COM-303 - Signal Processing for Communications

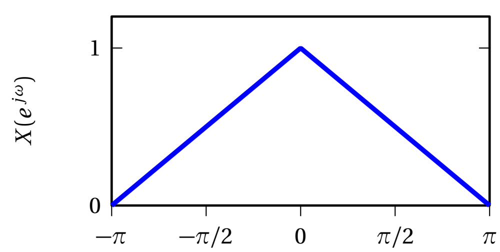
Solutions for Homework #9

Solution 1. Zero Order Hold

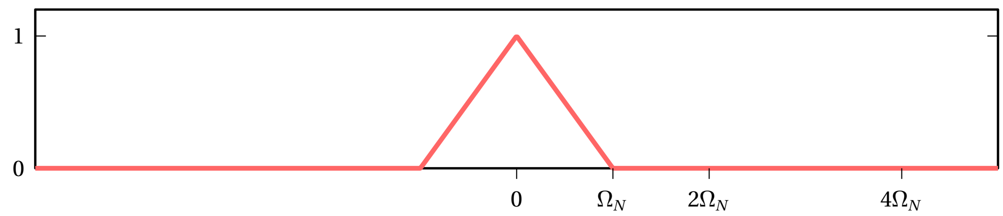
(a) We have

$$
\begin{aligned}
X_0(j\Omega) &= \int_{-\infty}^{\infty} x_0(t)e^{-j\Omega t}\,dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}(t-n)e^{-j\Omega t}\,dt \\
&= \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty} \mathrm{rect}(t-n)e^{-j\Omega t}\,dt \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\int_{-1/2}^{1/2} e^{-j\Omega\tau}\,d\tau \\
&= \frac{\sin(\Omega/2)}{\Omega/2}X(e^{j\Omega}) \\
&= \mathrm{sinc}(\Omega/2\pi)X(e^{j\Omega}).
\end{aligned}
$$

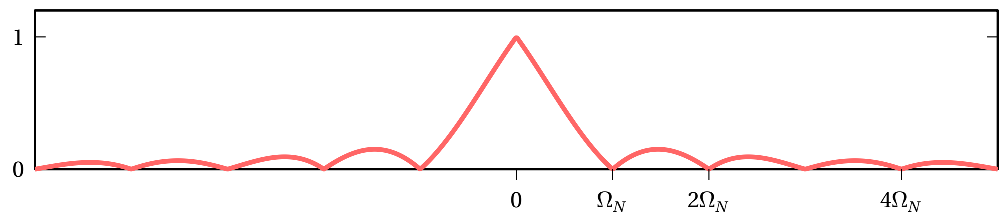
(b) To illustrate the effects of the zero-order hold interpolator consider for instance a discrete-time signal with a triangular spectrum likse so:



Sinc interpolation will produce us a continuous-time signal which is strictly bandlimited to the $[-\Omega_N, \Omega_N]$ interval (with $\Omega_N = \pi/T_s = \pi$) and whose shape is exactly triangular:



Conversely, the spectrum of the continuous-time signal interpolated by the zero-order hold is:



There are two main problems in the zero-order hold interpolation as compared to the sinc interpolation:

- The zero-order hold interpolation is NOT bandlimited: the 2π-periodic replicas of the digital spectrum leak through in the continuous-time signal as high frequency components. This is due to the sidelobes of the interpolation function in the frequency domain (rect in time $\longleftrightarrow$ sinc in frequency) and it represents an undesirable high-frequency content which is typical of all local interpolation schemes.

- There is a distortion in the main portion of the spectrum (that between $-\Omega_N$ and Ω_N, with $\Omega_N = \pi$) due to the non-flat frequency response of the interpolation function. This is evident in the example by the non-straight sides in the "triangular" basedband portion of the spectrum.

