Lab 1: Generalized linear models

March 31, 2016

Generalized linear models

– Specify distribution for response variable
– Specify linear predictor
– Specify link function
    - Calculates expected response given linear predictor

- Example
    – Counts for local densities

$$c_i \sim Poisson(\lambda_i)$$
$$\log(\lambda_i) = \mathbf{x}_i \boldsymbol{\beta}$$

Common distributions for data

– Discrete

Name	Notation	Domain	Range
Bernoulli	$B \sim Bernoulli(p)$	$0 \leq p \leq 1$	B = {0, 1}
Binomial	$N \sim Binomial(p, n)$	$0 \leq p \leq 1$	N = {0, 1, …, n}
Poisson	$N \sim Poisson(\lambda)$	$\lambda > 0$	N = {0, 1, …,∞}
Negative binomial	$N \sim NegativeBinomial(\lambda, \theta)$	$\lambda > 0$ $\theta > 0$	N = {0, 1, …,∞}
Conway-Maxwell-Poisson	$N \sim CMP(\mu, \nu)$	$\mu > 0$ $\nu > 0$	N = {0, 1, …,∞}

Common distributions for data

– Continuous

Name	Notation	Domain	Range
Normal	$Y \sim Normal(\mu, \sigma^2)$	$\sigma^2 > 0$	Unrestricted
Lognormal	$\log(Y) \sim Normal(\mu, \sigma^2)$	$\sigma^2 > 0$	$Y > 0$
Gamma	$Y \sim Gamma(\mu, CV)$	$\mu > 0$ $CV > 0$	$Y > 0$
Beta	$p \sim Beta(\alpha, \beta)$	$\alpha > 0$, $\beta > 0$	$0 < p < 1$

How to chose a distribution for data?

- Choice 1 – is it *continuous* or *discrete?*
  - Continuous: normal, lognormal, beta, gamma
  - Discrete: Bernoulli, binomial, poisson, negative binomial
- Choice 2 – what is the range of possible values?
  - E.g., if discrete:
    - If is is 0 or 1, then its Bernoulli
    - If its between 0 and N, where N is the number of trials, then its Binomial
- Choice 3 – How flexible do you want it?

How to chose a distribution for data?

- Frequent null models:
  1. Binomial
  2. Poisson
  3. Normal

How to chose a distribution for data?

- Binomial
  - If you have one or more binary events:

$$B_i \sim \text{Bernoulli}(p)$$

  - Then the sum of successes…

$$N = \sum_{i=1}^{n_i} B_i$$

… follows a binomial distribution

$$N \sim \text{Binomial}(p, n)$$

  - Characteristics:

$$\mathbb{E}(N) = pn$$

$$\mathbb{V}(N) = np(1 - p)$$

How to chose a distribution for data?

- Poisson
  - If you have a lot of independent events, each with low probability:

$$N \sim \text{Binomial}(p, n)$$

where $pn \gg 0$ and $p \ll 1$

  - Then the number of successes follows a Poisson distribution

$$N \sim \text{Poisson}(np)$$

  - Characteristics:

$$\mathbb{E}(N) = np$$
$$\mathbb{V}(N) = np$$

How to chose a distribution for data?

- Normal
  - If you have one or more events:

$$B_i \sim g(\boldsymbol{\theta})$$

where $g(\boldsymbol{\theta})$ is some unknown density function

  - Then the sum of outcomes …

$$N = \sum_{i=1}^{n_i} b_i$$

… will converge on a normal distribution

$$N \sim \mathrm{Normal}(\mu, \sigma_b^2)$$

… as the number of events gets large $n_i \to \infty$

$$\mathbb{E}(N) = \mu = n_i \mathbb{E}\big(g(\boldsymbol{\theta})\big)$$

$$\mathbb{V}(N) = \sigma_b^2 = n_i^2 \mathbb{V}\big(g(\boldsymbol{\theta})\big)$$

Review:

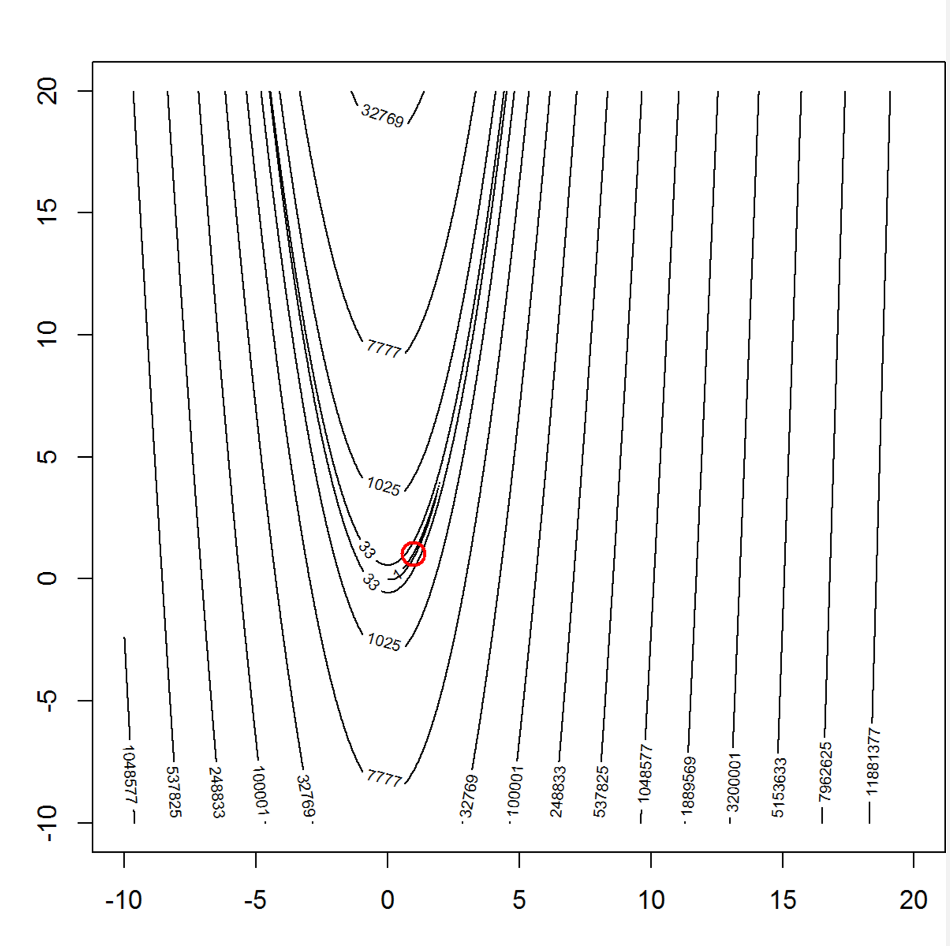
- Maximum likelihood estimation (MLE)

$$\widehat{\boldsymbol{\theta}} = \operatorname{argmax}_{\boldsymbol{\theta}}(L(\boldsymbol{\theta}; \mathbf{y}))$$

  - Where $\widehat{\boldsymbol{\theta}}$ is the MLE estimate of parameters
  - Where $\operatorname{argmax}_{\boldsymbol{\theta}}(L(\boldsymbol{\theta}; \mathbf{y}))$ is the maximum value for $L(\boldsymbol{\theta}; \mathbf{y})$ that can be achieved for any value of $\boldsymbol{\theta}$
  - *argmax* is done using maximization algorithms

