Lecture 7: Spatio-temporal models

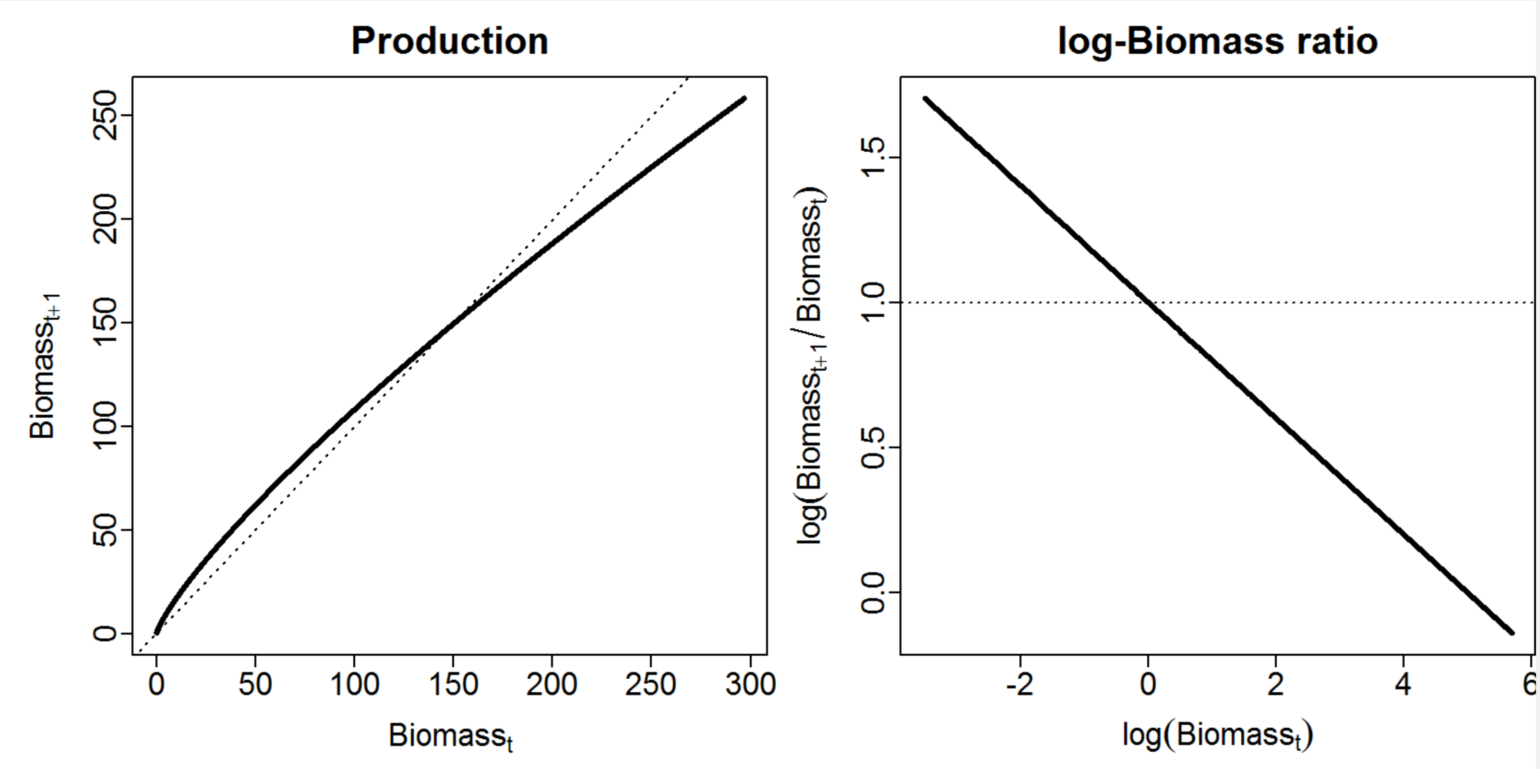
May 12, 2016

Gompertz model

$$d_{t+1} = d_t \times \exp(\alpha) \times \exp(-\beta \log(d_t)) \times \exp(\varepsilon_t)$$

$$\varepsilon_t \sim Normal(0, \sigma_\varepsilon^2)$$

$$\log(b_t) \sim Normal\left(\log(d_t), \sigma_b^2\right)$$



Gompertz model

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$$\log(b_t) \sim Normal\left(\log(d_t), \sigma_b^2\right)$$

