COM-303 - Signal Processing for Communications

Solutions for Homework #3

Solution 1. DFT in matrix form

Recall the DFT (analysis) formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

We can define an $N \times N$ square matrix $\mathbf{W}$ by stacking the conjugates of the basis vectors $\{\mathbf{w}^{(k)}\}_{k=0,\ldots,N-1}$:

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}^{*(0)} \\ \mathbf{w}^{*(1)} \\ \mathbf{w}^{*(2)} \\ \ldots \\ \mathbf{w}^{*(N-1)} \end{bmatrix}$$

and get the analysis formula in matrix - vector multiplication form:

$$\mathbf{X} = \mathbf{W}\mathbf{x}.$$

Then, knowing that:

$$\mathbf{W}\mathbf{W}^H = \mathbf{W}^H\mathbf{W} = N\mathbf{I}$$

we obtain the synthesis formula in matrix-vector multiplication form:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}.$$

A matrix $\mathbf{A}$ is hermitian when

$$\mathbf{A} = \mathbf{A}^H$$

Therefore, for $\mathbf{W}$ to be hermitian, we would need:

$$\mathbf{W}_{nk} = \mathbf{W}^*_{kn}$$

for all $n, k \in \{0, 1, \dots N-1\}$. This translates to having:

$$e^{-j\frac{2\pi}{N}nk} = e^{j\frac{2\pi}{N}kn}.$$

for all $n, k \in \{0, 1, \dots N-1\}$, which is generally not the case.

Consider, for example, the case when $n = k = 1$. We would need to have:

$$e^{-j\frac{2\pi}{N}} = e^{j\frac{2\pi}{N}}$$

which is equivalent to:

$$e^{j\frac{4\pi}{N}} = 1.$$

Clearly, this can only happen when $N = 2$.

Solution 2. DFS and DFT

Consider the analysis formula for DFS,

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk} \quad k \in \mathbb{Z} \\
&= \sum_{n=0}^{N-1} x[n \bmod N] e^{-j\frac{2\pi}{N}nk} \quad k \in \mathbb{Z} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} \quad k \in \mathbb{Z}
\end{aligned}$$

now, consider $k = m + lN$,where $m \in [0, 1, ..N-1], l \in \mathbb{Z}$

$$\begin{aligned}
\tilde{X}[m + lN] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}n(m+lN)} \quad m \in [0, 1, ..N-1], l \in \mathbb{Z} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nm} \quad m \in [0, 1, ..N-1] \\
&= X[m] \quad m \in [0, 1, ..N-1]
\end{aligned}$$

Thus, $\tilde{X}[k + lN] = X[k]$, for all $l \in \mathbb{Z}$ and $k = 0, ..., N-1$ which can also be represented with $\tilde{X}[k] = X[k \bmod N]$, for all $l \in \mathbb{Z}$.

Solution 3. Derivative in frequency

(a) There are two ways to solve this problem. You can either replace $X(e^{j\omega})$ by its iDTFT expansion and then take the derivative to obtain:

$$j\frac{d}{d\omega}X(e^{j\omega}) = j\frac{d}{d\omega}[\sum_n x(n)e^{-j\omega n}] = j\sum_n x(n)\frac{d}{d\omega}e^{-j\omega n} = \sum_n n\,x(n)e^{-j\omega n}$$

so that iDTFT$\left\{j\frac{d}{d\omega}X(e^{j\omega})\right\} = nx(n)$.

Alternatively, you can first take the IDTFT and then use integration by part:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} j\frac{d}{d\omega}X(e^{j\omega})e^{j\omega n}\,d\omega \\
&= \frac{j}{2\pi}X(e^{j\omega})e^{j\omega n}\Big|_{-\pi}^{\pi} - \frac{j}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})(jne^{j\omega n})\,d\omega \\
&= n\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = nx[n]
\end{aligned}$$

where the first equality uses integration by part and the second the periodicity of $X(e^{j\omega})e^{j\omega n}$.

(b) Because of linearity, the inverse DTFT will be equal to π/j times the result in previous question minus the inverse DTFT of the constant 2, which easily enough is $2\delta[n]$.