How to maximize the likelihood function

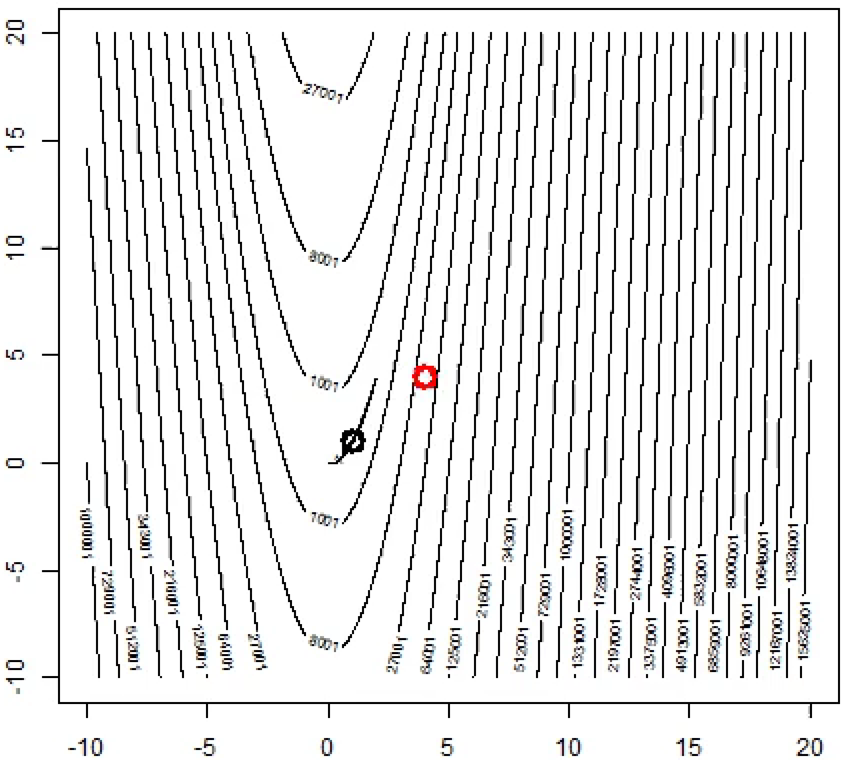
- Nonlinear minimizers
- Test using Rosenbrook “Banana” function



