Homogeneous coordinates

Converting *from* homogeneous coordinates:

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

(this makes homogenous coordinates invariant under scale)

Perspective Projection

- Projection is a matrix multiply using homogeneous coordinates (and |z|):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ |z| \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ |z/f| \end{bmatrix} \Rightarrow \left(f\frac{x}{|z|}, f\frac{y}{|z|} \right)$$

$$\Rightarrow (u, v)$$

S. Seitz

Geometric Camera calibration

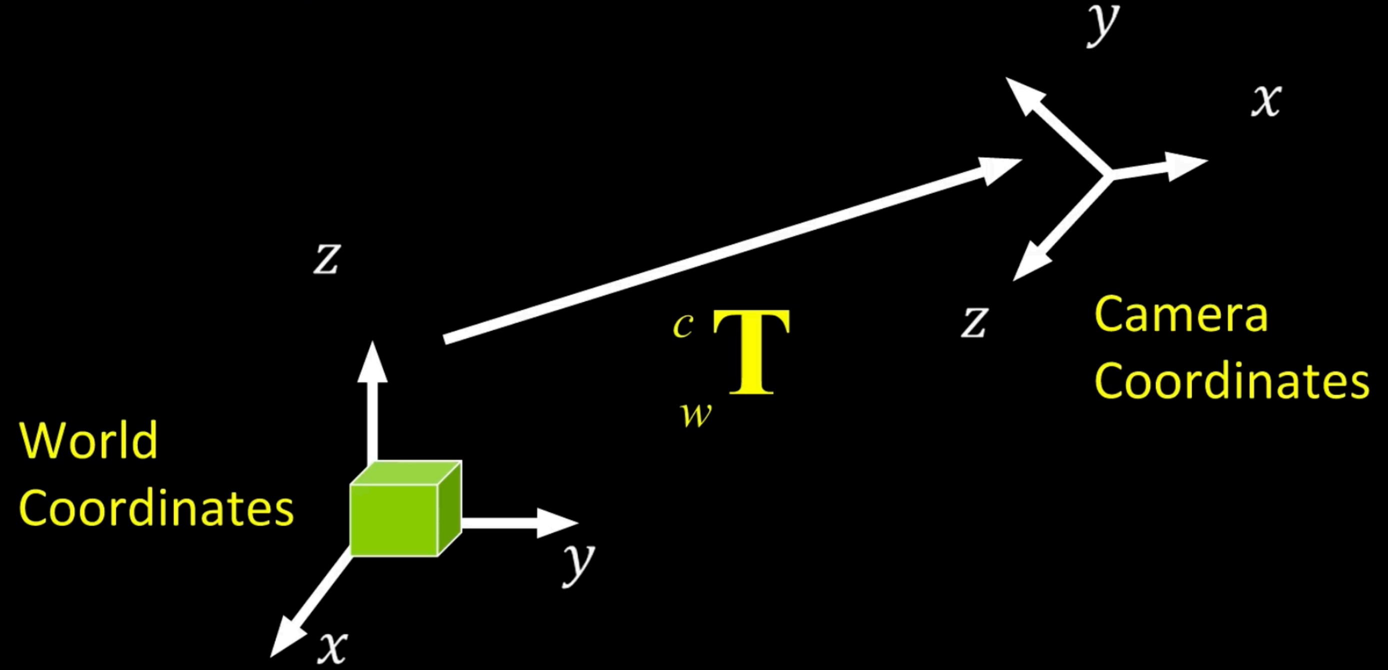
- We want to use the camera to tell us things about the world.
  - So we need the relationship between coordinates in the world and coordinates in the image: ***geometric camera calibration***
- For reference see Forsyth and Ponce, sections 1.2 and 1.3.

Geometric Camera calibration

Composed of 2 transformations:

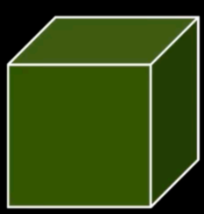
- From some (arbitrary) world coordinate system to the camera's 3D coordinate system. *Extrinsic parameters (or camera pose)*
- From the 3D coordinates in the camera frame to the 2D image plane via projection. *Intrinsic parameters*

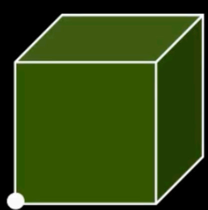
Camera Pose

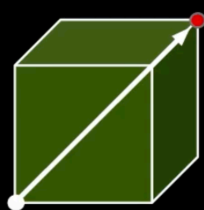


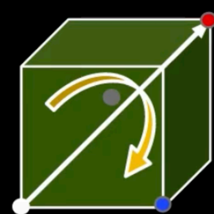
Rigid Body Transformations

Need a way to specify the six degrees-of-freedom of a rigid body. Why 6?









