Solutions of the last three exercices of Homework 1

Exercise 3. **a)** The transition matrix is given by

$$P = \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-q & 0 & q & 0 \\ 0 & p & 0 & 1-p \\ 1-q & 0 & q & 0 \end{pmatrix}.$$

Case 1. $p = q = 1$

b) There are three equivalence classes: $\{1\}$, $\{4\}$ and $\{2,3\}$.

c) The class $\{2,3\}$ is periodic of period 2.

d) The classes $\{1\}$ and $\{4\}$ are transient, the class $\{2,3\}$ is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

$$\pi = (0, 1/2, 1/2, 0).$$

Case 2. $p = 1, q = 0$

b) There are three equivalence classes: $\{3\}$, $\{4\}$ and $\{1,2\}$.

c) The class $\{1,2\}$ is periodic of period 2.

d) The classes $\{3\}$ and $\{4\}$ are transient, the class $\{1,2\}$ is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

$$\pi = (1/2, 1/2, 0, 0).$$

Case 3. $0 < p, q < 1$

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) Because the chain is irreducible, the stationary distribution is unique and given by

$$\pi = \left(\frac{1-q}{2}, \frac{p}{2}, \frac{q}{2}, \frac{1-p}{2}\right).$$

(note that this expression matches the former two cases).

f) Because the chain is periodic, the stationary distribution is not a a limiting distribution.

g) The detailed balance equations are satisfied for all values of $0 < p, q < 1$.

Exercise 4. **a)** The transition matrix is given by

$$P = \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-q & 0 & q & 0 \\ 0 & p & 0 & 1-p \\ q & 0 & 1-q & 0 \end{pmatrix}.$$

Case 1. $p = q = 1$

b) There are three equivalence classes: $\{1\}$, $\{4\}$ and $\{2,3\}$ (but note the graph is different from ex. 3, same case).

c) The class $\{2,3\}$ is periodic of period 2.

d) The classes $\{1\}$ and $\{4\}$ are transient, the class $\{2,3\}$ is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

$$\pi = (0, 1/2, 1/2, 0).$$

Case 2. $p = 1, q = 0$

b) There are three equivalence classes: $\{3\}$, $\{4\}$ and $\{1,2\}$ (but note the graph is different from ex. 3, same case).

c) The class $\{1,2\}$ is periodic of period 2.

d) The classes $\{3\}$ and $\{4\}$ are transient, the class $\{1,2\}$ is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

$$\pi = (1/2, 1/2, 0, 0).$$

Case 3. $0 < p, q < 1$

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) Because the chain is irreducible, the stationary distribution is unique and given by

$$\pi = \left(\frac{p+q-2pq}{2}, \frac{p}{2}, \frac{1-p-q+2pq}{2}, \frac{1-p}{2}\right).$$

(note that this expression matches the former two cases).

f) Because the chain is periodic, the stationary distribution is not a a limiting distribution.

g) The detailed balance equations are satisfied for all values of $0 < p < 1$, but $q = \frac{1}{2}$ only.

Exercise 5. **a)** The transition matrix is given by

$$P = \begin{pmatrix} 0 & 1-p & p & 0 \\ q & 0 & 0 & 1-q \\ 1-q & 0 & 0 & q \\ 0 & p & 1-p & 0 \end{pmatrix}.$$

Case 1. $p = q = 1$

b) The chain is irreducible.

c) The chain is periodic of period 4.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) The matrix P is doubly stochastic and the chain is irreducible. Hence, the stationary distribution is unique and it is the uniform distribution, i.e.,

$$\pi = (1/4, 1/4, 1/4, 1/4).$$

Case 2. $p = 1, q = 0$

b) There are two equivalence classes: $\{1, 3\}$ and $\{2, 4\}$.

c) Both equivalence classes are periodic of period 2.

d) Both equivalence classes are recurrent.

e) The matrix P is doubly stochastic, but the chain is not irreducible, so there are multiple stationary distributions, given by

$$\pi = (\alpha/2, \beta/2, \alpha/2, \beta/2).$$

with $0 \leq \alpha, \beta \leq 1$, $\alpha + \beta = 1$.

Case 3. $0 < p, q < 1$

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) The matrix P is doubly stochastic and the chain is irreducible. Hence, the stationary distribution is unique and it is the uniform distribution, i.e.,

$$\pi = (1/4, 1/4, 1/4, 1/4).$$

f) Because the chain is periodic, the stationary distribution is not a a limiting distribution.

g) Since the stationary distribution is the uniform distribution, the detailed balance equations are satisfied if and only if $p + q = 1$.

Answer to the final question. Of course, the matrix P itself and the equivalence classes do depend on the labelling of the states, as well as the expression of the stationary distribution(s) π. But the questions related to periodicity, recurrence, existence and uniqueness of the stationary distribution, limiting distribution and detailed balance are independent of the labelling of the states.