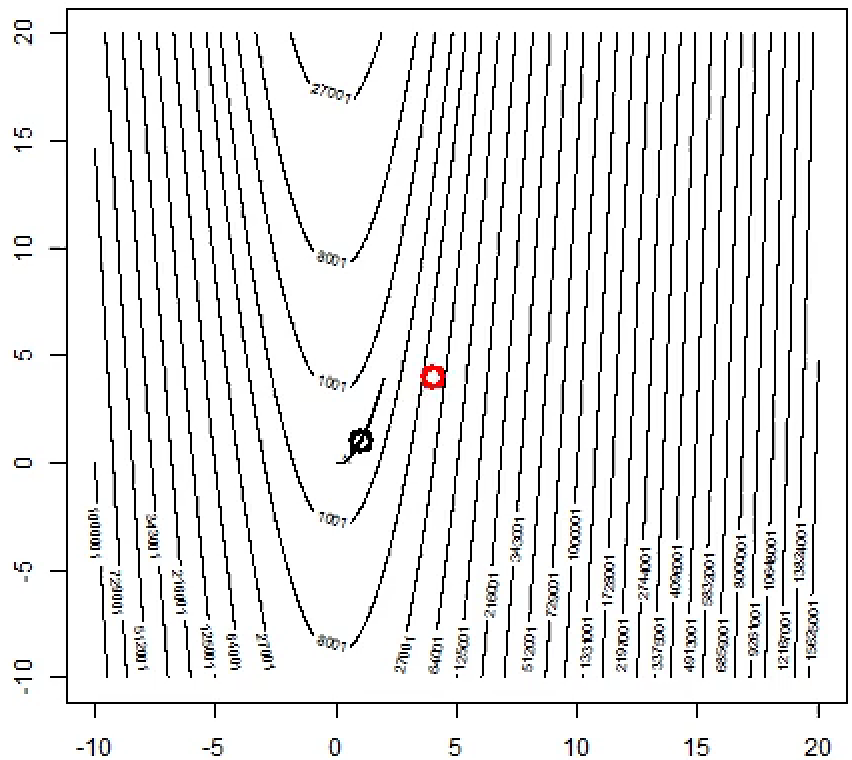
How to maximize the likelihood function

– Methods without gradients are slow

Quasi-Newton



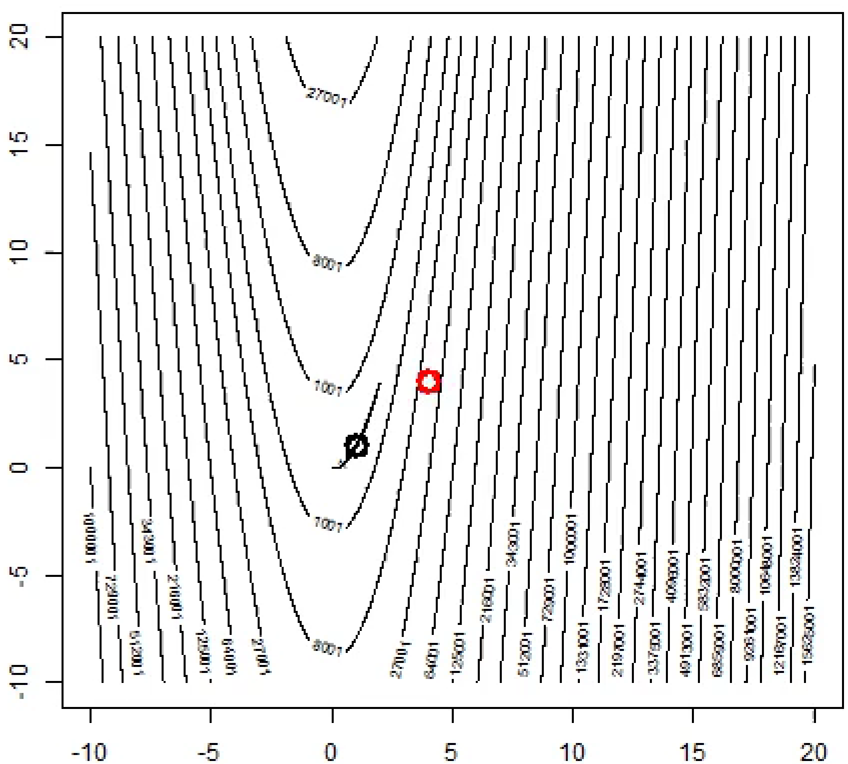
Nelder-Mead



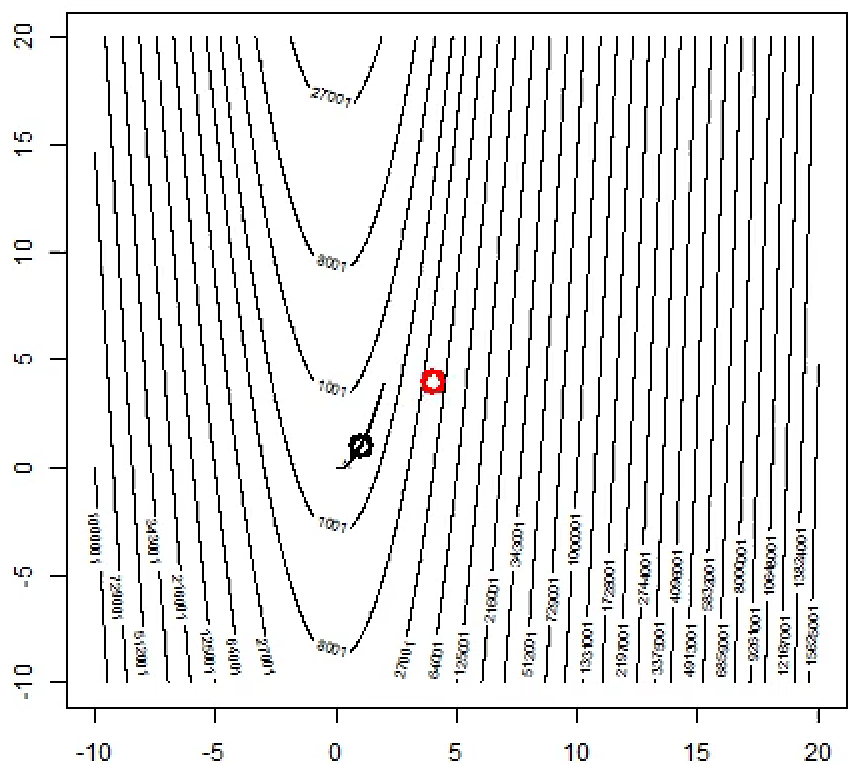
How to maximize the likelihood function

– Methods with gradients are much faster!

