Invariant densities of dynamical systems with small noise

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Introduction

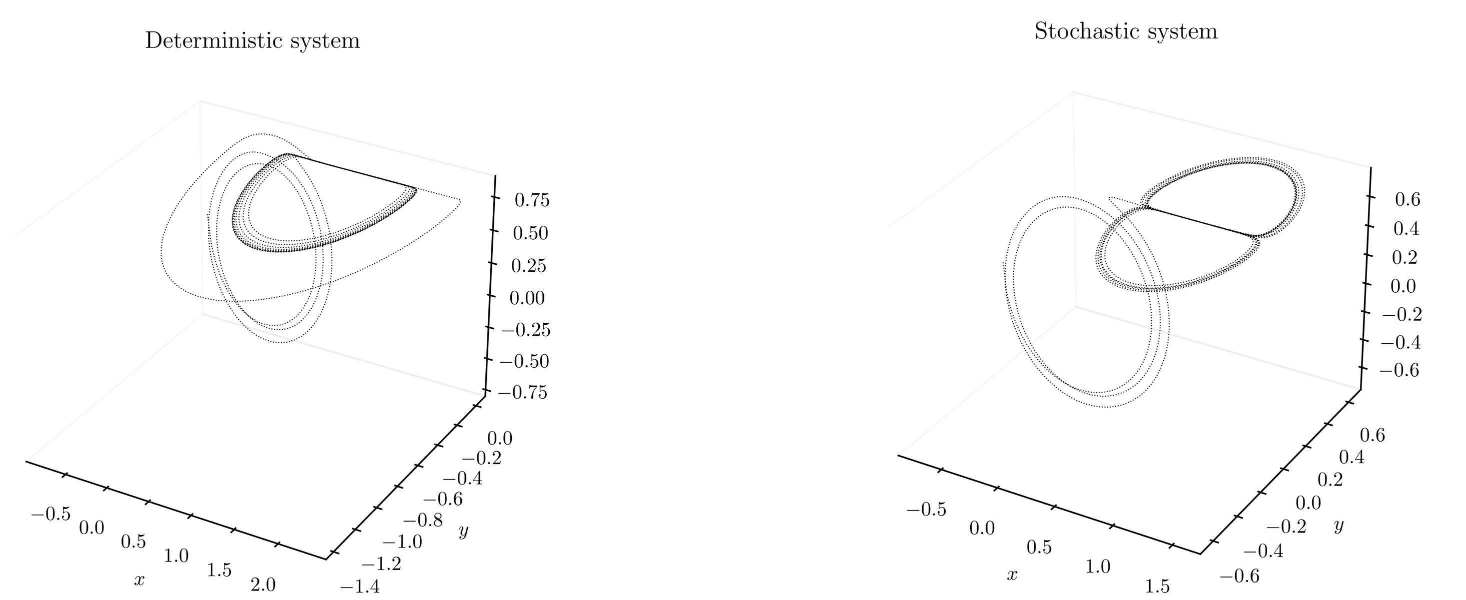
We consider a chaotic dynamical system, with a small noise term,

$$d\mathbf{x} = \mathbf{U}(\mathbf{x})dt + \sqrt{2}\epsilon d\mathbf{W}. \tag{1}$$

The dynamics are explicitly given by

$$dx = (\mu x - y^2 + 2z^2 - \delta z)dt,\ dy = y(x-1)dt + \sqrt{2}\epsilon dW,\ dz = (\mu z + \delta x - 2xz)dt, \tag{2}$$

where δ, μ, ϵ are parameters of the model and $W : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a standard Wiener process.



(a) Trajectory of the deterministic model, with $\mu = 0.01$, $\delta = 1.5$. The system quickly converges to its attractor in the plane $z = \delta/2$.

(b) Trajectory of the probabilistic model given the parameter values $\mu = 0.01$, $\delta = 1.5$, $\epsilon = 5 \cdot 10^{-4}$

We are interested in determining the invariant probability density of the dynamical system (1), $p(x, y)$.

Asymptotic expansion

The invariant density is constructed for three regions in [1] using asymptotic expansion. It is done by relying on the following observations and assumptions.

