104A: Homework 1

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Question 1

0.1 Intermediate Value Theorem

Suppose f is continuous over the interval $[a, b]$. Then

$$\forall\ p \in \left[\min_{[a,b]} f(x), \max_{[a,b]} f(x)\right],\ \exists\, c \in [a, b] \text{ such that } f(c) = p$$

0.2 Mean Value Theorem

Suppose f is continuous over $[a, b]$ and differentiable over (a, b). Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

0.3 Rolle's Theorem

Suppose f is continuous over $[a, b]$ and differentiable over (a, b). Also suppose that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

0.4 Mean Value Theorem for Integrals

Suppose f is continuous over $[a, b]$. Then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b - a)$$

0.5 Weighted Mean Value Theorem for Integrals

Suppose f is continuous over $[a, b]$, and that g is integrable and doesn't change sign over $[a, b]$. Then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Question 4

From the computer program, we find that a sufficient h is $\frac{1}{1054}$

Let $f(x) = \mathrm{e}^{-x^2}$. Since the error of the Composite Trapezoidal Method is $O(h^2)$, we have

$$E_h(f) = O(h^2) = ch^2 + o(h^2)$$

Equivalently, since $E_h(f) = I(f) - T_h(f)$

$$I(f) = T_h(f) + ch^2 + o(h^2)$$
$$\implies I(f) = T_{h/2}(f) + c\left(\frac{h}{2}\right)^2 + o(h^2)$$

Combining these and solving for ch^2 yields

$$ch^2 = \frac{4}{3}\left(T_{h/2}(f) - T_h(f)\right) + o(h^2)$$

So we find that $E_h = \frac{4}{3}\left(T_{h/2}(f) - T_h(f)\right) + o(h^2)$. Since we choose a sufficiently large h such that $o(h^2)$ is negligent, we find

$$E_h(f) \approx 5.5191609504845474 \cdot 10^{-8}$$

Now let's define and evaluate

$$\begin{aligned} S_h(f) &= T_h(f) + \frac{4}{3}\left(T_{h/2}(f) - T_h(f)\right) \\ &= 0.7468241328124265 \end{aligned}$$

Now to argue why $S_h(f)$ is a better approximation than $T_h(f)$, we have that

$$\begin{aligned} I(f) &= T_h(f) + E_h(f) \\ &= T_h(f) + O(h^2) \\ &= S_h(f) + o(h^2) \end{aligned}$$

So the error for S_h is $o(h^2)$, while the error for T_h is $O(h^2)$. Thus, S_h is a superior approximation.