# COM303: Digital Signal Processing

## Lecture 12: Filter design

# Overview

- two more ideal filters
- filter design: problem statement
- IIR design

# a couple more ideal filters

# Overview

- the fractional delay
- the Hilbert filter

# consider a simple delay...

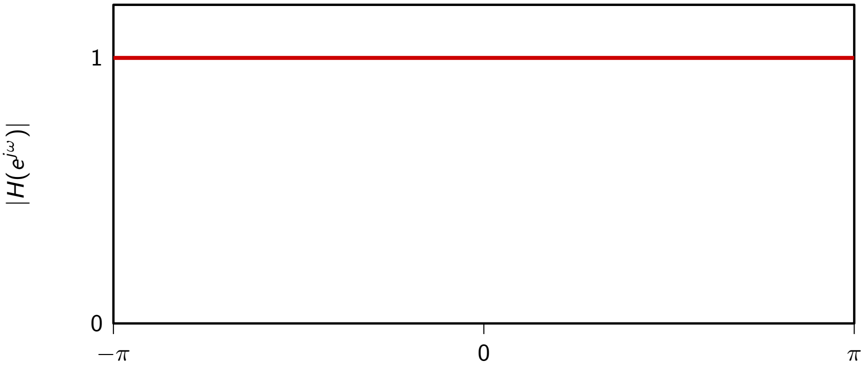
$$x[n] \longrightarrow \boxed{z^{-d}} \longrightarrow x[n-d]$$

$$H(e^{j\omega}) = e^{-j\omega d} \qquad d \in \mathbb{Z}$$

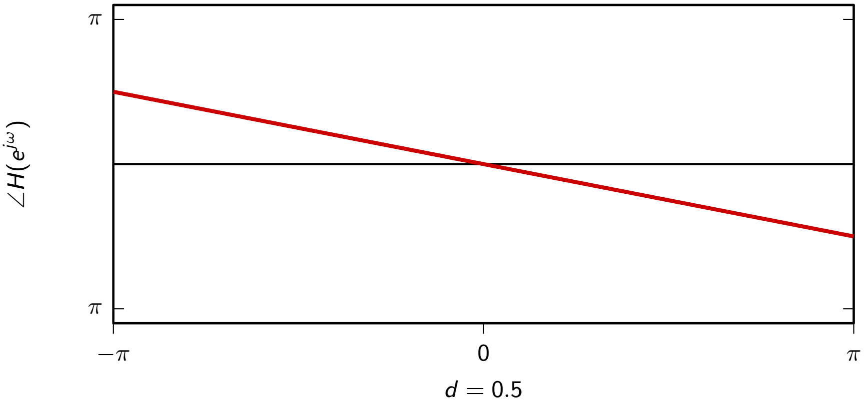
# question

what happens if, in $H(e^{j\omega})$ we use a non-integer $d \in \mathbb{R}$?

# Fractional delay: magnitude response



# Fractional delay: phase response



# impulse response

$$
\begin{aligned}
h[n] &= \text{IDTFT}\left\{e^{-j\omega d}\right\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\
&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\
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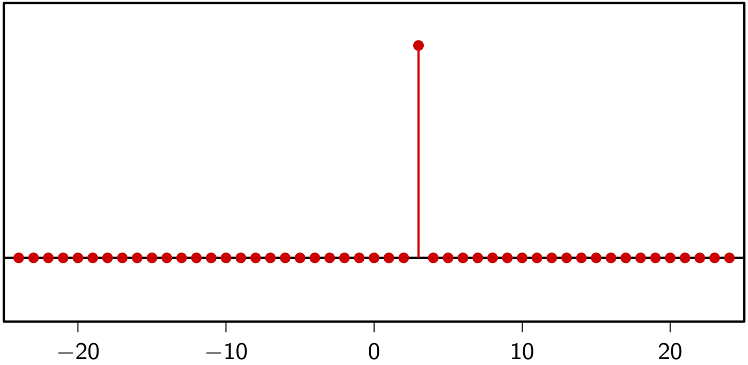
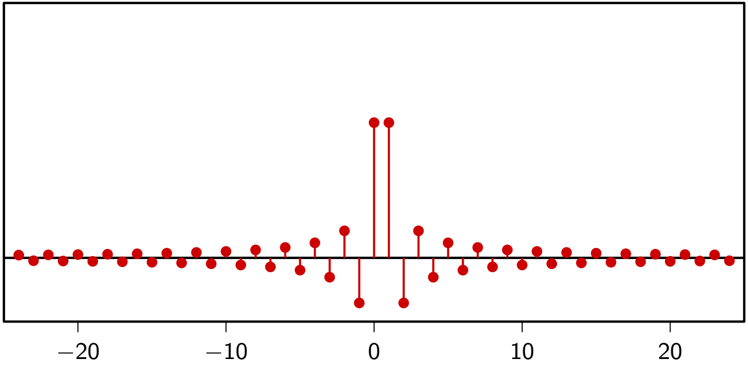
# Fractional delay: impulse response

$d = 3$



# Fractional delay: impulse response

$d = 0.5$



# Fractional delay: impulse response

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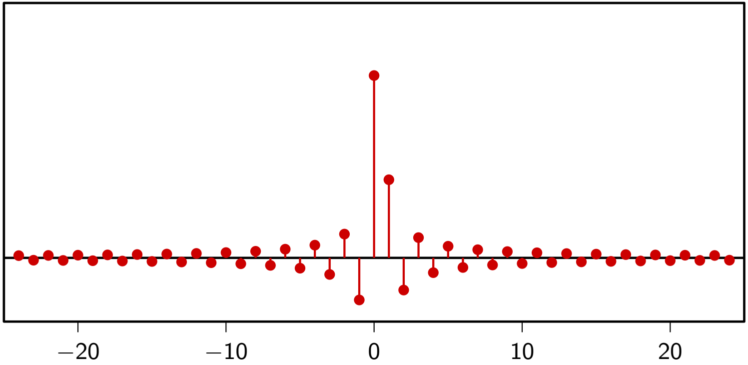
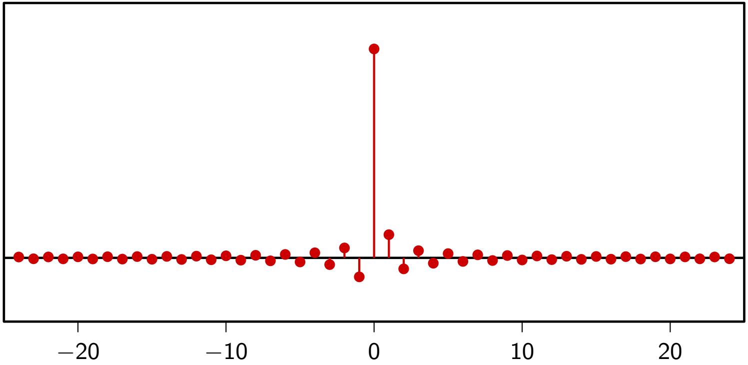


# Fractional delay: impulse response

$d = 0.1$



# fractional delay

- fractional delay computes “in-between” values for samples
- it is an ideal filter!
- often approximated with local interpolation
- all will be clear when we study the sampling theorem

# the Hilbert filter

# a quirky machine

$$\cos(\omega_0 n) \longrightarrow \boxed{\mathrm{H(z)}} \longrightarrow \sin(\omega_0 n)$$

can we build such a thing?

