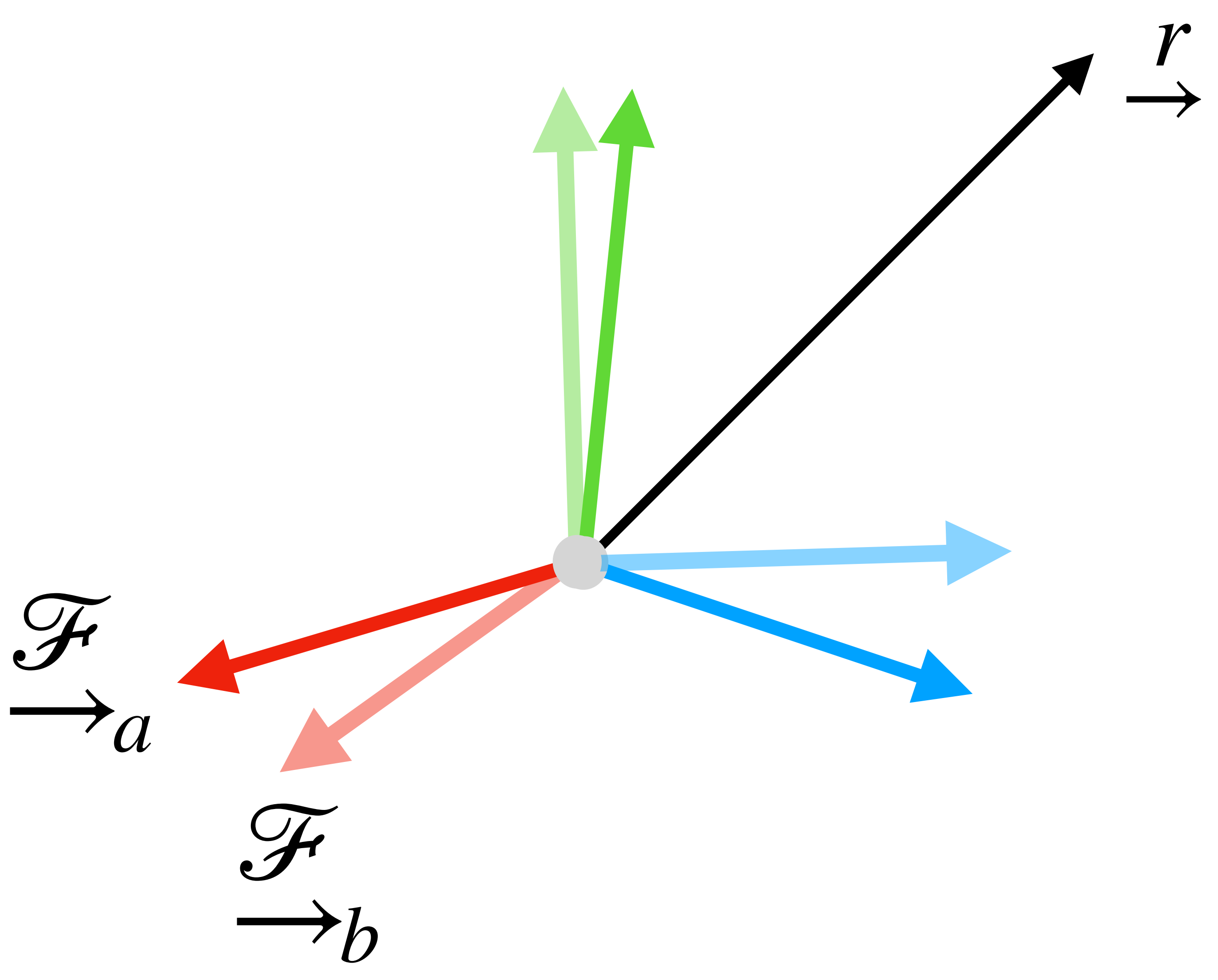
Module 3 | Lesson 1

3D GEOMETRY AND REFERENCE FRAMES

Coordinate Rotations

Vectors can be expressed in different coordinate frames:

$$\underset{\rightarrow}{r} \longrightarrow \begin{cases} \underset{\rightarrow}{\mathscr{F}}_a \longrightarrow \mathbf{r}_a \\ \underset{\rightarrow}{\mathscr{F}}_b \longrightarrow \mathbf{r}_b \end{cases}$$