TMB using Nelder-Mead



Nelder-Mead



Example #1 – What is the mean density of canary rockfish in the California Current?

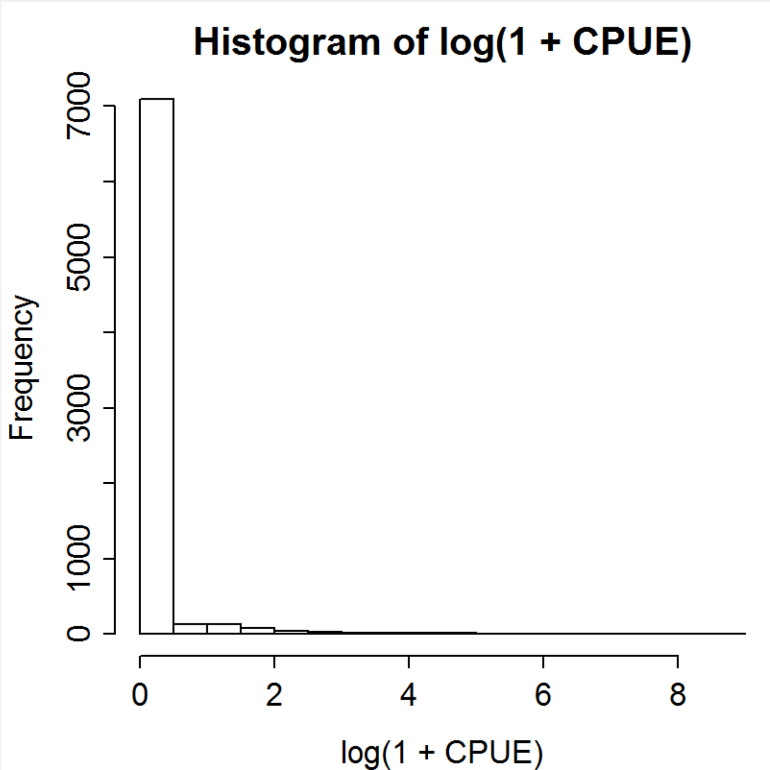
- Define linear predictor matrix

$$x_i = 1$$

  - i.e.,

$$\mathbf{X} = \mathbf{1}$$

  - We call **X** an intercept matrix

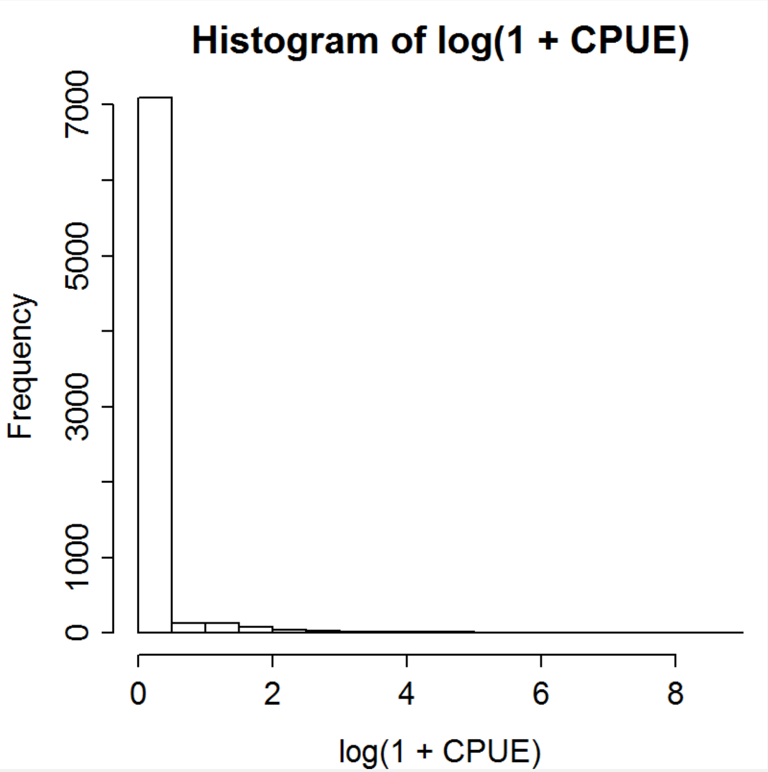


- Generalized linear models
  - Specify distribution for response variable
  - Specify function for expected value