(c) Observe that $X(j\Omega)$ can be expressed as

$$X(j\Omega) = \begin{cases} X(e^{j\Omega}) & \text{if } \Omega \in [-\pi, \pi] \\ 0 & \text{otherwise,} \end{cases}$$

where $X(e^{j\Omega})$ is the DTFT of the sequence $x[n]$ evaluated at $\omega = \Omega$. So

$$X(j\Omega) = X(e^{j\Omega})\mathrm{rect}\left(\frac{\Omega}{2\pi}\right) = X_0(j\Omega)\mathrm{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{2\pi}\right).$$

Hence

$$G(j\Omega) = \mathrm{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{2\pi}\right).$$

(d) A first solution is to compensate for the distortion introduced by $G(j\Omega)$ in the discrete-time domain. This is equivalent to pre-filtering $x[n]$ with a discrete-time filter of magnitude $1/G(e^{j\Omega})$ which can even be designed with the Parks-McClellan optimization technique (see Oppenheim-Schafer, Section 7.7.2). The advantages of this method is that digital filters such as this one are very easy to design and that the filtering can be done in the discrete-time domain. The disadvantage is that this approach does not eliminate or attenuate the high frequency leakage outside of the baseband.

Alternatively, one can cascade the interpolator with an analog lowpass filter to eliminate the leakage. The disadvantage is that it is hard to design an analog lowpass which can also compensate for the in-band distortion introduced by $G(j\Omega)$; such a filter will also introduce unavoidable phase distortion (no analog filter has linear phase).

Solution 2. Another View of Sampling

We have that

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

and, by using the modulation theorem,

$$\begin{aligned}
X_s(j\Omega) &= X(j\Omega) * P(j\Omega) \\
&= \int_{\mathbb{R}} X(j\tilde{\Omega}) P(j(\Omega - \tilde{\Omega})) d\tilde{\Omega} = \frac{2\pi}{T_s} \int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k\in\mathbb{Z}} \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} \\
&= \frac{2\pi}{T_s} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} X(j\tilde{\Omega}) \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} = \frac{2\pi}{T_s} \sum_{k\in\mathbb{Z}} X\left(j\left(\Omega - k\frac{2\pi}{T_s}\right)\right).
\end{aligned}$$

In other words, the spectrum of the delta-modulated signal is just the periodic repetition (with period $(2\pi/T_s)$ of the original spectrum. If the latter is bandlimited to (π/T_s) there will be no overlap and therefore $x(t)$ can be obtained simply by lowpass filtering $x_s(t)$ (in the continuous-time domain).

Solution 3. Aliasing Can Be Good!

(a) According to our definition of bandlimited functions, the highest nonzero frequency is $2\Omega_0$ and therefore $x_c(t)$ is $2\Omega_0$-bandlimited for a total bandwidth of $4\Omega_0$. The maximum sampling period (i.e. the inverse of the *minimum* sampling frequency) which satisfies the sampling theorem is therefore $T_s = \pi/(2\Omega_0)$. Note however that the total support over which the (positive) spectrum is nonzero is the interval $[\Omega_0, 2\Omega_0]$ so that one could say that the total *effective* positive bandwidth of the signal is just Ω_0; this will be useful later.

(b) The digital spectrum will be the rescaled version of the periodized continuous-time spectrum

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_0)).$$

The general term $X_c(j\Omega - j2k\Omega_0)$ is nonzero only for

$$\Omega_0 \le |\Omega - 2k\Omega_0| \le 2\Omega_0 \quad \text{for } k \in \mathbb{Z}.$$

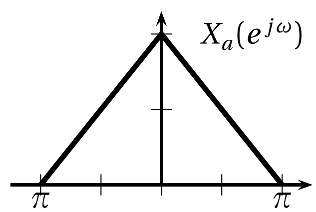
This translates to

$$\begin{aligned}(2k+1)\Omega_0 &\le \Omega \le (2k+2)\Omega_0\\ (2k-2)\Omega_0 &\le \Omega \le (2k-1)\Omega_0\end{aligned}$$

which are non-overlapping intervals! Therefore, there will be no disruptive superpositions of the copies of the spectrum. The digital spectrum will be simply

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$$

and it will look like this (with 2π-periodicity, of course):



(c) Here's a possible scheme (verify that it works):

- Sinc-interpolate $x_a[n]$ with period T_s to obtain $x_b(t)$
- Multiply $x_b(t)$ by $\cos(2\Omega_0 t)$ in the continuous time domain to obtain $x_p(t)$ (i.e. modulate by a carrier at frequency (Ω_0/π) Hz).
- Bandpass filter $x_p(t)$ with an ideal bandpass filter with (positive) passband equal to $[\Omega_0, 2\Omega_0]$ to obtain $x_c(t)$.