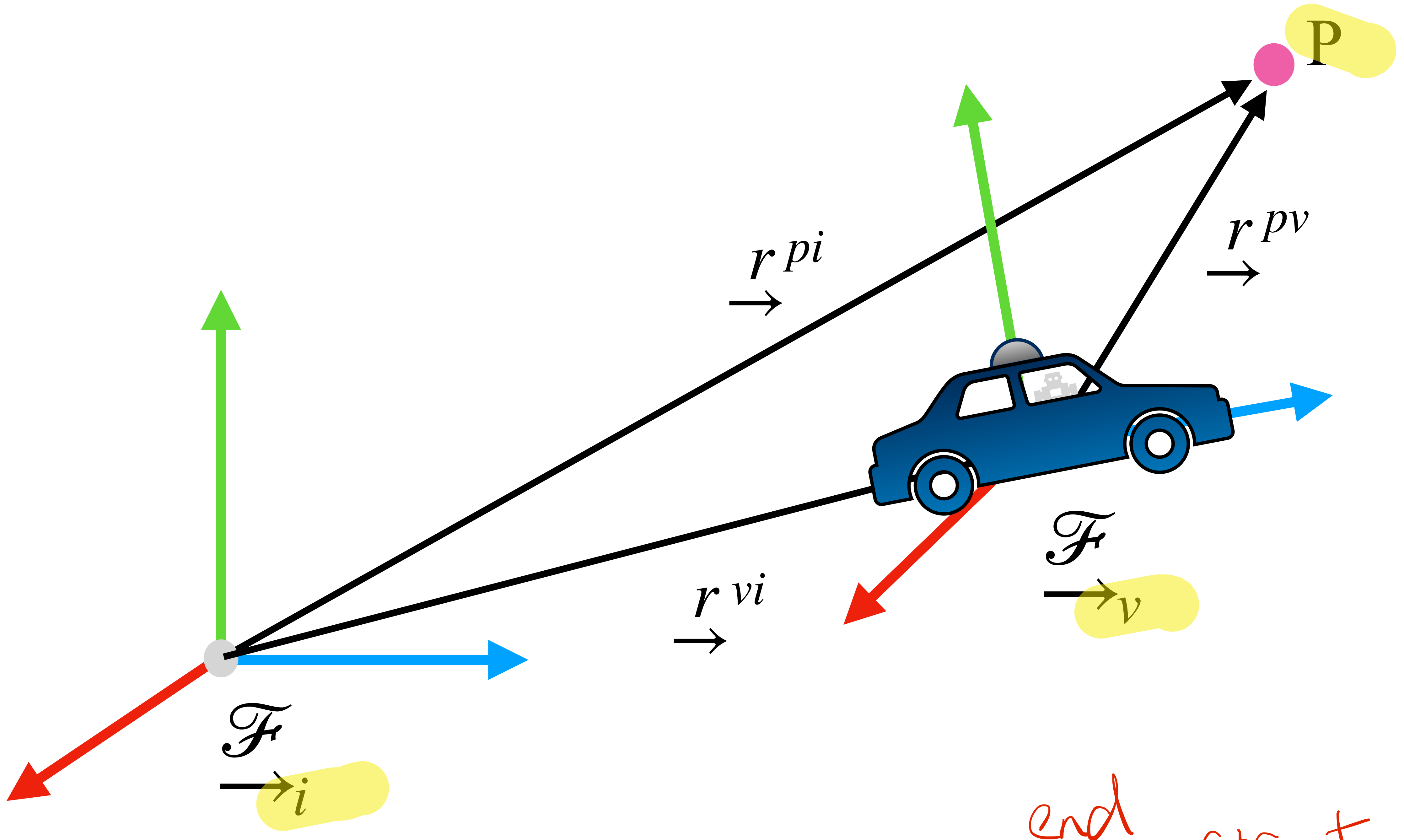


The coordinates of the vector are related through a *rotation matrix*:

$$\mathbf{r}_b = \mathbf{C}_{ba}\mathbf{r}_a$$

takes coordinates in frame *a* and rotates them into frame *b*

Transformations



end start

$$\underrightarrow{r}^{pi} = \underrightarrow{r}^{vi} + \underrightarrow{r}^{pv} \longrightarrow \mathbf{r}_i^{pi} = \mathbf{r}_i^{vi} + \mathbf{C}_{iv}\mathbf{r}_v^{pv}$$

r_i^{pv}

How Can We Represent a Rotation?

