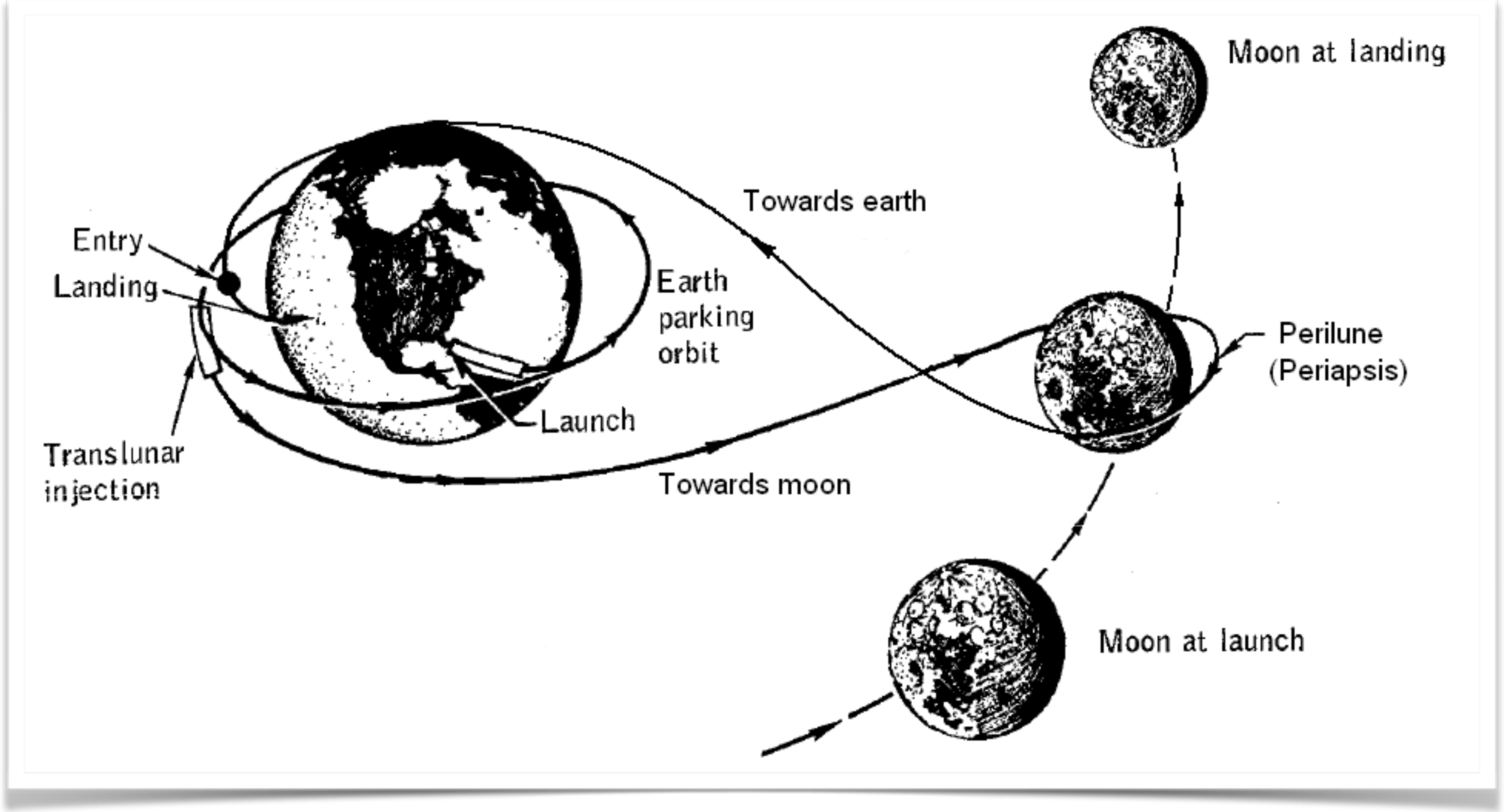
MODULE 2 LESSON 1

THE KALMAN FILTER

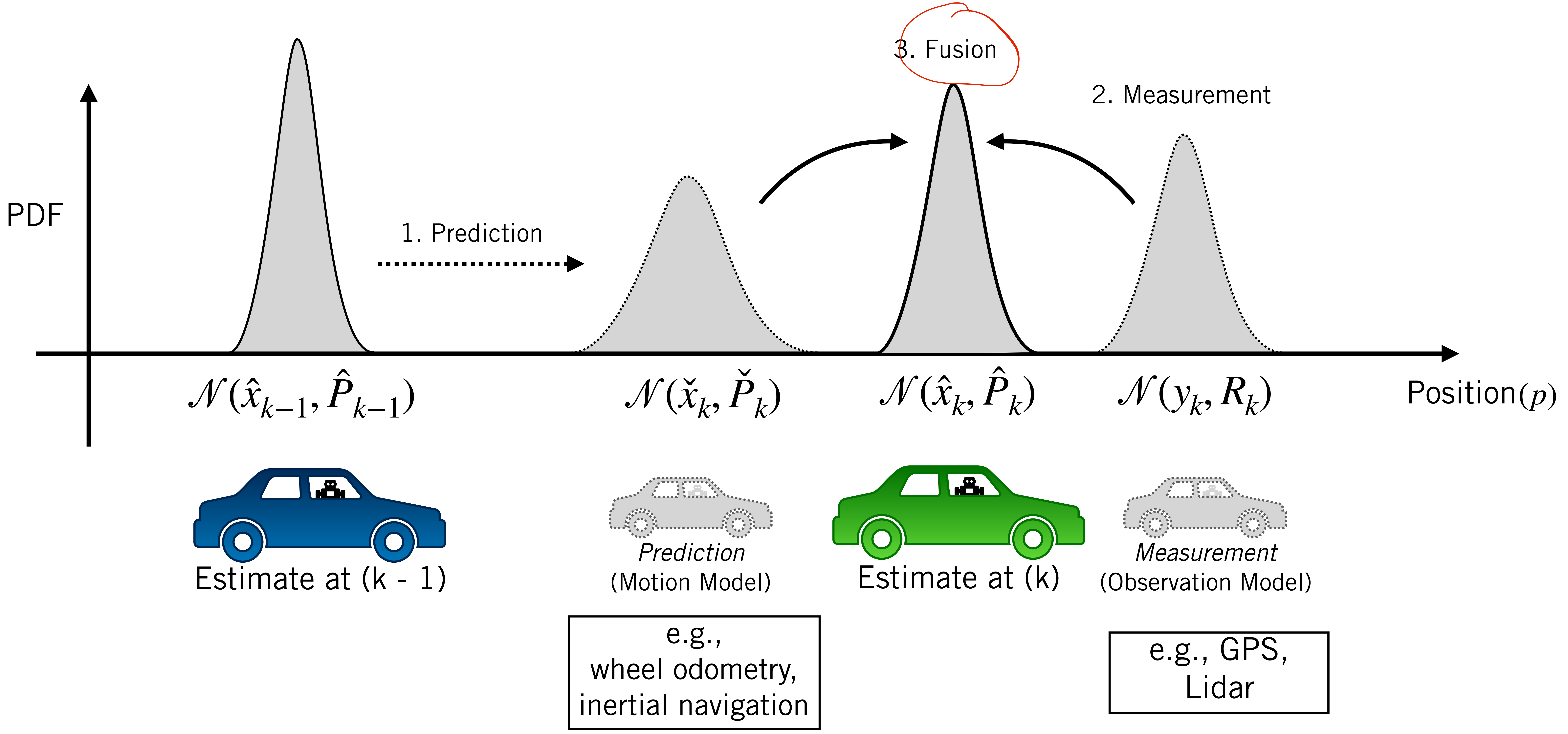
The Kalman Filter

Apollo Guidance Computer



The (extended) Kalman Filter became widely known after its use in the Apollo Guidance Computer for circumlunar navigation.

