Lecture 6: Species distributions and 2D spatial models

May 3, 2016

Why might we care about 2D spatial models?

1. Experimental plots
   – Often uses grid
2. Observational analysis
   – Fish surveys

2D spatial models

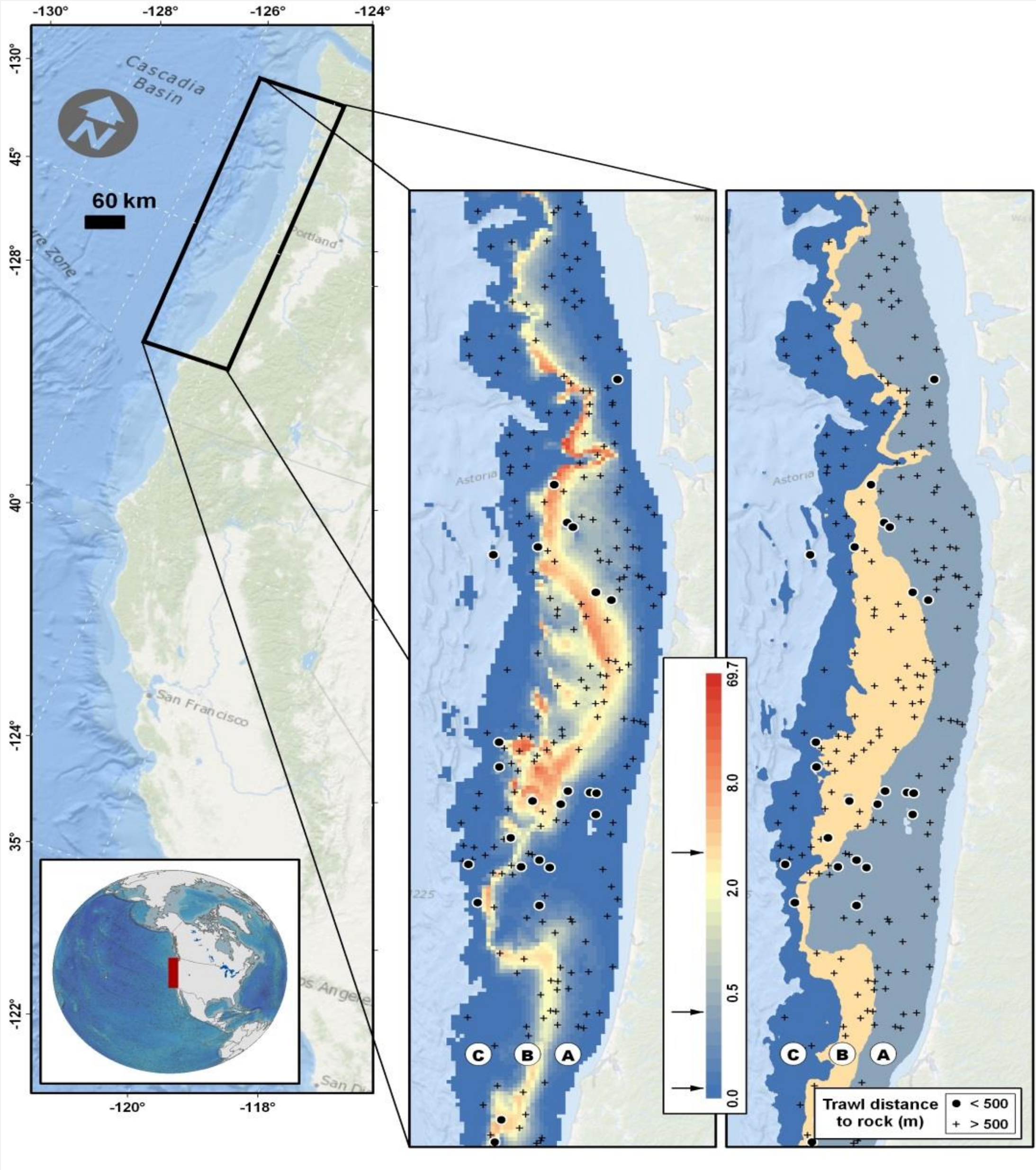
Species distribution models

- Widely used tool in zoology, conservation-planning, and invasion biology
- Synonyms:
  - Climate envelop model – Fit model to species observations, and then identify other areas with suitable conditions
  - Species density model – Sometimes used when fitting to density rather than presence/absence data
- Often used to infer “Grinnellian niche”
  - Distribution is a product of niche, dispersal ability, and species interactions

2D spatial models

Spatial models are useful for estimating species distribution

- Capture fine-scale variation
- Decrease residual variation -> Decrease standard errors
- Shelton Thorson Ward Feist (2014) CJFAS



