

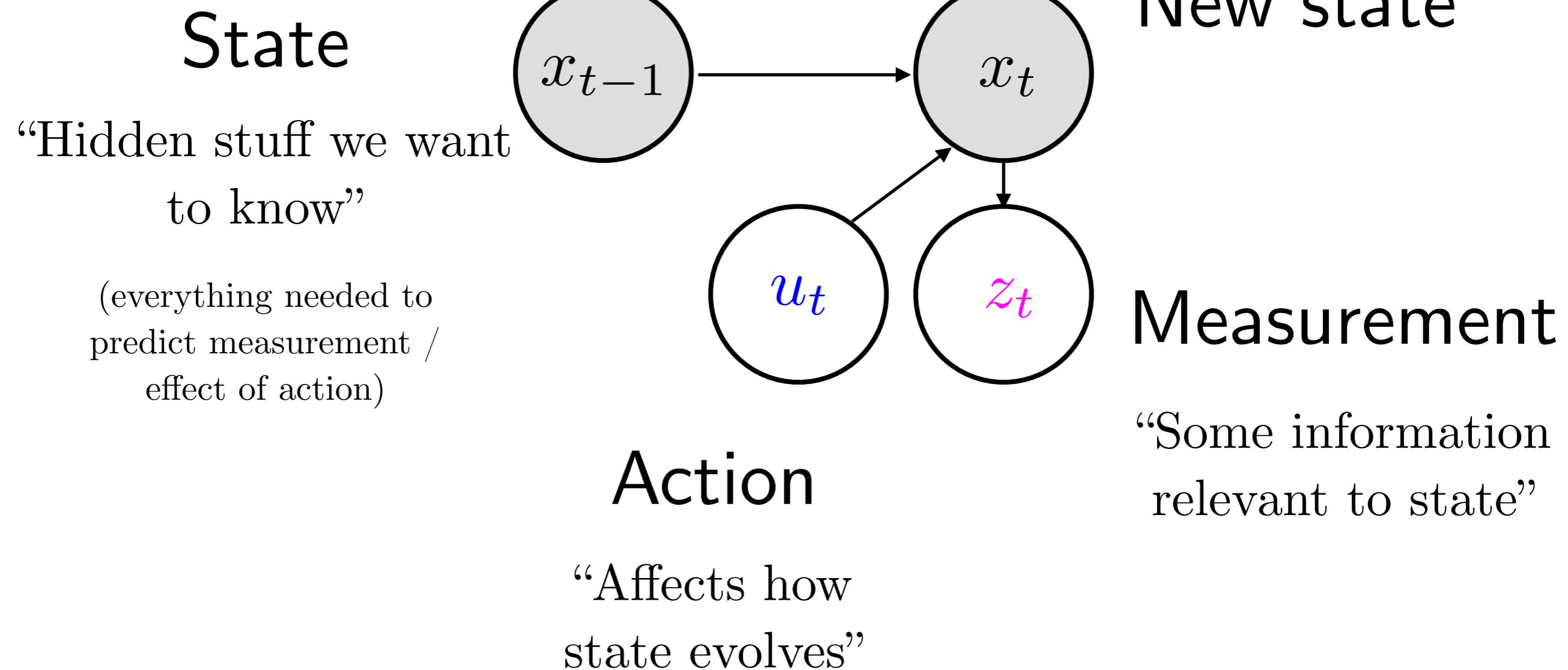
Bayes filtering : A deeper dive

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*Slides based on or adapted from Sanjiban Choudhury

Recap: Key players in a Bayes filter



Today's objective

1. Examples of **nonparametric** Bayes filtering
2. Work through derivation
3. Question assumptions along the way

States and beliefs

State

Discrete (Binary)