1

Rotation matrix

$$\mathbf{C}_{ba} = \begin{bmatrix} \underrightarrow{b}_1 \\ \underrightarrow{b}_2 \\ \underrightarrow{b}_3 \end{bmatrix} \begin{bmatrix} \underrightarrow{a}_1 & \underrightarrow{a}_2 & \underrightarrow{a}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \underrightarrow{b}_1 \cdot \underrightarrow{a}_1 & \underrightarrow{b}_1 \cdot \underrightarrow{a}_2 & \underrightarrow{b}_1 \cdot \underrightarrow{a}_3 \\ \underrightarrow{b}_2 \cdot \underrightarrow{a}_1 & \underrightarrow{b}_2 \cdot \underrightarrow{a}_2 & \underrightarrow{b}_2 \cdot \underrightarrow{a}_3 \\ \underrightarrow{b}_3 \cdot \underrightarrow{a}_1 & \underrightarrow{b}_3 \cdot \underrightarrow{a}_2 & \underrightarrow{b}_3 \cdot \underrightarrow{a}_3 \end{bmatrix}$$

"direction cosine matrix" (DCM)

$$\mathbf{C}_{ba} \in \mathscr{R}^{3\times 3}$$

$$\mathbf{r}_b = \mathbf{C}_{ba}\mathbf{r}_a$$

$$\mathbf{C}_{ba}\mathbf{C}_{ba}^T = \mathbf{C}_{ba}\mathbf{C}_{ab} = \mathbf{1}$$

$C_{ba}^{-1} = C_{ba}^{T}$

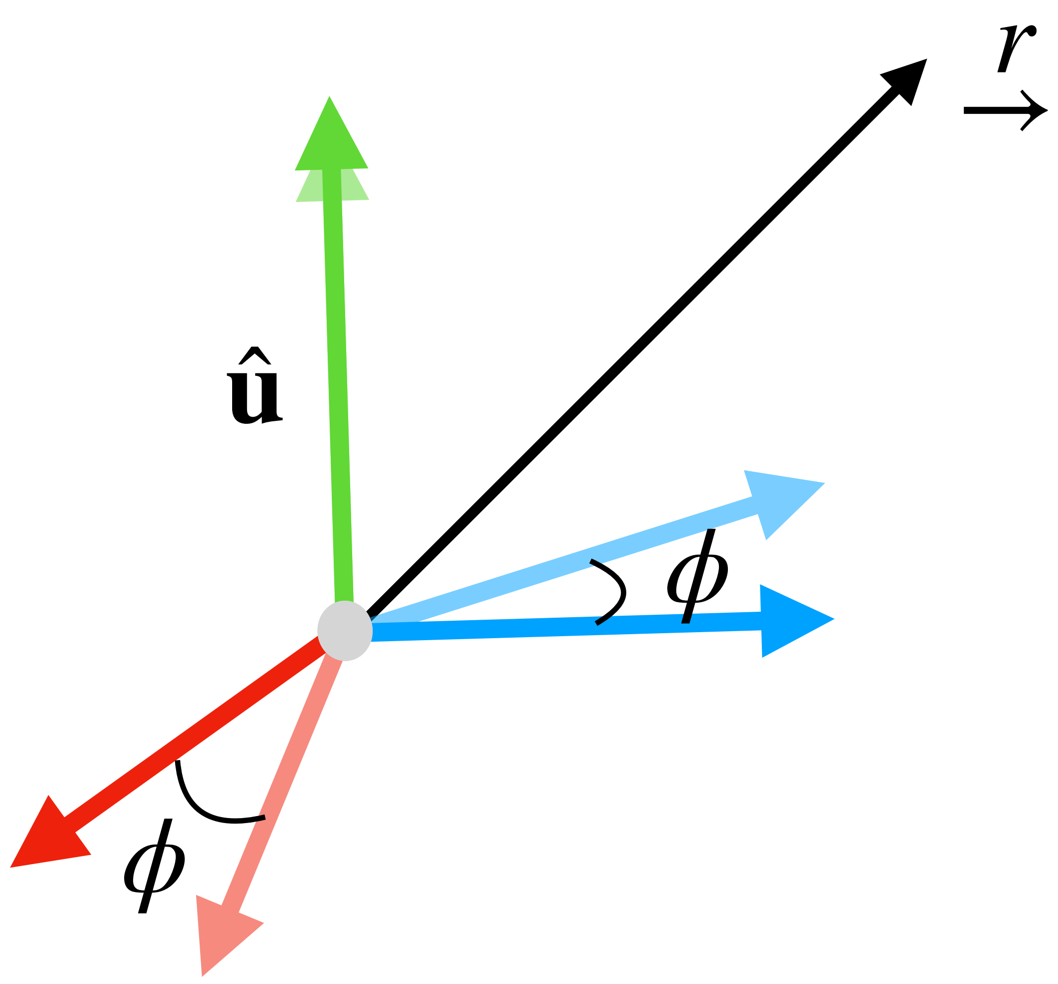
How Can We Represent a Rotation?