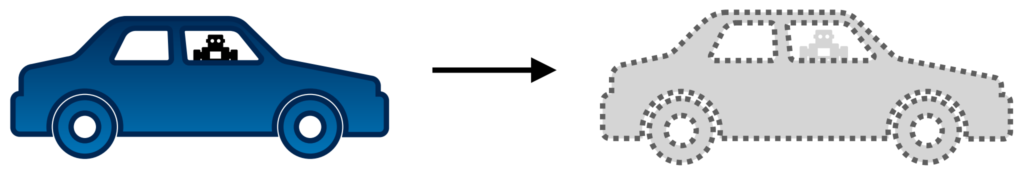
The Kalman Filter | Prediction and Correction



The Kalman Filter | Linear Dynamical System

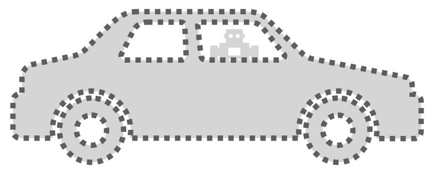
- The Kalman Filter requires the following motion and measurement models:

Motion model: $$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\underbrace{\mathbf{u}_{k-1}}_{\textit{input}} + \underbrace{\mathbf{w}_{k-1}}_{\textit{noise}}$$



Measurement model: $$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \underbrace{\mathbf{v}_k}_{\textit{noise}}$$

ex. wheel torque



- With the following noise properties:

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

Measurement Noise

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$$

Process or Motion Noise

The Kalman Filter | Recursive Least Squares + Process Model

- The Kalman filter is a recursive least squares estimator that also includes a motion model

1 Prediction

$$\check{\mathbf{x}}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1}$$
$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}$$

2a Optimal Gain

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

2b Correction

$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k)$$
$$\hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{P}}_k$$

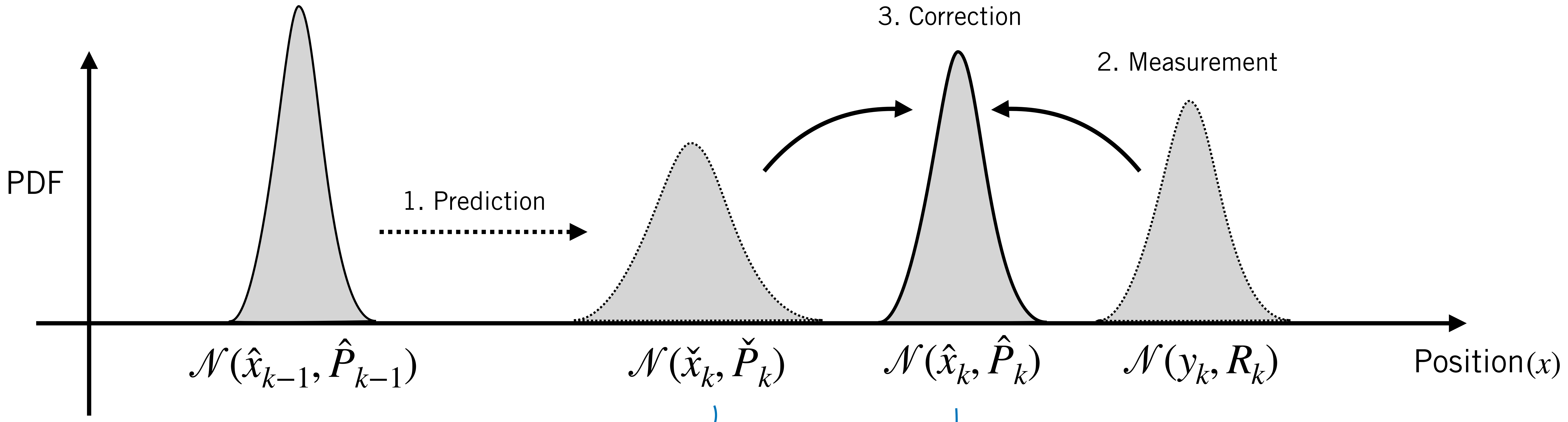
$(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k)$

is often called the 'innovation'

$\check{\mathbf{x}}_k$ **Prediction** *(given motion model) at time k*

$\hat{\mathbf{x}}_k$ **Corrected prediction** (given measurement) at time k

The Kalman Filter | Prediction & Correction



1. Prediction

$$\begin{cases} \check{\mathbf{x}}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} \\ \check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1} \end{cases}$$