$$X = \{X_1, X_2\}$$

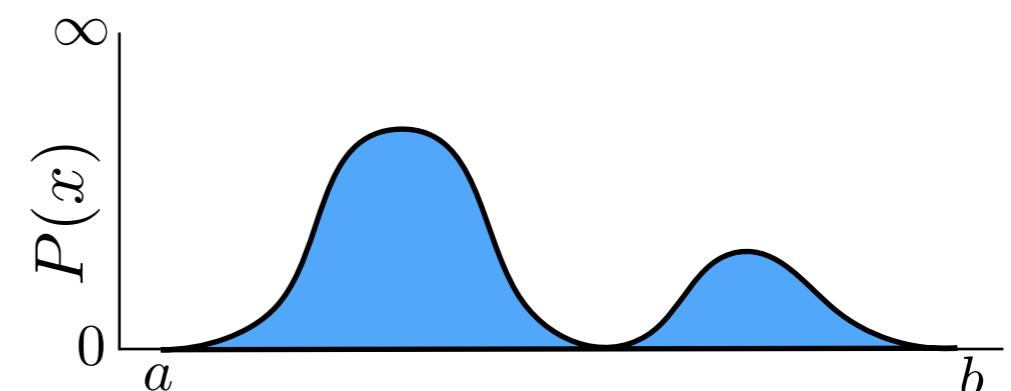
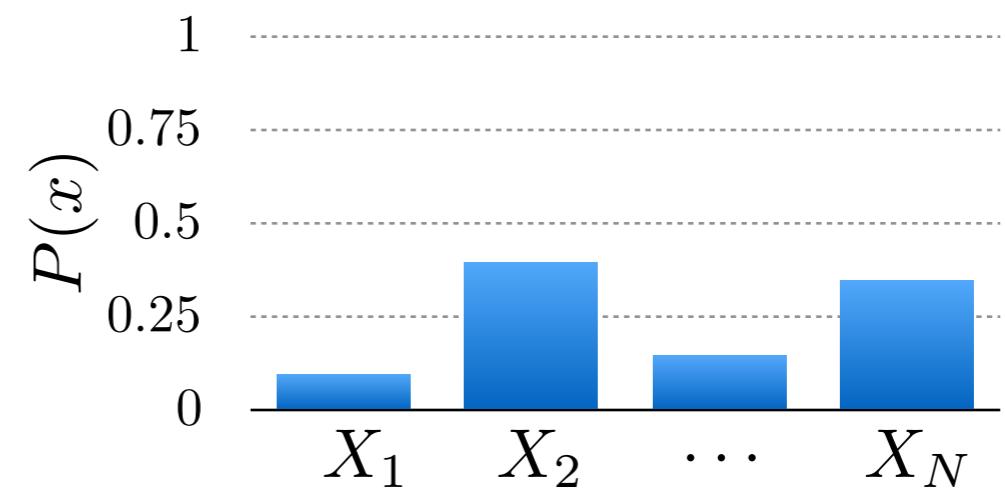
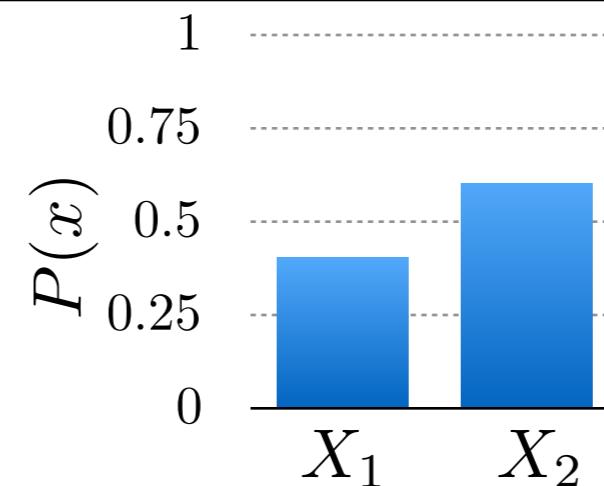
Discrete (More than 2)

$$X = \{X_1, X_2, \dots, X_N\}$$

Continuous

$$X = [a, b]$$

Belief



API of a general Bayes filter

Parameters of the Bayes filter:

Transition
model: $P(x_t | x_{t-1}, u_t)$

Measurement
model: $P(z_t | x_t)$

Input to the filter:

Old belief: $bel(x_{t-1})$

Action: u_t

Measurement: z_t

Output of the filter:

Updated belief: $bel(x_t)$

2 simple steps:

1. Predict belief after action
2. Correct belief after measurement

Discrete

Example 1: Robot opening door



Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

Our robot can do two actions

$$A = \{ \text{Pull}, \text{Leave} \}$$



We define a transition model (note: our robot is clumsy)

$$P(x_t | x_{t-1}, u_t)$$

$$P(O | C, P) = 0.7 \quad P(C | C, P) = 0.3$$

..... and so on

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

Our robot can do two actions

$$A = \{ \text{Pull}, \text{Leave} \}$$



Rewrite the transition model as a matrix

$$\begin{bmatrix} P(x_t = \mathbf{O}|x_{t-1} = \mathbf{O}, \mathbf{u}_t) & P(x_t = \mathbf{O}|x_{t-1} = \mathbf{C}, \mathbf{u}_t) \\ P(x_t = \mathbf{C}|x_{t-1} = \mathbf{O}, \mathbf{u}_t) & P(x_t = \mathbf{C}|x_{t-1} = \mathbf{C}, \mathbf{u}_t) \end{bmatrix}$$

$$P(\cdot|., \mathbf{P}) = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \quad P(\cdot|., \mathbf{L}) = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

Our robot can do two actions

$$A = \{ \text{Pull}, \text{Leave} \}$$



We have a door detector sensor. The sensor is kinda buggy!

$$Z = \{ \text{Open}, \text{Closed} \}$$

$$P(z_t | x_t)$$

.... let's use our matrix format

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Rewrite the measurement model as a vector

$$\begin{bmatrix} P(z_t | O) \\ P(z_t | C) \end{bmatrix}$$

$$P(O|.) = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix}$$

$$P(C|.) = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Let's get ready to Bayes filter!