(d) The effective *positive* bandwidth of such a signal is $\Omega_\Delta = (\Omega_1 - \Omega_0)$. Clearly, the sampling frequency must be at least equal to the effective total bandwidth so we have a first condition on the maximum allowable sampling period: $T_{\max} < \pi/\Omega_\Delta$.

Now, to make things simpler, assume that the upper frequency Ω_1 is a multiple of the bandwidth, i.e. $\Omega_1 = M\Omega_\Delta$ for some integer M (in the previous case, it was $M = 2$). In this case, the argument we made in the previous point can be easily generalized: if we pick $T_s = \pi/\Omega_\Delta$ and sample we have that

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_\Delta)).$$

The general term $X_c(j\Omega - j2k\Omega_\Delta)$ is nonzero only for

$$\Omega_0 \le |\Omega - 2k\Omega_\Delta| \le \Omega_1 \quad \text{for } k \in \mathbb{Z}.$$

Since $\Omega_0 = \Omega_1 - \Omega_\Delta = (M-1)\Omega_\Delta$, this translates to

$$\begin{aligned}(2k+M-1)\Omega_\Delta &\le \Omega \le (2k+M)\Omega_\Delta\\ (2k-M)\Omega_\Delta &\le \Omega \le (2k-M+1)\Omega_\Delta\end{aligned}$$

which are again non-overlapping intervals.

If Ω_1 is *not* a multiple of the bandwidth, then the easiest thing to do is to change the lower frequency Ω_0 to a new frequency Ω_0' so that the new bandwidth $\Omega_1 - \Omega_0'$ divides Ω_1 exactly. In other words we set a new lower frequency Ω_0' so that it will be $\Omega_1 = M(\Omega_1 - \Omega_0')$ for some integer M; it is easy to see that

$$M = \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor.$$

since this is the maximum number of copies of the Ω_Δ-wide spectrum which fit *with no overlap* in the $[0, \Omega_0]$ interval. Note also that, if $\Omega_\Delta > \Omega_0$ we cannot hope to reduce the sampling frequency and we have to use normal sampling. This artificial change of frequency will leave a small empty "gap" in the new bandwidth $[\Omega_0', \Omega_1]$, but that's no problem. Now we can use the previous result and sample with $T_s = \pi/(\Omega_1 - \Omega_0')$ with no overlap. Since $(\Omega_1 - \Omega_0') = \Omega_1/M$, we have that, in conclusion, the maximum sampling period is

$$T_{\max} = \frac{\pi}{\Omega_1} \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor$$

i.e. we can obtain a sampling frequency reduction factor of $\lfloor \Omega_1/(\Omega_1 - \Omega_0) \rfloor$.

Solution 4. Sampling and Aliasing

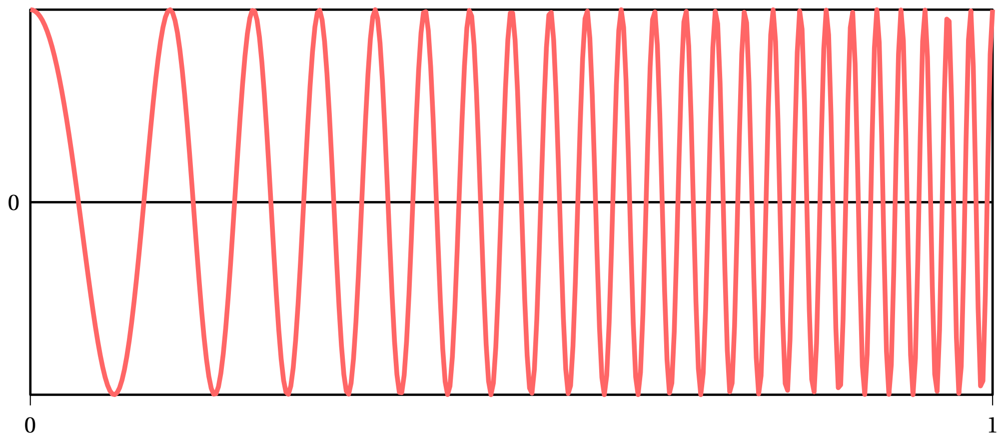
(a) The instantaneous angular frequency of the chirp at time t is given by $\omega(t) = \varphi'(t) = 2\pi(at + b)$ (or, in Hertz, the instantaneous frequency at time t is given by $f(t) = at + b$). We can see that the frequency of the chirp is linearly increasing with time. Therefore, independently of the chosen sampling frequency, aliasing will occur as soon as the frequency of the chirp increases beyond half the sampling frequency.