# in the frequency domain

$$H(e^{j\omega})[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)] = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]$$

we can derive two values:

$$\begin{cases} H(e^{j\omega_0}) & = -j \\ H(e^{-j\omega_0}) & = +j \end{cases}$$

# in the frequency domain

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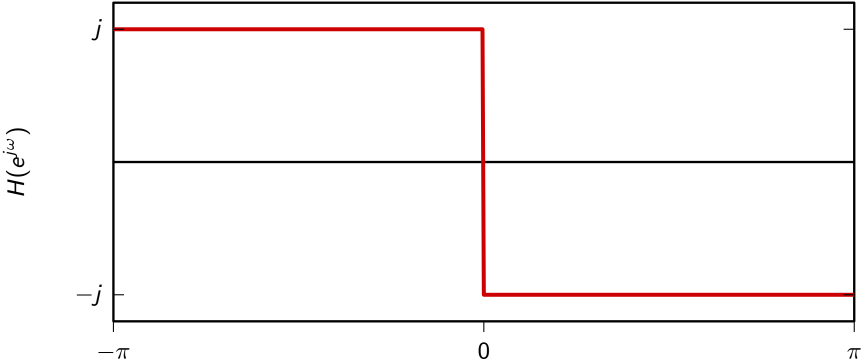
# in the frequency domain

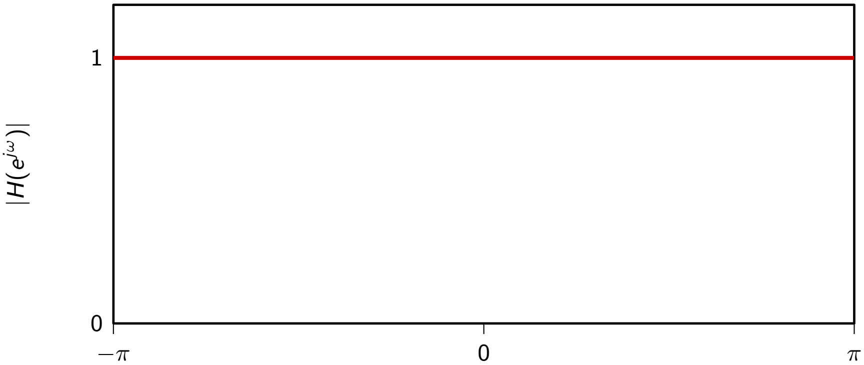
for the machine to work at all frequencies:

$$H(e^{j\omega_0}) = \begin{cases} -j & \text{for } 0 \leq \omega < \pi \\ +j & \text{for } -\pi \leq \omega < 0 \end{cases} \qquad (2\pi\text{-periodic})$$

# Hilbert filter



# Hilbert filter is an allpass



# impulse response

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{0} je^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0}^{\pi} -je^{j\omega n} d\omega$$

$$= \frac{1}{2\pi n}[1 - e^{-j\pi n} - e^{-j\pi n} + 1]$$

$$= \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

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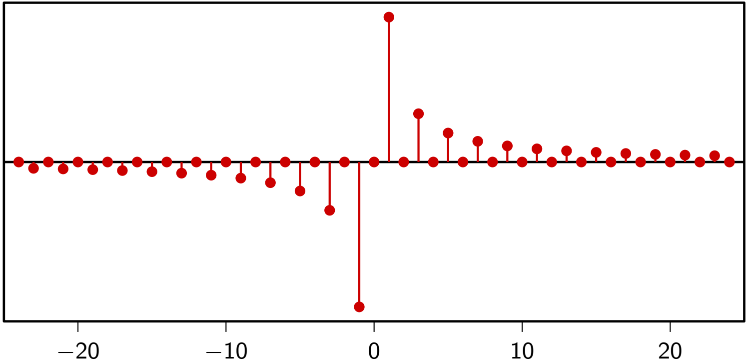
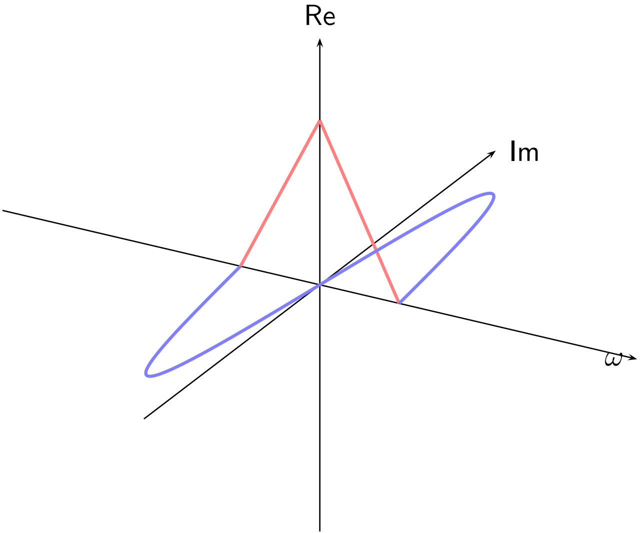


# what does the Hilbert filter do?



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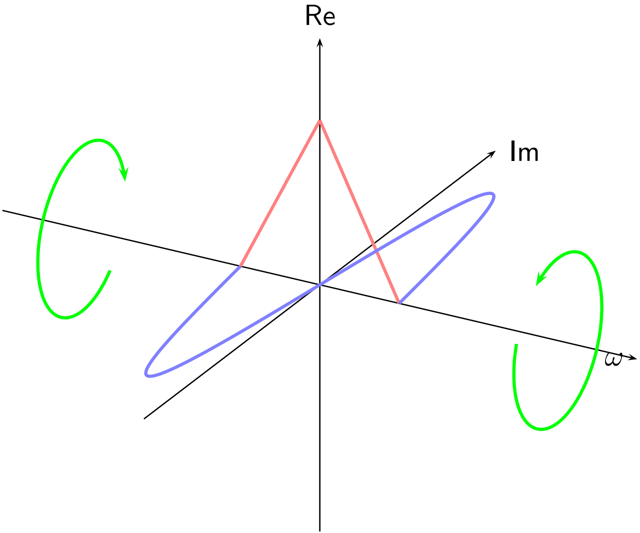
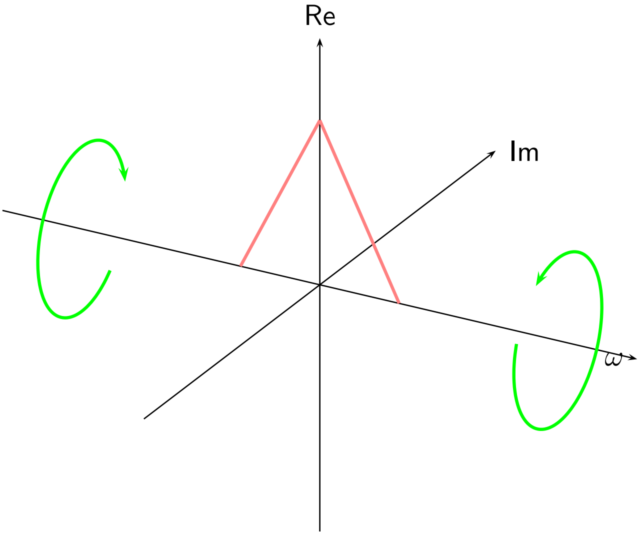


# effect of the Hilbert filter (real part)



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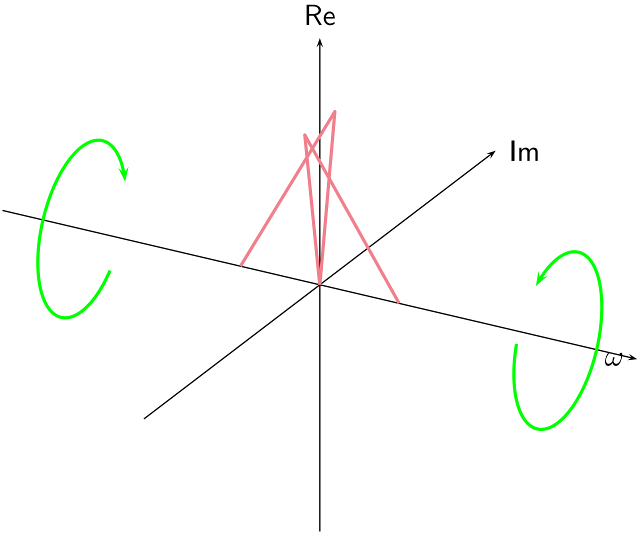