A rigid body is a collection of points whose positions relative to each other can't change

Fix one point, three DOF

Fix second point, two more DOF (must maintain distance constraint)

Third point adds one more DOF, for rotation around line

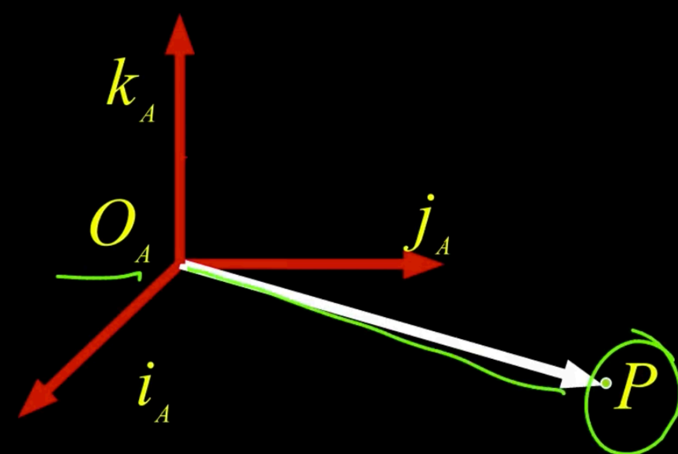
3

+2

+1

Notation (from F&P)

- Superscript references coordinate frame
- ${}^{A}P$ is coordinates of P in frame A
- ${}^{B}P$ is coordinates of P in frame B



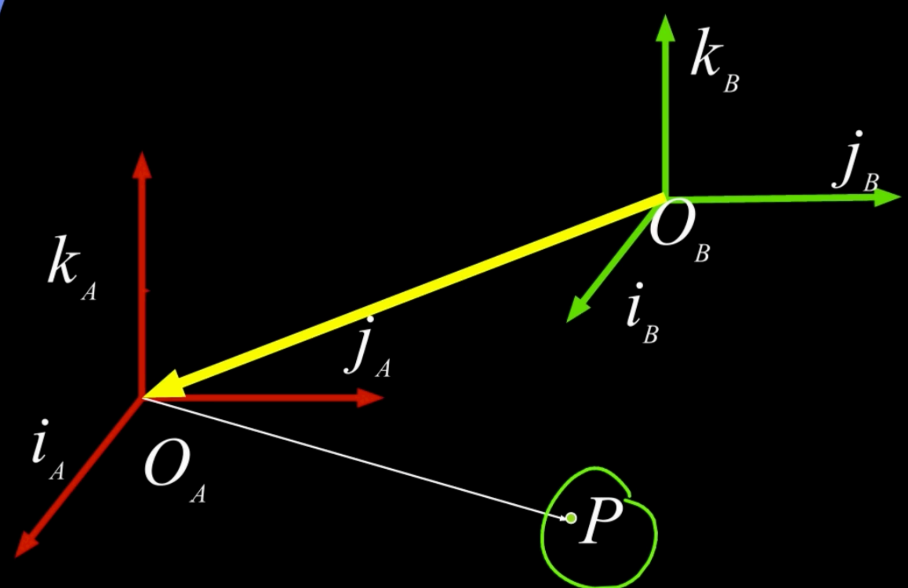
$${}^{A}P = \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{pmatrix} \Leftrightarrow \overline{OP} = \left({}^{A}x \cdot \overline{i_A}\right) + \left({}^{A}y \cdot \overline{j_A}\right) + \left({}^{A}z \cdot \overline{k_A}\right)$$

Translation Only

$$^{B}P = {}^{A}P + {}^{B}(O_A)$$

or

$$^{B}P = {}^{B}(O_A) + {}^{A}P$$



Translation

Using homogeneous coordinates, translation can be expressed as a matrix multiplication.

