Quantum Docs

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Welcome to Quantum Docs! This document is an opportunity to show one of my many studying strategies. I recently took the time to review the course notes from an archived course on edX called Quantum Mechanics and Quantum Computation. This course was offered by Berkeley University. My goal was to retype the course notes on this document while rephrasing and summarizing the information so that it would make more sense to me. I also allowed myself to add additional concepts where it seemed appropriate. This document was of course also an opportunity to practice my LaTeX skills! Please feel free to read through this document if you are interested in quantum mechanics and would like to learn more about it.

Contents

1 Principles of Quantum Mechanics

Quantum mechanics is very mysterious and counter intuitive in that it completely contradicts our understanding of what happens in the macroscopic world. Many of the concepts that we discuss in quantum mechanics lead to absurd implications that completely defy any logic. While its counter intuitiveness makes it difficult for any one to accept: It has many wonderful applications that are constantly being developed and studied such as quantum computing and quantum optics. Features of quantum mechanics include:

One cannot have complete knowledge of the state of a system.

Any measurement that it made on a particle will disturb is state.

Quantum objects behave like a particle in some ways and like a wave in other ways.

2 The Double Slit Experiment

It has been established that light is an electromagnetic wave propagating through space. Yet it has also been established light is composed of individual particles called photons. What is the reason for this wave particle duality?

Imagine randomly firing marbles toward a barrier. Certain marbles would pass through Slit 1 while other marbles would pass through Slit 2. And some would be blocked by the barrier. Any marble that passes the barrier lands and is traced on the observing screen behind the barrier.

We would imagine that if we closed slit 2: The marbles would only be able to travel through Slit 1. We would thus have a small peak behind Slit 1. A graph showing the number of times marbles hit a position y on the observing screen would produce a normal distribution behind Slit 1. Note that while some would land directly behind Slit 1: Others would land with a certain offset due to the random initial directions and collisions with the edges of the slit. A similar thing would happen if we closed Slit 1 and opened Slit 2.

If analysed the distribution pattern based on position y with both slits open: We would see that sum of the graph/normal distribution from Slit 1 and the graph/normal distribution from Slit 2. These graphs can also be thought of as representing the probability that a bullet will land at a particular spot y on the screen. We let $P_1(y))$ represent the probability that a marble will land at position y when only Slit 1 is open and let $P_2(y)$) represent the probability that a marble will land at position y when only Slit 2 is open. Additionally: We let $P_{12}(y)$ represent the probability that the marble lands at position y when both slits are open. Then it would follow that $P_{12}(y) = P_1(y) + P_2(y)$.

Let us now consider what would happen if we allowed waves to travel through the slits. When the waves pass through the slits and diffract: An interference pattern occurs. This can be seen by plotting the intensity or the amount of energy carried by the waves at each position y on the observing screen. The dark spots of the interference pattern occur when the wave from Slit 1 arrives completely out of phase with the wave from Slit 2. That is that the path difference is a half integer multiple of the wavelength ($PD = n\lambda \setminus 2$ where n is any odd integer). The bright spots occur when the wave from Slit 1 arrives completely in phase with the wave from Slit 2. That is that the path difference is an integer multiple of the wavelength ($PD = n\lambda$ where n is any integer multiple). The bright spot in the middle is bright because the path length difference between the source at Slit 1 and the source at Slit 2 is equal to 0 (the two waves travel the exact same distance). The first dark spots occur when the path length difference is half of a wavelength (one wave travels a distance of half a wavelength more than the other). Note that it is not the intensities/energies of the waves that add but the heights. We therefore have $I_{12}(y) \neq I_1(y) + I_2(y)$ and $h_{12}(y) = h_1(y) + h_2(y)$ where $I_{12}(y) = h(y)^2$.

We understand that lowering the intensity of the marbles means lowering the rate at which the marbles are fired. In other words: It is the frequency with which the marbles hit the screen and not the energy that it transferred to the screen after each collision that is lowered.