(b) Here is a simple Python implementation

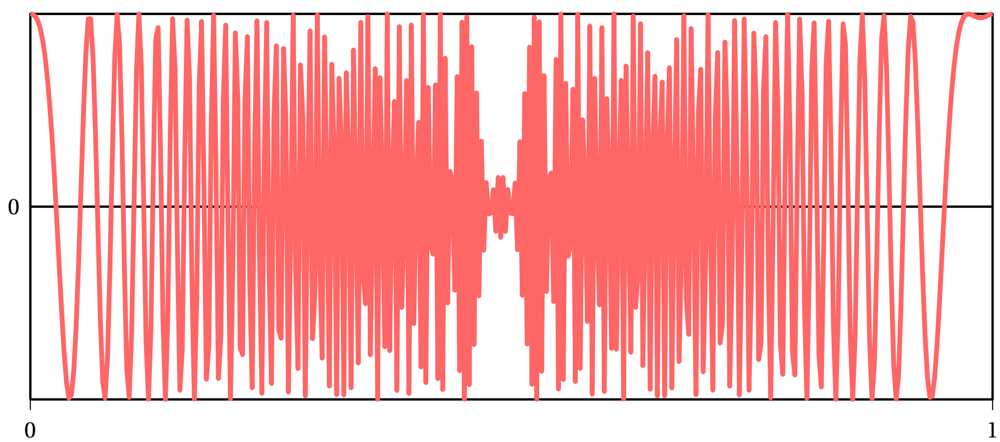
```
import numpy as np

def chirp(a, b, sf, len=1):
    t = np.arange(0, len, 1.0/sf)
    return cos(np.pi * a * (t * t) + 2 * np.pi * b * t)
```

(c) We can see from this plot that when $a = 40$, a sampling frequency of 400 [samples/second] is enough to prevent aliasing:



When $a = 400$, the sampling frequency is too small and aliasing occurs:



(d) Since the Nyquist criterion must be satisfied until $t = 1$ second, the minimum sampling frequency is twice the frequency at $t = 1$ second. Thus $f_s = 2(a + b) = 88$ [samples/second].

(e) We can play the chirp for a duration of 10 seconds by executing

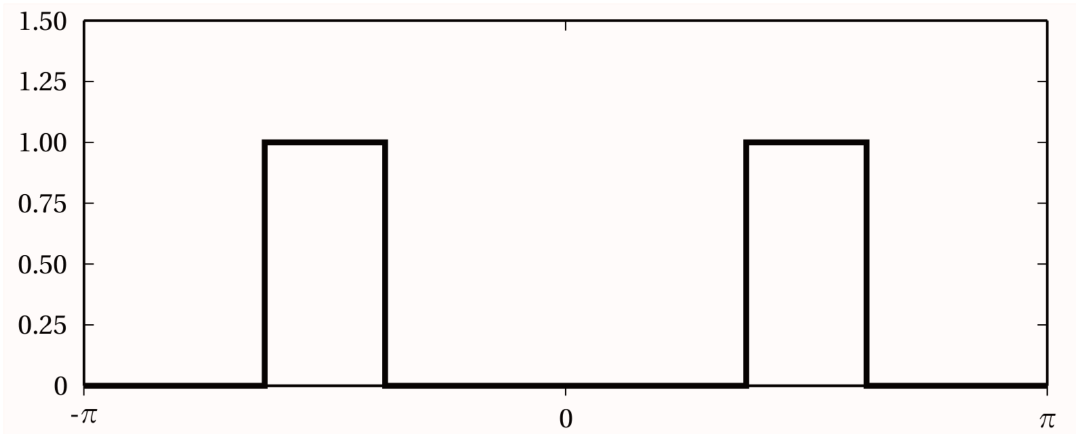
```
import IPython
IPython.display.Audio(chirp(40, 4, 900, 10), rate=900)
```

