Parametric Curves

Course 4, Module 7, Lesson 1

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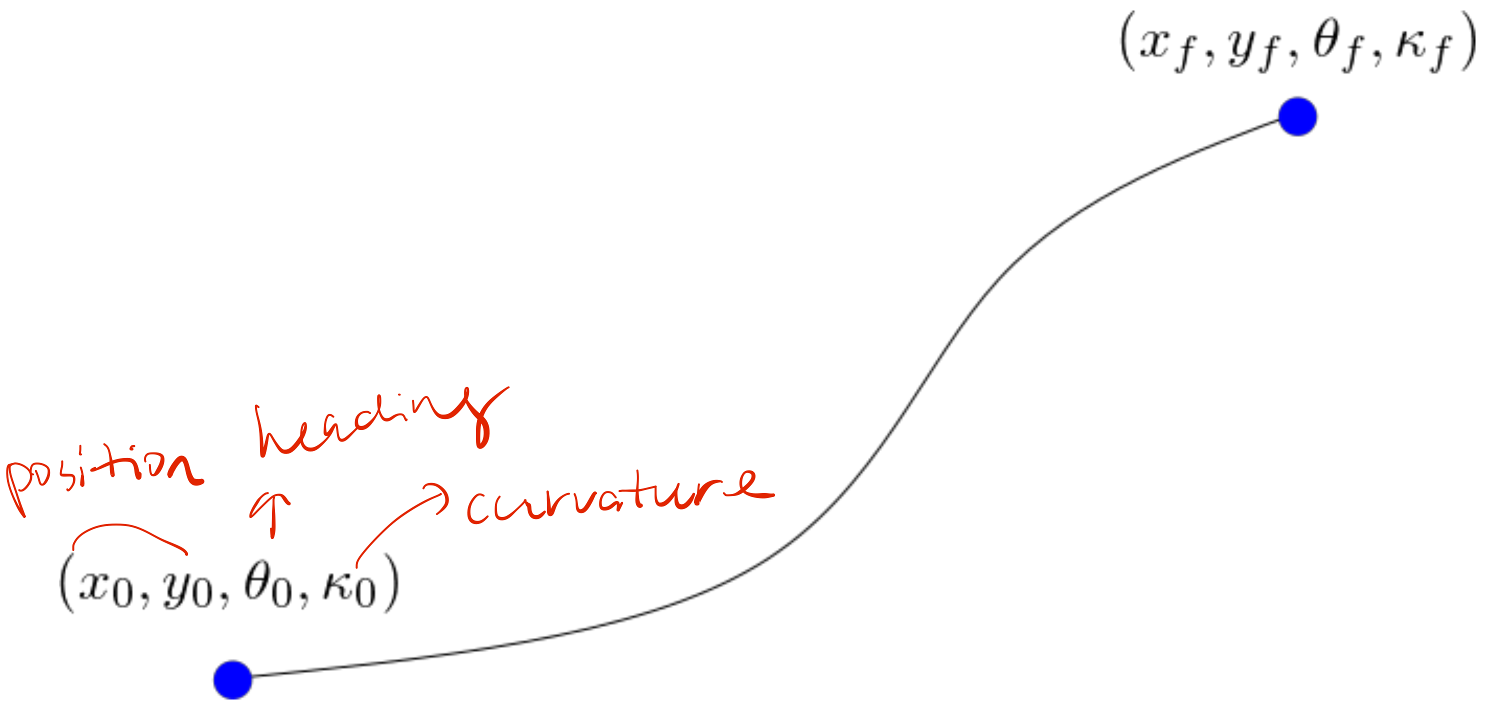
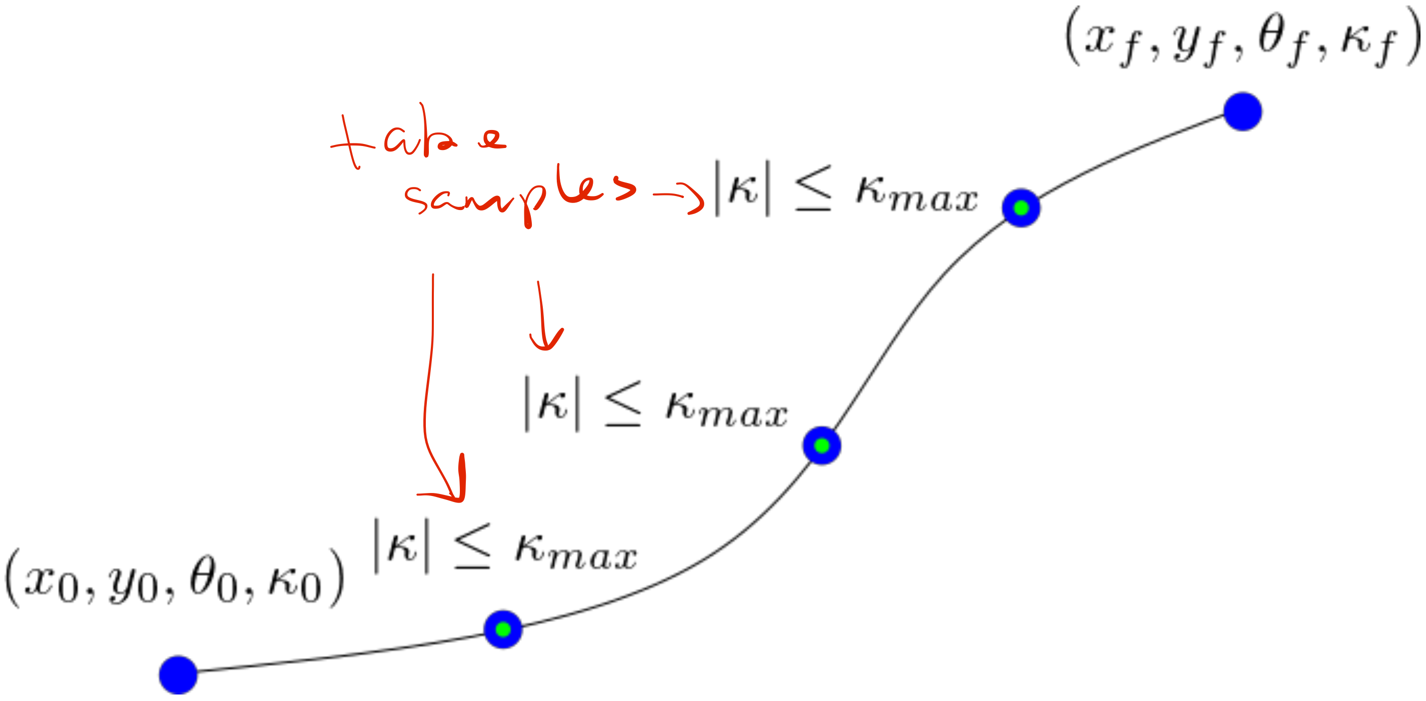
Boundary Conditions