Lowering the intensity of waves means decreasing the amplitudes. In this case: It is the energy that is transferred to the screen and not the frequency of the waves hitting the screen that is lowered.

When Young performed this experiment with light: An interference pattern was produced. This would suggest to us that light is a wave. But let us imagine that the observing screen consisted of thousands of tiny photo detectors. High intensities of light would cause the photo detectors to absorb a lot of energy while low intensities would cause the photo detectors to absorb less energy. If we produce a graph that represents the intensity at each position y on the screen: We absorb the same interference pattern as described earlier.

Now imagine that we turn down the intensity of the light. To begin with: The intensity amplitudes on the distribution graph would gradually decrease as expected. However: If we lower the energy enough: We will reach a point when all of the photo detectors will record the same minimum energy E_0 but at different rates. This energy E_0 corresponds to the energy carried by a single photon.

Recall that we established that marble hitting a screen would always have the same energy but different frequencies. Photo detectors in bright spots of the interference pattern would record energy E_0 very frequently while dark spots would record energy E_0 a lot less frequently. The behaviour resembles the behaviour of marbles or particles rather than waves. At this point: Photons seems to have behaviour that resembles that of a wave and behaviour that resembles that of a particle. This is the wave particle duality phenomenon that was described earlier.

Let us try something different. We will lower the intensity to the point where only one photo detector will record something each second. This means that the source only sends one photon at a time. We plot a graph that resembles the number of photons that were absorbed by each photo detector. The resulting distribution will represent the probability that a single photon will land at a particular position y.

We would expect that the photon would go through either one slit or the other. Then similar to the marbles: The probability that the photon would land on position y would be $P_{12}(y) = P_1(y) + P_2(y)$. This would correspond to a two peaked distribution where each corresponds to either Slit 1 or Slit 2. However: This is not what we see! Instead we see the same interference pattern as before.

We understand that in order to have an interference pattern: Light from one slit must interference with light from the other slit. However: There is only photon passing the barrier each time. The current explanation is that the photon travels through both slits and then interferes with itself. We establish that the probability $P(y)$ that a photon will be recorded at position y is proportional to the square of the probability amplitude $a(y)$. We have that $a_{12}(y) = a_1(y) + a_2(y)$ and that $P_{12}(y) = |a_{12}(y)|^2 = P_1(y) + P_2(y)$.

It would make sense for us to try to determine which slit each photon went through. We may do this by placing a photo detector at each slit so that each time a photon travelled through the barrier: We would be able to record which slit it went through and where it hit the screen. The amazing thing is that if we perform this experiment: There is no longer an interference pattern! The very act of observation causes a particle like behaviour rather than a wave like behaviour. This is an example of how measurement and observation alters a quantum system.

This is referred to wave particle duality. Amazingly: Performing the same experiment with electrons would produce the same result.

3 Axioms of Quantum Mechanics

The superposition principle establishes that a particle can be in two different states at the same time.

The measurement principle establishes that the act of measuring a particle influences its state.

The unitary evolution axiom describes how the state of a quantum system evolves with time.

4 Schrödinger's Cat

Schrödinger's Cat experiment was a thought experiment designed by Schrdinger with the purpose of helping to establish the absurd and counter intuitive implications of quantum mechanics. We recently established the axioms of quantum mechanics. The superposition principle essentially states that while you are not looking: An object can be in more than one state at once. The measurement rule essentially states that when we measure the related property such as position: The object has to choose to be in just one state.

In Young's Double Slit Experiment: We established that the photon or electron passes through both slits at the same time and therefore is in more than one state at once. This is the superposition principle. We also established that the photon or electron immediately begins passing through one slit each time and therefore only occupies one state as soon as we start looking. This is the measurement rule.