- Fits to an index of abundance, $\mathbf{b}$
- β is the strength of density dependence
  - Linear impact of $\log(d_t)$ on per-capita productivity
- ε_t is a lognormally distributed process error
- σ_ε^2 is the variance of log-process errors
- σ_b^2 is the variance of log-observation errors

Gompertz model (Version #2)

$$\log(d_{t+1}) = \alpha + \rho \log(d_t) + \varepsilon_t$$

$$\varepsilon_t \sim Normal(0, \sigma_\varepsilon^2)$$

$$\log(b_t) \sim Normal(d_t, \sigma_b^2)$$

- Log-density follows an autoregressive process over time
- $\rho = 1 - \beta$ is "density dependence"
  - $\rho = 0$ means each year fluctuates independently
  - $\rho = 1$ means the population follows a random-walk with no equilibrium

Gompertz model (version #3)

$$\mathbf{d} = \alpha + \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} \sim \mathrm{MVN}(\mathbf{0}, \sigma_{\varepsilon}^{2}\mathbf{R})$$

$$\log(\mathbf{b}) \sim \mathrm{MVN}(\mathbf{d}, \sigma_{b}^{2}\mathbf{I})$$

- Where $\mathbf{R}$ is the correlation matrix for an AR1 process

$$\mathbf{R} = \frac{1}{1-\rho^{2}} \begin{bmatrix} 1 & \rho & \rho^{2} & \dots & \rho^{T-1} \\ \rho & 1 & \rho & \dots & \rho^{T-2} \\ \rho^{2} & \rho & 1 & \dots & \rho^{T-3} \\ \dots & \dots & \dots & \dots & \dots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \dots & 1 \end{bmatrix}$$

REVIEW: Four ways to code

1. Stochastic process

$$\varepsilon_{s+1} \sim Normal(\rho \varepsilon_s, \sigma_\varepsilon^2)$$

2. Via precision matrix Q

$$\Pr(\boldsymbol{\varepsilon}|\boldsymbol{\Sigma}) \propto \sqrt{\det(\mathbf{Q})}\exp\left(-0.5\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{Q}\boldsymbol{\varepsilon}\right)$$

3. Via multivariate normal density function in TMB

$$\boldsymbol{\varepsilon} \sim MVN(0, \boldsymbol{\Sigma})$$

4. Via autoregressive function in TMB

$$\boldsymbol{\varepsilon} \sim f(\rho, \sigma_\varepsilon^2)$$

Spatial Gompertz model

$$\log\left(d_{t+1}(s)\right) = \alpha + \omega(s) + \rho\log\left(d_t(s)\right) + \varepsilon_t(s)$$

$$\boldsymbol{\omega} \sim \mathrm{MVN}(0, \sigma_\omega^2 \mathbf{C}_\omega)$$

$$\boldsymbol{\varepsilon}_{t+1} \sim MVN(\rho\boldsymbol{\varepsilon}_t, \sigma_\varepsilon^2 \mathbf{C}_\varepsilon)$$

$$c_i \sim \mathrm{Poisson}\left(d_{t_i}(s_i)\right)$$

- Very similar to standard Gompertz model
- Fits to spatially referenced count data

Implications

- Expectation:

$$\mathbb{E}(\log(\mathbf{d})) = \frac{\alpha}{1-\rho}$$

$$\mathbb{E}(\log(d(s))) = \frac{\alpha + \omega(s)}{1-\rho}$$

Variance

$$\mathbb{V}(\log(d(s))) = \sigma_\omega^2 + \frac{\sigma_\varepsilon^2}{1-\rho^2}$$

Spatial Gompertz model:

- Can use same trick as "Gompertz model version #3"

$$c_i \sim \text{Poisson}(\exp(d_{t_i}(s_i)))$$

$$\mathbf{d} = \alpha + \boldsymbol{\omega} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\omega} \sim \text{MVN}(\mathbf{0}, \frac{\sigma_\omega^2}{1-\rho}\mathbf{C}_\omega)$$

$$\text{vec}(\mathbf{E}) \sim \text{MVN}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{C}_\varepsilon \otimes \mathbf{R})$$

Where $\mathbf{R}$ is the correlation matrix for 1st order autocorrelation

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{T-1} \\ \rho & 1 & \rho & \dots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{T-3} \\ \dots & \dots & \dots & \dots & \dots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \dots & 1 \end{bmatrix}$$

Spatial Gompertz model:

Innovations parameterization

$$\log\left(d_{t+1}(s)\right) = \alpha + \omega(s) + \rho\log\left(d_t(s)\right) + \varepsilon_t(s)$$

$$\boldsymbol{\omega} \sim \mathrm{MVN}(\mathbf{0}, \sigma_\omega^2 \mathbf{C}_\omega)$$

$$\boldsymbol{\varepsilon}_t \sim \mathrm{MVN}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{C}_\varepsilon)$$

State-space parameterization

$$\log(d_{t+1}^*(s)) = \alpha + \omega(s) + \rho\log\left(d_t(s)\right)$$

$$\log(\mathbf{d}_t) \sim \begin{cases} \mathrm{MVN}\left(\dfrac{\alpha + \boldsymbol{\omega}}{1-\rho}, \sigma_\varepsilon^2 \mathbf{C}_\varepsilon\right) & \text{if } t = 1 \\ \mathrm{MVN}(\log(\mathbf{d}_{t-1}^*), \sigma_\varepsilon^2 \mathbf{C}_\varepsilon) & \text{if } t > 1 \end{cases}$$

$$\boldsymbol{\omega} \sim \mathrm{MVN}(\alpha\mathbf{1}, \sigma_\omega^2 \mathbf{C}_\omega)$$

… where both use the same measurement model

$$c_i \sim \mathrm{Poisson}\left(d_{t_i}(s_i)\right)$$

REVIEW: Four ways to code

1. **Stochastic process**

$$\boldsymbol{\varepsilon_{s+1} \sim Normal(\rho \varepsilon_s, \sigma_\varepsilon^2)}$$

2. Via precision matrix Q

$$\Pr(\boldsymbol{\varepsilon}|\boldsymbol{\Sigma}) \propto \sqrt{\det(\mathbf{Q})}\exp\left(-0.5\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{Q}\boldsymbol{\varepsilon}\right)$$

3. Via multivariate normal density function in TMB

$$\boldsymbol{\varepsilon} \sim MVN(0, \boldsymbol{\Sigma})$$

4. **Via autoregressive function in TMB**

$$\boldsymbol{\varepsilon \sim f(\rho, \sigma_\varepsilon^2)}$$

Without spatial

		Temporal effect		
Spatio-temporal effect		**None**	**Independent**	**Smoothed**
	None	$\mathrm{gam}(c \sim 1)$	$\mathrm{gam}\left(c \sim 1 + factor(t)\right)$	$\mathrm{gam}\left(c \sim 1 + s(t)\right)$
	Independent	$\mathrm{gam}(c \sim 1$	$\mathrm{gam}(c \sim 1 + factor(t)$	$\mathrm{gam}(c \sim 1 + s(t)$
	Smoothed	$\mathrm{gam}(c \sim 1 + s(x, y, t))$	$\mathrm{gam}(c \sim 1 + factor(t)$	$\mathrm{gam}(c \sim 1 + s(t) + s(x, y, t))$

With spatial

		Temporal effect		
Spatio-temporal effect		**None**	**Independent**	**Smoothed**
	None	$\mathrm{gam}(c \sim 1 + s(x, y))$	$\mathrm{gam}(c \sim 1 + s(x, y)$	$\mathrm{gam}\left(c \sim 1 + s(x, y) + s(t)\right)$
	Independent	$\mathrm{gam}(c \sim 1 + s(x, y)$	$\mathrm{gam}(c \sim 1 + factor(t)$	$\mathrm{gam}(c \sim 1 + s(x, y) + s(t)$
	Smoothed	$\mathrm{gam}(c \sim 1 + s(x, y)$	$\mathrm{gam}(c \sim 1 + s(x, y)$	$\mathrm{gam}(c \sim 1 + s(x, y) + s(t)$

What to do if you have 1000s of unique locations?

1. Get a bigger computer
   - TMB allows parallelization
2. Decrease spatial resolution
   - Define number of “knots”
   - Run k-means algorithm to identify placement of knots
     - Minimize distance between locations and nearest knot
   - Associate each sample with the nearest knot
   - Assumes that density is constant at finer scales than the distance between knots
3. Run in batches
   - Run model on smaller scales
   - Run “meta-analysis” model on results from each small-scale model

In-class exercise:

Convert to using a different production function

Hint:

- Gompertz production function

$$\log(d_{t+1}) = \alpha + (1 - \beta)\log(d_t)$$

In other words…

$$d_{t+1} = d_t \times \exp(\alpha - \beta \log(d_t))$$

- Moran-Ricker production function

$$d_{t+1} = d_t \times \exp(\alpha - \beta d_t)$$