2

Unit quaternions

$$\mathbf{q} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \hat{\mathbf{u}} \sin\frac{\phi}{2} \end{bmatrix}$$

$$\|\mathbf{q}\| = 1$$



$$\mathbf{r}_b = \mathbf{C}(\mathbf{q}_{ba})\mathbf{r}_a$$

$$\mathbf{C}(\mathbf{q}) = (q_w^2 - \mathbf{q}_v^T\mathbf{q}_v)\mathbf{1} + 2\mathbf{q}_v\mathbf{q}_v^T + 2q_w[\mathbf{q}_v]_\times$$

where $$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Quaternion Multiplication and Rotations

Quaternions multiplication is a special operation that is *associative* but is not *commutative* in general (just like matrix multiplication!):

The order matters

$$\mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} p_w q_w - \mathbf{p}_v^T \mathbf{q}_v \\ p_w \mathbf{q}_v + q_w \mathbf{p}_v + [\mathbf{p}_v]_\times \mathbf{q}_v \end{bmatrix}$$

quaternion product operator

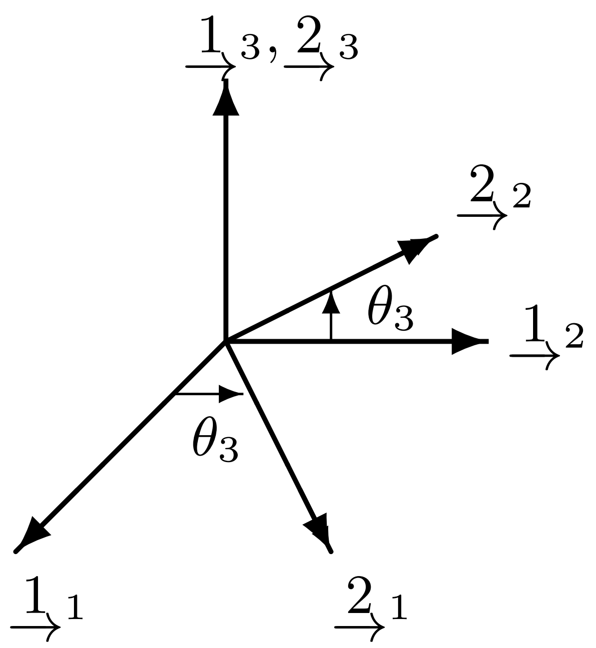
Sequential rotation operations can also be performed by taking advantage of quaternion multiplication:

$$\mathbf{C}(\mathbf{p} \otimes \mathbf{q}) = \mathbf{C}(\mathbf{p})\,\mathbf{C}(\mathbf{q})$$