Indeed, it sounds like a smooth sinusoid with linearly increasing frequency. If now we play `IPython.display.Audio(chirp(40, 4, 700, 10), rate=700)`, we can hear aliasing under the form of a frequency decrease at the end. Thus a sampling frequency of 700 [samples/second] is not enough to allow perfect reconstruction of the original signal over the first 10 seconds.

Solution 5. Digital processing of continuous-time signals.

(a) The digital frequencies are always $\omega = 2\pi(f/F_s)$ so that the digitized AM band resides in the $[0.5\pi, 0.6\pi]$ interval. Each 20KHz channel occupies a slice 0.01π-wide.

(b) the modulation moves the center frequency of the filter to $\pi/2$ so that the lowpass characteristic becomes as in the following figure:

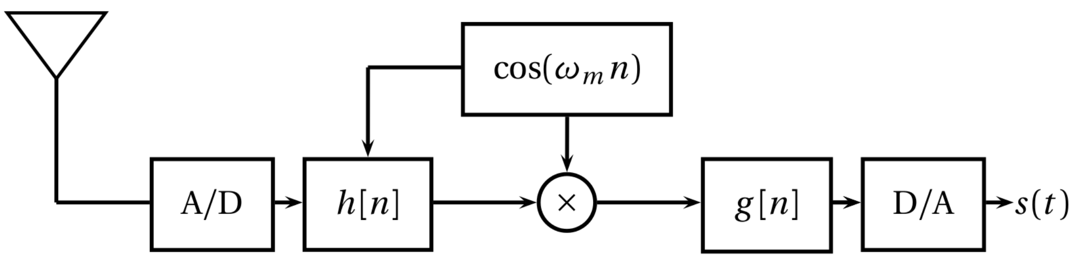


Note that the (positive) spectral support of the passband is $\pi/4$, i.e. twice the cutoff frequency.

(c) The bandwidth of a digitized AM channel is 0.01π so that we need a cutoff frequency $\omega_c = 0.005\pi$.

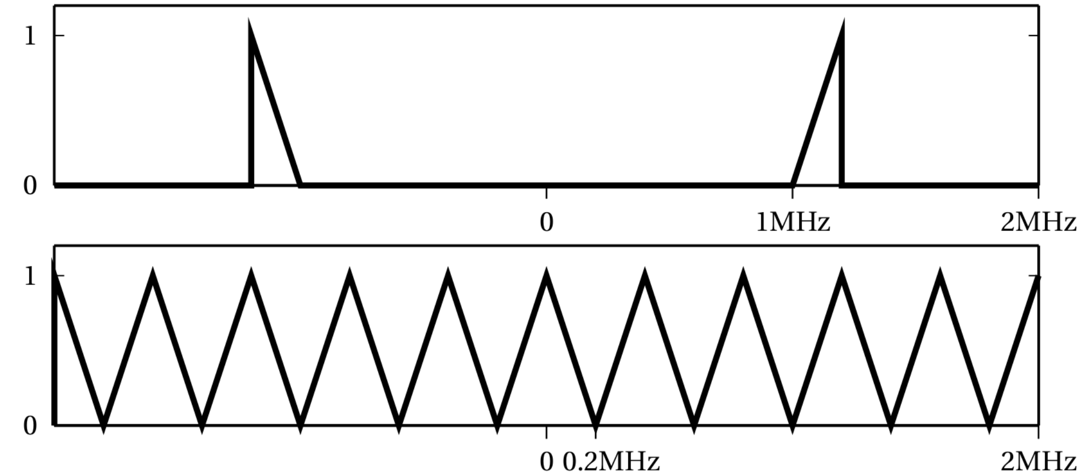
(d) The tuning frequencies are 0.505π, 0.545π and 0.595π respectively.