Covariance

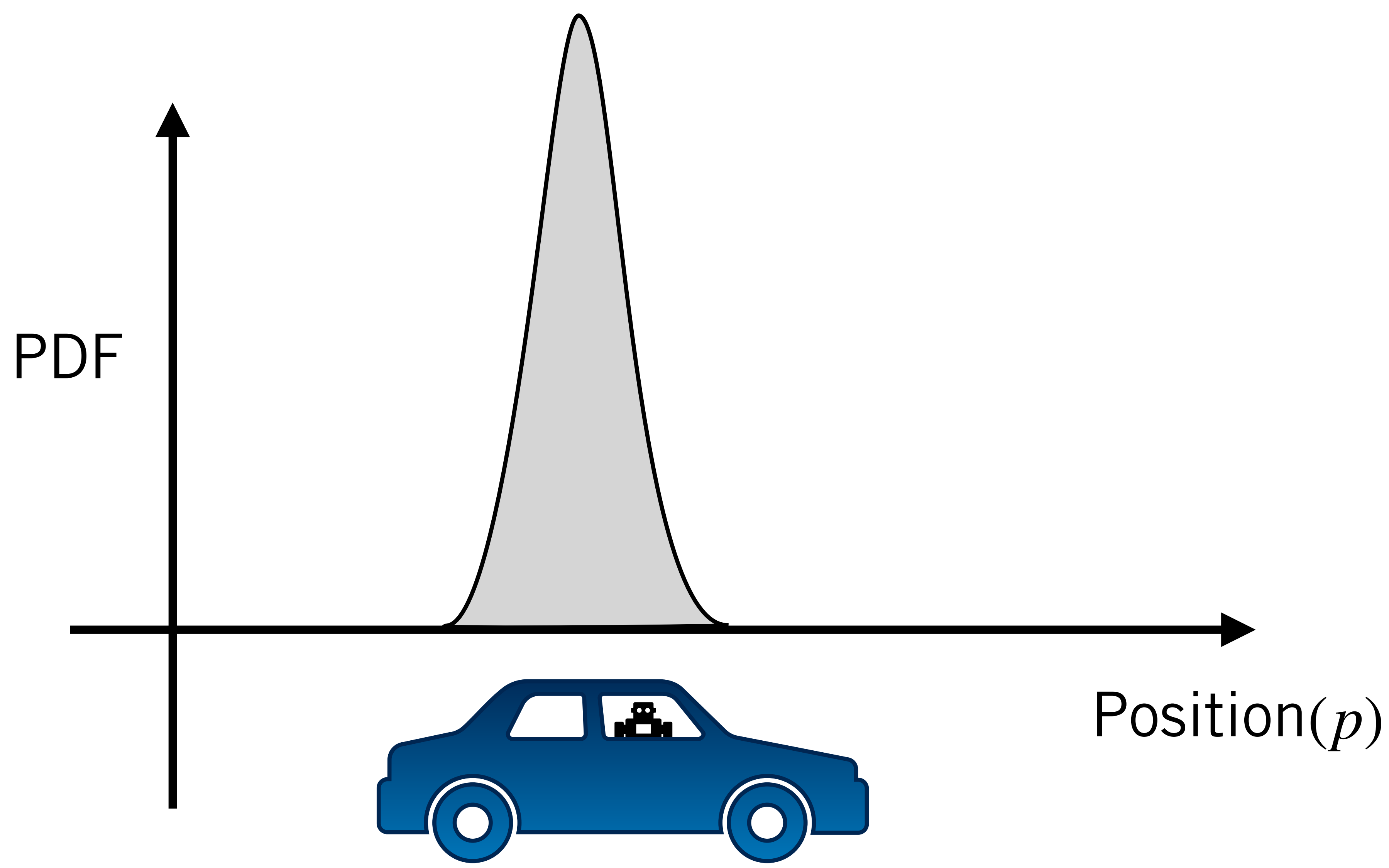
2,3. Measurement & Correction

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Kalman gain

$$\begin{cases} \hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k) \\ \hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{P}}_k \end{cases}$$