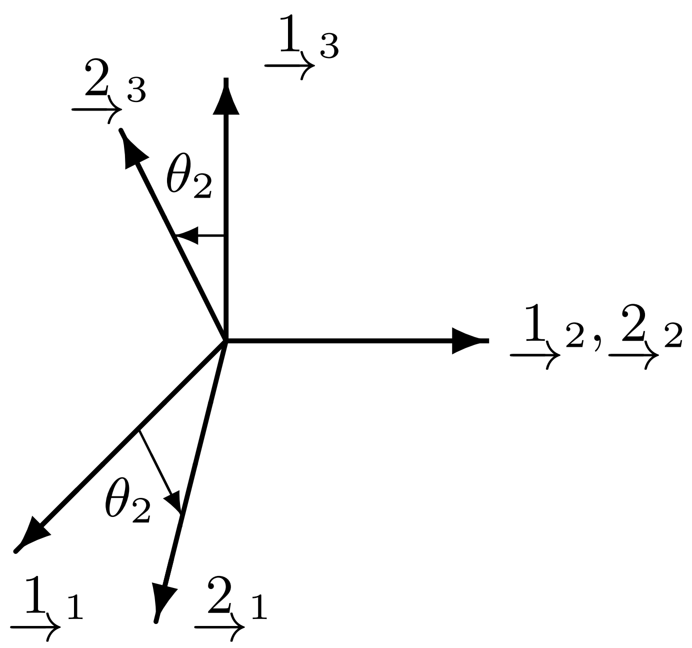
How Can We Represent a Rotation?

3 Euler angles

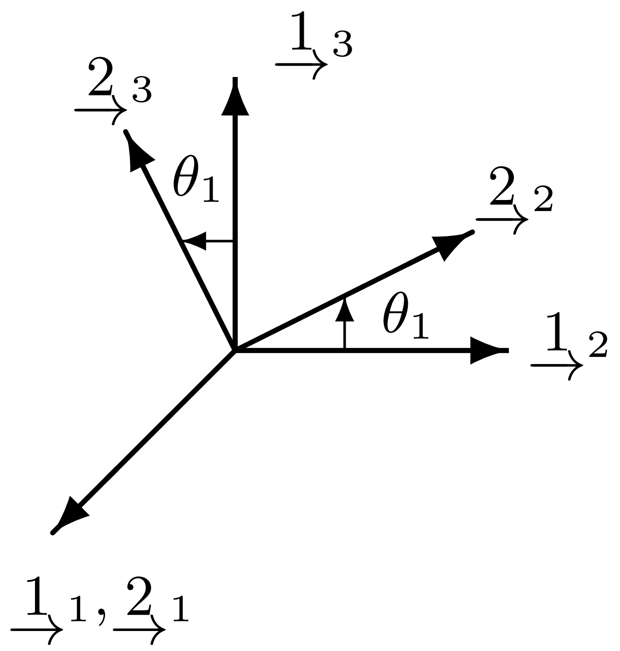
$$\mathbf{C}(\theta_3, \theta_2, \theta_1) = \mathbf{C}_3(\theta_3)\,\mathbf{C}_2(\theta_2)\,\mathbf{C}_1(\theta_1)$$