- Canary catch rates

$$\log(\lambda_i) = \mathbf{x}_i \boldsymbol{\beta}$$

$$\Pr(C = c_i) = \begin{cases} \theta_1 & \text{if } c_i = 0 \\ (1-\theta_1) Lognormal(\lambda_i, \theta_2) & \text{if } c_i > 0 \end{cases}$$



Hint

If:

$$\Pr(c_i) = \begin{cases} \theta_1 & \text{if } c_i = 0 \\ (1-\theta_1)Lognormal(\lambda_i, \theta_2) & \text{if } c_i > 0 \end{cases}$$

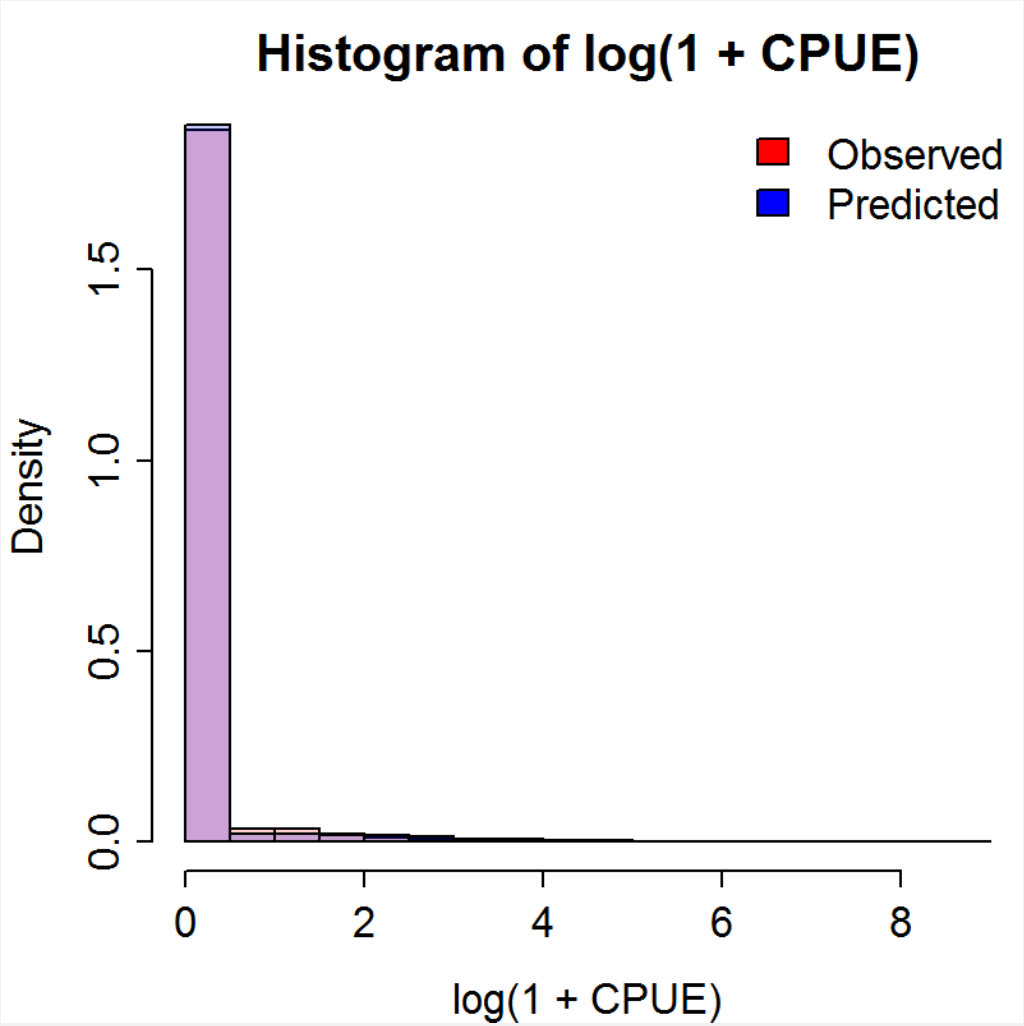
Then:

$$\log(\Pr(c_i)) = \begin{cases} \log(\theta_1) & \text{if } c_i = 0 \\ \log(1-\theta_1) + \log(Lognormal(\lambda_i, \theta_2)) & \text{if } c_i > 0 \end{cases}$$

[Work on TMB code in groups of 2 for 20 minutes]

Conclusion

– Decent fit…



How do we assess fit?

- We want expected predictive loss
  - Assume there's a true "data-generating process" (DGP)

$$f(y_i)$$

  - Where $p(\mathbf{y}|\boldsymbol{\theta}) = L(\boldsymbol{\theta}; \mathbf{y})$ is your specified probability distribution

$$\text{predictive probability} = \int p\left(y^*\middle|\hat{\boldsymbol{\theta}}\right) f(y^*)\, dy^*$$

  - Where
    - y^* is some future data

Then

$$\text{expected predictive log. probability} = \sum_{j=1}^{J} \log\left(p\left(y_j\middle|\hat{\boldsymbol{\theta}}\right)\right)$$

  - Where
    - y_j is some data that were "held out" when estimating parameters $\hat{\boldsymbol{\theta}}$

More reading: Gelman, A., Hwang, J. & Vehtari, A. (2014). Understanding predictive information criteria for Bayesian models. *Stat. Comput.*, 24, 997–1016.

How do we assess fit?

- K-fold crossvalidation
  1. Divide data set into *K* even partitions
  2. Calculate predictive probability for 1st partition
     - For each piece K, fit the model to all data except data in that partition
     - Calculate the predictive probability of data in partition K using this model
     - Record predictive probability
  3. Repeat step 2 for all K partitions
  4. Chose the model with the highest predictive probability

Confidence interval:

- Parameter estimates are normally distributed

- Computation

$$CI_{x\%}(\hat{\theta}) = \hat{\theta} \pm \widehat{\mathrm{SE}}(\hat{\theta}) \times \Phi^{-1}\left(\frac{x}{2}\right)$$

  - Where $CI_{x\%}$ contains the true value x% of the time if the model is correct
  - Φ^{-1} is the inverse cumulative distribution for a normal distribution
  - $\hat{\theta}$ is the estimate for parameter θ
  - $\widehat{\mathrm{SE}}(\hat{\theta})$ is the estimated standard error for parameter θ

Confidence interval coverage

- *Coverage* – the expected proportion of times that an estimated x% confidence interval contains the true value given an estimation model and true "data-generating process"

Estimation:

1. Simulate data with a known value for parameter θ
2. Record true parameter values
3. Apply estimator
4. Record confidence interval $CI_{x\%}(\hat{\theta})$ for parameter θ
5. Repeat steps 1-4 hundreds of times
6. Compute the proportion of times where $CI_{x\%}(\hat{\theta})$ contains the true value for parameter θ

[Work on TMB code in groups of 2 for 20 more minutes]

Homework assignment:

- Due at beginning of Lab #2
- Must turn in your own code
- Cannot cut-paste any code from other students
  - You can hand-write your own code while working with someone else, or looking at my example code