- The invariant density is known to satisfy the Fokker-Planck equation,

$$\begin{aligned}\partial_t p(t,\mathbf{x}) &= \epsilon^2\partial_y^2 p(t,\mathbf{x}) - \partial_x\big((\mu x - y^2 + 2z^2 - \delta z)p(t,\mathbf{x})\big) \\ &\quad - \partial_y\big(y(x-1)p(t,\mathbf{x})\big) - \partial_z\big(\mu z + \delta x - 2xz)p(t,\mathbf{x})\big) \\ &= \mathcal{L}^* p(t,\mathbf{x}) = 0\end{aligned}$$

- The invariant density is assumed to depend only on x and y since the attractor of the deterministic system is in the $z = \delta/2-$ plane.
- The invariant density is concentrated near the plane $y = 0, z = \delta/2$ for certain values of the parameters μ and δ, justifying a change of variables

$$z = \frac{\delta}{2} + \mu\frac{\delta}{4x} + \zeta. \tag{3}$$

  Note that ζ is coupled to x by the following relation,

$$d\zeta \sim \mu\zeta - 2x\zeta dt. \tag{4}$$

- For $x > 0$ and $\mu \ll 1$, ζ decays exponentially, and is thus assumed to be small in this region.
- The invariant density in the new variables is assumed to be separable

$$P(x, y, \zeta) = q_1(x, y)q_2(x, \zeta), \tag{5}$$

  with q_1 satisfying,

$$\epsilon^2\partial_y^2 q_1 - \partial_x\left(\left(\mu x + \mu\frac{\delta^2}{4x} - y^2\right)q_1\right) \tag{6}$$

Explicit probability density

Using the outlined assumptions, it is shown in [1] that an asymptotic expansion can be constructed for the probability density in different regions.

$$p(x,y) = \begin{cases} K_1(x)\exp\left[-\left(y - F_1(x) - \frac{2\epsilon}{g_{1,n}}\right)^2 g_1(x)/(2\epsilon^2)\right], & \text{if } (x-1) \ll \sqrt{\mu},\ y \ll 1 \\ K_2\left(\frac{x-1}{\sqrt{\mu}}\right)\exp\left[-\left(\frac{y}{\sqrt{\mu}} - F_2(\frac{x-1}{\sqrt{\mu}}) + \frac{2\epsilon}{\sqrt{g_{2,n}}}\right)^2/(2\epsilon^2)\right], & \text{if } (x-1) = \mathcal{O}(\sqrt{\mu}),\ y = \mathcal{O}(\mu^{1/2}) \\ K_3(x)\exp\left[-\left(-(y - F_3(x) - \frac{2\epsilon}{\sqrt{g_{3,n}}}\right)^2 g_3(x)/(2\epsilon^2)\right] & \text{if } x - 1 = \mathcal{O}(1),\ y = \mathcal{O}(1) \end{cases}$$

Where $K_i, F_i, g_{i,n}$, for $i = 1, 2, 3$ are functions that can all be computed analytically.

A data driven approach

The asymptotic expansion gives us a good tool to analyze the probability density, but can be very hard to actually construct. Alternatively we can also estimate the probability density numerically. In statistical theory there are many different ways of doing this. We have implemented the Kernel Density Estimate (KDE) and will compare these results to that of the asymptotic expansion.

Definition: Let $\mathbf{X} : \Omega \to \mathbb{R}^d$ be a random variable in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and $p(\mathbf{x})$ be the pdf of $\mathbf{X}$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a sample from $p(x)$ for some $n \in \mathbb{Z}^+$. Then the Kernel density estimate of p is defined as

$$\hat{p}_{\mathbf{H}}(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n} K_{\mathbf{H}}(\mathbf{x} - \mathbf{x}_j), \tag{7}$$

where

- $\mathbf{x} = (x_1, \ldots, x_d)^T$, $\mathbf{x_j} = (x_{j1}, \ldots, x_{jd})$, for $j = 1, \ldots, n$,
- $\mathbf{H} \in \mathbb{R}^{d\times d}$ is the symmetric and positive definite bandwidth matrix,
- $K_{\mathbf{H}}$ is a multivariate density known as a "kernel",

$$K_{\mathbf{H}}(x) = |\mathbf{H}|^{-1/2}K(\mathbf{H}^{-1/2}\mathbf{x}) \tag{8}$$

As an example of a kernel we consider the Multivariate Gaussian kernel,

$$K_{\mathbf{H}}(\mathbf{x}) = (2\pi)^{-d/2}\mathbf{H}^{-1/2}e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{H}^{-\mathbf{1}}\mathbf{x}}. \tag{9}$$