about the 3-axis



about the 2-axis



about the 1-axis

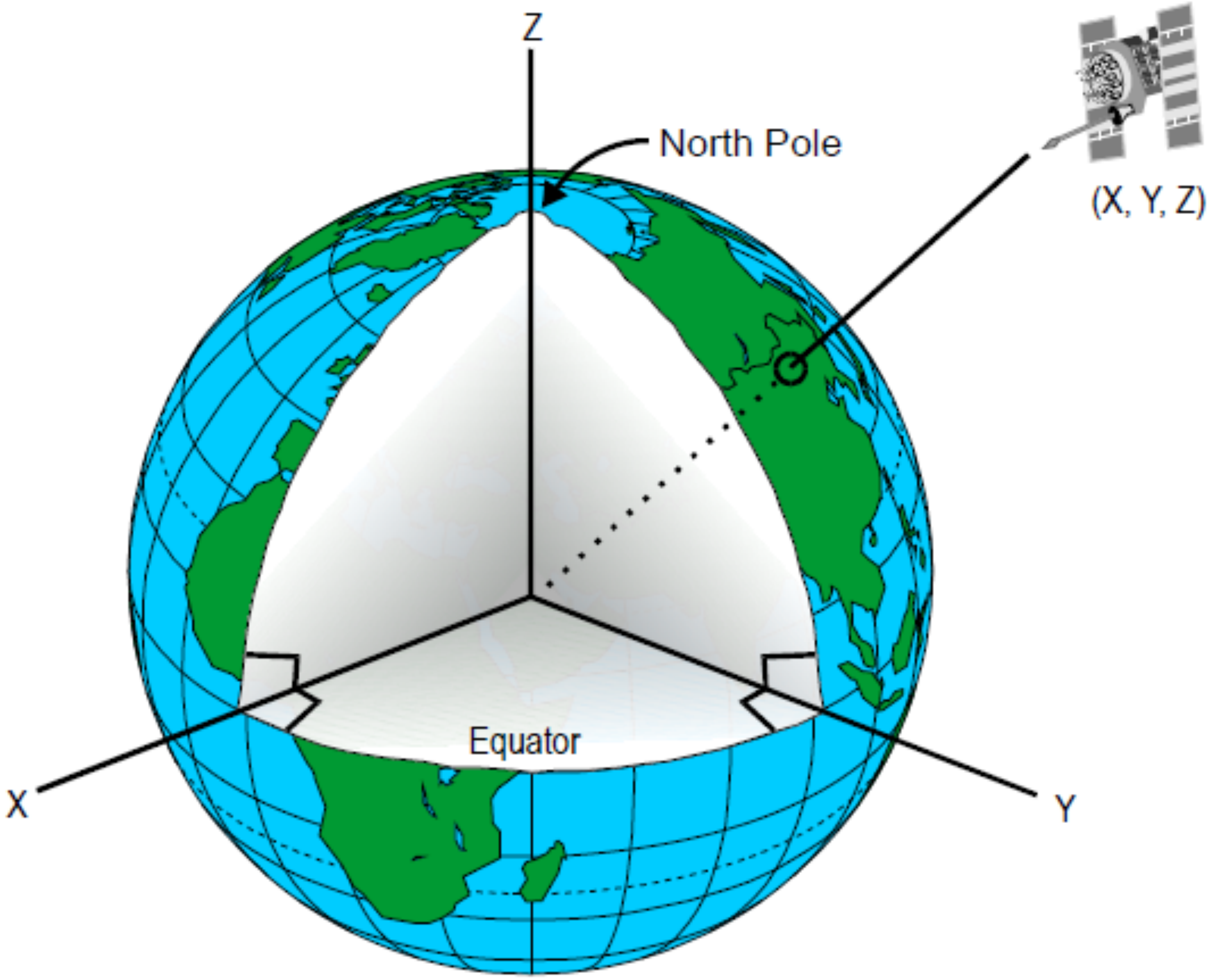
$$\mathbf{C}_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Which Rotation Representation Should I Use?

	Rotation Matrix	Unit quaternion	Euler angles
Expression	$\mathbf{C}$	$\mathbf{q} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \hat{\mathbf{u}}\sin\frac{\phi}{2} \end{bmatrix}$	$\{\theta_3, \theta_2, \theta_1\}$
Parameters	9	4	3
Constraints	$\mathbf{CC}^T = \mathbf{1}$	$\lvert\mathbf{q}\rvert = 1$	*None**
Singularities?	No	No	*Yes!*

For particular rotations, two Euler angles are indistinguishable.