2D spatial models

Two potential treatments

1. Equally spaced grid
2. Unequal or sporadic spacing

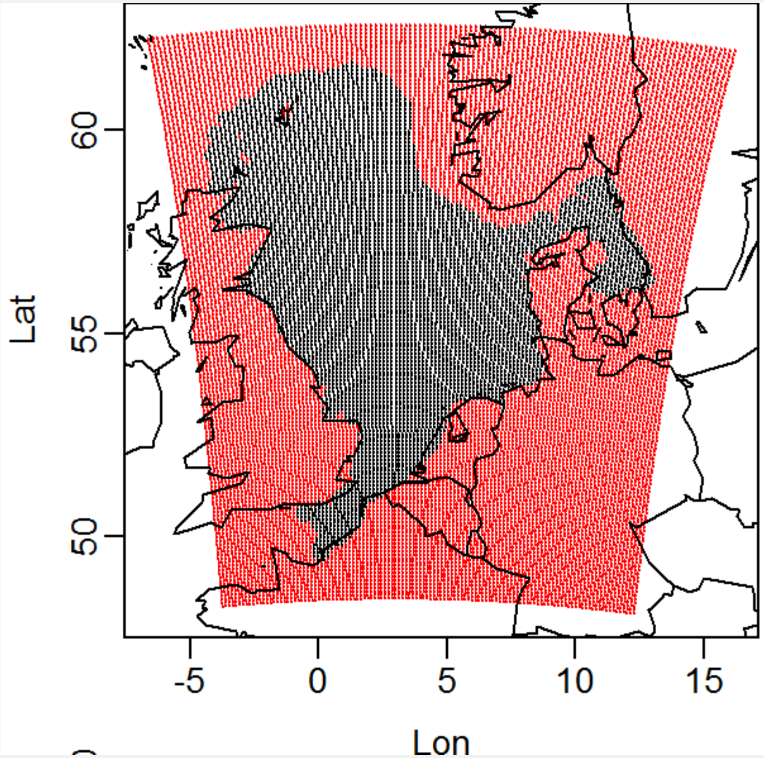
2D spatial models

Two potential treatments

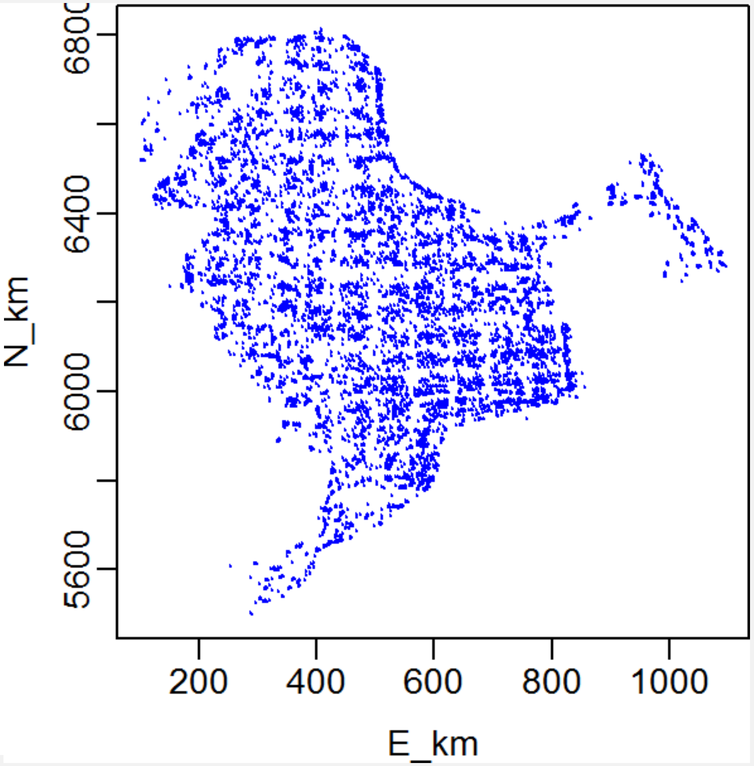
1. **Equally spaced grid**

Example #1: Divide north sea into grid

Domain of samples (black: Included)



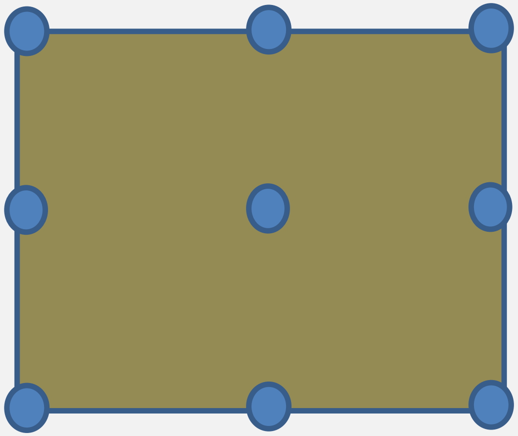
Location of samples



2D spatial models

Imagine a 3x3 grid

- Equal spacing among sample locations
- ε_{ij} is the value for row *i* and column *j*
- $\boldsymbol{\varepsilon}_i$ is the value for row *i* and all columns
- $\mathbf{E}$ is the value for all rows and columns



