Exercise Set 11

Saliba, May 22, 2019

Exercise 1. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Find the stationary distribution of the process

(ii). Give an intuitive explanation of the result.

Exercise 2. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2\mu_2 & -\lambda_2 & \lambda_2(1-\mu_2) & 0 & \cdots & 0 & 0 \\ \lambda_3\mu_3 & 0 & -\lambda_3 & \lambda_3(1-\mu_3) & \cdots & 0 & 0 \\ \lambda_4\mu_4 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1}\mu_{n-1} & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1}(1-\mu_{n-1}) \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Guess the expression $\pi_1 = \lim_{t\to\infty} \mathbb{P}(X(t) = 1)$.

(ii). Find the stationary distribution of the process.

Exercise 3. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be an irreducible Markov process on a finite space of n states, with generator Q. Let us take λ so that $\lambda > \max_i\{-Q_{ii}\}$ and define the matrix:

$$P = I + \frac{1}{\lambda}Q \qquad \text{(where } I \text{ is the identity matrix).}$$

(i). Show that P is a transition matrix, and that its stationary distribution is identical to the one of Q.

(ii). Let $\{N(t) \mid t \in \mathbb{R}^+\}$ be a Poisson process with parameter λ and let $\{Y_k \mid k \in \mathbb{N}\}$ be a Markov chain with matrix P, independent from $\{N(t)\}$. Let $T_0 = 0$, and $T_1, T_2, \cdots$ denote the arrival times in the Poisson process.

We define the process $\{Z(t) \mid t \in \mathbb{R}^+\}$ as follows:

$$Z(t) = Y_n \qquad \forall t \in [T_n, T_{n+1}[.$$

Show that $\{Z(t)\}$ is a Markov process with generator Q.

Exercise 4. Show that the renewal function $R(t)$ satisfy a renewal equation, and specify the corresponding function $g(t)$.

Exercise 5. Let $S_1, S_2, \ldots$ be the successive times at which cars cross a certain fixed position on the highway. We assume that the intervals of time $W_1, W_2, \ldots$ between each renewal are i.i.d. with cumulative distribution $F(\cdot)$. Suppose that at time $t = 0$, a pedestrian arrives at this fixed position, and wants to cross the road. Assume that he needs τ units of time to cross it. Let L be the time that the pedestrian has to wait before starting to cross the road.

(a) Find the distribution of L and its expectation.

(b) Same questions if we assume that the arrivals of cars follow a Poisson process with parameter λ.