The Kalman Filter | Short Example



$$\mathbf{x} = \begin{bmatrix} p \\ \frac{dp}{dt} = \dot{p} \end{bmatrix} \qquad \mathbf{u} = a = \frac{d^2p}{dt^2}$$

Motion/Process Model

$$\mathbf{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} \mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

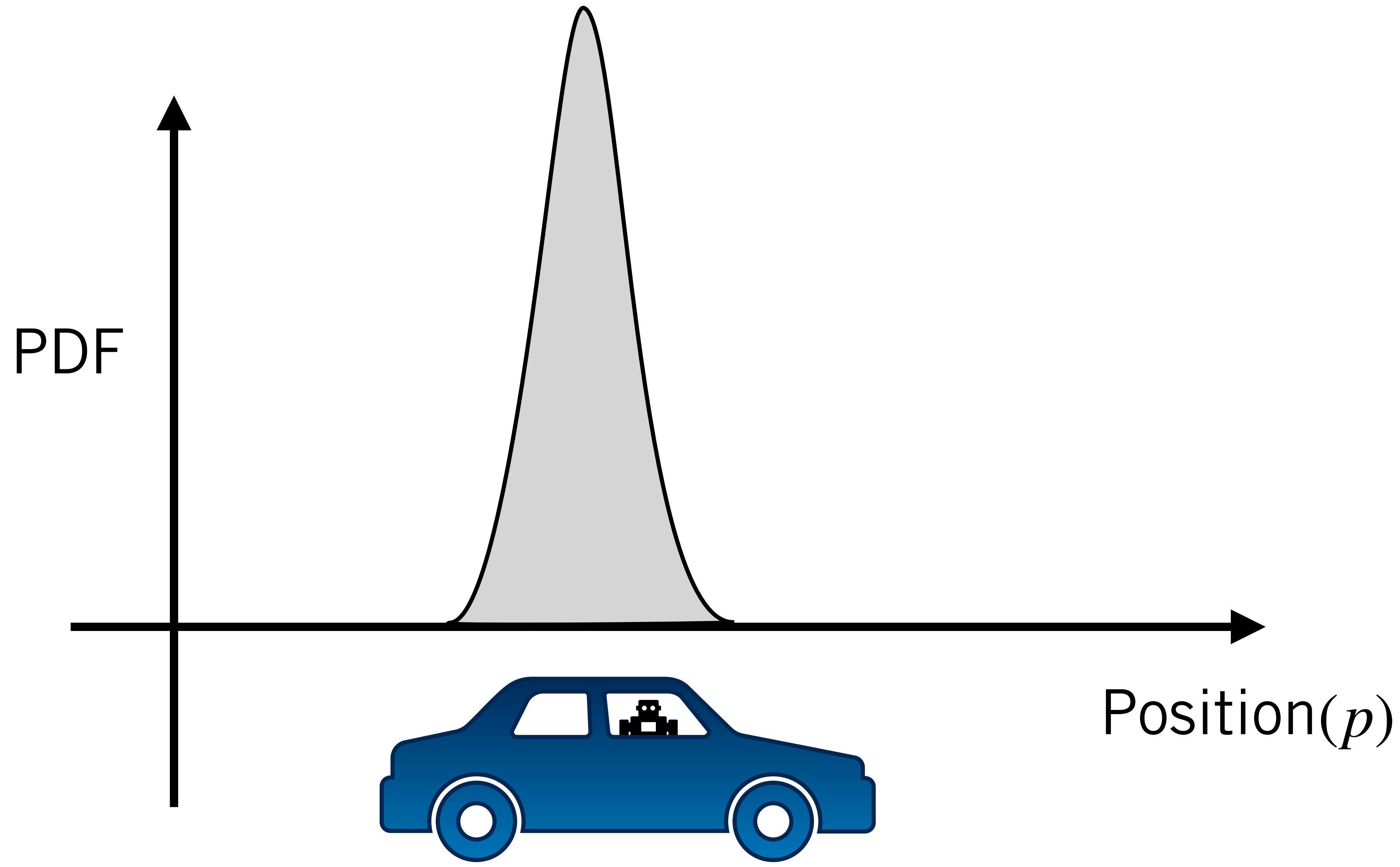
Position Observation

$$y_k = [1 \quad 0]\, \mathbf{x}_k + v_k$$

Noise Densities

$$v_k \sim \mathcal{N}(0,\, 0.05) \qquad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, (0.1)\mathbf{1}_{2\times 2})$$

