned}$$

- usually $x[n]$ baseband
- ω_c is the *carrier* frequency

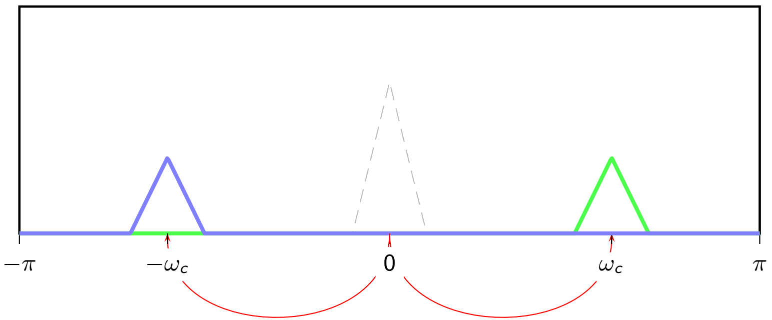
Example



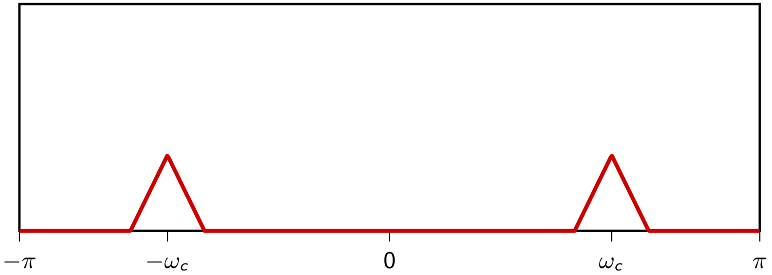
Example



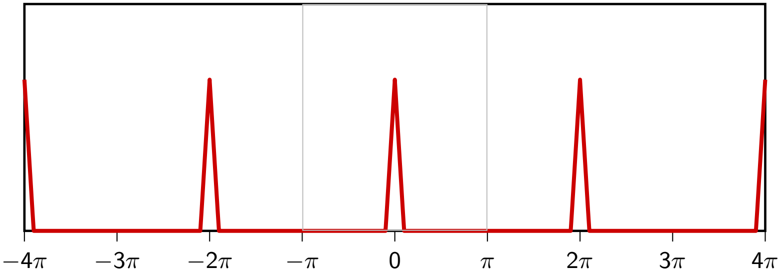
Example



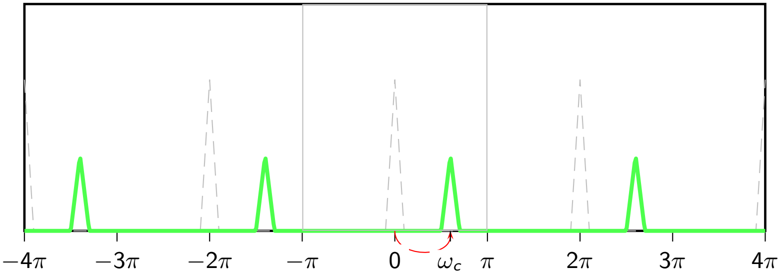
Example



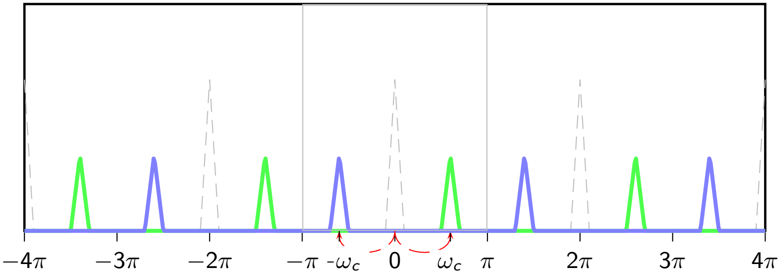
Again, explicitly showing the periodicity of the spectrum



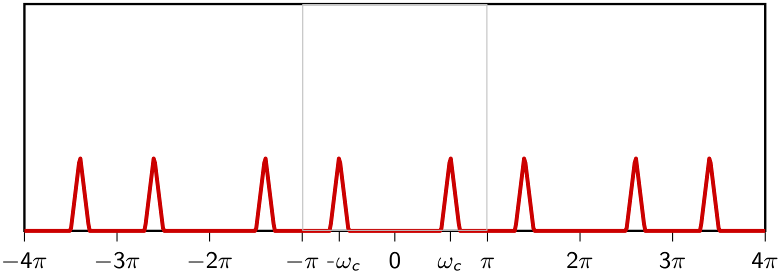
Again, explicitly showing the periodicity of the spectrum



