1.1

$$p(x,y|z) = \frac{p(x,y,z)}{p(z)} = \frac{p(x|y,z)\,p(y,z)}{p(z)}$$

$$= \frac{p(x|y,z)\,p(y|z)\,\cancel{p(z)}}{\cancel{p(z)}}$$

$$= p(x|y,z)\,p(y|z)$$

Recall $p(x,y|z) = p(x|z)\,p(y|z)$ ①

We get $p(x|y,z) = p(x|z)$ //

Similarly,

$$p(x,y|z) = \frac{p(x,y,z)}{p(z)} = \frac{p(y|x,z)\,p(x,z)}{p(z)}$$

$$= \frac{p(y|x,z)\,p(x|z)\,\cancel{p(z)}}{\cancel{p(z)}}$$

$$= p(y|x,z)\,p(x|z)$$

From ①, we get $p(y|z) = p(y|x,z)$ //

1.2

$$\begin{cases} P(z_t = clean \mid x_t = dirty) = 0.2 \\ P(z_t = clean \mid x_t = clean) = 0.6 \end{cases}$$

$$\begin{cases} P(x_{t+1} = clean \mid u_{t+1} = act,\ x_t = dirty) = 0.6 \\ P(x_{t+1} = dirty \mid u_{t+1} = act,\ x_t = dirty) = 0.4 \\ P(x_{t+1} = clean \mid u_{t+1} = act,\ x_t = clean) = 1 \\ P(x_{t+1} = dirty \mid u_{t+1} = act,\ x_t = clean) = 0 \end{cases}$$

$P(x_t = \text{clean}) = c$. $P(x_t = \text{dirty}) = 1-c$

Take action $u_{t+1} = \text{act}$.

$\overline{bel}(x_{t+1} = \text{clean})$

$= P(x_{t+1} = \text{clean} \mid u_{t+1} = \text{act}, x_t = \text{clean})\, bel(x_t = \text{clean})$

$+ P(x_{t+1} = \text{clean} \mid u_{t+1} = \text{act}, x_t = \text{dirty})\, bel(x_t = \text{dirty})$

$= c + 0.6(1-c) = 0.4c + 0.6$

$\overline{bel}(x_{t+1} = \text{dirty})$

$= P(x_{t+1} = \text{dirty} \mid u_{t+1} = \text{act}, x_t = \text{clean})\, bel(x_t = \text{clean})$

$+ P(x_{t+1} = \text{dirty} \mid u_{t+1} = \text{act}, x_t = \text{dirty}) \cdot bel(x = \text{dirty})$

$= 0 + 0.4(1-c) = 0.4(1-c)$

Incorporating the measurements:

$bel(x_{t+1} = \text{clean})$

$= \eta\, P(z_{t+1} = \text{clean} \mid x_{t+1} = \text{clean})\, \overline{bel}(x_{t+1} = \text{clean})$