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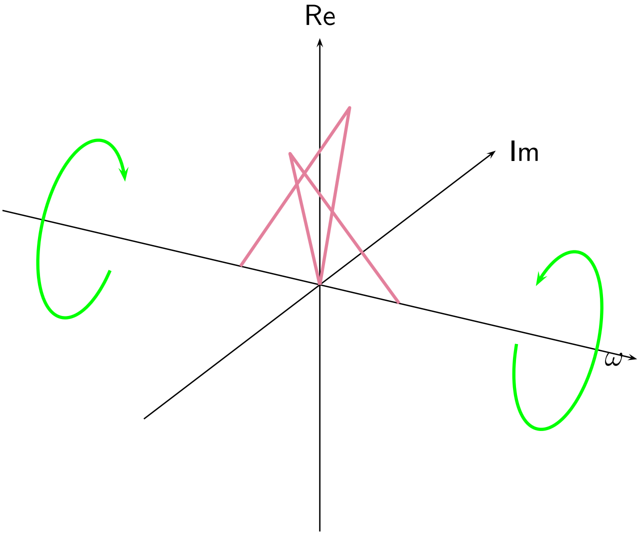

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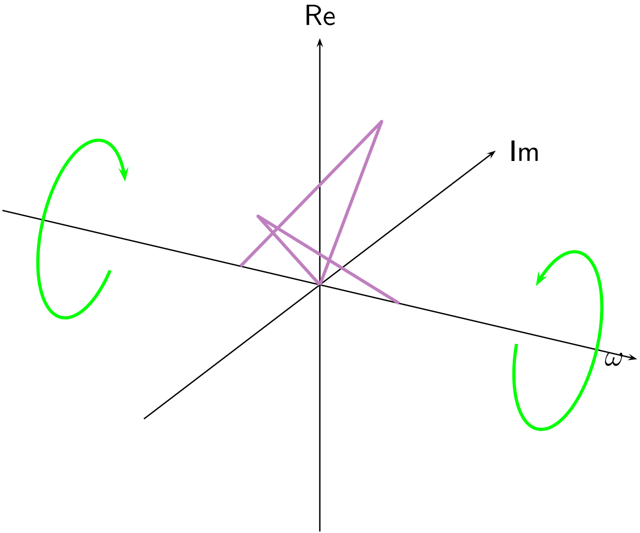

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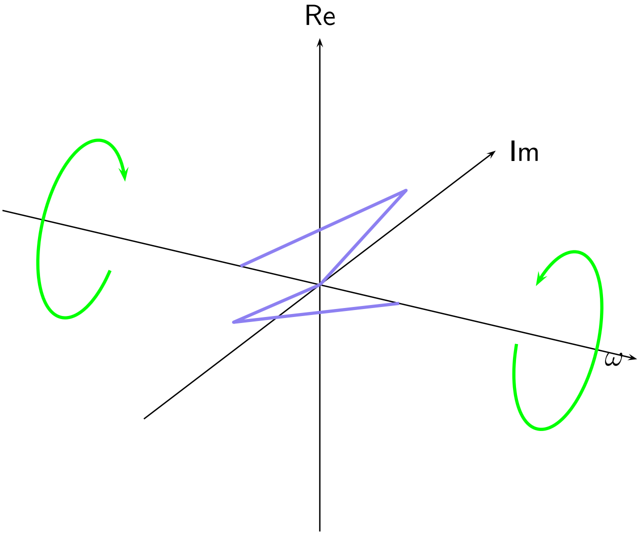

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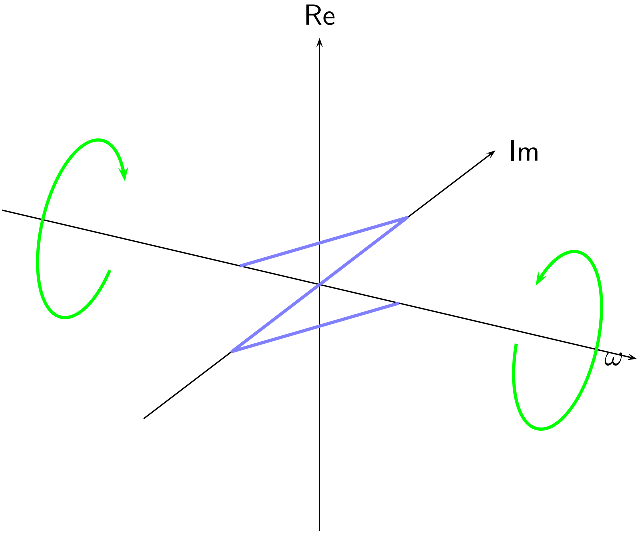

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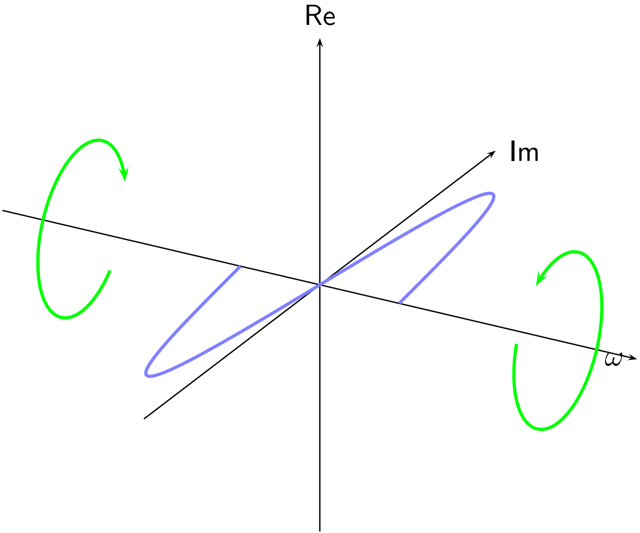

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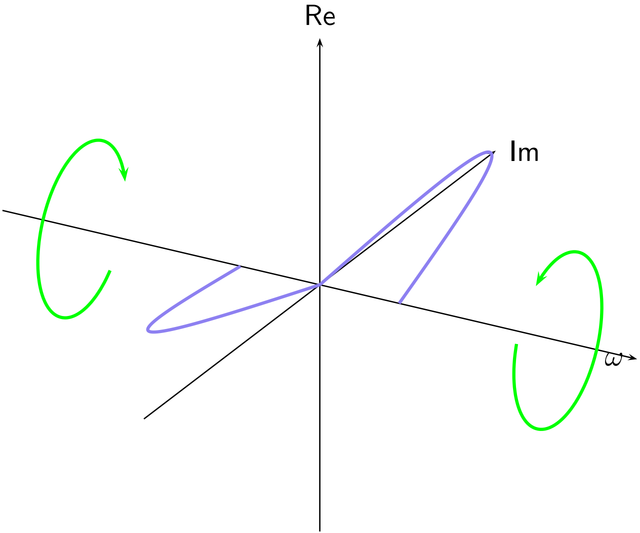

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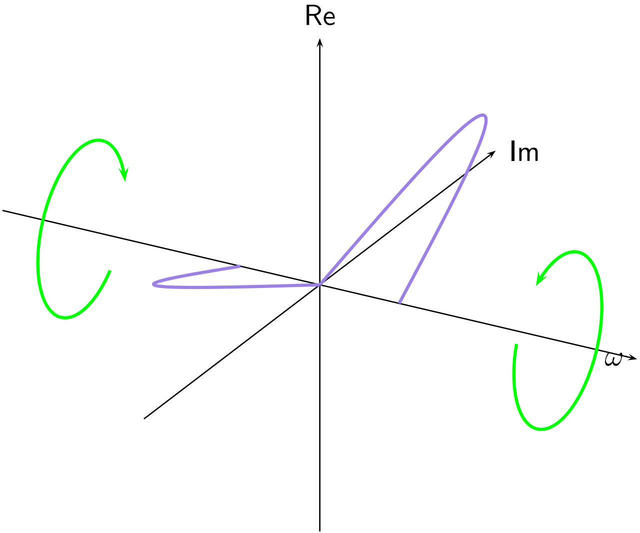