(e) The same sinusoidal oscillator can be used both to modulate the passband filter and to demodulated the extracted band to baseband; the demodulation centers the channel band around zero. The receiver looks like so:



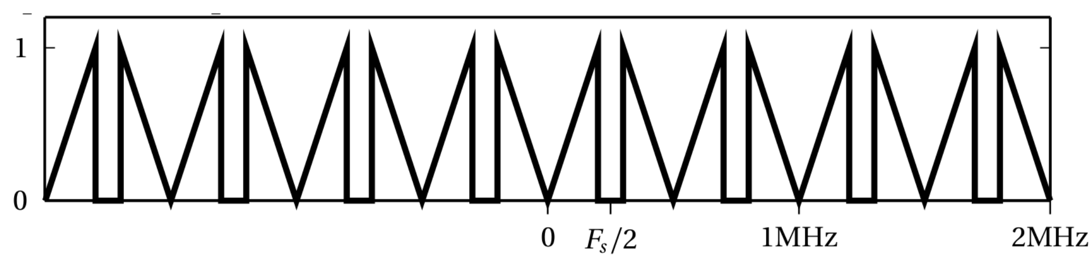
In the figure, $h[n]$ is used again after demodulation to remove the cross-modulation components. The D/A operates at the same frequency as the A/D, i.e. 4MHz; this is of course a waste, considering that the analog signal contains frequencies only up to 10KHz; clearly by using a suitable downsampler we could reduce such frequency.

(f) Assume the spectrum is empty except for the AM band. Since this band starts at 1MHz and is 0.2MHz wide and since 1MHz is an integer multiple of 0.2MHz, we can use bandpass sampling, i.e. we can sample at a frequency as low as twice the bandwidth of the passband signal, i.e. 0.4MHz. There will be aliasing but the copies will not overlap with each other as can be readily seen from the following figure, which represent the original AM spectrum $X(j\Omega)$ and the intermediate periodized spectrum $\tilde{X}(j\Omega)$ as created by the sampler:



This minimal frequency, however, will result in a swap of the positive and negative portions of the spectrum around baseband which greatly complicates the receiver.

In order to avoid this, we can sample at $F_s = 0.5MHz$, and the AM band will be correctly downshifted to baseband as seen in the following figure showing the periodized spectrum:



After this, the scheme will proceed as before, with the following differences:

- the digitized AM band will occupy the $[0, 2\pi/3]$ positive band;
- each channel is $(\pi/15)$-wide;
- the tunable filter will have a cutoff frequency $\omega_c = \pi/30$
- the modulation frequency for both the tunable filter and the demodulation will start at $\pi/15$.

Note however that the spectrum is not empty outside of the AM band and therefore, in order to use bandpass sampling, we need to filter the signal *in the analog domain* with a sharp bandpass which kills everything outside the 1MHz-1.2MHz interval. This custom analog filter is precisely what one would like to avoid in a digital design.

Solution 6. Aliasing in time?

$$
\begin{aligned}
\tilde{y}[n] &= \frac{2}{N}\sum_{k=0}^{N/2-1} \tilde{Y}[k]e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{k=0}^{N/2-1} \tilde{X}[2k]e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{k=0}^{N/2-1}\sum_{i=0}^{N-1} \tilde{x}[i]e^{-j\frac{2\pi}{N}(2k)i}\, e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{i=0}^{N-1}\tilde{x}[i]\sum_{k=0}^{N/2-1} e^{j\frac{2\pi}{N/2}(n-i)k}
\end{aligned}
$$

Now

$$
\sum_{k=0}^{N/2-1} e^{j\frac{2\pi}{N/2}(n-i)k} = \begin{cases} N/2 & \text{if } (n-i) \text{ is a multiple of } (N/2) \\ 0 & \text{otherwise} \end{cases}
$$

so that the only nonzero terms in the outer sum (that for index i) are those for $i = n$ and $i = n + N/2$. In the end