The setup for this experiment is as follows. Suppose we have an atom that has a 50 percent of radioactively decaying within the next minute. We then set up a Geiger counter that will fire if the atom decays and will not fire if the atoms does not decay. If the Geiger counter fires: Poisonous gas will be released and the cat will be dead. If the Geiger counter does not fire: Poisonous gas will not be released and the cat will be alive.

We place this setup in a box with a cat. The box is sealed shut for one minute so that there is no way of observing what is happening inside the box. This means that we cannot see what happened to the atom. The atom is either decayed or not decayed each with a 50 percent chance of occurring. The superposition principle of quantum mechanics establishes that while the box is closed: The atom is in all possible states at the same time. That is that the atom is both decayed and not decayed. The Geiger counter is therefore both fired and not fired. The poisonous gas is both released and not released. The cat is both dead and alive.

When you open the box: We are measuring the state of the atom and that cat therefore must be either dead or alive. This is the measurement principle in effect. But how could the cat have been both dead and alive at the same time before we opened the box?

Now suppose that we were able to complete the same exact experiment with a time span of 8 hours instead of one minute. If we find that the cat is alive: The cat would be hungry considering that it has not eaten for 8 hours. If we find that the cat is dead: An examination by a veterinary forensic pathologist would determine that the cat died eight hours ago. Our observation not only determines the current reality but also creates the history appropriate to that reality.

5 The Superposition Principle

Let us consider a system with k distinguishable states. For example: An electron in a hydrogen atom may only occupy one of a discrete set of energy levels. The very first energy level is the ground state and the energy levels that follow are the first excited state followed by the second excited state and so on. If we establish that there are k different states or energy levels then the electron will occupy the ground state or one of $(k-1)$ excited states. We may use the state of this system to store a number between 0 and $(k-1)$.

The superposition principle establishes that if a quantum system can be in one of two states then is can also be placed in a linear superposition of these two states with complex coefficients. We represent the ground state of our k system by $|0\rangle$ and the excited states by $|1\rangle ... |k-1\rangle$. The electron has k possible classical states. Based on the superposition principle: We have that the quantum state of the electron is given by $\alpha_0|0\rangle + ... + \alpha_{k-1}|k-1\rangle$ where $\alpha_0 ... \alpha_{k-1}$ are complex numbers that are normalized so that $\sum_j |\alpha_j|^2 = 1$.

Note that α_j is called the amplitude of the state $|j\rangle$. For example: If $k = 3$ then the state of the electron could be:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle \tag{1}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle \tag{2}$$

$$|\psi\rangle = \frac{1+i}{3}|0\rangle - \frac{1-i}{3}|1\rangle + \frac{1+2i}{3}|2\rangle \tag{3}$$

Superposition essentially claims that the electron cannot decide whether it is in the ground state or one of the $(k-1)$ excited states. The amplitude α_0 is a measure of the inclination toward the ground state. Note that α_0 as well as the other complex amplitudes cannot be thought of as the probability that the electron is in each of the states as they can be negative and imaginary.

Note that the state of the system can also be written as a k dimensional vector:

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}$$

The normalization on the complex amplitudes means that the state of the system is a unit vector in a k dimensional complex vector space. This complex vector space is called a Hilbert space.

The previous notation that was used to express the quantum state $(\alpha_0|0\rangle + ... + \alpha_{k-1}|k-1\rangle)$ is Dirac's ket notation. Dirac's ket notation is another way of expressing a vector:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad |k-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

We see that the k distinguishable classical states are represented by mutually orthogonal unit vectors in a k dimensional complex vector space. This means that they form an orthonormal basis for that space. Furthermore:

Given any two states $\alpha_0|0\rangle + ... + \alpha_{k-1}|k-1\rangle$ and $\beta_0|0\rangle + ... + \beta_{k-1}|k-1\rangle$ we can compute the inner product of these two vectors by $\sum_{j=0}^{k-1} \alpha_j^* \beta_j$. Note that the absolute value of the inner product is the cosine of the angle between these two vectors in Hilbert space since each of these vectors are unit vectors of length one.

