Homework 6 (due Friday, November 2)

Exercise 1. Let P be the $N \times N$ transition matrix of a Markov chain, which we moreover assume to be *circulant*, that is: P is of the form

$$P = \begin{pmatrix} c_0 & c_1 & c_2 & & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \ddots & & c_{N-2} \\ c_{N-2} & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & & \ddots & c_{N-1} & c_0 & c_1 \\ c_1 & c_2 & & c_{N-2} & c_{N-1} & c_0 \end{pmatrix}$$

where $c_j \geq 0$ and $\sum_{j=0}^{N-1} c_j = 1$. Notation: $P = \mathrm{circ}(c_0, c_1, \ldots, c_{N-1})$.

a) Show (by a direct verification) that the eigenvalues λ_k and (unnormalized) eigenvectors $\phi^{(k)}$ of P are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \exp(2\pi ijk/N), \quad \phi_j^{(k)} = \exp(2\pi ijk/N)$$

Except for λ_0 which is equal to 1, notice that the eigenvalues λ_k are not necessarily ordered here, nor even real-valued (recall that in order for the latter to hold, we would need the further assumption that the detailed balance equation is satisfied).

b) Deduce from there the eigenvalues of the matrices seen in class:

b1) $P = \dfrac{1}{2}\,\mathrm{circ}(0, 1, 0, \ldots, 0, 1)$ (random walk on the circle, N odd)

b2) $P = \dfrac{1}{N-1}\,\mathrm{circ}(0, 1, \ldots, 1)$ (random walk on the complete graph)

c) One can possibly enlarge the spectral gap γ by adding self-loops of weight $\alpha \geq 0$ to each state ("lazy" random walk), so that the new transition matrix becomes:

$$\widetilde{P} = \alpha\, I + (1 - \alpha)\, P$$

What is the value of α providing the largest possible value of γ in each of the two cases above?

please turn the page %

Exercise 2. Let $K, M \geq 1$, $N = KM$ and let $S = \{0, \dots, K-1\} \times \{0, \dots, M-1\}$ be a *torus*. We consider a random walk on this torus with transition probabilities

$$p_{(ij),(kl)} = \begin{cases} \frac{1}{4} & \text{if } k = i \pm 1 (\text{mod } K) \text{ and } l = j \\ \frac{1}{4} & \text{if } l = j \pm 1 (\text{mod } M) \text{ and } k = i \\ 0 & \text{otherwise} \end{cases}$$

a) Compute the stationary distribution π. For what values of K and M is it also a limiting distribution?

b) By a reasoning similar to that of Exercise 1, one can show that the eigenvalues of P are given by

$$\lambda_{(km)} = \frac{1}{2}\left(\cos\left(\frac{2\pi k}{K}\right) + \cos\left(\frac{2\pi m}{M}\right)\right), \quad k = 0, \dots, K-1, \quad m = 0, \dots, M-1$$

Compute the spectral gap γ of this random walk (for K, M large).

c) In the case where $K = M$, deduce an upper bound on the mixing time

$$T_\varepsilon = \inf\{n \geq 1 : \max_{(ij) \in S} \|P^n_{(ij)} - \pi\|_{\text{TV}} \leq \varepsilon\}$$

where $\varepsilon > 0$.