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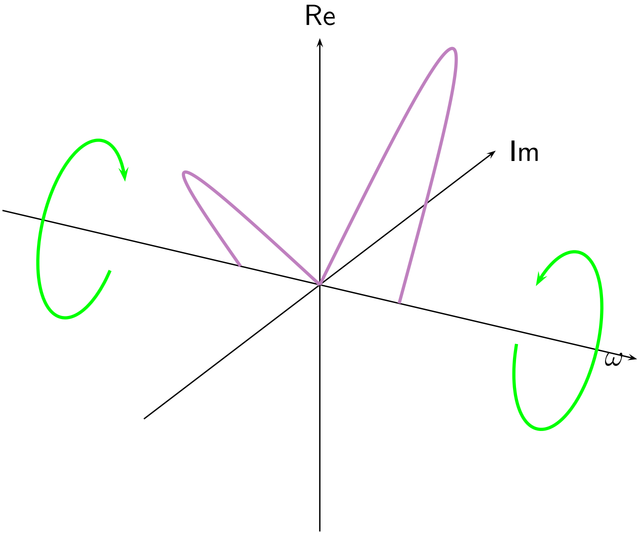

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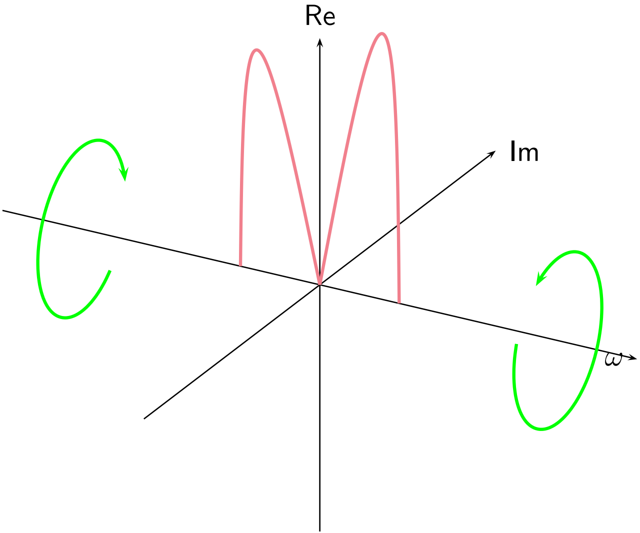

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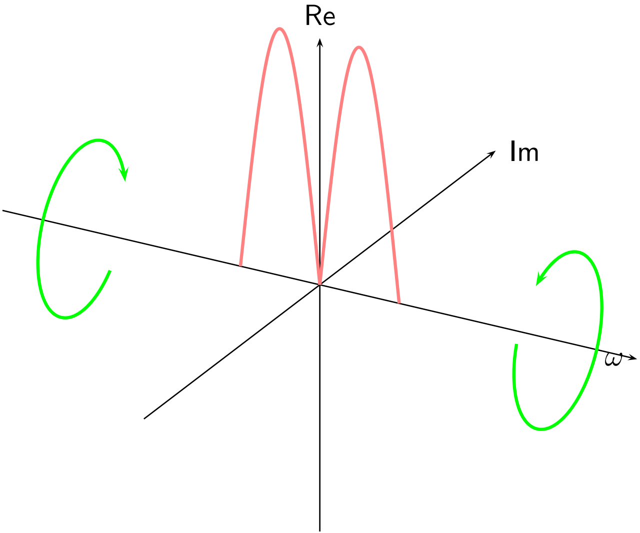

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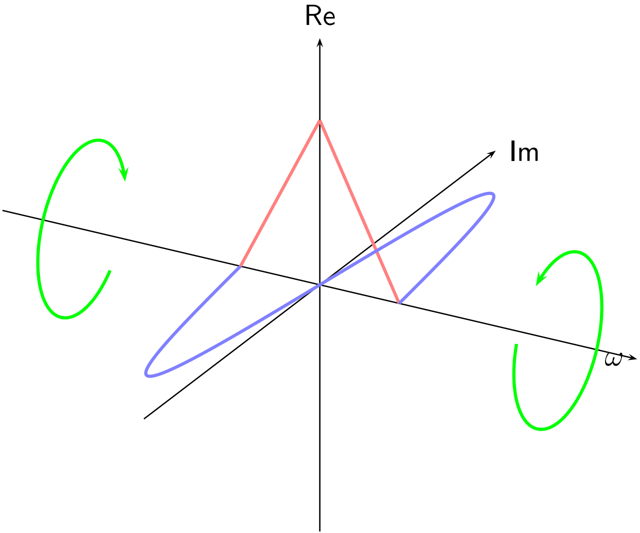

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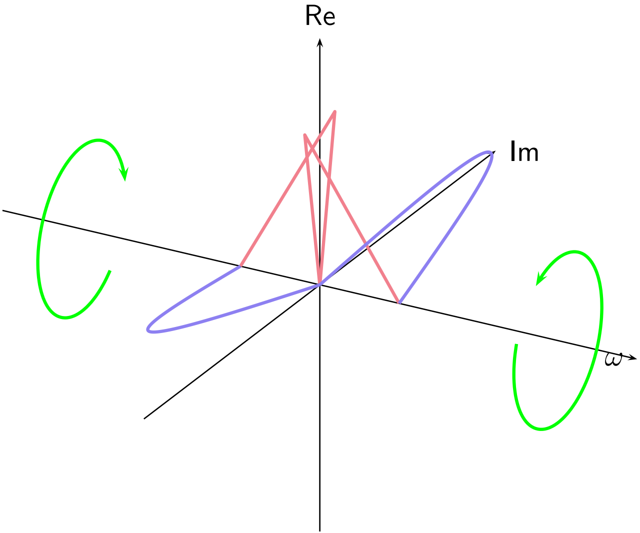

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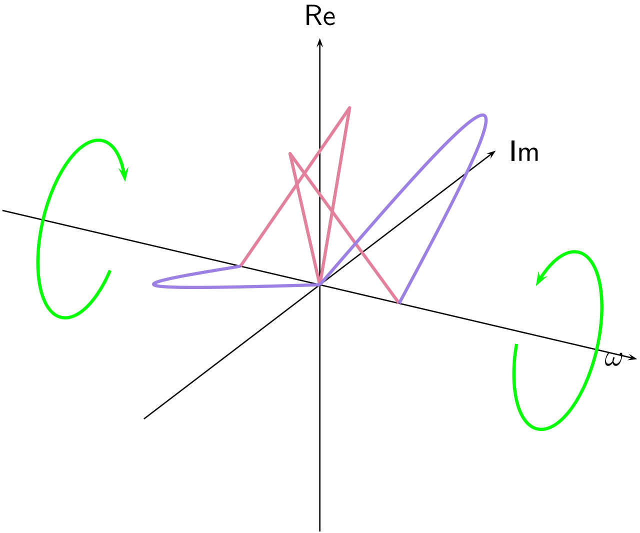

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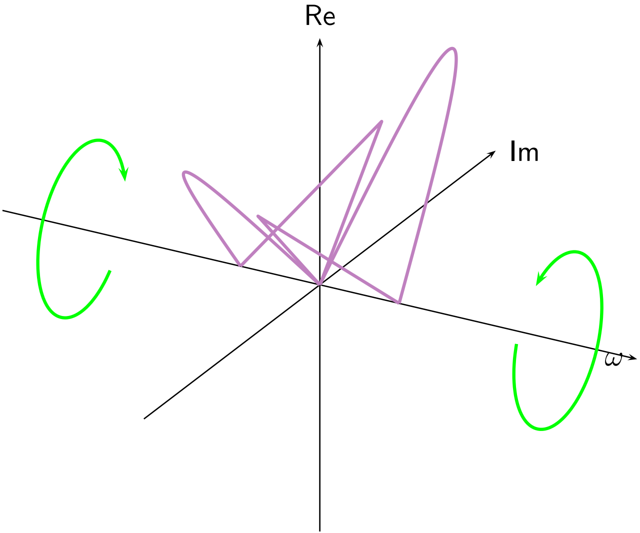