The Kalman Filter | Short Example



Data

$$\hat{\mathbf{x}}_0 \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\Delta t = 0.5\text{s}$$

$$u_0 = -2 \ [m/s^2] \qquad y_1 = 2.2 \ [m]$$

The Kalman Filter | Short Example Solution

Prediction

$$\check{\mathbf{x}}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1}$$

$$\begin{bmatrix} \check{p}_1 \\ \check{\dot{p}}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}(-2) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}$$

$$\check{\mathbf{P}}_1 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

The Kalman Filter | Short Example Solution

<u>Correction</u>

$$\mathbf{K}_1 = \check{\mathbf{P}}_1\mathbf{H}_1^T(\mathbf{H}_1\check{\mathbf{P}}_1\mathbf{H}_1^T + \mathbf{R}_1)^{-1}$$

$$= \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.05 \right)^{-1}$$

$$= \begin{bmatrix} 0.88 \\ 1.22 \end{bmatrix}$$

$$\hat{\mathbf{x}}_1 = \check{\mathbf{x}}_1 + \mathbf{K}_1(\mathbf{y}_1 - \mathbf{H}_1\check{\mathbf{x}}_1)$$

$$\begin{bmatrix} \hat{p}_1 \\ \hat{\dot{p}}_1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0.88 \\ 1.22 \end{bmatrix} \left(2.2 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2.24 \\ 3.63 \end{bmatrix}$$

state covariance

Bonus! Smaller

$$\hat{\mathbf{P}}_1 = (\mathbf{1} - \mathbf{K}_1\mathbf{H}_1)\check{\mathbf{P}}_1$$

$$= \begin{bmatrix} 0.04 & 0.06 \\ 0.06 & 0.49 \end{bmatrix}$$

Summary | The Kalman Filter

- The Kalman Filter is very similar to RLS but includes a *motion model* that tells us how the state evolves over time

- The Kalman Filter updates a state estimate through two stages:

  1. *prediction* using the motion model

  2. *correction* using the measurement model

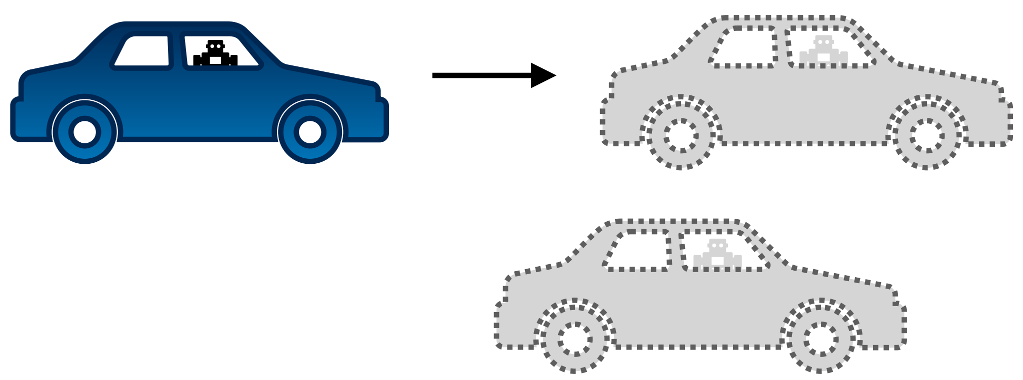
MODULE 2 LESSON 2

THE KALMAN FILTER: BIAS BLUES

Bias in State Estimation

Process/Motion: $$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

Measurement: $$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k$$

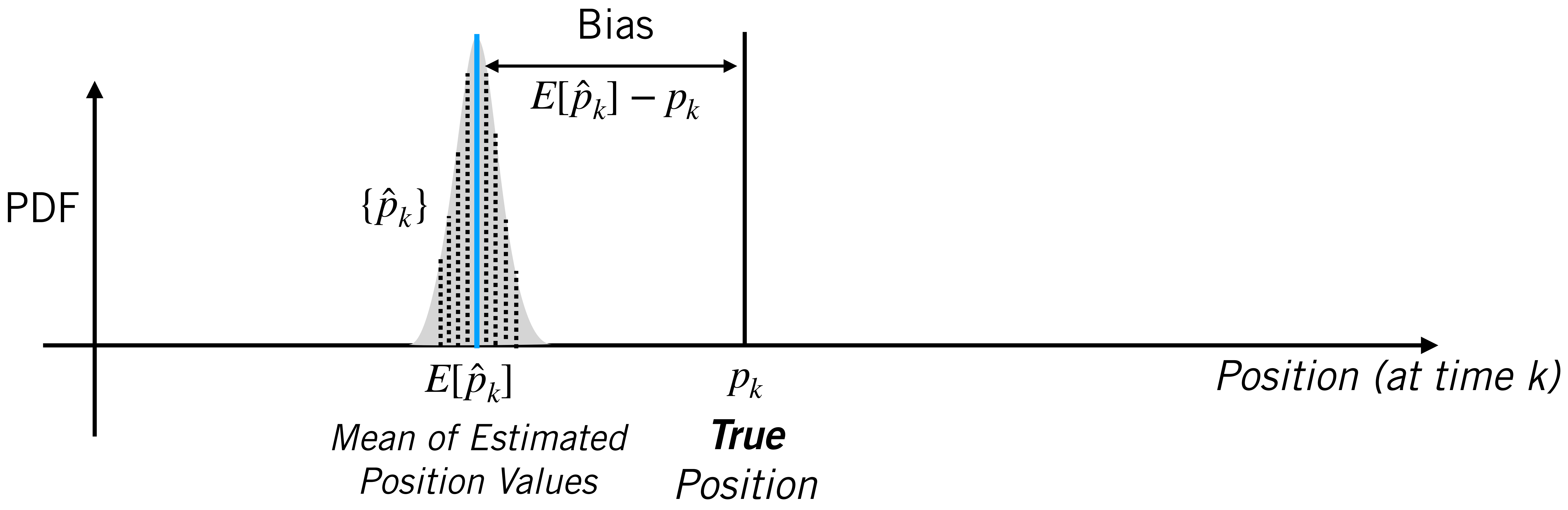


$k = 0,\ 1,\ ....\ K$

Drive the car for K time steps, record estimation error, repeat.....

- We say an estimator or filter is unbiased if it produces an 'average' error of zero at a particular time step *k*, over many trials

Bias in State Estimation



This filter is an unbiased if **for all *k***,

$$E[\hat{e}_k] = E[\hat{p}_k - p_k] = E[\hat{p}_k] - p_k = 0$$

Bias in State Estimation

- How can we compute this analytically for the Kalman filter?
- Consider the *error dynamics*:

$$\check{\mathbf{e}}_k = \check{\mathbf{x}}_k - \mathbf{x}_k$$

Predicted state error

$$\hat{\mathbf{e}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k$$

Corrected estimate error

- Using the Kalman Filter equations, we can derive:

$$\check{\mathbf{e}}_k = \mathbf{F}_{k-1}\check{\mathbf{e}}_{k-1} - \mathbf{w}_k$$

$$\hat{\mathbf{e}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{e}}_k + \mathbf{K}_k\mathbf{v}_k$$

Bias in State Estimation

- For the Kalman filter, for all k,

$$\begin{aligned} E[\check{\mathbf{e}}_k] &= E[\mathbf{F}_{k-1}\check{\mathbf{e}}_{k-1} - \mathbf{w}_k] \\ &= \mathbf{F}_{k-1}E[\check{\mathbf{e}}_{k-1}] - E[\mathbf{w}_k] \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} E[\hat{\mathbf{e}}_k] &= E[(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{e}}_k + \mathbf{K}_k\mathbf{v}_k] \\ &= (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)E[\check{\mathbf{e}}_k] + \mathbf{K}_kE[\mathbf{v}_k] \\ &= \mathbf{0} \end{aligned}$$

