Pstat 222C Spring 2025 – Asn 1

Note. **Hand in all Code used for all computations .**

Problem 1: Simulating Heston

The Heston model is an example of a stochastic volatility model with the dynamics

$$dS_t = rS_t dt + S_t \sqrt{V_t} dW_t^1 \tag{1}$$
$$dV_t = \kappa(\theta - V_t)\, dt + \eta \sqrt{V_t} dW_t^2. \tag{2}$$

Above V_t is the volatility process and (S_t) is the asset price; the driving Wiener processes W^1, W^2 have constant correlation ρ.

(a) Write down the Milstein scheme for the volatility process (V_t).

(b) We will implement two schemes: [A] Euler for (S_t) + Euler for (V_t); [B] Euler for (S_t) and Milstein for (V_t). We will then price two different contracts:

- A Put option with payoff $\Phi(S_T) = e^{-rT}(K - S_T)_+$
- A discretely monitored Asian option with payoff $\Phi(S_{[0,T]}) = e^{-rT}\left(S_T - \frac{1}{N_T}\sum_{n=1}^{N_T} S_{T_n}\right)_+$ where $T_n = n/52$ are weekly

The model parameters are: $r = 0.05; \kappa = 1; \theta = 0.2; \eta = 0.5; \rho = -0.4$ with initial conditions $S_0 = 100, V_0 = 0.25$ and option parameters $T = 1, K = 100, N_T = 52$.

Use $\Delta t = \frac{1}{52 \cdot 2^r}$ for $r = 1, 2, 3, 4$ and $M = 10^5$ Monte Carlo simulations to estimate prices of the above two contracts. For each run report (i) 95% confidence interval; (ii) 99% confidence interval; (iii) running time of your scheme.

In total you're doing 16 runs (2 options x 2 schemes x 4 time-steps). *Hint:* for the Put option, my reference reports a true option price of 15.60.

(c) Summarize your findings in terms of comparing the two schemes and the convergence as $\Delta t \to 0$.

Problem 2: Wealth Maximization by Monte Carlo

The Merton problem aims to maximize utility of terminal wealth by finding the optimal investment portfolio. We will consider the simplest case where there is a single risky asset (S_t) and the savings account $B_t := e^{rt}$ which earns constant interest rate r. Let π be the fixed *proportion* of wealth invested in the risky asset. Then the wealth process satisfies

$$dX_t = \pi X_t (dS_t/S_t) + (1 - \pi) X_t (dB_t/B_t).$$

We will assume that

$$dS_t = \mu S_t dt + \sigma S_t dW_t - S_{t-} dJ_t$$

follows the Geometric Brownian motion with jumps model. Above, $J_t = \sum_{j=1}^{N_t} (e^{Y_j} - 1)$ is a compound Poisson process with constant intensity λ: $N_t \sim Poisson(\lambda t)$ and exponential jumps $Y_j \sim Exp(\zeta)$ i.i.d. This means that at Poisson times $\tau_1, \ldots,$ the stock price experiences instantaneous drops in value of Y_j percent: $S_{\tau_j} = S_{\tau_j -}(2 - e^{Y_j})$. Note that the drift of (S_t) is μ

because we are under the physical measure $\mathbb{P}$, not risk-neutral $\mathbb{Q}$. Potentially $S_{\tau_j} < 0$, so make sure to round up to zero if this ever happens.

We will use the parameters $T = 2, r = 0.05, \mu = 0.25, \sigma = 0.25, \lambda = 6, \zeta = 50$ where the last parameter means that average jumps are 2% down.

(a) Fix $\pi = 0.5$. Use Monte Carlo to evaluate the expected utility of terminal wealth $\mathbb{E}[U(X_T)]$ where $U(x) = x^\gamma/\gamma$, $\gamma = -1.5$ is of power-type. To simulate, use a *non-uniform grid*, namely the union of the regular grid with mesh size $\Delta t = 0.02$ and the jump times $\tau_1, \ldots$. Since the Poisson process (N_t) is independent, you should simulate (τ_j, Y_j) *first*, and then use the resulting path-dependent time grid to implement an Euler scheme for (X_t).

(b) By trying different π's, find π^* which (empirically) maximizes $\mathbb{E}[U(X_T)]$. Include a plot showing your estimates of $\pi \mapsto \mathbb{E}[U(X_T^{(\pi)})]$ for the values of π you tried.

Hint 1: without jumps ($\lambda = 0$), it is known that $\pi^* = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}$. Hint 2: you may wish to use the same driving Brownian motion increments ξ_k when simulating $X_T^{(\pi)}$ for different π's as a way to reduce variance among the respective estimates.

Problem 3: Multi Level Monte Carlo

For a fixed time-step Δt, the Monte Carlo estimate using $X^{(\Delta t)}$ will be biased since $\mathbb{E}[\Phi(X_T)] \neq \mathbb{E}[\Phi(X_T^{(\Delta t)})]$.

Consider the **corridor** option with payoff 1 if $X_t \in [L_1, L_2]$ for all $t \leq T$ and payoff zero otherwise. Thus, the asset price is required to remain between L_1 and L_2 the entire time until maturity.

We use the CEV model

$$dX_t = rX_t dt + \sigma X_t^\gamma dW_t$$

where we select $\gamma = 0.8$ (note that $\gamma = 1$ corresponds to the Black-Scholes model) with parameters $X_0 = 20, \sigma = 0.4, r = 0.04, T = 1/2$ and $L_1 = 15, L_2 = 25$.

(a) Using $\Delta t = 0.01$ implement the Euler scheme for (X_t) and estimate via Monte Carlo (based on X_{t_k} at discrete times only) with $M = 10^5$ the 95% confidence interval for the price of the corridor option.

(b) Implement antithetic sampling for this example (for the plain Euler scheme, no Multilevel) – how much is variance reduced?

(c) Implement the multi level Monte Carlo scheme with 3 levels and $N_1 = 50,000; N_2 = 15,000; N_3 = 5000$, halving the time step at each level. To this end, use

$$X_{t_{k+1}}^{(h/2)} = X_{t_k}^{(h/2)} + a(X_{t_k}^{(h/2)})(h/2) + b(X_{t_k}^{(h/2)})\sqrt{h/2} \cdot \xi_{k+1}, \qquad k = 1, \ldots, 2T/h$$

and

$$X_{t_{k'+1}}^{(h)} = X_{t_{k'}}^{(h)} + a(X_{t_{k'}}^{(h)})h + b(X_{t_{k'}}^{(h)})\sqrt{h} \cdot \xi'_{k'+1}, \qquad k' = 1, \ldots, T/h$$

we set $\xi'_{k+1} = \frac{\xi_{2k+1}+\xi_{2k+2}}{\sqrt{2}}$ so the increments from the more refined scheme are *re-used* in the coarser scheme (the $\sqrt{2}$ is to match $\xi, \xi' \sim \mathcal{N}(0, 1)$ all being standard normals).

Compare the running time and the variance of the MLMC scheme to the schemes from part a) and b).