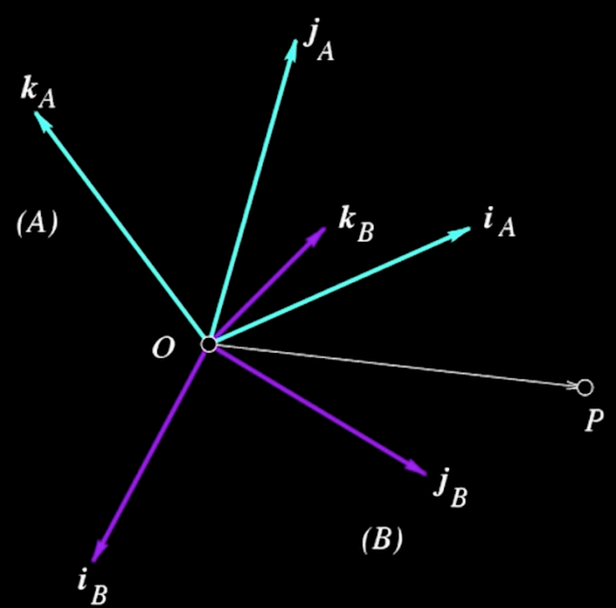
$$^{B}P = {}^{A}P + {}^{B}O_{A}$$

3x3 identity

$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} I & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

(Translation is commutative)

$$\overrightarrow{OP} = \begin{pmatrix} i_A & j_A & k_A \end{pmatrix} \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{pmatrix} = \begin{pmatrix} i_B & j_B & k_B \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{pmatrix}$$

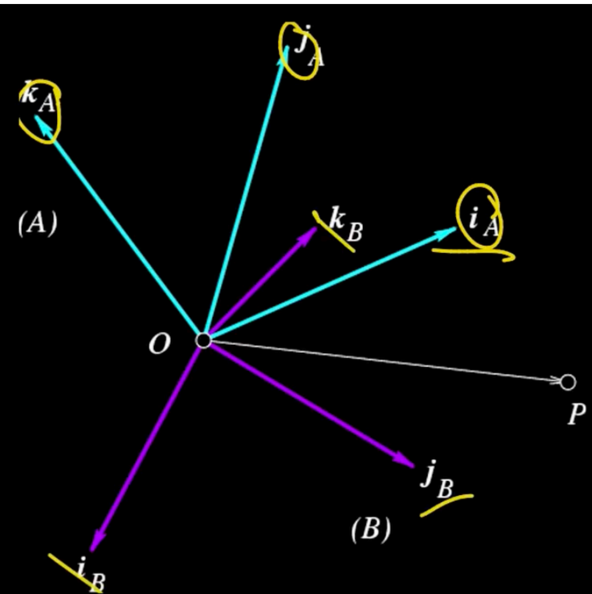


$${}^{B}P = {}^{B}_{A}R\,{}^{A}P$$

${}^{B}_{A}R$ means describing frame A in the coordinate system of frame B

$$
{}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A . \mathbf{i}_B & \mathbf{j}_A . \mathbf{i}_B & \mathbf{k}_A . \mathbf{i}_B \\ \mathbf{i}_A . \mathbf{j}_B & \mathbf{j}_A . \mathbf{j}_B & \mathbf{k}_A . \mathbf{j}_B \\ \mathbf{i}_A . \mathbf{k}_B & \mathbf{j}_A . \mathbf{k}_B & \mathbf{k}_A . \mathbf{k}_B \end{bmatrix}
$$

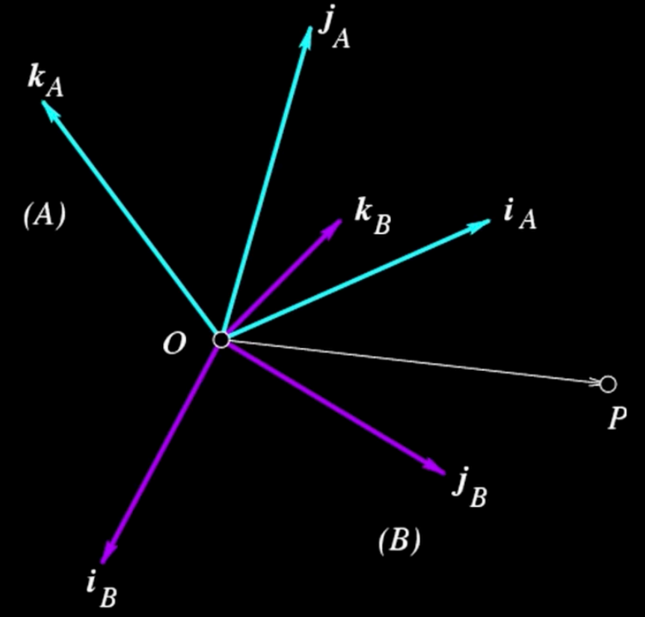
$$
= \begin{bmatrix} {}^{\mathbf{B}}\mathbf{i}_{\mathbf{A}} & {}^{\mathbf{B}}\mathbf{j}_{\mathbf{A}} & {}^{\mathbf{B}}\mathbf{k}_{\mathbf{A}} \end{bmatrix}
$$