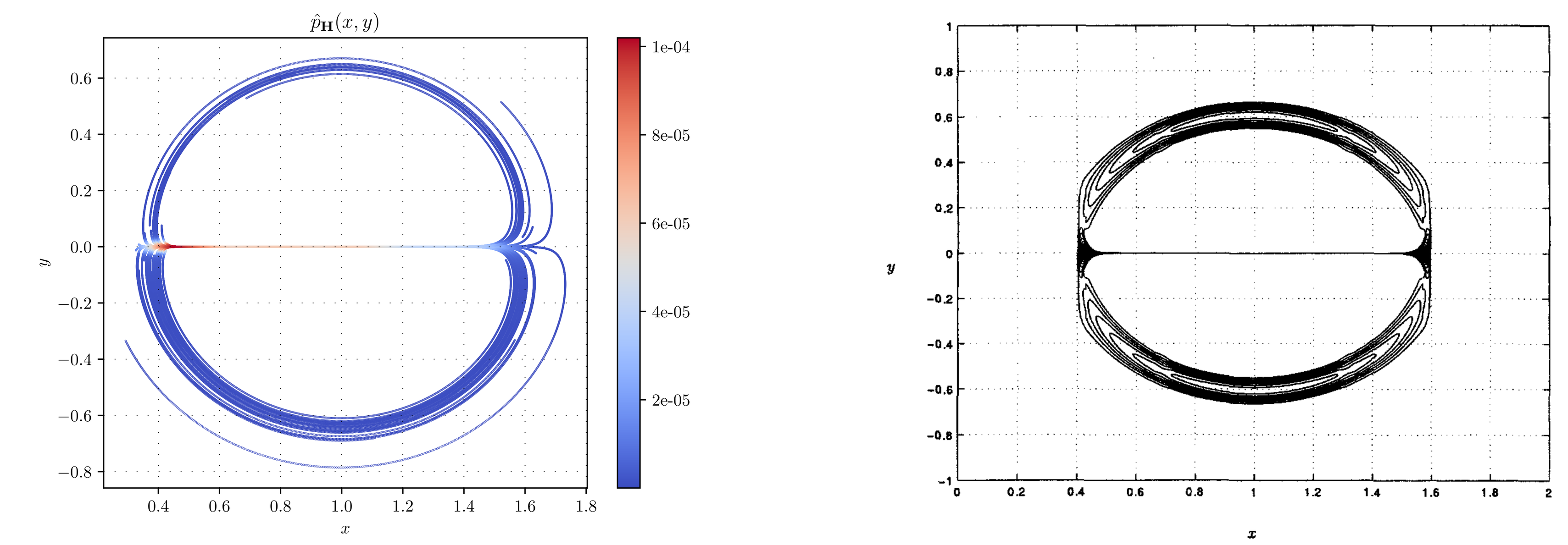
Algorithm 0: Estimation of invariant density

Pick some (large) $T \in \mathbb{R}^+$, $N \in \mathbb{Z}^+$, $M \in \mathbb{Z}^+$ and define $\Delta t = T/N$.
for $k = 1, \ldots, M$ **do**
  Generate random $\boldsymbol{\xi}_k \in \mathbb{R}^3$.
  Initialize empty $\mathbf{X}_k \in \mathbb{R}^{N+1}$
  $\mathbf{X_k(0)} \leftarrow \boldsymbol{\xi_k}$
  for $n = 1, \ldots, N$ **do**
    $\mathbf{X_k(n\Delta t)} = \mathbf{X_k([n-1]\Delta t)} + \Delta t\mathbf{M}[\mathbf{X_k([n-1]\Delta t)}]$
  Store $\mathbf{X_k}$
Pick some $K \in \mathbb{Z}^+$ such that $1 \ll K < N$
for $k = 1, \ldots, M$ **do**
  Let $\widetilde{\mathbf{X}}_k = [\mathbf{X}_k([N-K]\Delta t), \ldots, \mathbf{X}_k(T)]$
Initialize empty $\mathbf{Y} \in \mathbb{R}^{(N-K)\cdot M\times 3}$
Set $\mathbf{Y} = [\widetilde{\mathbf{X}}_1, \ldots, \widetilde{\mathbf{X}}_M]^T$
Compute KDE:

$$\hat{p}_{\mathbf{H}}(\mathbf{x}) = \frac{1}{(N-K)\cdot M}\sum_{j=1}^{(N-K)\cdot M} K_{\mathbf{H}}(\mathbf{x} - \mathbf{Y_j})$$

