Math 104A Homework #1 *

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General Instructions: Please write your homework papers neatly. You need to turn in both your codes and descriptions on the appropriate runs you made by compressing and emailing them to TA with your name and ID number as the file name. Write your own code, individually. Do not copy codes!

1. Review and state the following theorems of Calculus:

   (a) The Intermediate Value Theorem.

   (b) The Mean Value Theorem.

   (c) Rolle's Theorem.

   (d) The Mean Value Theorem for Integrals.

   (e) The Weighted Mean Value Theorem for Integrals.

2. Write a computer code to implement the Composite Trapezoidal Rule quadrature

$$T_h[f] = h\left(\frac{1}{2}f(x_0) + f(x_1) + ... + f(x_{N-1}) + \frac{1}{2}f(x_N)\right), \tag{1}$$

   to approximate the definite integral

$$I[f] = \int_a^b f(x)dx, \tag{2}$$

   using the equally spaced points $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, . . . , $x_N = b$, where $h = (b-a)/N$. Make sure that **all your codes** have a preamble which describes the purpose of the code, all the input variables, the expected output, your name, and the date of the last time you modified the code.

3. To test your code, take $f(x) = xe^{x^2}$ in $[0, 1]$, compute the error $|I[f] - T_h[f]|$ for $h = 1/10$, $1/20$, $1/40$, and verify that T_h has a convergent trend at the expected quadratic rate.

4. Consider the definite integral

$$I[e^{-x^2}] = \int_0^1 e^{-x^2} dx, \tag{3}$$

We cannot calculate its exact value but we can compute accurate approximations to it using $T_h[e^{-x^2}]$. Let

$$q(h) = \frac{T_{h/2}[e^{-x^2}] - T_h[e^{-x^2}]}{T_{h/4}[e^{-x^2}] - T_{h/2}[e^{-x^2}]}. \tag{4}$$

Using your code, find a value of h for which $q(h)$ is approximately equal to 4. (a) Get an *approximation* of the error, $I[e^{-x^2}] - T_h[e^{-x^2}]$, for that particular value of h. (b) Use this error approximation to obtain the *extrapolated*, improved, approximation

$$S_h[e^{-x^2}] = T_h[e^{-x^2}] + \frac{4}{3}\left(T_{h/2}[e^{-x^2}] - T_h[e^{-x^2}]\right). \tag{5}$$

Explain why $S_h[e^{-x^2}]$ is more accurate and converges faster to $I[e^{-x^2}]$ than $T_h[e^{-x^2}]$.