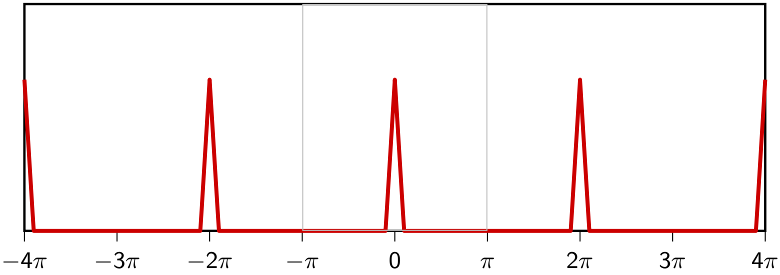
Again, explicitly showing the periodicity of the spectrum



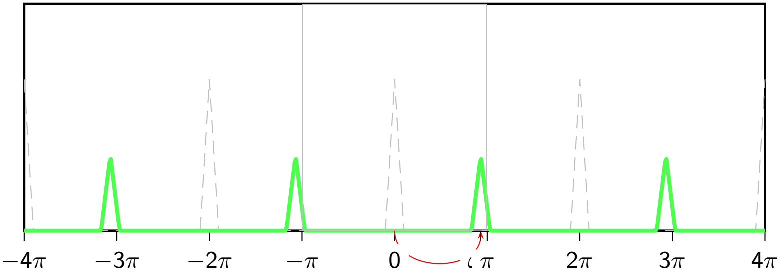
Again, explicitly showing the periodicity of the spectrum



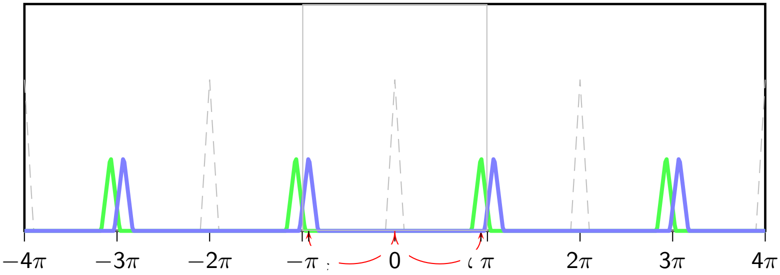
Careful when the modulation frequency is too large!



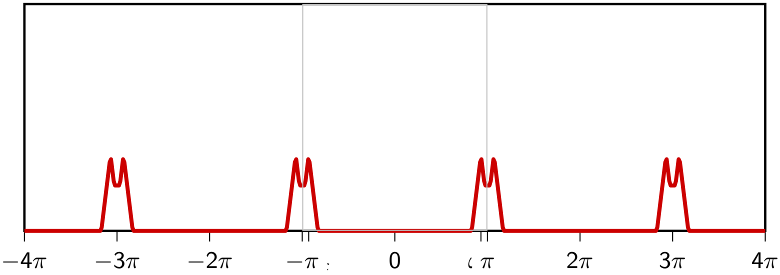
Careful when the modulation frequency is too large!



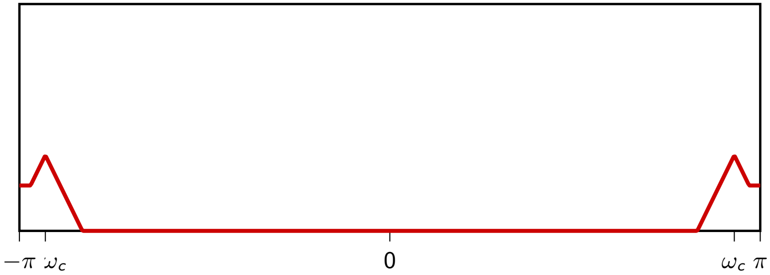
Careful when the modulation frequency is too large!



Careful when the modulation frequency is too large!



Careful when the modulation frequency is too large!



Sinusoidal modulation: applications

- voice and music are lowpass signals
- radio channels are bandpass, in much higher frequencies
- modulation brings the baseband signal in the transmission band
- demodulation at the receiver brings it back

Sinusoidal demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\text{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
&= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\
&= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]
\end{aligned}$$

Sinusoidal demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\text{DTFT}\left\{y[n] \cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
&= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\
&= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]
\end{aligned}$$

Sinusoidal demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}\text{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\ &= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\ &= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]\end{aligned}$$

Sinusoidal demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\mathrm{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
&= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\
&= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]
\end{aligned}$$

