Solutions 8

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Exercise 1. A *simple birth process* $(X(t))_{t\geq 0}$ on $\{0, 1, 2, \dots\}$ is a generalisation of a Poisson process by introducing a correlation between the parameter λ and the actual state of the process. More precisely, if the process is in state i, it will go to state $i+1$ after an exponential random time of parameter $0 \leq \delta_i < \infty$. This process is also a Markov process (using similar arguments as those used to prove that a Poisson process is Markovian.)

(i). Find the Q-matrix corresponding to the simple birth process.

(ii). Let $X(0) = 0$ and T_i be the time when the ith jump occurs. Find an example of a simple birth process such that $\lim\limits_{i\to\infty} T_i < \infty$ a.s. This phenomena is called "the explosion". Find a general condition on the δ_i's so that the process explodes almost surely in a finite amount of time.

(iii). Do we have an explosion in the Poisson process case?

(iv). More generally, use the strong law of large numbers to show that if $\sup\limits_{i\in\mathbb{N}}\delta_i < \infty$, then $\lim_{i\to\infty} T_i = \infty$ almost surely.

Solution. (i). The transition rates matrix Q on $\{0, 1, 2, \cdots\}$ is given by

$$Q = \begin{pmatrix} -\delta_0 & \delta_0 & 0 & 0 & \cdots \\ 0 & -\delta_1 & \delta_1 & 0 & \cdots \\ 0 & 0 & -\delta_2 & \delta_2 & \ddots \\ 0 & 0 & 0 & -\delta_3 & \delta_3 \\ \vdots & \vdots & \vdots & \ddots & -\delta_4 \end{pmatrix}$$

(ii). We let $\delta_i = i^2$ for all $i \geq 0$. We can easily verify that $\lim\limits_{i\to\infty} T_i < \infty$ almost surely by computing its expectation:

$$\mathbb{E}[\lim_{i\to\infty} T_i] = \mathbb{E}\left[\sum_{i=1}^{\infty}(T_i - T_{i-1})\right] = \sum_{i=1}^{\infty}\delta_{i-1}^{-1} = \sum_{i=1}^{\infty}\frac{1}{(i-1)^2} < \infty.$$

Similarly, the process explodes almost surely if

$$\mathbb{E}[\lim_{i\to\infty} T_i] = \mathbb{E}\left[\sum_{i=1}^{\infty}(T_i - T_{i-1})\right] = \sum_{i=1}^{\infty}\delta_{i-1}^{-1} < \infty. \tag{1}$$

(iii)-(iv). Let $S_{i-1} = T_i - T_{i-1}$ the waiting time between the $(i-1)$ and the ith jump, that follows an exponential law of parameter δ_{i-1} (since we start at $X_0 = 0$). It is easy to see that $\delta_i S_i$ follows an exponential law of parameter 1. By the strong law of large numbers, we obtain:

$$\frac{\sum_{i=0}^{n-1}\delta_i S_i}{n} \xrightarrow{a.s.} 1.$$

For n sufficiently large, we thus get

$$\frac{\sum_{i=0}^{n-1}\delta_i S_i}{n} > \frac{1}{2}.$$

This is equivalent to $\sum_{i=0}^{n-1} \delta_i S_i > \frac{n}{2}$ for n sufficiently large, and this implies that $\sum_{i=0}^{n-1} \delta_i S_i \to \infty$ a.s.
If $\sup_{j\in\mathbb{N}} \delta_j < \infty$, we can write

$$\sum_{i=0}^{n-1} S_i \geq \frac{1}{\sup_j \delta_j} \sum_{i=0}^{n-1} \delta_i S_i \to \infty.$$

Thus we have that $\lim_{i\to\infty} T_i = \sum_{i=1}^{\infty}(T_i - T_{i-1}) = \sum_{i=0}^{\infty} S_i = \infty$ almost surely and so the process does not explode in this case.
In particular, the Poisson process does not explode since all the δ_i's are equal in this case (to some $\lambda < \infty$) and so we have $\sup_{j\in\mathbb{N}} \delta_j < \infty$.
We can even obtain a stronger result that shows that if the sum in (1) is infinite, then the process does not explode. (see theorem 2.3.2 in Norris)

Exercise 2. A radioactive source emits particles according to a Poisson process of rate λ. The particles are spread in random directions independently from each other. A Geiger counter placed next to the source measures a fraction p of the emitted particles. What is the distribution of the number of particles detected by time t?