$= \eta\, 0.6\,(0.4c + 0.6) = \eta(0.24c + 0.36)$

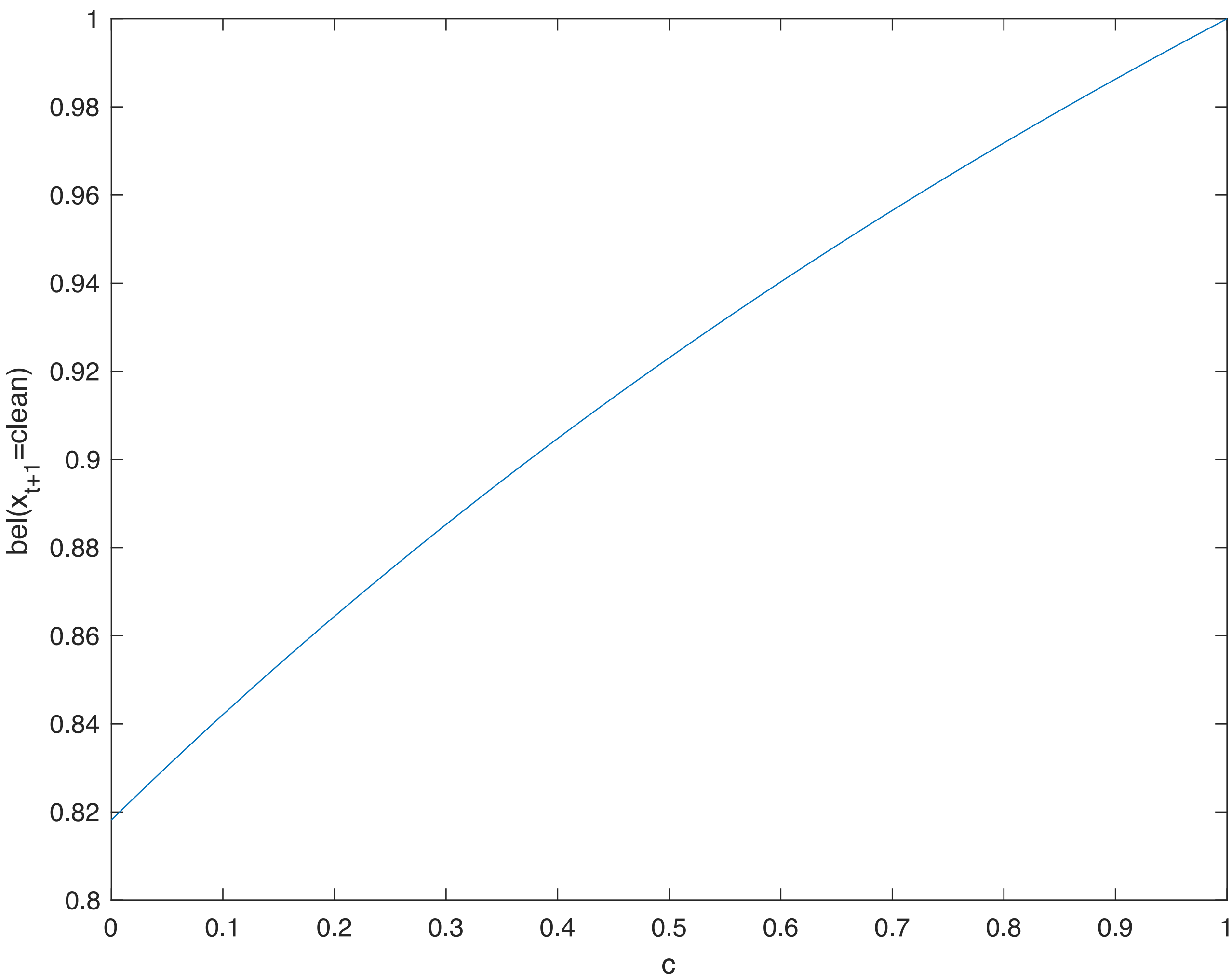
$bel(x_{t+1} = \text{dirty})$

$= \eta\, P(z_{t+1} = \text{clean} \mid x_{t+1} = \text{dirty})\, \overline{bel}(x_{t+1} = \text{dirty})$

$= \eta\, 0.2 \times 0.4(1-c) = \eta\, 0.08(1-c)$

$$\eta = \frac{1}{0.08(1-c) + 0.24c + 0.36} = \frac{1}{0.16c + 0.44}$$

$$P(x_{t+1} = \text{clean} \mid z_{t+1} = \text{clean}) = \frac{0.24c + 0.36}{0.16c + 0.44}$$ //



1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
0.82
0.8
bel(x_{t+1}=clean)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
c

1.3.

$$p(x,y) = p(x|y)\,p(y)$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{\frac{1}{2\pi\sqrt{|\Sigma|}}\exp\left\{-\frac{1}{2}\left(\begin{bmatrix}x-\mu_x\\y-\mu_y\end{bmatrix}^T\Sigma^{-1}\begin{bmatrix}x-\mu_x\\y-\mu_y\end{bmatrix}\right)\right\}}{\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)}$$

since $p(y) = \mathcal{N}(\mu_y, \sigma_y^2)$

Looking at the items before the exponential functions:

$$\frac{\frac{1}{2\pi\sqrt{|\Sigma|}}}{\frac{1}{\sqrt{2\pi}\sigma_y}} = \frac{\frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2-\sigma_{xy}^4}}}{\frac{1}{\sqrt{2\pi}\sigma_y}} = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2-\frac{\sigma_{xy}^4}{\sigma_y^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_{x|y}}$$

Thus we have:

$$\sigma_{x|y}^2 = \sigma_x^2 - \frac{\sigma_{xy}^4}{\sigma_y^2} \quad //$$

Next looking into the exponential functions.

We have $\Sigma^{-1} = \frac{1}{\sigma_x^2\sigma_y^2-\sigma_{xy}^4}\begin{bmatrix}\sigma_y^2 & -\sigma_{xy}^2\\-\sigma_{xy}^2 & \sigma_x^2\end{bmatrix}$

$$\exp\left\{-\frac{1}{2}\cdot\frac{1}{\sigma_x^2\sigma_y^2-\sigma_{xy}^4}\begin{bmatrix}x-\mu_x\\y-\mu_y\end{bmatrix}^T\begin{bmatrix}\sigma_y^2 & -\sigma_{xy}^2\\-\sigma_{xy}^2 & \sigma_x^2\end{bmatrix}\begin{bmatrix}x-\mu_x\\y-\mu_y\end{bmatrix} + \frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

$$= \exp\left\{-\frac{1}{2\left(\sigma_x^2-\frac{\sigma_{xy}^4}{\sigma_y^2}\right)}\cdot A\right\} = \exp\left\{-\frac{A}{2\sigma_{x|y}^2}\right\}$$

We'll next evaluate A.