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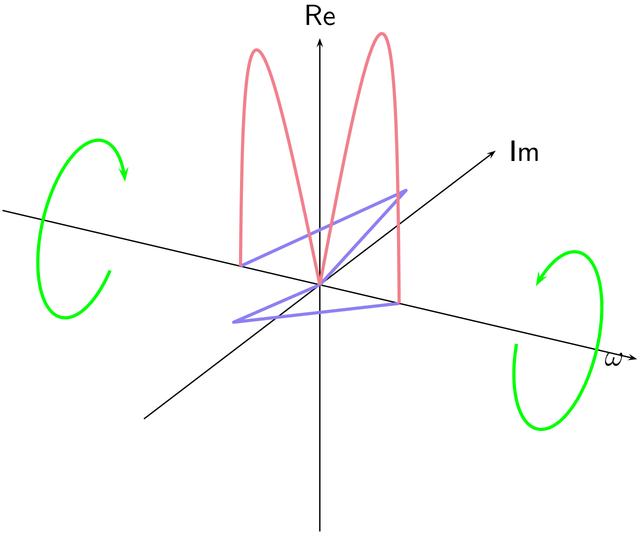

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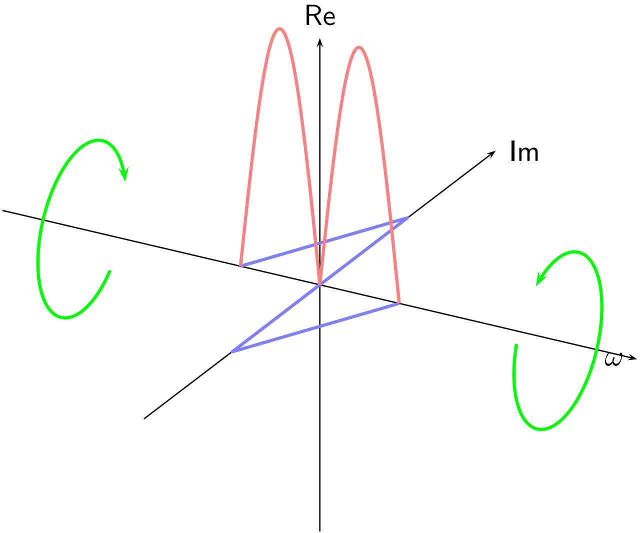
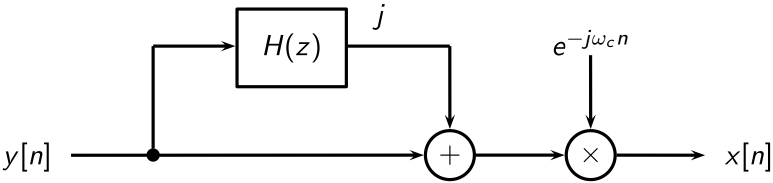
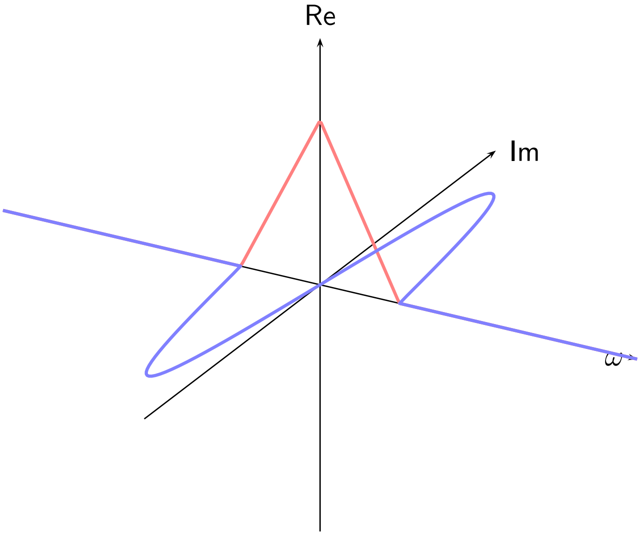


# Hilbert demodulation



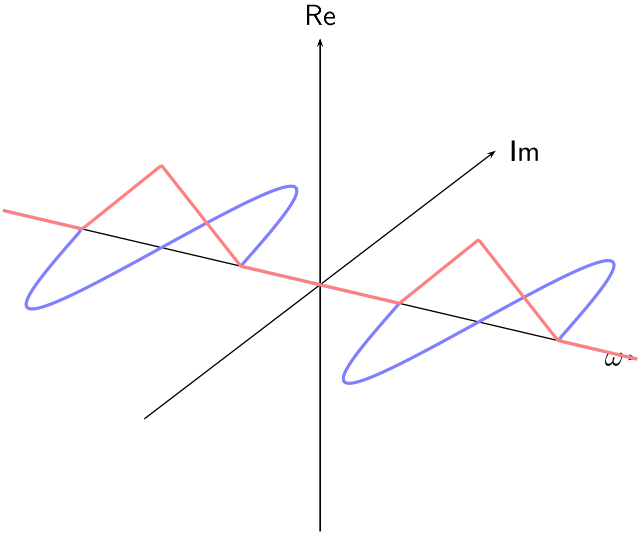
# Hilbert demodulation

$x[n]$



# Hilbert demodulation

$x[n]\cos(\omega_0 n) = y[n]$



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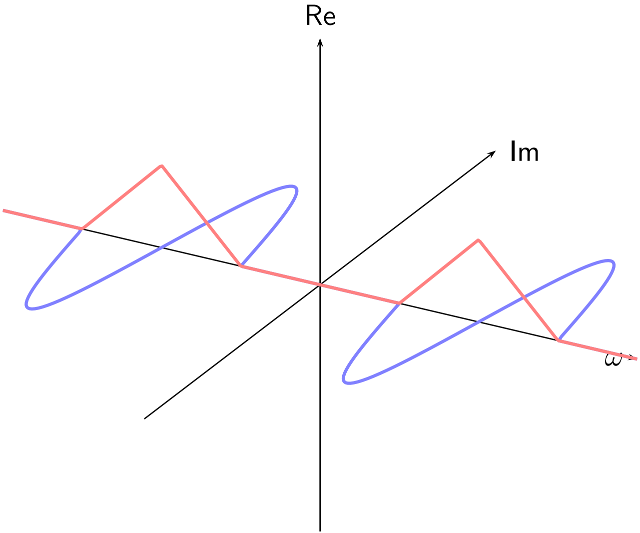

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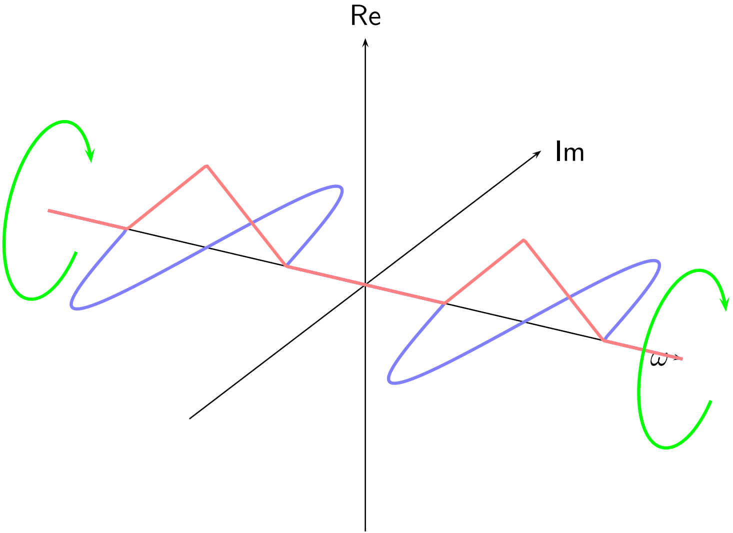