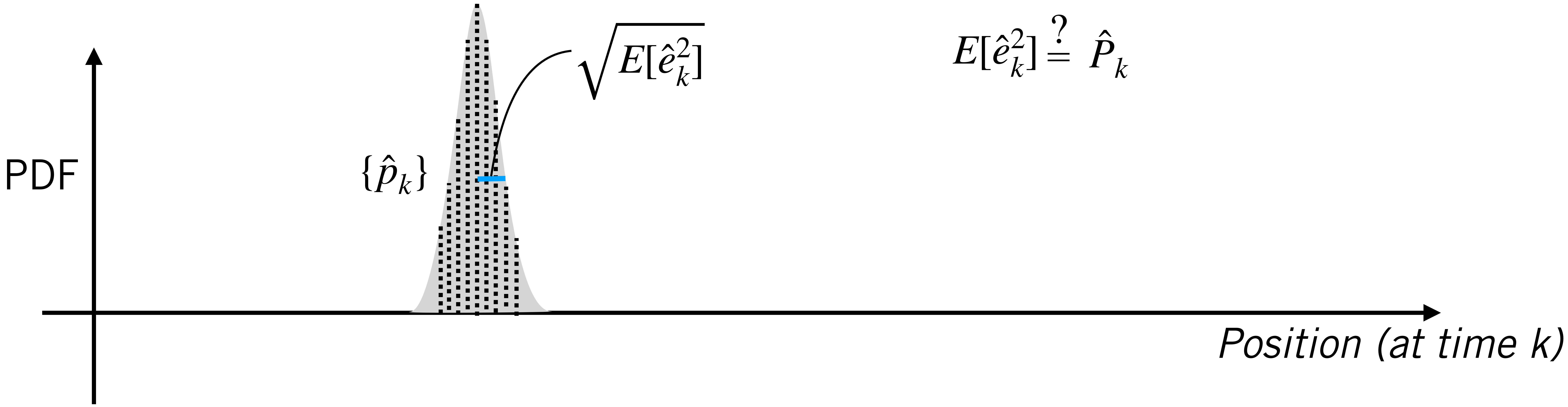
Unbiased *predictions!*

So long as $E[\hat{\mathbf{e}}_0] = \mathbf{0}$ $\quad E[\mathbf{v}] = \mathbf{0}$ $\quad E[\mathbf{w}] = \mathbf{0}$

+ white, uncorrelated noise

Note: this does *not* mean that the error on a *given* trial will be zero, but that, with enough trials, our expected error is zero!

Consistency in State Estimation



$$E[\hat{e}_k^2] \stackrel{?}{=} \hat{P}_k$$

This filter is consistent if **for all *k***,

$$E[\hat{e}_k^2] = E[(\hat{p}_k - p_k)^2] = \hat{P}_k$$

Consistency in State Estimation

- One can also show (with more algebra!) that for all *k*,

$$E[\check{\mathbf{e}}_k \check{\mathbf{e}}_k^T] = \check{\mathbf{P}}_k \qquad\qquad E[\hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T] = \hat{\mathbf{P}}_k$$

***Consistent** predictions!*

- Provided,

$$E[\hat{\mathbf{e}}_0 \hat{\mathbf{e}}_0^T] = \check{\mathbf{P}}_0 \qquad E[\mathbf{v}] = \mathbf{0} \qquad E[\mathbf{w}] = \mathbf{0}$$

+ white noise

The Kalman Filter is the BLUE
Best Linear Unbiased Estimator

- We've shown that given our linear formulation, and zero-mean, white noise the Kalman Filter is ***unbiased***

- We can also say that the filter is ***consistent:***

$$E[\hat{\mathbf{e}}_k] = \mathbf{0}$$
$$E[\hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T] = \hat{\mathbf{P}}_k$$

- In general, if we have white, uncorrelated zero-mean noise, the Kalman filter is the best (i.e., lowest variance) unbiased estimator that uses only a linear combination of measurements

- For this reason, we call it the **BLUE** (best linear unbiased estimator)