Solution. The number of emitted particles up to time t follows has a law $Poisson\,(\lambda t)$. For each emitted particle, the probability that it is detected by the counter is p. Knowing that the number of emitted particles until time t is equal to n,the number of detected particles has a law $Binomial\,(n, p)$. We use this to obtain

$$\begin{aligned}
\mathbb{P}\left(\#\text{ detected particles} = r\right) &= \sum_{n=r}^{\infty} \mathbb{P}\left(\#\text{of particles} = n\right) \binom{n}{r} p^r \left(1-p\right)^{n-r} = \\
&= \sum_{n=r}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{n!}{r!\,(n-r)!} p^r \left(1-p\right)^{n-r} = \\
&= \frac{e^{-\lambda t} p^r}{r!} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-r)!} \left(1-p\right)^{n-r} = \\
&= \frac{e^{-\lambda t} p^r \left(\lambda t\right)^r}{r!} \sum_{n=r}^{\infty} \frac{\lambda t\left(1-p\right)^{n-r}}{(n-r)!} = \\
&= \frac{e^{-\lambda t} \left(\lambda t p\right)^r}{r!} \sum_{n=0}^{\infty} \frac{\left(\lambda t\left(1-p\right)\right)^n}{n!} = \\
&= \frac{e^{-\lambda t} \left(\lambda t p\right)^r}{r!} e^{\lambda t (1-p)} = \\
&= e^{-\lambda t p} \frac{\left(p\lambda t\right)^r}{r!}.
\end{aligned}$$

Exercise 3. The arrival times of the bus number 1 are modeled by a Poisson process with an average frequency of one bus per hour, whereas the arrival times of the bus number 7 are modeled by a Poisson process independent of the first one with frequency of 7 buses per hour.

(1) What is the probability that we see exactly 3 buses (no matter which buses) in one hour?

(2) What is the probability that we see exactly 3 buses 7 while we are waiting for the bus 1?

Solution. (1) By a result seen in class, we know that that buses are arriving according to a Poisson process of rate $\lambda = 1 + 7 = 8$. By exercise 1, The number of buses passing in one hour follows a Poisson distribution of mean 8. Then the probability that we see exactly 3 buses is given by $e^{-8}8^3/3!$.

(2) Let $E_i \sim Exp(1)$ and $E'_i \sim Exp(7)$ be the waiting times between two consecutive arrivals of buses 1 and 7 respectively. The probability that the first bus is 7 is given by

$$\mathbb{P}(E'_1 < E_1) = \int_0^\infty f_{E'_1}(s)\,\mathbb{P}(E_1 > s)ds = \int_0^\infty 7e^{-7s}e^{-s}ds = \frac{7}{8}\int_0^\infty 8e^{-8s}ds = \frac{7}{8}.$$

Using memorylessness and independence of E_1, E'_1 and E'_2, the probability that the second bus is also the 7 is given by

$$\begin{aligned}\mathbb{P}(E_1 > E'_1 + E'_2 \mid E_1 > E'_1) &= \frac{\int_0^\infty \int_0^\infty \mathbb{P}(E_1 > s+t) f_{E'_1}(s) f_{E'_2}(t)dsdt}{\int_0^\infty \mathbb{P}(E_1 > s) f_{E'_1}(s)ds} \\ &= \frac{\int_0^\infty \int_0^\infty e^{-(s+t)} f_{E'_1}(s) f_{E'_2}(t)dsdt}{\int_0^\infty e^{-s} f_{E'_1}(s)ds} \\ &= \frac{\int_0^\infty e^{-s} f_{E'_1}(s)ds \int_0^\infty e^{-t} f_{E'_2}(t)dt}{\int_0^\infty e^{-s} f_{E'_1}(s)ds} \\ &= \int_0^\infty e^{-t} f_{E'_2}(t)dt = \mathbb{P}(E_1 > E'_2) = \frac{7}{8}.\end{aligned}$$