$$A = \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{\sigma_{XY}^2}{\sigma_Y^2} \\ -\frac{\sigma_{XY}^2}{\sigma_Y^2} & \frac{\sigma_X^2}{\sigma_Y^2} \end{bmatrix} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix} - \left(\frac{\sigma_X^2}{\sigma_Y^2} - \frac{\sigma_{XY}^4}{\sigma_Y^4}\right)(y-\mu_Y)^2$$

$$= \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{\sigma_{XY}^2}{\sigma_Y^2} \\ -\frac{\sigma_{XY}^2}{\sigma_Y^2} & \frac{\sigma_X^2}{\sigma_Y^2} \end{bmatrix} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix} - \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{\sigma_X^2}{\sigma_Y^2} - \frac{\sigma_{XY}^4}{\sigma_Y^4} \end{bmatrix} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}$$

$$= \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{\sigma_{XY}^2}{\sigma_Y^2} \\ -\frac{\sigma_{XY}^2}{\sigma_Y^2} & \frac{\sigma_{XY}^4}{\sigma_Y^4} \end{bmatrix} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}$$

$$= (x-\mu_X)^2 - 2\cdot\frac{\sigma_{XY}^2}{\sigma_Y^2}(x-\mu_X)(y-\mu_Y) + \frac{\sigma_{XY}^4}{\sigma_Y^4}(y-\mu_Y)^2$$

$$= \left[(x-\mu_X) - \frac{\sigma_{XY}^2}{\sigma_Y^2}(y-\mu_Y)\right]^2$$

$$= \left\{x - \left[\mu_X + \frac{\sigma_{XY}^2}{\sigma_Y^2}(y-\mu_Y)\right]\right\}^2$$

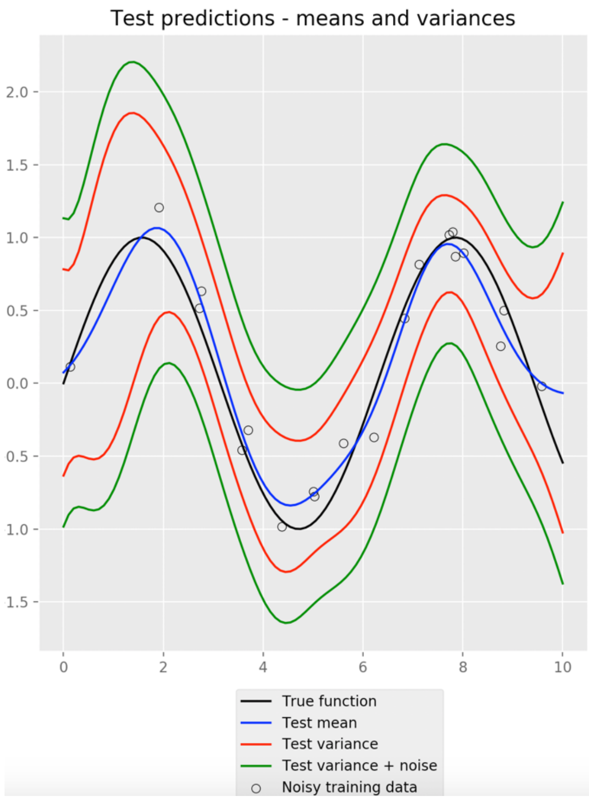
Thus, $\mu_{X|Y} = \mu_X + \frac{\sigma_{XY}^2}{\sigma_Y^2}(y-\mu_Y)$ //

$p(x|y) = \mathcal{N}(\mu_{X|Y}, \sigma_{X|Y}^2)$ and

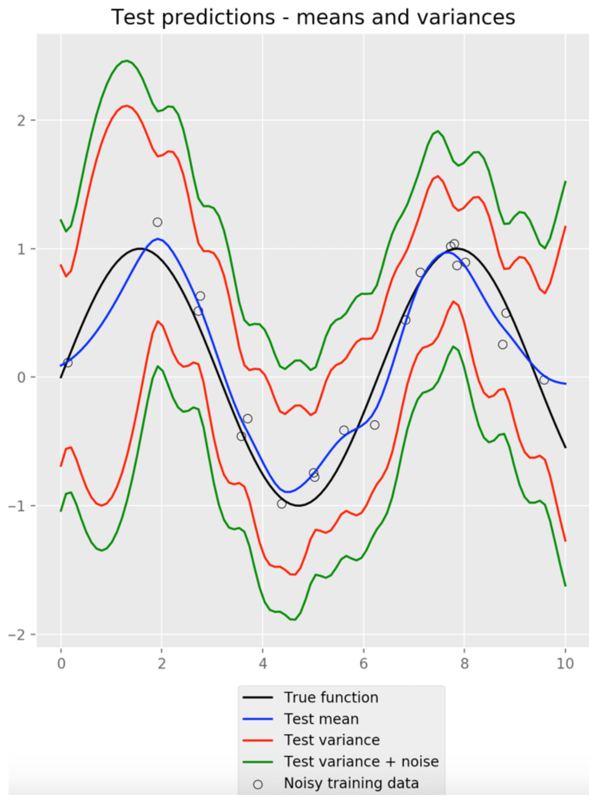
$\mu_{X|Y}$ & $\sigma_{X|Y}^2$ are given above.

2.2 Gaussian Process Predictions

Plot using the default parameters.



SE kernel



Matern Kernel