- Boundary conditions must hold on either endpoint of a path
  - The starting and ending conditions of the path



Kinematic Constraints

- Maximum curvature along path cannot be exceeded
- Ensures that car can drive along path



Parametric Curves

- Parametric curve $\mathbf{r}$ can be described by a set of parameterized equations
- Parameter denotes path traversal, can be arc length or unitless
- Example: Cubic spline formulation for x and y

$$\mathbf{r}(u) = \langle x(u), y(u) \rangle$$
$$u \in [0,1]$$

$$x(u) = \alpha_3 u^3 + \alpha_2 u^2 + \alpha_1 u + \alpha_0$$
$$y(u) = \beta_3 u^3 + \beta_2 u^2 + \beta_1 u + \beta_0$$

Path Optimization

- Want to optimize path according to cost functional f
- Parametric curves allow for optimizing over parameter space, which simplifies optimization formulation

kinematic constraints

$$\underbrace{\min f(\mathbf{r}(u))}_{\text{objective functional}} \text{ s.t. } \begin{cases} c(\mathbf{r}(u)) \leq \alpha, & \forall u \in [0,1] \\ \mathbf{r}(0) = \beta_0 \\ \mathbf{r}(u_f) = \beta_f \end{cases}$$

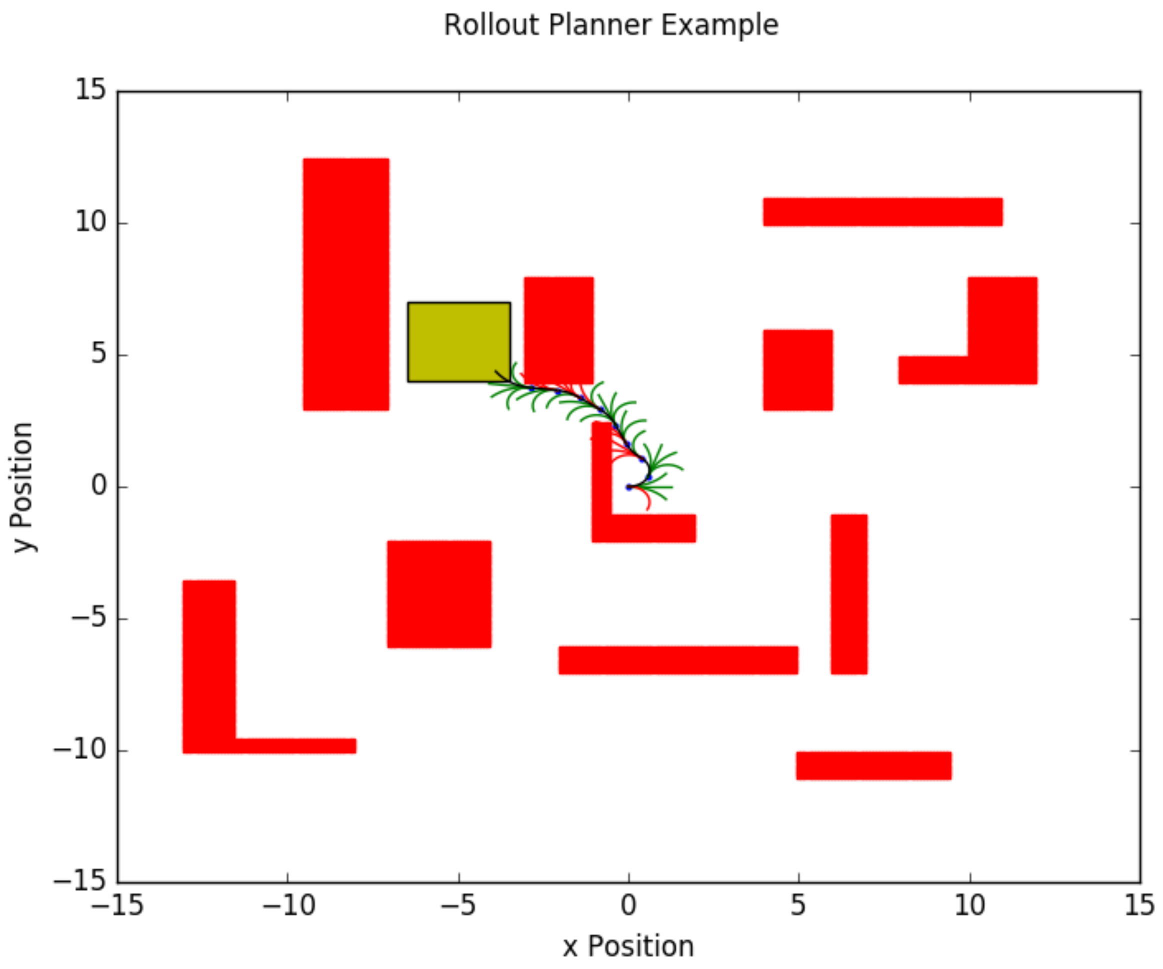
boundary conditions

objective functional

take a function as argument & return a real value

Non-Parametric Path

- Reactive planner used non-parametric paths underlying each trajectory
  - Path was represented as a sequence of points rather than parameterized curves