Sinusoidal demodulation

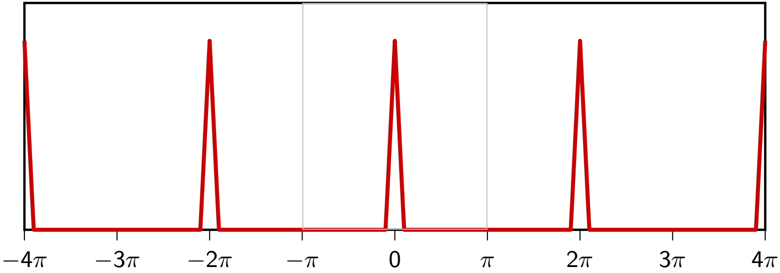
just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\text{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
&= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\
&= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]
\end{aligned}$$

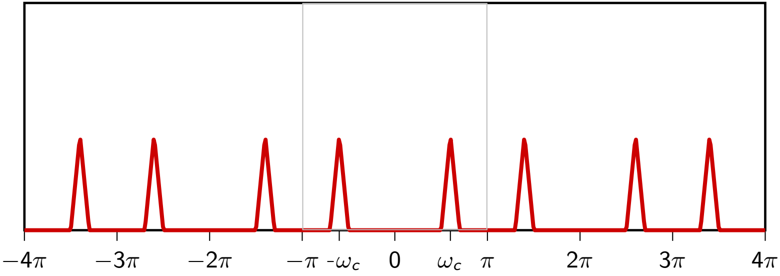
Demodulation in the frequency domain

DTFT $\{x[n]\}$



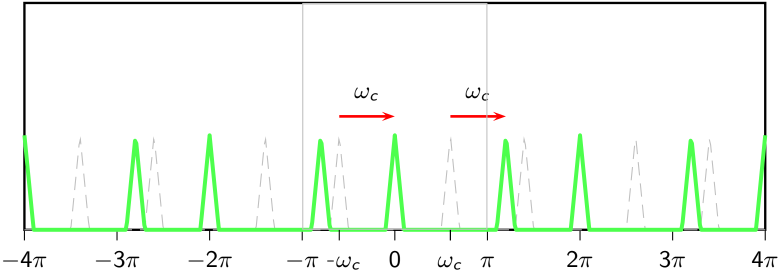
Demodulation in the frequency domain

$$\mathrm{DTFT}\{y[n]\} = \mathrm{DTFT}\{x[n]\cos\omega_c n\}$$



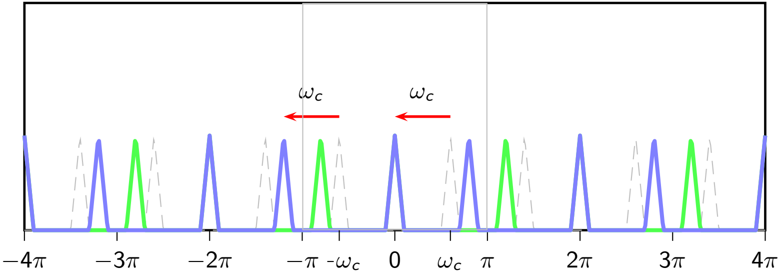
Demodulation in the frequency domain

$$\text{DTFT}\left\{y[n]\, 2\cos\omega_c n\right\}$$



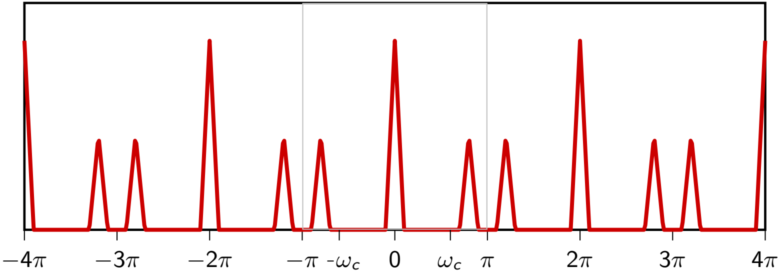
Demodulation in the frequency domain

$$\mathrm{DTFT}\left\{y[n]\,2\cos\omega_c n\right\}$$



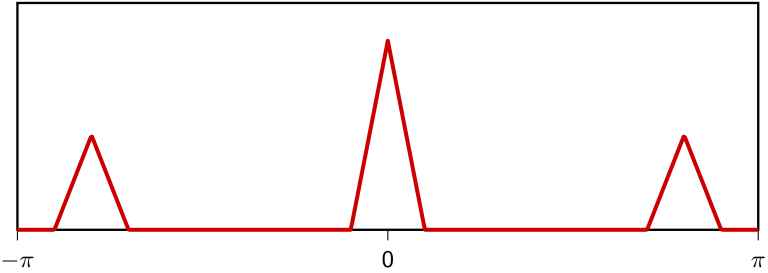
Demodulation in the frequency domain

$$\text{DTFT}\left\{y[n]\,2\cos\omega_c n\right\}$$



Demodulation in the frequency domain

$$\text{DTFT}\{y[n]\cos\omega_c n\}$$



Demodulation in the frequency domain

- we recovered the baseband signal exactly...
- but we have some spurious high-frequency components
- in the next lectures we will learn how to get rid of them!