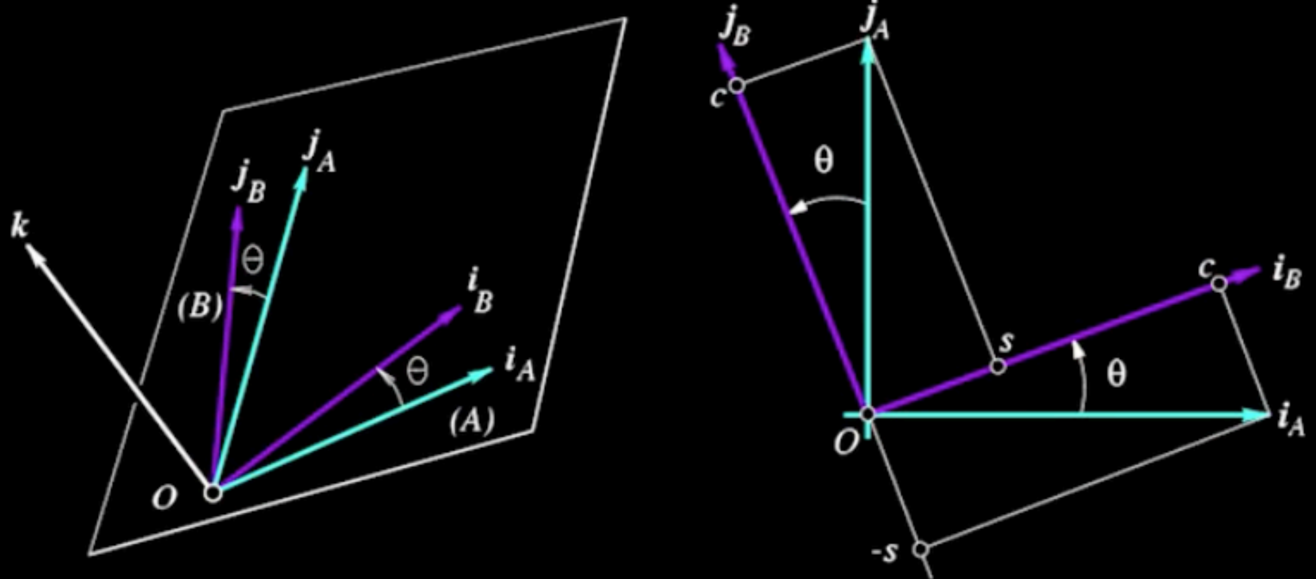
$$
{}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A . \mathbf{i}_B & \mathbf{j}_A . \mathbf{i}_B & \mathbf{k}_A . \mathbf{i}_B \\ \mathbf{i}_A . \mathbf{j}_B & \mathbf{j}_A . \mathbf{j}_B & \mathbf{k}_A . \mathbf{j}_B \\ \mathbf{i}_A . \mathbf{k}_B & \mathbf{j}_A . \mathbf{k}_B & \mathbf{k}_A . \mathbf{k}_B \end{bmatrix}
$$

$$
= \begin{bmatrix} {}^{\mathbf{B}}\mathbf{i}_{\mathbf{A}} & {}^{\mathbf{B}}\mathbf{j}_{\mathbf{A}} & {}^{\mathbf{B}}\mathbf{k}_{\mathbf{A}} \end{bmatrix}
$$

$$
= \begin{bmatrix} {}^{\mathbf{A}}\mathbf{i}_{\mathbf{B}}{}^{T} \\ {}^{\mathbf{A}}\mathbf{j}_{\mathbf{B}}{}^{T} \\ {}^{\mathbf{A}}\mathbf{k}_{\mathbf{B}}{}^{T} \end{bmatrix}
$$



Example: Rotation about z axis



What is the rotation matrix?