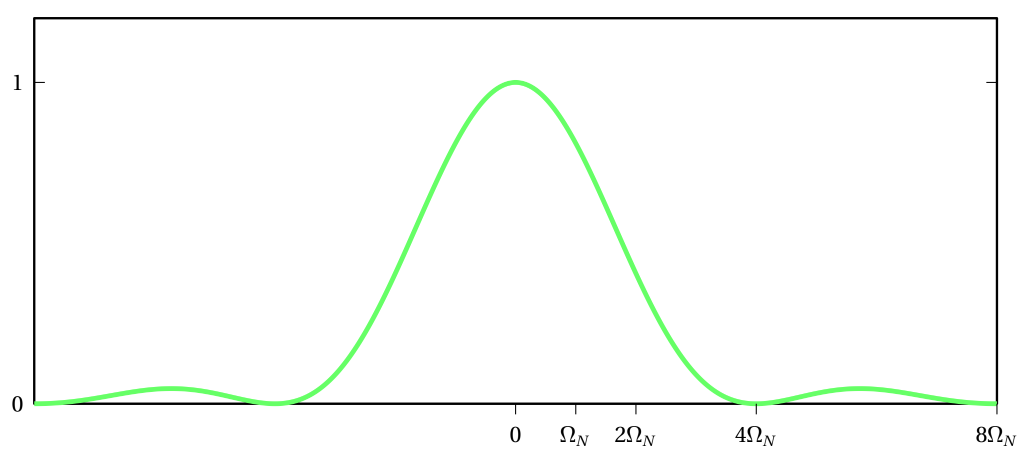
$$
\tilde{y}[n] = \tilde{x}[n] + \tilde{x}[n + N/2].
$$

Since $\tilde{x}[n]$ is N-periodic, this defines an $(N/2)$-periodic sequence obtained by summing two translated versions of $\tilde{x}[n]$. It's exactly like aliasing in the frequency domain: since we are not using enough DFS samples for the reconstruction, then the time-domain signal gets aliased in time.

Solution 7. Other interpolators

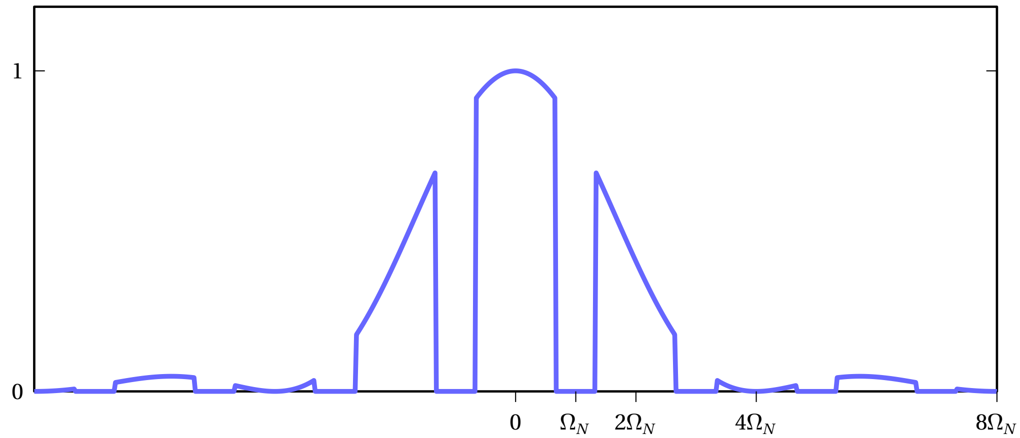
Since the triangular characteristic of the interpolator can be expressed as the convolution of $\mathrm{rect}(2t)$ with itself, we know that

$$
I(j\Omega) = \frac{1}{2}\mathrm{sinc}^2\left(\frac{\Omega}{4\pi}\right)
$$



The interpolated signal in the Fourier domain is

$$X(j\Omega) = T_s \cdot X(e^{j\Omega T_s}) \cdot I(j\Omega T_s)$$



Solution 8. Time and frequency.

No. The signal is time-limited and therefore it is not bandlimited. Consequently, there will always be a certain amount of aliasing in the sampled version regardless of how high the sampling frequency is.