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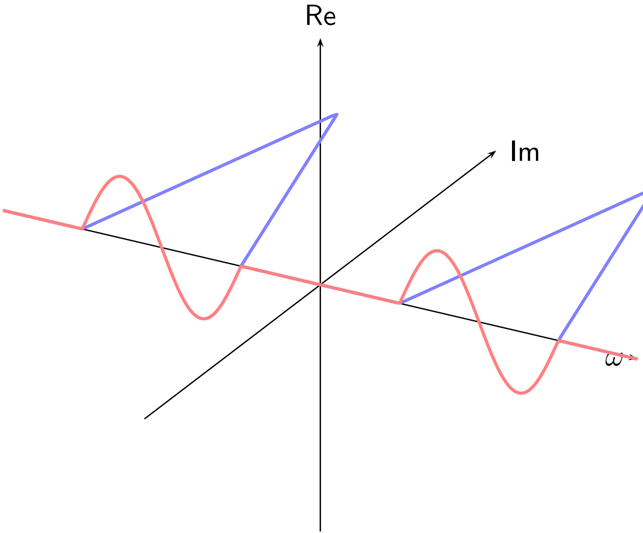

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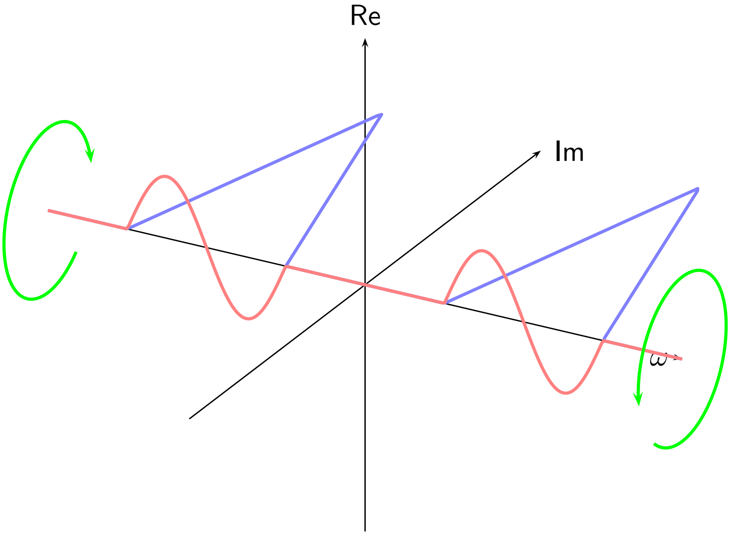

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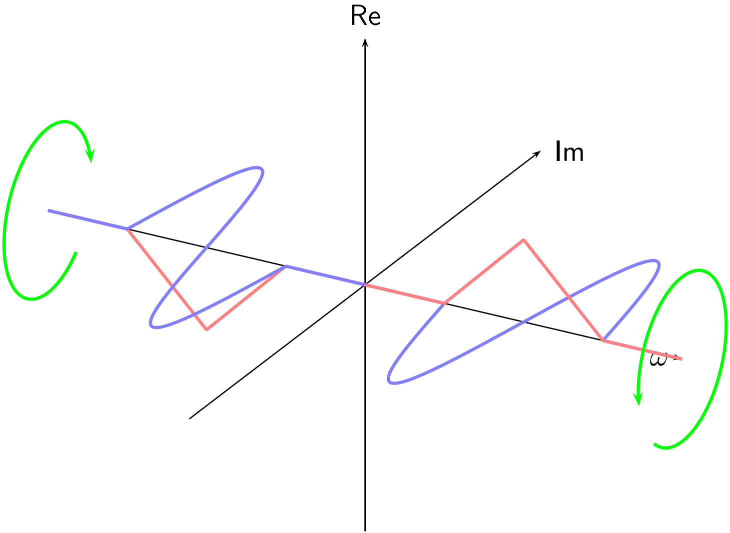
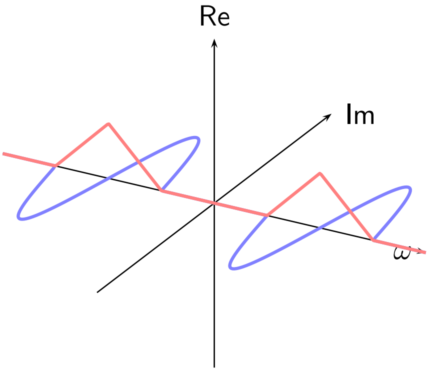
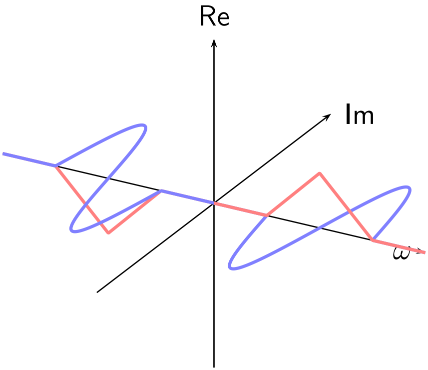


# Hilbert demodulation: $jy[n] * h[n] + y[n]$

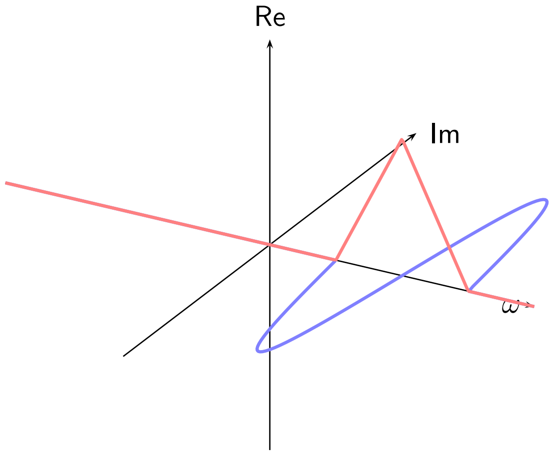



$y[n]$




$jy[n] * h[n]$

# Hilbert demodulation: $jy[n] * h[n] + y[n] = x[n]e^{j\omega_0 n}$



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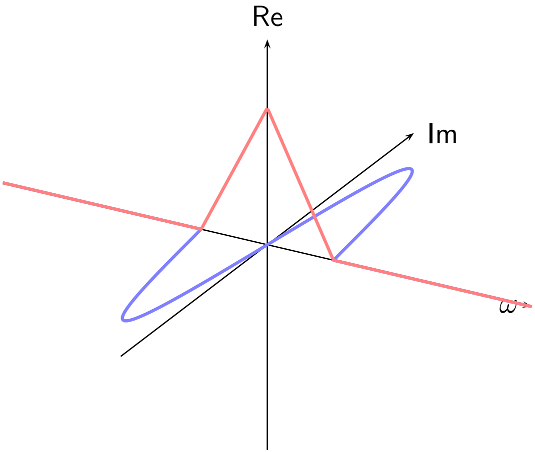

# filter design

# The filter design problem

You are given a set of requirements:

- frequency response: passband(s) and stopband(s)
- phase: overall delay, linearity
- some limit on computational resources and/or numerical precision