Following the same reasoning, The probability that the third bus is again the 7 is $\frac{7}{8}$. Letting A_i be the event "the i^{th} bus is the 7", we finally get

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4^c) = \mathbb{P}(A_1)\cdot\mathbb{P}(A_2|A_1)\cdot\mathbb{P}(A_3|A_1\cap A_2)\cdot(1-\mathbb{P}(A_4|A_1\cap A_2\cap A_3)) = \left(\frac{7}{8}\right)^3 \cdot \frac{1}{8}.$$

Alternative method: Let $S_n := \sum_{i=1}^n E_i$ and $S'_m := \sum_{j=1}^m E'_j$ where $E_i \sim Exp(1)$ and $E'_j \sim Exp(7)$. The probability of seeing exactly 3 buses 7 before the arrival of bus 1 is given by

$$\mathbb{P}(S'_3 \le S_1 \le S'_4).$$

Using that S'_3 has an Erlang distribution (see exercise 4 in Serie 7) with parameters $n = 3$ and $\lambda = 7$, we get

$$\begin{aligned}\mathbb{P}(S'_3 \le S_1 \le S'_4) &= \int_0^\infty \mathbb{P}(s \le S_1 \le s + E'_4)\frac{7^3}{2}t^2e^{-7s}ds \\ &= \int_0^\infty \left(\int_0^\infty \mathbb{P}(s \le E_1 \le s+t)\ 7e^{7t}dt\right)\frac{7^3}{2}t^2e^{-7s}ds \\ &= \int_0^\infty \frac{e^{-s}}{8} \times \frac{7^3}{2}t^2e^{-7s}ds \\ &= \int_0^\infty \frac{7^3}{16} \times t^2e^{-8s}ds = \frac{7^3}{8^4}.\end{aligned}$$

Exercise 4. Hockey teams 1 and 2 score goals at times of Poisson processes with rates 1 and 2. Suppose that $N_1(0) = 3$ and $N_2(0) = 1$.

a) What is the probability that $N_1(t)$ will reach 5 before $N_2(t)$ does?

b) Answer part a) for Poisson processes with rates λ_1 and λ_2.

Solution. a) The probabiliy that team 2 (T_2) scores the first goal is $\frac{2}{3}$ (using the exercise 2 of serie 1). Therefore, using the memorlyness property of the exponenial distribution, the probability that at least 4 of the 5 next goals are scored by Team 2 is

$$\begin{aligned}\mathbb{P}(\text{Team 2 wins}) &= \mathbb{P}(T_2 \text{ scores at least 4 of the next 5 goals})\\ &= \binom{5}{4}\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} + \binom{5}{5}\left(\frac{2}{3}\right)^5.\end{aligned}$$

Hence, the probability the Team 1 (T_1) wins is

$$\mathbb{P}(T_1 \text{ wins}) = 1 - \binom{5}{4}\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} + \binom{5}{5}\left(\frac{2}{3}\right)^5 = \frac{131}{243}.$$

Another way of computing this probability is by noticing that the waiting time for T_1 to score 2 goals has the Erlang (or Gamma) distribution with parameters 2 and 1. Likewise, the wating time for team T_2 to score 4 goals is distributed as $\Gamma(4, 2)$. Therefore, the probability that we are looking for is given by

$$\mathbb{P}\left(\Gamma(2,1) \leq \Gamma(4,2)\right) = \int_0^\infty te^{-t}\left(\int_t^\infty \frac{2e^{-2s}(2s)^3}{6} ds\right) \, dt.$$

After some computations, we gett

$$\mathbb{P}(T_1 \text{ wins}) = \mathbb{P}\left(\Gamma(2,1) \leq \Gamma(4,2)\right) = \int_0^\infty e^{-3t}\left(\frac{4}{3}t^4 + 2t^3 + 2t^2 + 3t\right) dt = \frac{131}{243}.$$

b) Similarly to part a), the probability that T_1 wins is

$$\mathbb{P}(T_1 \text{ wins}) = 1 - 5\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^4 \frac{\lambda_1}{\lambda_1 + \lambda_2} - \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^5.$$