Demodulation in the frequency domain

- we recovered the baseband signal exactly...
- but we have some spurious high-frequency components
- in the next lectures we will learn how to get rid of them!

Demodulation in the frequency domain

- we recovered the baseband signal exactly...
- but we have some spurious high-frequency components
- in the next lectures we will learn how to get rid of them!

Another application: tuning a guitar

Problem (abstraction):

- reference sinusoid at frequency ω_0
- tunable sinusoid of frequency ω
- make $\omega = \omega_0$ "by ear"

Another application: tuning a guitar

Problem (abstraction):

- reference sinusoid at frequency ω_0
- tunable sinusoid of frequency ω
- make $\omega = \omega_0$ "by ear"

Another application: tuning a guitar

Problem (abstraction):

- reference sinusoid at frequency ω_0
- tunable sinusoid of frequency ω
- make $\omega = \omega_0$ "by ear"

The procedure

1. bring ω close to ω_0 (easy)
2. when $\omega \approx \omega_0$ play both sinusoids together
3. trigonometry comes to the rescue:

$$x[n] = \cos(\omega_0 n) + \cos(\omega n)$$
$$= 2\cos\left(\frac{\omega_0 + \omega}{2} n\right)\cos\left(\frac{\omega_0 - \omega}{2} n\right)$$
$$\approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)$$

The procedure

1. bring ω close to ω_0 (easy)
2. when $\omega \approx \omega_0$ play both sinusoids together
3. trigonometry comes to the rescue:

$$\begin{aligned} x[n] &= \cos(\omega_0 n) + \cos(\omega n) \\ &= 2\cos\left(\frac{\omega_0 + \omega}{2} n\right)\cos\left(\frac{\omega_0 - \omega}{2} n\right) \\ &\approx 2\cos(\Delta_\omega n)\cos(\omega_0 n) \end{aligned}$$

The procedure

1. bring ω close to ω_0 (easy)
2. when $\omega \approx \omega_0$ play both sinusoids together
3. trigonometry comes to the rescue:

$$
\begin{aligned}
x[n] &= \cos(\omega_0 n) + \cos(\omega n) \\
&= 2\cos\left(\frac{\omega_0 + \omega}{2} n\right) \cos\left(\frac{\omega_0 - \omega}{2} n\right) \\
&\approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)
\end{aligned}
$$

Let's see what's happening

$$x[n] \approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)$$

- "error" signal
- modulation at ω_0
- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

Let's see what's happening

$$x[n] \approx \boxed{2\cos(\Delta_\omega n)} \cdot \boxed{\cos(\omega_0 n)}$$

- "error" signal
- modulation at ω_0
- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

Let's see what's happening

$$x[n] \approx \boxed{2\cos(\Delta_\omega n)} \cdot \boxed{\cos(\omega_0 n)}$$

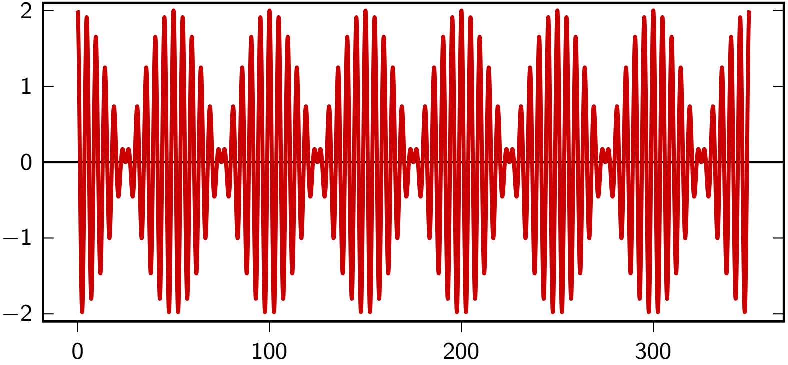
- "error" signal
- modulation at ω_0
- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