You must determine $N$, $M$, $a_k$'s and $b_k$'s in

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{-(N-1)}}$$

in order to best fulfill the requirements

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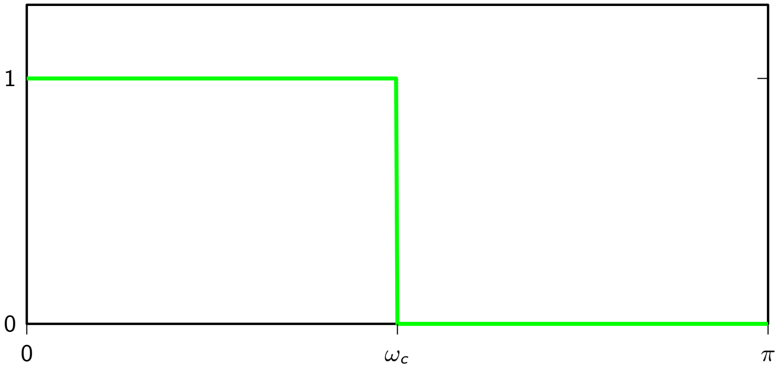
# Example: lowpass specs



# Practical limitations

- passband/stopband transitions cannot be infinitely sharp
  $\Rightarrow$ use *transition bands*
- magnitude response cannot be constant over an interval
  $\Rightarrow$ specify magnitude *tolerances* over bands
- in general:
  - smaller transition bands $\Rightarrow$ higher filter order
  - smaller error tolerances $\Rightarrow$ higher filter order
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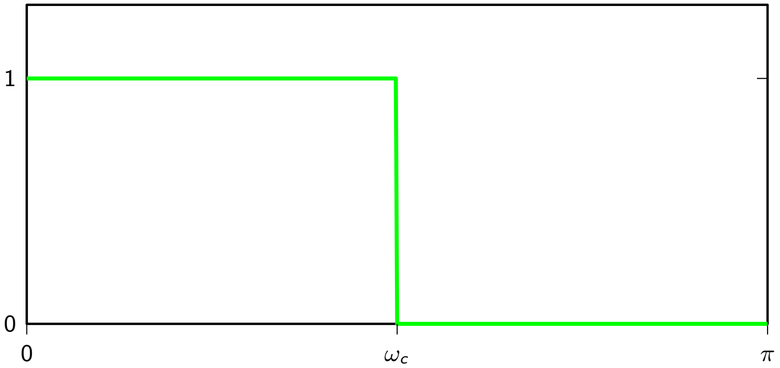
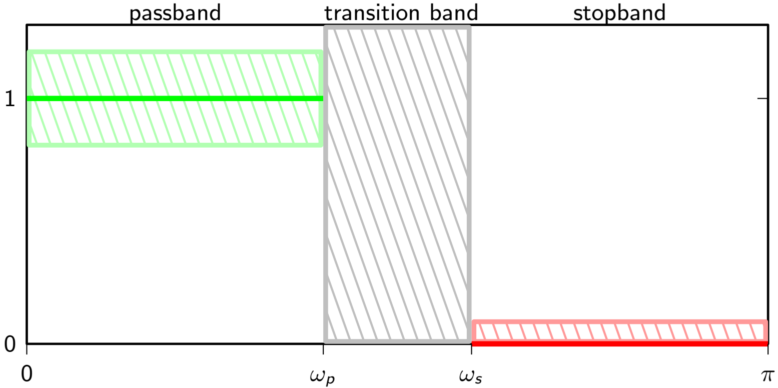
# Example: lowpass specs



# Realistic specs



# Why we can't have a "vertical" transition

$$H(z) = B(z)/A(z) \quad \text{is } C^\infty$$

# Why we can't have a flat response

$$H(z) = B(z)/A(z), \qquad \text{with } A \text{ and } B \text{ polynomials}$$

$$\begin{aligned} H(e^{j\omega}) = c \text{ over an interval} \quad &\Rightarrow \quad B(z) - cA(z) = 0 \text{ over an interval} \\ &\Rightarrow \quad B(z) - cA(z) \text{ has an infinite number of roots} \\ &\Rightarrow \quad B(z) - cA(z) = 0 \text{ for all values of } z \\ &\Rightarrow \quad H(e^{j\omega}) = c \text{ over the entire } [-\pi, \pi] \text{ interval.} \end{aligned}$$

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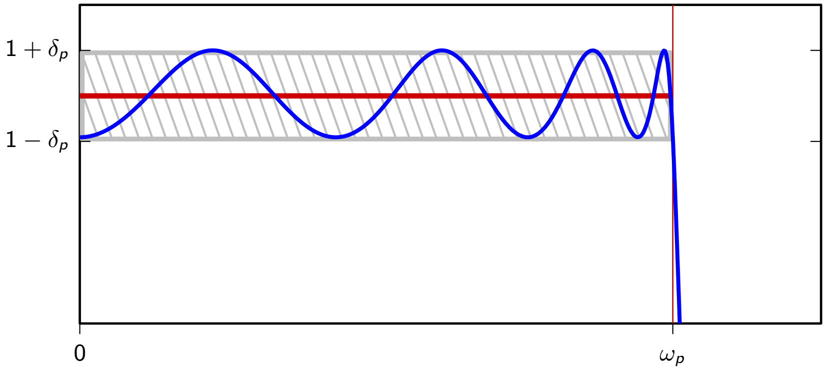
# Important case: equiripple error



# The big questions

- IIR or FIR?
- how to determine the coefficients?
- how to evaluate the performance?

# IIRs: pros and cons

Pros:

- computationally efficient
- strong attenuation easy
- good for audio

Cons:

- stability issues
- difficult to design for arbitrary response
- nonlinear phase

# FIRs: pros and cons

Pros:

- always stable
- optimal design techniques exist
- can be designed with linear phase

Cons:

- computationally much more expensive
- may “sound” harsh

## The design methods

- finding $N$, $M$, $a_k$'s and $b_k$'s from specs is a hard nonlinear problem
- established methods:
  - IIR: conversion of analog design
  - FIR: optimal minimax filter design

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# IIR design

# IIR: conversion of analog design

Filter design was an established art long before digital processing appeared

- lots of nice analog filters exist
- methods exist to "translate" the analog design into a rational transfer function
- most numerical packages (Matlab, etc.) provide ready-made routines
- design involves specifying some parameters and testing that the specs are fulfilled

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# Butterworth lowpass

Magnitude response:

- maximally flat
- monotonic over $[0, \pi]$

Design parameters:

- order $N$ ($N$ poles and $N$ zeros)
- cutoff frequency

Design test criterion:

- width of transition band
- passband error

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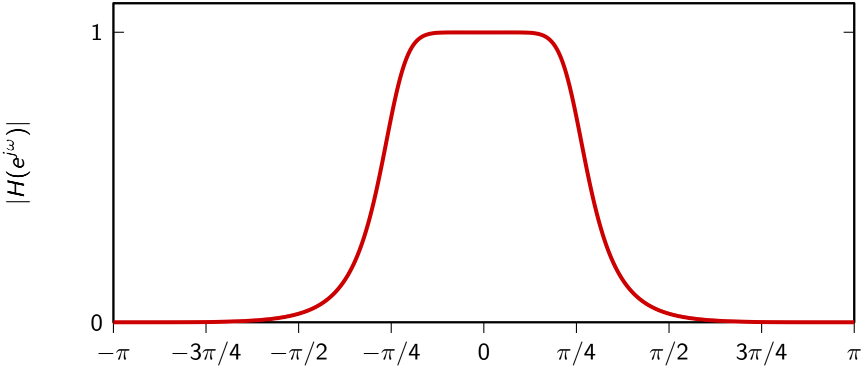
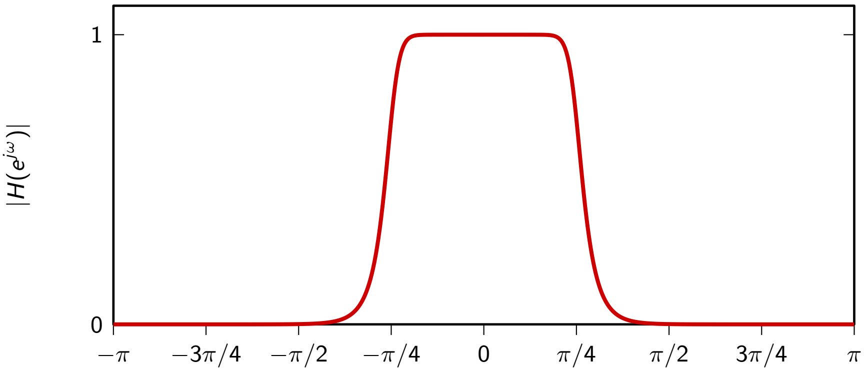
# Butterworth lowpass example

$$N = 4, \omega_c = \pi/4$$



# Butterworth lowpass example

$$N = 8, \omega_c = \pi/4$$



# Chebyshev lowpass

Magnitude response:

- equiripple in passband
- monotonic in stopband

Design parameters:

- order $N$ ($N$ poles and $N$ zeros)
- passband max error
- cutoff frequency

Design test criterion:

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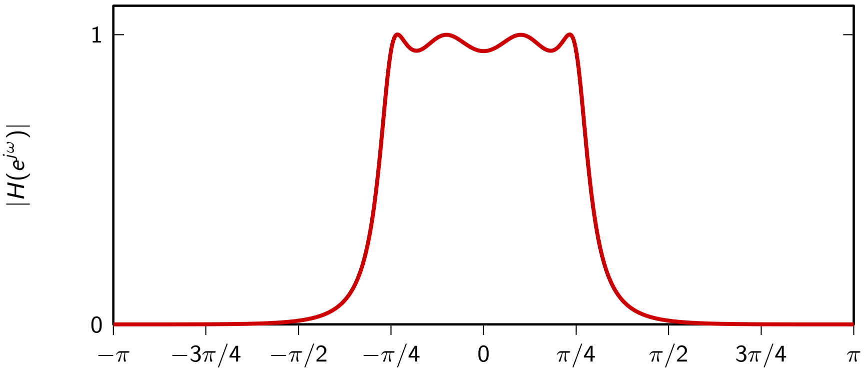
# Chebyshev lowpass example

$$N = 4, \omega_c = \pi/4, e_{\mathrm{max}} = 12\%$$



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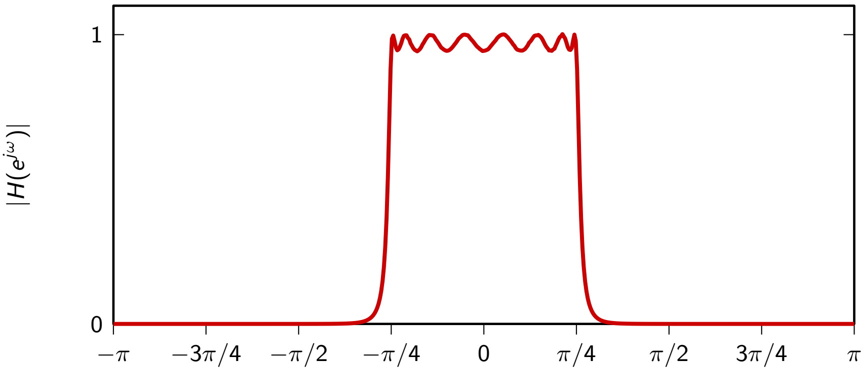


# Elliptic lowpass

Magnitude response:

- equiripple in passband and stopband

Design parameters:

- order $N$
- cutoff frequency
- passband max error
- stopband min attenuation

Design test criterion:

- width of transition band

# Elliptic lowpass

Magnitude response:

- equiripple in passband and stopband

Design parameters:

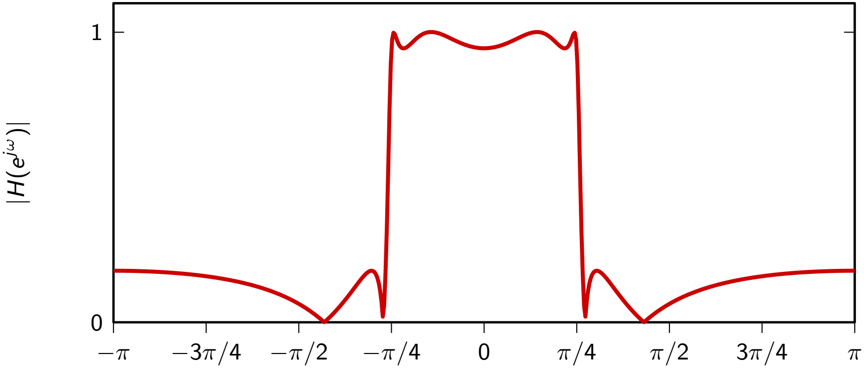
- order $N$
- cutoff frequency
- passband max error
- stopband min attenuation

Design test criterion:

- width of transition band

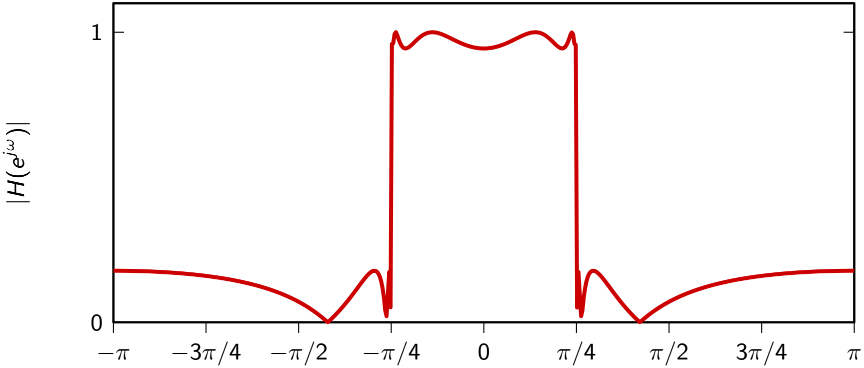
# Elliptic lowpass example

$$N = 4, \omega_c = \pi/4, e_{\mathrm{max}} = 12\%, \mathrm{att}_{\mathrm{min}} = 0.03$$

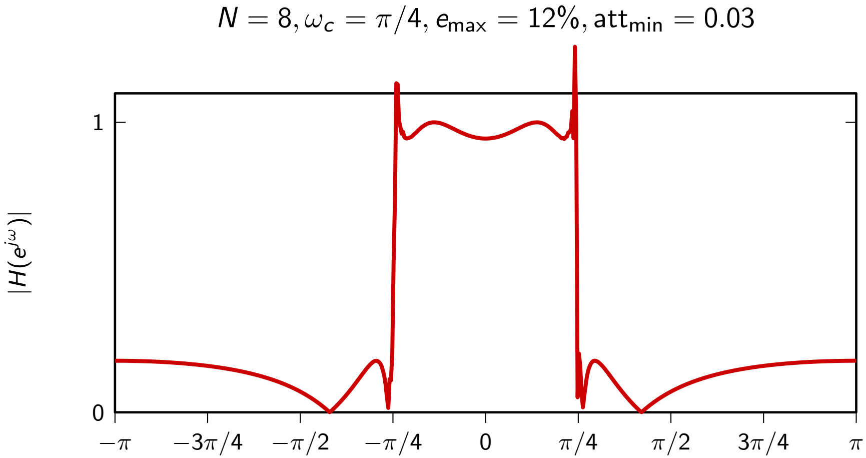


# Elliptic lowpass example

$$N = 6, \omega_c = \pi/4, e_{\mathrm{max}} = 12\%, \mathrm{att}_{\mathrm{min}} = 0.03$$



# Elliptic lowpass example



# Magnitude response in decibels

- filter max passband magnitude $G$
- filter attenuation expressed in decibels as:

$$A_{\mathrm{dB}} = 20\log_{10}(|H(e^{j\omega})|/G)$$

- useful to compare attenuations between filters

# 4-th order lowpass comparison