Consider $|\psi\rangle = \sum_k a_k|k\rangle$ where the kets $|k\rangle$ form a basis and are therefore orthogonal. We may also write this state as a column vector:

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

The inner product of $|\psi\rangle$ with itself is:

$$\langle \psi, \psi \rangle = (a_0^* \;\; a_1^* \ldots a_{N-1}^*) \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} = \sum_{k=0}^{N-1} a_k^* a_k = \sum_{k=0}^{N-1} |a_k|^2$$

The complex conjugation step is important so that when we take the inner product of a vector with itself we get a real number that we can associate with a length. This makes sense as each positive complex term will cancel out with a corresponding negative complex term.

Dirac defined an object called a bra to represent the conjugate transpose of a ket:

$$\langle \psi| = |\psi\rangle^\dagger = \sum_k a_k^* \langle k| \tag{4}$$

Note the following rule: $\langle j||k\rangle \equiv \langle j|k\rangle = \delta_{jk}$

The inner product of $|\psi\rangle$ with itself can be written as:

$$\langle\psi|\psi\rangle = (\sum_j a_j^*\langle j|)(\sum_k a_k|k\rangle) \quad (5)$$

$$= \sum_{j,k} a_j^* a_k \langle j|k\rangle \quad (6)$$

$$= \sum_{j,k} a_j^* a_k \delta_{jk} \quad (7)$$

$$= \sum_k |a_k|^2 \quad (8)$$

Based on the above: We have that $a_j^*\delta_{jk} = a_k^*$. We may now use the same concepts to write the inner product of any two states $|\psi\rangle$ and $|\phi\rangle$ where $|\phi\rangle = \sum_k b_k|k\rangle$. The inner product is:

$$\langle\psi|\phi\rangle = \sum_{j,k} a_j^* b_k \langle j|k\rangle = \sum_k a_k^* b_k \quad (9)$$

Note that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \in \mathbb{C}$

6 The Measurement Principle

The linear superposition $|\psi\rangle = \sum_{j=0}^{k-1} \alpha_j|j\rangle$ is part of the private world of the electron. Access to the information describing the state is very limited: We cannot actually measure the complex amplitudes α_j. A measurement on this k state system yields one of at most k possible outcomes. In other words: It yields an integer between 0 and $k-1$. Measuring $|\psi\rangle$ in the standard basis yields j with probability $|\alpha_j|^2$.

One important aspect of the measurement process it that it alters the state of the quantum system. The effect of the measurement is that the new state is exactly the outcome of the measurement. If the outcome of the measurement is j then following the measurement: The qubit is in state $|j\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement.

In general we will have an orthonormal basis $|e_0\rangle \ldots |e_{k-1}\rangle$. The outcome of the measurement is $|e_j\rangle$ with probability equal to the square of the length of the projection of the state vector ψ on $|e_j\rangle$. A consequence of performing the measurement is that the new state vector is $|e_j\rangle$. Therefore: Measurement may be regarded as a probabilistic rule for projecting the state vector onto one of the vectors of the orthonormal measurement basis.

7 Qubits

Qubits are 2 state systems. For example: If we set $k = 2$: The electron in a Hydrogen atom can be in the ground state or the first excited state or any superposition of the two.

The state of a qubit can be written as a unit vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{C}^2$. In Dirac notation this may be written as: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha\beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the electron. For us to know the electron's state: We must make a measurement. Making a measurement gives us a single classical bit of information: 0 or 1. Measuring $|\psi\rangle$ in the standard basis $\{|0\rangle, |1\rangle\}$ yields 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$.

Similar to the situation previously described: One important aspect of the measurement process is that it alters the state of the qubit. The effect of the measurement is that the new state is exactly the outcome of the measurement. If the outcome of the measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ yields 0 then following the measurement the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about α and β by repeating the measurement.