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combine 3 to get arbitrary rotation

- Euler angles: Z, X', Z''
- Or heading, pitch roll: world Z, new X, new Y …
- Or roll, pitch and yaw …
- Or Azimuth, elevation, roll…
- Three basic matrices: order matters, but we'll not focus on that

$$R_X(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

$$R_Z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_Y(\kappa) = \begin{bmatrix} \cos(\kappa) & 0 & -\sin(\kappa) \\ 0 & 1 & 0 \\ \sin(\kappa) & 0 & \cos(\kappa) \end{bmatrix}$$

Rotation in homogeneous coordinates

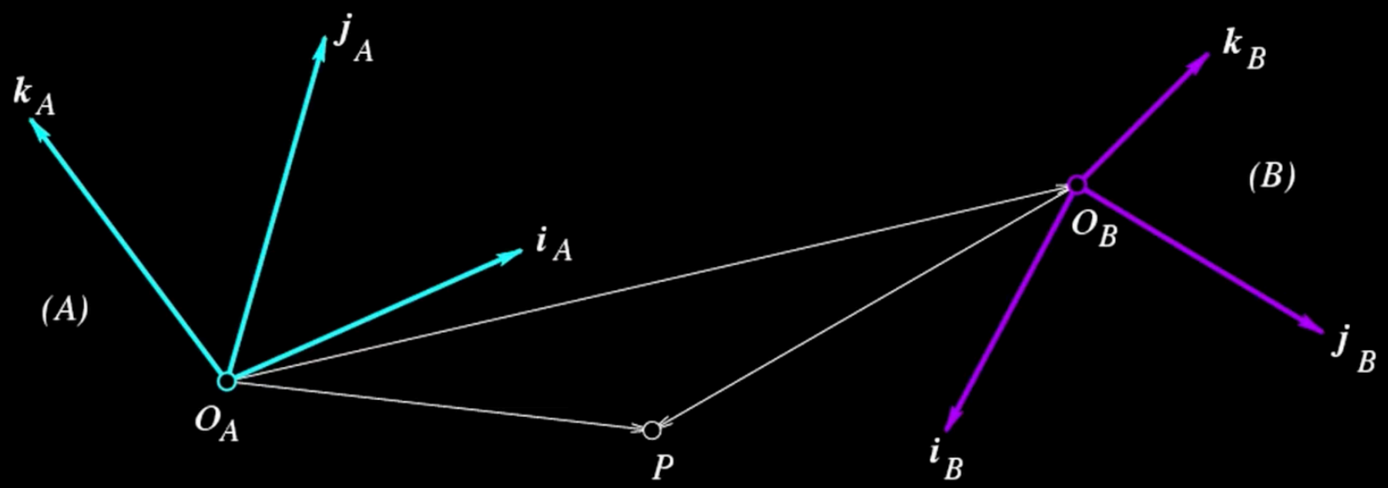
- Using homogeneous coordinates, rotation can be expressed as a matrix multiplication.

$$ {}^{B}P = {}^{B}_{A}R \, {}^{A}P $$

$$ \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} \overset{3\times 3}{{}^{B}_{A}R} & 0 \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} $$

(Rotation is ***not*** commutative)

Rigid transformations



$$ {}^{B}P = {}^{B}_{A}R \, {}^{A}P + {}^{B}O_{A} $$

Rigid transformations (con't)

Unified treatment using homogeneous coordinates:

$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & {}^{B}O_A \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^{B}_{A}R & 0 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_A \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

Rigid transformations (con't)

And even better:

$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

so

Invertible!

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \left({}^{B}_{A}T \right)^{-1} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

Translation and rotation

Homogeneous coordinates:

$$ {}^{B}\vec{p} = {}^{B}_{A}T \; {}^{A}\vec{p} $$

$$ {}^{B}\vec{p} = \begin{pmatrix} \begin{pmatrix} & & \\ & {}^{B}_{A}R & \\ & & \end{pmatrix} & \begin{matrix} | \\ {}^{B}_{A}\vec{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

Homogenous coordinates allows us to write coordinate transforms as a single matrix!

From World to Camera

Rotation from world to camera frame

Translation from world to camera frame

$$ {}^{C}\vec{p} = {}^{C}_{W}R \; {}^{W}\vec{p} + {}^{C}_{W}\vec{t} $$

Non-homogeneous coordinates

Point now in camera frame

Point in world frame

From World to Camera

$$\begin{pmatrix} \\ {}^{C}\vec{p} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} - & - & - & | \\ - & {}^{C}_{W}R & - & {}^{C}_{W}\vec{t} \\ - & - & - & | \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \\ {}^{W}\vec{p} \\ \\ \\ \end{pmatrix}$$

Homogeneous coordinates

From world to camera is the extrinsic parameter matrix (4x4)