Let's see what's happening

$$x[n] \approx \boxed{2\cos(\Delta_\omega n)} \cdot \boxed{\cos(\omega_0 n)}$$

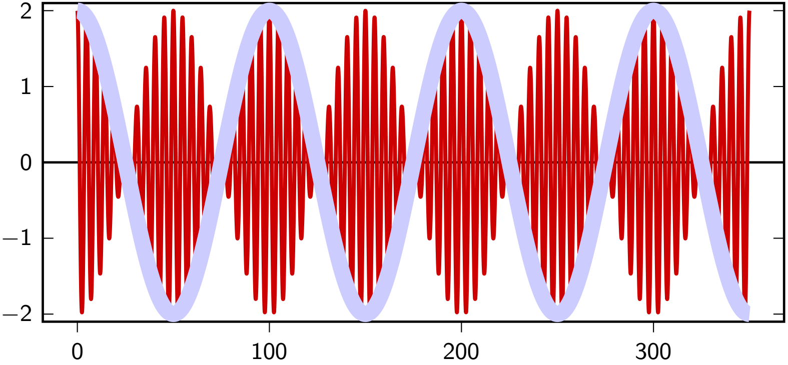
- "error" signal
- modulation at ω_0
- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

In the time domain...



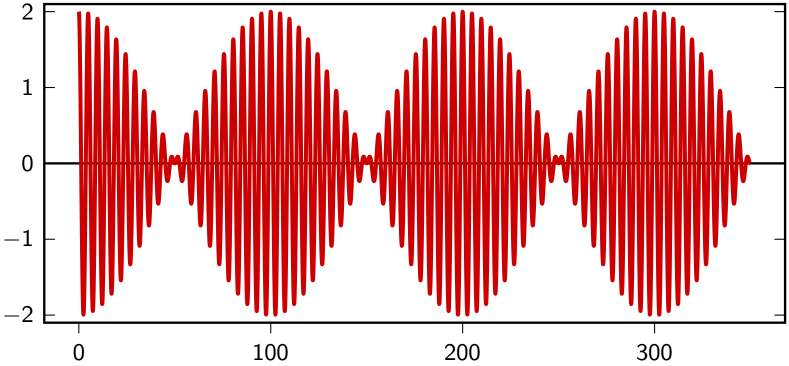
$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.22, \quad \Delta_\omega = 2\pi \cdot 0.0100$$

In the time domain...



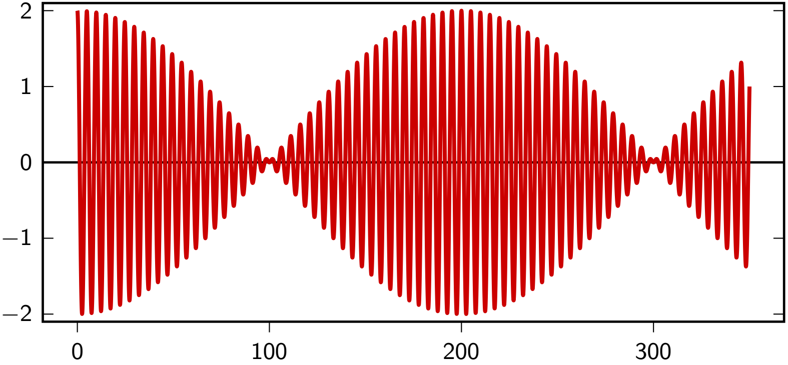
$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.22, \quad \Delta_\omega = 2\pi \cdot 0.0100$$

In the time domain...



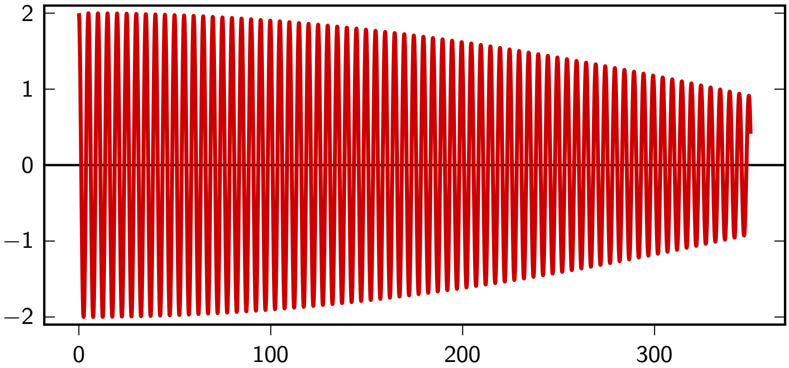
$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.21, \quad \Delta_\omega = 2\pi \cdot 0.0050$$

In the time domain...



$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.205, \quad \Delta_\omega = 2\pi \cdot 0.0025$$

In the time domain...



$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.201, \quad \Delta_\omega = 2\pi \cdot 0.0005$$

demonstration