Solution 4. DTFT visual inspection

(a) From the plots one can see that $X(e^{j\omega})$ is 0 at the origin;

$$X^*(e^{-j\omega})|_{\omega=0} = \sum_{n\in\mathbb{Z}} x[n] = 0$$

so $x[n]$ is 0-mean.

(b) From the plots one can see that the real part of $X(e^{j\omega})$ is symmetric at the origin, and its imaginary part is antisymmetric. Since

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

Then

$$X^*(e^{-j\omega}) = \Re\left(X\left(e^{-j\omega}\right)\right) - j\Im\left(X\left(e^{-j\omega}\right)\right) = \Re\left(X\left(e^{j\omega}\right)\right) + j\Im\left(X\left(e^{j\omega}\right)\right) = X\left(e^{j\omega}\right)$$

Therefore, $x[n] = x^*[n]$.

Solution 5. DTFT properties

(a) time reversal:

$$\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{j\omega m} = \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega})$$

with the change of variable $m = -n$. Therefore the DTFT of the time-reversed sequence $x[-n]$ is $X(e^{-j\omega})$

(b) time shift:

$$\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = e^{-j\omega n_0} X(e^{j\omega})$$

with the change of variable $m = n - n_0$. Therefore the DTFT of the time-shifted sequence $x[n-n_0]$ is $e^{-j\omega n_0}X(e^{j\omega})$

Solution 6. Plancherel-Parseval Equality

(a) The inner product in $l_2(\mathbb{Z})$ is defined as

$$\langle x[n], y[n]\rangle = \Sigma_n x^*[n]y[n],$$

and in $L_2([-\pi,\pi])$ as

$$\langle X(e^{jw}), Y(e^{jw})\rangle = \int_{-\pi}^{\pi} X^*(e^{jw})Y(e^{jw})dw.$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})Y(e^{jw})dw &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\Sigma_n x[n]e^{-jwn}\right)^* \Sigma_m y[m]e^{-jwm} dw \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_n x^*[n]e^{jwn}\Sigma_m y[m]e^{-jwm} dw \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_n\Sigma_m x^*[n]y[m]e^{jw(n-m)} dw \\
&\overset{(2)}{=} \frac{1}{2\pi}\Sigma_n\Sigma_m x^*[n]y[m]\int_{-\pi}^{\pi} e^{jw(n-m)} dw \\
&\overset{(3)}{=} \Sigma_n x^*[n]y[n],
\end{aligned}$$

where (1) follows from the properties of the complex conjugate, (2) follows from swapping the integral and the sums and (3) fromthe fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-m)}dw = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

(b) If $x[n] = y[n]$, then $\langle x[n], x[n]\rangle$ corresponds to the energy of the signal in the time domain and $\langle X(e^{jw}), X(e^{jw})\rangle$ to the energy of the signal in the frequency domain. In this case, the Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the *Parseval theorem*.

Solution 7. DTFT,DFT, and numerical computations

The analytical expression for the DTFT of the signal is

$$X(e^{j\omega}) = \frac{\sin((M/2)\omega)}{\sin(\omega/2)} e^{j\frac{N-1}{2}\omega}$$

so that the magnitude can be easily plotted by any numerical package. Here is a Python Notebook code snippet that provides the analysis requested by the exercise:

```
%pylab inline
import matplotlib
import matplotlib.pyplot as plt
import numpy as np

M = 20

# plot the analytic DTFT over 10,000 points
w = np.linspace(-np.pi, np.pi, 10000)
X = np.abs(np.sin((M / 2.0) * w) / np.sin(w / 2.0))
plt.plot(w, X);
plt.xlim([-np.pi, np.pi])  # fit the x-axis
plt.show();

# now compute the FFT-based approximations
N = [30, 50, 100, 1000]
for pts in N:
    # build the signal
    x = np.zeros(pts)
    x[0:M] = 1
    # compute the DFT and shift it to align it with the DTFT
    X_a = np.fft.fftshift(np.fft.fft(x))
    plt.plot(np.abs(X_a))
    plt.show()
```