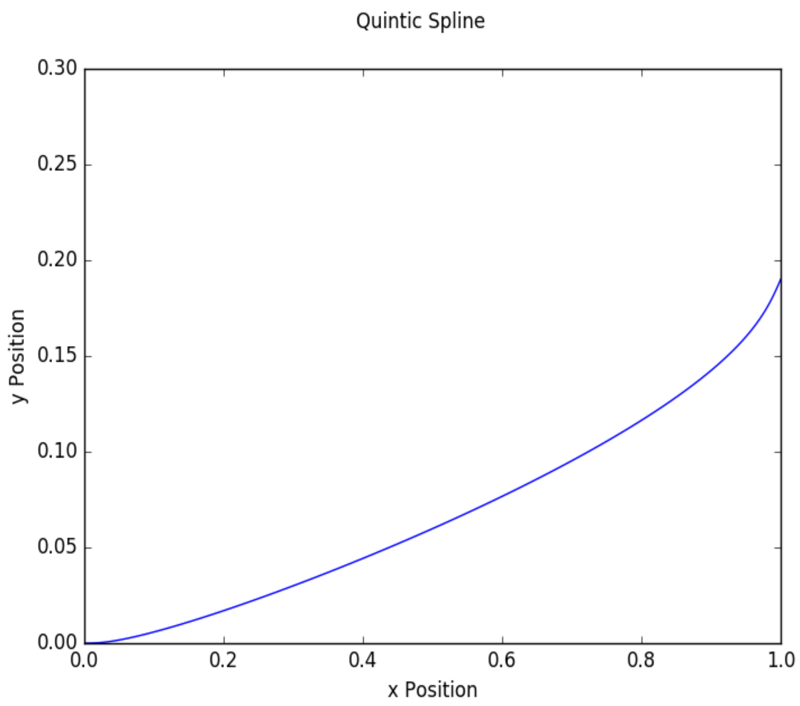
Path Parameterization Examples

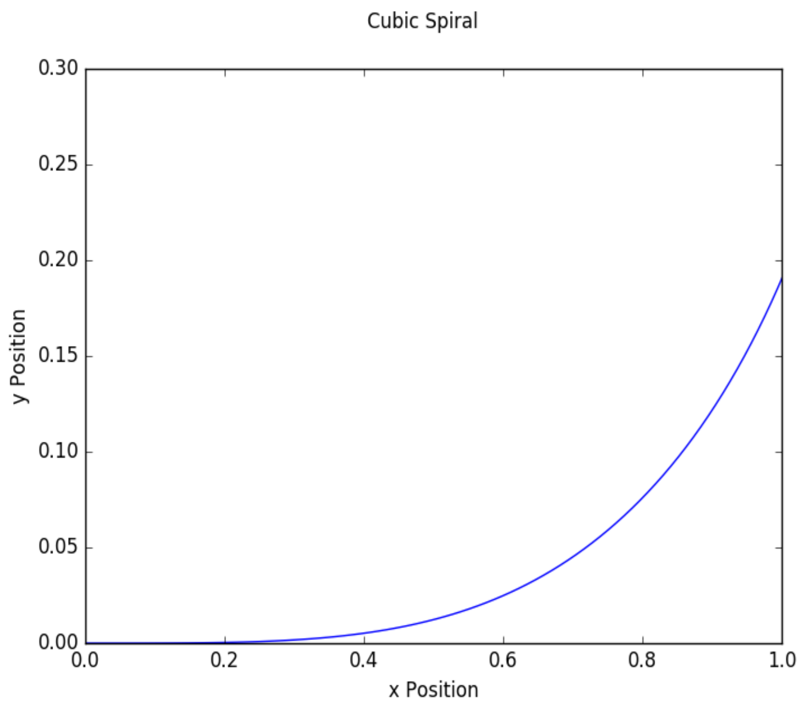
5th order of polynomial functions of x, y position of the car

- Two common parameterized curves are quintic splines and cubic spirals

polynomial curvature func. wrt arc length

- Both allow us to satisfy boundary conditions, and can be optimized parametrically





Quintic Splines

- x and y are defined by 5th order splines
- Closed form solution available for (x, y, θ, κ) boundary conditions

$$x(u) = \alpha_5 u^5 + \alpha_4 u^4 + \alpha_3 u^3 + \alpha_2 u^2 + \alpha_1 u + \alpha_0$$

$$y(u) = \beta_5 u^5 + \beta_4 u^4 + \beta_3 u^3 + \beta_2 u^2 + \beta_1 u + \beta_0$$

start end

$$u \in [0,1]$$

traversal parameter

Quintic Splines Curvature

- Challenging to constrain curvature due to nature of spline's curvature
  - Due to potential discontinuities in curvature or its derivatives

$$\kappa(u) = \frac{x'(u)y''(u) - y'(u)x''(u)}{(x'(u)^2 + y'(u)^2)^{\frac{3}{2}}}$$

not a polynomial

difficult to satisfy curvature constraints

Polynomial Spirals

- Spirals are defined by their curvature as a function of arc length
- Closed form curvature definition allows for simple curvature constraint checking
  - Curvature is well-behaved between sampled points as well due to polynomial formulation

$$\kappa(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$\theta(s) = \theta_0 + \int_0^s a_3 s'^3 + a_2 s'^2 + a_1 s' + a_0 ds'$$
$$= \theta_0 + a_3 \frac{s^4}{4} + a_2 \frac{s^3}{3} + a_1 \frac{s^2}{2} + a_0 s$$

$$x(s) = x_0 + \int_0^s \cos(\theta(s'))ds'$$
$$y(s) = y_0 + \int_0^s \sin(\theta(s'))ds'$$

Polynomial Spiral Position

- Spiral position does not have a closed form solution of x, y, & θ
- Fresnel integrals need to be evaluated numerically
  - This can be done using Simpson's rule

no closed-form solution

$$x(s) = x_0 + \int_0^s \cos(\theta(s'))ds'$$

$$y(s) = y_0 + \int_0^s \sin(\theta(s'))ds'$$

$$\boxed{\int_0^s f(s')ds' \approx \frac{s}{3n}\left(f(0) + 4f\left(\frac{s}{n}\right) + 2f\left(\frac{2s}{n}\right) + \cdots + f(s)\right)}$$

Simpson's Rule

Summary

- Discussed boundary conditions and constraints in path planning problem
- Introduced parametric curves
- Discussed differences between spirals and splines in the context of path planning

spiral: easier implementation of curvature constraints.

spline: computational efficiency