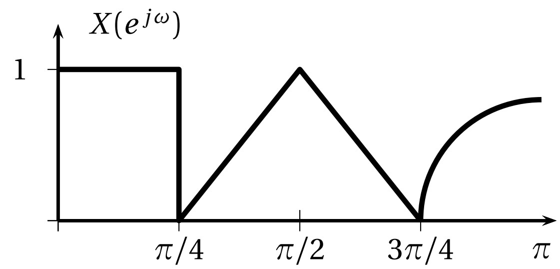
COM-303 - Signal Processing for Communications

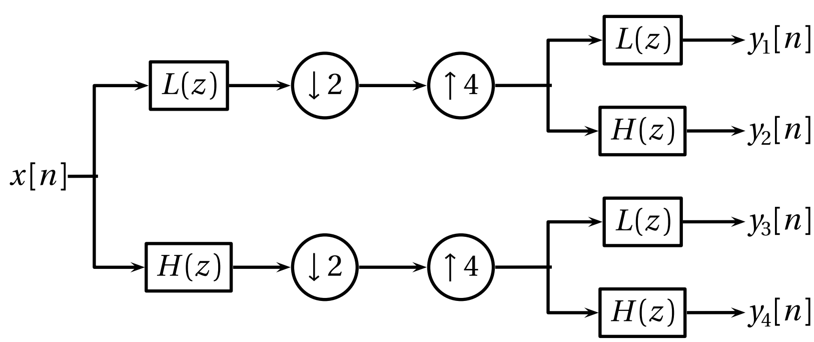
Homework #10

Exercise 1. Multirate Signal Processing

Consider a discrete-time signal $x[n]$ with the following spectrum:



Now consider the following multirate processing scheme in which $L(z)$ is an ideal *lowpass* filter with cutoff frequency $\pi/2$ and $H(z)$ is an ideal *highpass* filter with cutoff frequency $\pi/2$:



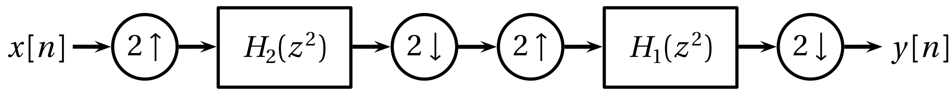
Plot the four spectra $Y_1(e^{j\omega})$, $Y_2(e^{j\omega})$, $Y_3(e^{j\omega})$, $Y_4(e^{j\omega})$.

Exercise 2. Multirate identities

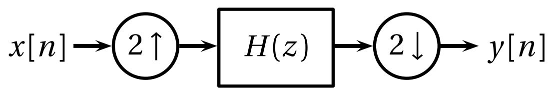
(a) Prove the following two identities:

- Downsampling by 2 followed by filtering by $H(z)$ is equivalent to filtering by $H(z^2)$ followed by downsampling by 2.
- Filtering by $H(z)$ followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering by $H(z^2)$.

(b) Find the overall transfer function of the following system:



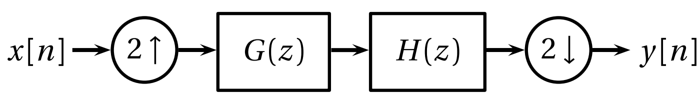
(c) Assume $H(z) = A(z^2) + z^{-1}B(z^2)$ for arbitrary $A(z)$ and $B(z)$. Show that the transfer function of the following system is equal to $A(z)$:

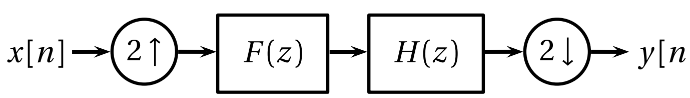


(d) Let $H(z), F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2$$
$$H(z)F(z) + H(-z)F(-z) = 0.$$

Prove the transfer functions of the following systems are one and zero:





Exercise 3. Quantization

The standard model for the error introduced by quantization is that of additive white noise, where the noise is i.i.d. and uncorrelated with the signal:

$$\mathcal{Q}\{x[n]\} = x[n] + e[n].$$

If the quantizer is uniform and the input signal is also uniformly distributed, the probability distribution of the noise samples is also uniform over the interval $[-\Delta/2, \Delta/2]$ where Δ is the quantization step size.

The quantized signal $y[n]$ is processed by a digital filter with impulse response

$$h[n] = \frac{1}{2}[a^n + (-a)^n]u[n].$$

Determine the variance of the noise at the output of the filter and determine the signal-to-noise ratio at the output.

Exercise 4. Digital processing of continuous-time signals

In your grandmother's attic you just found a treasure: a collection of super-rare 78rpm vinyl jazz records. The first thing you want to do is to transfer the recordings to compact discs, so you can listen to them without wearing out the originals. Your idea is obviously to play the record on a turntable and use an A/D converter to convert the line-out signal into a discrete-time sequence, which you can then burn onto a CD. The problem is, you only have a "modern" turntable, which plays records at 33rpm. Since you're a DSP wizard, you know you can just go ahead, play the 78rpm record at 33rpm and sample the output of the turntable at 44.1 KHz. You can then manipulate the signal in the discrete-time domain so that, when the signal is recorded on a CD and played back, it will sound right.

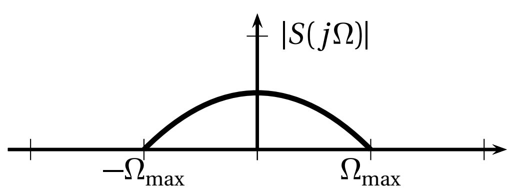
Design a system which performs the above conversion. If you need to get on the right track, consider the following:

- Call $s(t)$ the continuous-time signal encoded on the 78rpm vinyl (the jazz music)
- Call $x(t)$ the continuous-time signal you obtain when you play the record at 33rpm on the modern turntable
- Let $x[n] = x(nT_s)$, with $T_s = 1/44100$.

and answer the following questions:

(a) Express $x(t)$ in terms of $s(t)$.

(b) Sketch the Fourier transform $X(j\Omega)$ when $S(j\Omega)$ is as in the following figure. The highest nonzero frequency of $S(j\Omega)$ is $\Omega_{\text{max}} = (2\pi) \cdot 16000$ Hz (old records have a smaller bandwidth than modern ones).



(c) Design a system to convert $x[n]$ into a sequence $y[n]$ so that, when you interpolate $y[n]$ to a continuous-time signal $y(t)$ with interpolation period T_s, you obtain $Y(j\Omega) = S(j\Omega)$.

(d) What if you had a turntable which plays records at 45rpm? Would your system be different? Would it be better?

Exercise 5. Oversampled sequences

Consider a real-value sequence $x[n]$ for which:

$$X(e^{j\omega}) = 0 \qquad \frac{\pi}{3} \leq |\omega| \leq \pi$$

One sample of $x[n]$ may have been corrupted and we would like to approximately or exactly recover it. We denote n_0 the time index of the corrupted sample and $\hat{x}[n]$ the corresponding corrupted sequence.

(a) Specify a practical algorithm for exactly or approximately recovering $x[n]$ from $\hat{x}[n]$ if n_0 is known.

(b) What would you do if the value of n_0 is not known?

(c) Now suppose we have k corrupted samples: what is the condition that $X(e^{j\omega})$ must satisfy to be able to exactly recover $x[n]$? Describe the algorithm.