Reference Frames | ECIF



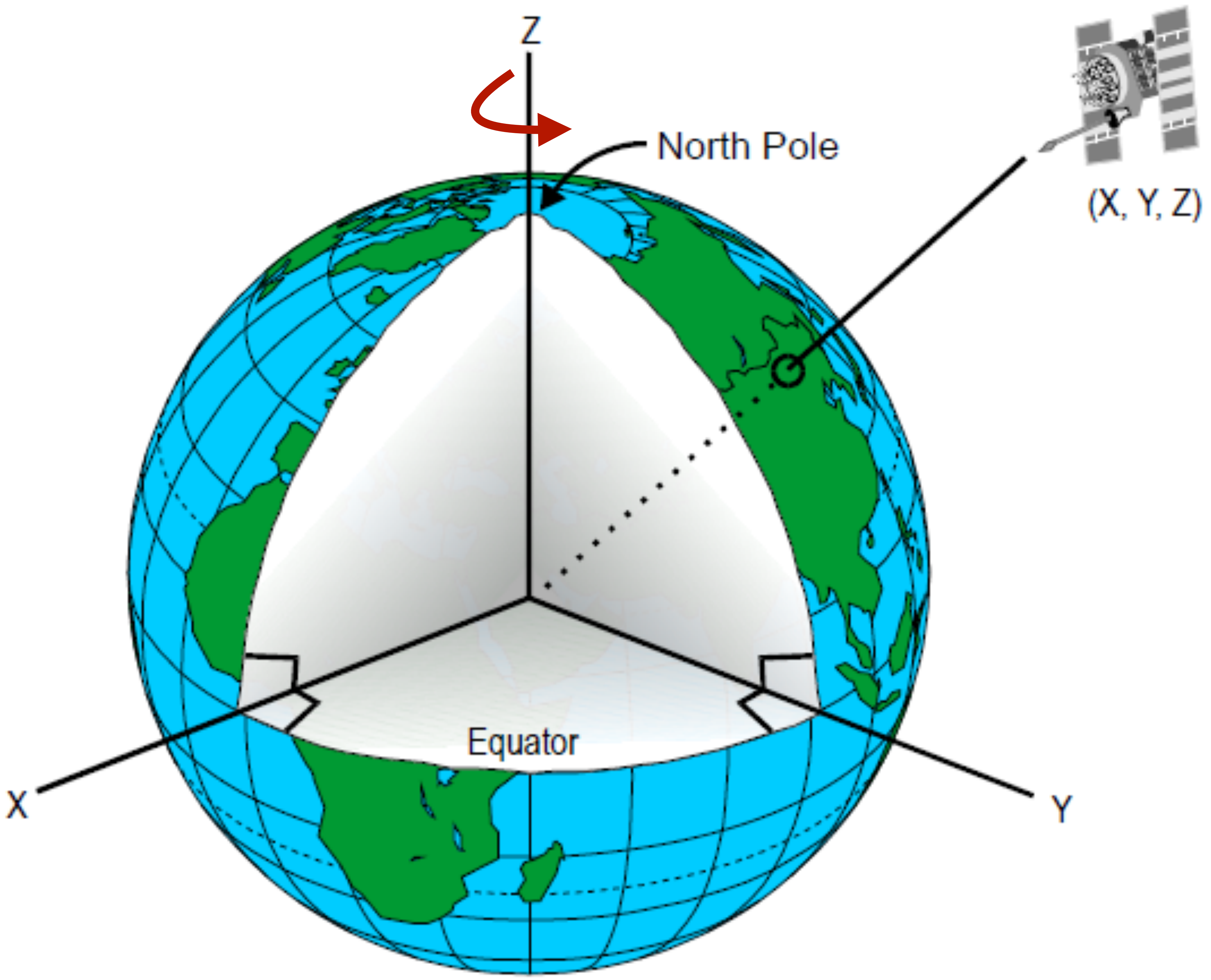
Earth-Centred Inertial Frame

ECIF coordinate frame is fixed,
Earth rotates about the z axis.

$$\underrightarrow{\mathscr{F}}_{\text{ECIF}}$$

x	*fixed w.r.t. stars*
y	*fixed w.r.t. stars*
z	true north

Reference Frames | ECEF



Earth-Centred Earth-Fixed Frame

ECEF coordinate frame rotates with the Earth.