Review

- Let's assume first-order autoregression:

$$\varepsilon_{s+1} \sim Normal(\rho\varepsilon_s, \sigma_\varepsilon^2)$$

- Then the joint distribution is multivariate normal

$$\boldsymbol{\varepsilon} \sim MVN(0, \boldsymbol{\Sigma})$$

  – where

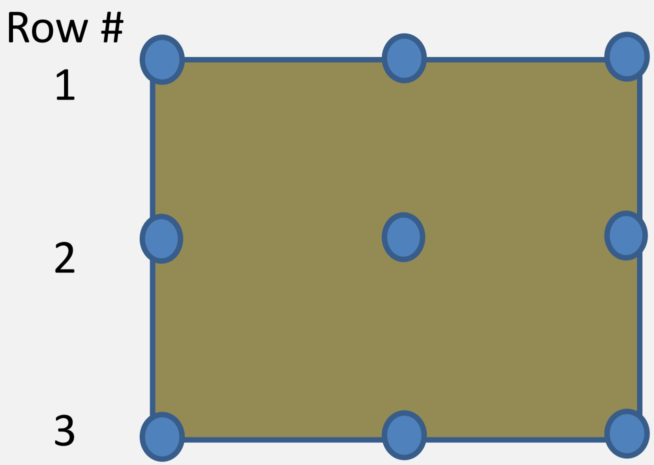
$$\boldsymbol{\Sigma} = \frac{{\sigma_\varepsilon}^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{{\sigma_\varepsilon}^2}\begin{bmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{bmatrix}$$

2D spatial models

Intuitive to extend this

- What is the distribution for Row #2 conditional on Row #1
- [Work through on board]



2D spatial models

Distribution for row *i+1* conditional on row *i*

$$\boldsymbol{\varepsilon}_{i+1} \sim MVN(\rho \boldsymbol{\varepsilon}_i, \boldsymbol{\Sigma})$$

– Where

$$\boldsymbol{\Sigma} = \frac{\sigma_{\varepsilon}^{\ 2}}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

– i.e., its identical to the 1D case, except replacing the normal distribution with a multivariate normal distribution

2D spatial models

Joint distribution of all sites

- Then:

$$\mathrm{vec}(\mathbf{E}) \sim MVN(0, \boldsymbol{\Sigma}_{\mathrm{total}})$$

- where

$$\boldsymbol{\Sigma}_{\mathrm{total}} = \frac{{\sigma_\varepsilon}^2}{1-\rho^2}\begin{bmatrix} \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \rho\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \rho^2\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \\ \rho\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \rho\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \\ \rho^2\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \rho\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} & \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \end{bmatrix}$$

2D spatial models

Joint distribution of all sites

- Then:

$$vec(\mathbf{E}) \sim MVN(0, \mathbf{\Sigma}_{\text{total}})$$

- where

$$\mathbf{\Sigma}_{\text{total}} = \mathbf{\Sigma} \otimes \mathbf{\Sigma}$$

Background:

- $\otimes$ is the Kroenecker product

$$\mathbf{A} \otimes \mathbf{B} \equiv \begin{matrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \dots & a_{nn}\mathbf{B} \end{matrix}$$

- Easy and compact way to explain spatio-temporal models

2D spatial models

- Kroenecker product inverse is easy

$$\boldsymbol{\Sigma}_{\text{total}}^{-1} = (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})^{-1} = \boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}$$

where we know how to calculate $\boldsymbol{\Sigma}^{-1}$

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma_\varepsilon{}^2} \begin{bmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{total}^{-1} = \frac{1}{\sigma_\varepsilon{}^2} \begin{bmatrix} 1+\rho^2 \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & -\rho \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & 0 \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} \\ -\rho \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & 1+\rho^2 \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & -\rho \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} \\ 0 \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & -\rho \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} & 1+\rho^2 \begin{pmatrix} 1+\rho^2 & -\rho & 0 \\ -\rho & 1+\rho^2 & -\rho \\ 0 & -\rho & 1+\rho^2 \end{pmatrix} \end{bmatrix}$$

2D spatial models

- Kroenecker product conserves “sparseness”
  - If $\mathbf{A}$ …
    - is n_A by n_A
    - It has n_A^2 elements
    - But has $m_A < n_A^2$ non-zero elements
    - It has “sparseness” = m_A/n_A^2
  - If $\mathbf{B}$ …
    - has $m_B < n_B^2$ non-zero elements
  - Then $\mathbf{A} \otimes \mathbf{B}$…
    - Has “sparseness” $(m_B m_A)/(n_A n_B)^2$

Look at GitHub functions

- AR1
  - http://kaskr.github.io/adcomp/classdensity_1_1AR1__t.html
- SEPERABLE
  - http://kaskr.github.io/adcomp/classdensity_1_1SEPARABLE__t.html

Modify code to avoid using Rmgauss

- How to do this?