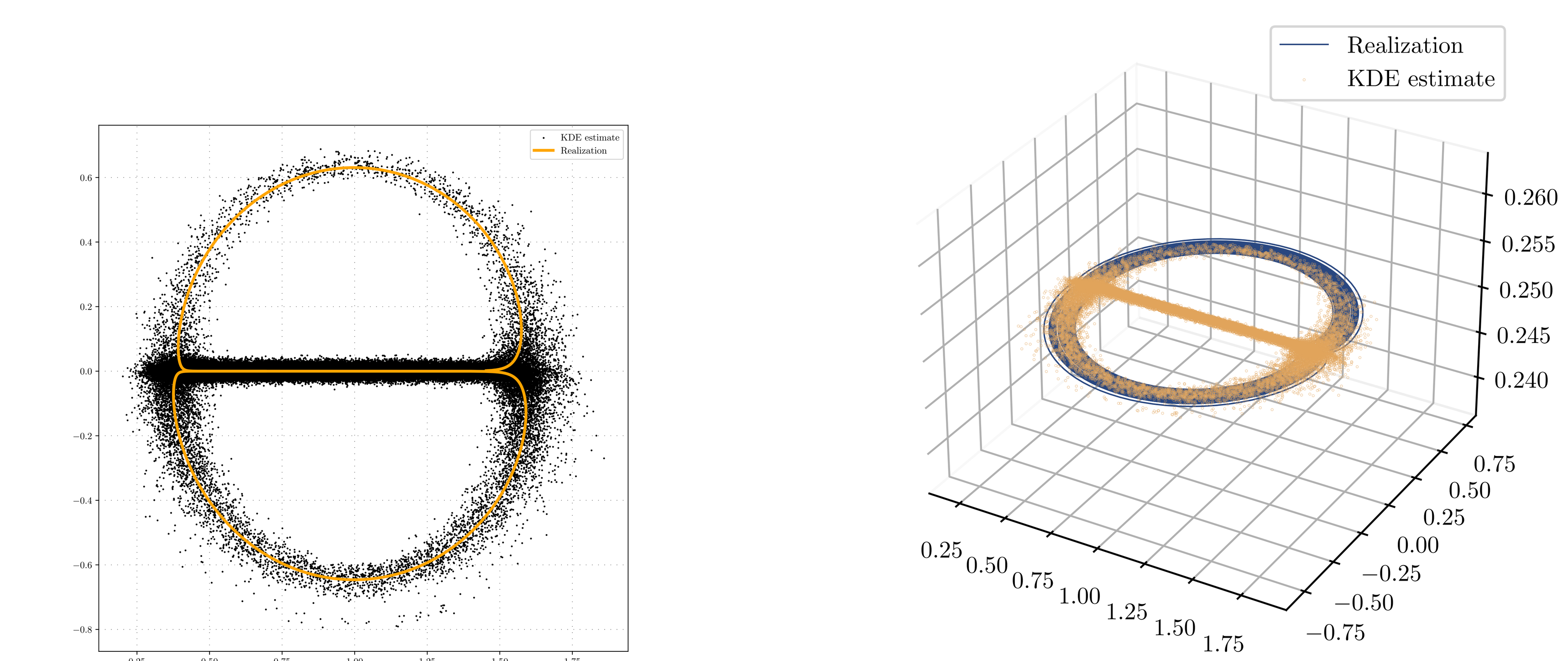
Results

By using Algorithm 0 we have a systematic way to generate points from the distribution $\hat{p}_{\mathbf{H}}$ which, given large enough M and N will approximate the invariant density p. A clear advantage of the data driven approach is that we can generate points from any arbitrary random dynamical system.



(a) Numerical estimate of density using a KDE with a Multivariate Gaussian kernel.

(b) Proposed analytical density



(c) 100000 Generated points and one realization of the stochastic dynamics. Here $\delta = 0.5, \mu = 0.1$.

(d) 100000 Generated points with a realization of the stochastic dynamics. Here $\delta = 0.5, \mu = 0.1$.

Conclusions

The KDE method does a seemingly good job at estimating the shape and size of the invariant probability density and moreover allows us to generate arbitrary points from the estimate.

References

[1] G. Papanicolaou R. Kuske.
The invariant density of a chaotic dynamical system with small noise.
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