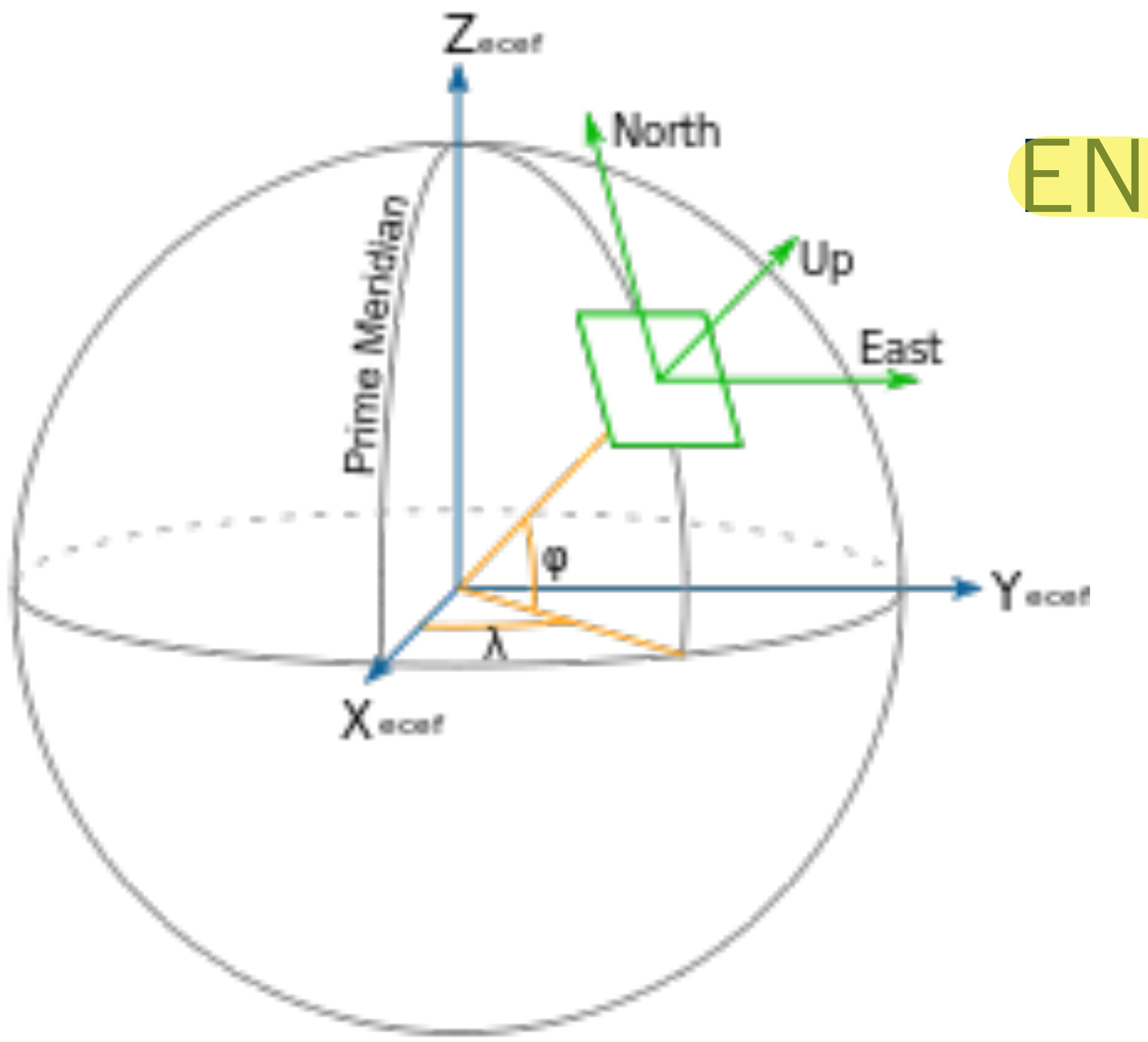
$\underrightarrow{\mathscr{F}}_{\text{ECEF}}$

x	prime meridian (on equator)
y	RHR right hand rule
z	true north

Reference Frames | Navigation

NED Frame

x	True North
y	True East
z	Down (along gravity)



ENU Frame

Reference Frames | Sensor & Vehicle