Solutions 2

1. a) We imitate the proof given in class. For the first identity, let $A_m = \{X_m = j\}$ and $B_m = \{X_m = j, X_r \neq j, \text{ for } 1 \leq r < m\}$. The B_m are disjoint, so that

$$\begin{aligned}\mathbb{P}(A_m|X_0 = i) &= \sum_{r=1}^{m} \mathbb{P}\left(A_m \bigcap B_r \mid X_0 = i\right) \\ &= \sum_{r=1}^{m} \mathbb{P}(A_m|B_r, X_0 = i)\, \mathbb{P}(B_r|X_0 = i) \\ &= \sum_{r=1}^{m} \mathbb{P}(A_m|X_r = j)\, \mathbb{P}(B_r|X_0 = i)\end{aligned}$$

where we used the Markov condition in the last equality. Hence

$$p_{ij}^{(m)} = \sum_{r=1}^{m} p_{jj}^{(m-r)}\, f_{ij}^{(r)}$$

Multiply by $s^m, |s| < 1$ and summing over m we find

$$P_{ij}(s) = P_{jj}(s)\, F_{ij}(s)$$

In order to prove the second identity we first define the events that the last visit to i prior to n took place at time k:

$$E_k = \{X_k = i, X_{k+1} \neq i, \ldots, X_{n-1} \neq i\} \quad \text{for } 0 \leq k \leq n-1$$

These events form a partition for the set of walks that satisfy $X_0 = i$ and $X_n = j$. Thus

$$\begin{aligned}\mathbb{P}(X_n = j|X_0 = i) &= \sum_{k=0}^{n-1} \mathbb{P}(X_n = j, E_k|X_0 = i) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(X_n = j, X_{n-1} \neq i \ldots X_{k+1} \neq i|X_k = i, X_0 = i)\, \mathbb{P}(X_k = i|X_0 = i)\end{aligned}$$

by conditioning. Now by the Markov property,

$$\mathbb{P}(X_n = j|X_0 = i) = \sum_{k=0}^{n-1} \mathbb{P}(X_n = j, X_{n-1} \neq i \ldots X_{k+1} \neq i|X_k = i)\, \mathbb{P}(X_k = i|X_0 = i)$$

Thus

$$p_{ij}^{(n)} = \sum_{k=0}^{n-1} l_{ij}^{(n-k)}\, p_{ii}^{(k)}$$

(note that the sum extends to n since $l_{ij}(0) = 0$ by definition). Multiplying by $s^m, |s| < 1$ and summing yields, for $i \neq j$

$$P_{ij}(s) = P_{ii}(s)\, L_{ij}(s)$$

b)

1. If j is recurrent we have $f_{jj} = 1$ by definition and we know from $P_{jj}(s) = 1 + P_{jj}(s)F_{jj}(s)$ that $P_{jj}(1) = +\infty$. Then $P_{ij}(1) = +\infty$ as long as $F_{ij}(1) > 0$. This means that $\sum_{n\geq 0} p_{ij}^{(n)} = +\infty$ for i s.t. $f_{ij} > 0$. (use Abel's theorem like in class to take $\lim\limits_{s\nearrow 1}$).

2. If j is transient $P_{jj}(1) < +\infty$ (because $f_{jj} < 1$) and since $F_{ij}(1) \leq 1$, we have that $P_{ij}(1) < +\infty$ which means $\sum_{n\geq 0} p_{ij}^{(n)} < +\infty$.

3. The two identities imply

$$L_{ij}(s) = \frac{P_{ij}(s)}{P_{ii}(s)} = F_{ij}(s)\frac{P_{jj}(s)}{P_{ii}(s)}$$

thus

$$L_{ij}(s) = F_{ij}(s)\frac{1 - F_{ii}(s)}{1 - F_{jj}(s)}$$

Let j be recurrent and i transient. Then $f_{jj} = 1$ and $f_{ii} < 1 \Rightarrow L_{ij}(1) = +\infty$ if $F_{ij}(1) > 0$. This means $\sum_{n\geq 0} l_{ij}^{(n)} = +\infty$ as long as $f_{ij} > 0$.

c) From the two identities, we conclude

$$F_{ij}(s)P_{jj}(s) = P_{ii}(s)L_{ij}(s)$$

so if $P_{ii}(s) = P_{jj}(s)$ for all $i \neq j$ and $|s| < 1$, then we have

$$F_{ij}(s) = L_{ij}(s) \quad \forall |s| < 1$$

thus $f_{ij}^{(n)} = l_{ij}^{(n)}$ for all $n \geq 1$.

2. a) Given that the walk starts at the origin and does i steps in direction e_1, j steps in direction e_2 and k steps in direction e_3, it will return to the origin in $2n$ steps if and only if $i + j + k = n$ and it takes i steps in direction $-e_1$, j steps in direction $-e_2$ and k steps in direction $-e_3$. There are $\frac{(2n)!}{(i!j!k!)^2}$ such walks, as the steps can be taken in any order, each with probability $\frac{1}{6^{2n}}$. Thus

$$\mathbb{P}(S_{2n} = (0,0,0)|S_0 = (0,0,0)) = \frac{1}{6^{2n}} \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2}$$

b) The sum equals

$$\frac{1}{2^{2n}}\binom{2n}{n} \sum_{i+j+k=n} \left(\frac{n!}{i!j!k!}\right)^2 \frac{1}{3^{2n}}$$

Hence it is upper bounded by

$$\frac{1}{2^{2n}}\binom{2n}{n} M \sum_{i+j+k=n} \frac{1}{3^n}\frac{n!}{i!j!k!}$$

c) Next, notice that the last sum is the expansion of $\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)^n$ and thus equals 1. Further, if the maximum is attained at $i=j=k\simeq\frac{n}{3}$, we get:

$$\mathbb{P}(S_{2n}=(0,0,0)|S_0=(0,0,0))\le\frac{1}{2^{2n}}\binom{2n}{n}\frac{n!}{3^n(\lfloor n/3\rfloor!)^3}$$

By Stirling's approximation (see Homework 1)

$$\frac{1}{2^{2n}}\binom{2n}{n}\simeq\frac{1}{\sqrt{\pi n}}$$

and

$$\frac{n!}{3^n(\lfloor n/3\rfloor)^{3n}}\sim\frac{\sqrt{2\pi n}(n/e)^n}{3^n\sqrt{2\pi n/3}^3\left(\frac{n}{3e}\right)^n}\simeq\frac{3\sqrt{3}}{2\pi}\left(\frac{1}{\sqrt{n}}\right)^2$$

So finally

$$\mathbb{P}(S_{2n}=(0,0,0)|S_0=(0,0,0))\le\frac{c}{n^{3/2}}$$

for $c\simeq\frac{3\sqrt{3}}{2\pi^{3/2}}$ (and n large).

d) The walk is transient because of the criterion proved in class:

$$\sum_n\mathbb{P}(S_{2n}=(0,0,0)|S_0=(0,0,0))\le\sum_n\frac{c}{n^{3/2}}<+\infty$$