More generally: We may choose any orthogonal basis $|v\rangle, |w\rangle$ and measure the qubit in that basis. To do this: We rewrite our state in that basis: $|\psi\rangle = \alpha'|v\rangle + \beta'|w\rangle$. The outcome is v with probability $|\alpha'|^2$ and $|w\rangle$ with probability $|\beta'|^2$. If the outcome of the measurement on $|\psi\rangle$ yields $|v\rangle$ then as before the qubit is then in state $|v\rangle$.

We will now look at examples of qubits. Let us first look at atomic orbitals. The electrons within an atom exist in quantized energy levels. Two such individual levels can be isolated to configure the basis states for a qubit. The ground state would correspond to state $|0\rangle$ and the first excited state would correspond to state $|1\rangle$.

Another example would be photon polarization. Classically: A photon may be described as a travelling electromagnetic wave. An electromagnetic wave has a polarization which describes the orientation of the electric field oscillations. For a given direction of photon motion: The photon's polarization axis may lie anywhere in a 2D plane perpendicular to that motion. It is thus natural to use an orthonormal 2D basis (such as $\overrightarrow{x}$ and $\overrightarrow{y}$ or vertical and horizontal) to describe the polarization state or polarization direction of a photon. The amplitude of the overall polarization state in each basis vector is just the projection of the polarization in that direction.

The polarization of a photon can be measured by using a polaroid film or a calcite crystal. A suitably oriented polaroid sheet transmits x polarized photons and absorbs y polarized photons. Thus a photon that is in a superposition $|\phi\rangle = \alpha|x\rangle + \beta|y\rangle$ is transmitted with probability $|\alpha|^2$ as that is the probability that it is x polarized. If the photon now encounters another polaroid sheet with the same orientation then it is transmitted with probability 1. On the other hand: If the second polaroid sheet has its axes crossed at right angles to the first one and the photon is transmitted by the first polaroid: Then it is definitely absorbed by the second sheet. This pair of polarized sheets at right angles thus blocks all of the light.

Like photon polarization: The spin of a spin 1/2 particle is a two state system and can be described by a qubit. Roughly speaking: The spin is a quantum description of the magnetic moment of an electron which behaves like a spinning charge. The two allowed states can roughly be thought of as clockwise rotations (spin up) and counter clockwise rotations (spin down).

We may now examine a system of two qubits. Consider the two electrons in two hydrogen atoms each regarded as a 2 state quantum system. Since each electron can either be in the ground state or the excited state: Classically we have that the two electrons are in one of four states: 00 or 01 or 10 or 11. Each state represents 2 bits of classical information. By the superposition principle: The quantum state of the two electrons can be any linear combination of these four classical states:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \tag{10}$$

Note that $\sum_{ij} |\alpha_{ij}|^2 = 1$. This is Dirac notation for a unit vector in $\mathbb{C}^4$:

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

As in the case of a single qubit: Even though the state of two qubits is specified by four complex numbers: Most of this information is not accessible by measurement. A measurement of a two qubit system can only reveal two bits of information. The probability that the outcome of the measurement is ij is $|\alpha_{ij}|^2$. Moreover: Following the measurement the state of the two qubits is $|ij\rangle$. This means that if the first bit is j and the second bit is k then following the measurement: The state of the first qubit is $|j\rangle$ and the state of the second qubit $|j\rangle$.

What if we only measure the first qubit? What is the probability that the outcome is 0? It is actually exactly the same as if we had measured both qubits: $\Pr\{1\text{st bit} = 0\} = \Pr\{00\} + \Pr\{01\} = |\alpha_{00}|^2 + |\alpha_{01}|^2$.

How does this partial measurement disturb the state of the system? The new superposition is obtained by eliminating all of the terms of $|\psi\rangle$ that are inconsistent with the outcome of the measurement (those whose first bit is 1). Of course the sum of the squared amplitudes is no longer 1. We therefore must renormalize to obtain a unit vector:

$$|\phi_{new}\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \tag{11}$$