Example 1: Robot opening door

There are two states that we are tracking

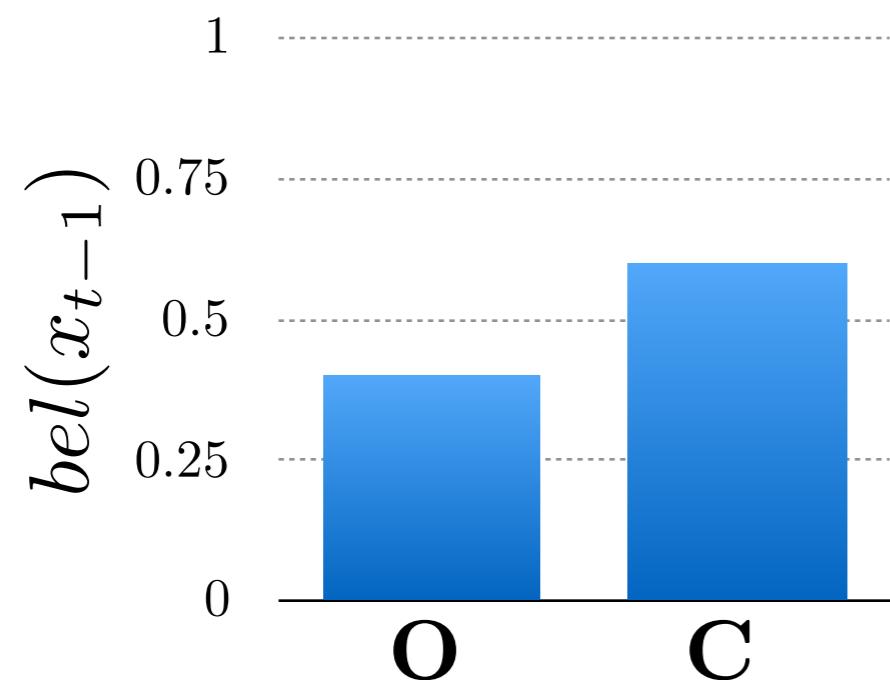
$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 0. Start with the belief at time step $t-1$



$$bel(x_{t-1}) = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Robot thinks the door is open with 0.4 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Robot executes action **Pull**

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum_{x_{t-1}} P(x_t | x_{t-1}, u_t) bel(x_{t-1})$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 1: Prediction - push belief through dynamics given action

$$\begin{bmatrix} P(x_t = O) \\ P(x_t = C) \end{bmatrix} = \begin{bmatrix} P(x_t = O | x_{t-1} = O, u_t) & P(x_t = O | x_{t-1} = C, u_t) \\ P(x_t = C | x_{t-1} = O, u_t) & P(x_t = C | x_{t-1} = C, u_t) \end{bmatrix} \begin{bmatrix} P(x_{t-1} = O) \\ P(x_{t-1} = C) \end{bmatrix}$$

$$\overline{bel}(x_t)$$

$$bel(x_{t-1})$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 1: Prediction - push belief through dynamics given action

$$\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

$$\overline{bel}(x_t) \quad P(., ., \mathbf{P}) \quad bel(x_{t-1})$$

Robot thinks the door is open with 0.74 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Robot receives measurement

Closed

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t | x_t) \overline{bel}(x_t)$$

(normalize)

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 2: Correction - apply Bayes rule given measurement

$$\begin{bmatrix} P(x_t = O) \\ P(x_t = C) \end{bmatrix} = \eta \begin{bmatrix} P(z_t | O) \\ P(z_t | C) \end{bmatrix} * \begin{bmatrix} P(x_t = O) \\ P(x_t = C) \end{bmatrix}$$

$$bel(x_t)$$

$$P(\text{C}|.)$$

element
wise

$$\overline{bel}(x_t)$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 2: Correction - apply Bayes rule given measurement

$$\begin{bmatrix} P(x_t = O) \\ P(x_t = C) \end{bmatrix} = \eta \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$$

$$bel(x_t)$$

$$\overline{bel}(x_t)$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 2: Correction - apply Bayes rule given measurement

$$\begin{bmatrix} P(x_t = O) \\ P(x_t = C) \end{bmatrix} = \eta \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \eta \begin{bmatrix} 0.296 \\ 0.208 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

$$bel(x_t)$$

$$\overline{bel}(x_t)$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

Robot thinks the door is open with 0.58 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \text{Open}, \text{Closed} \}$$

$$A = \{ \text{Pull}, \text{Leave} \}$$

$$Z = \{ \text{Open}, \text{Closed} \}$$



Let's summarize

Robot thought the door is open with 0.4 probability

Robot executed **Pull** action.

Robot thinks the door is open with 0.74 probability

Robot got **Closed** measurement.

Robot thinks the door is open with 0.58 probability

Continuous

Bayes filter in a nutshell

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

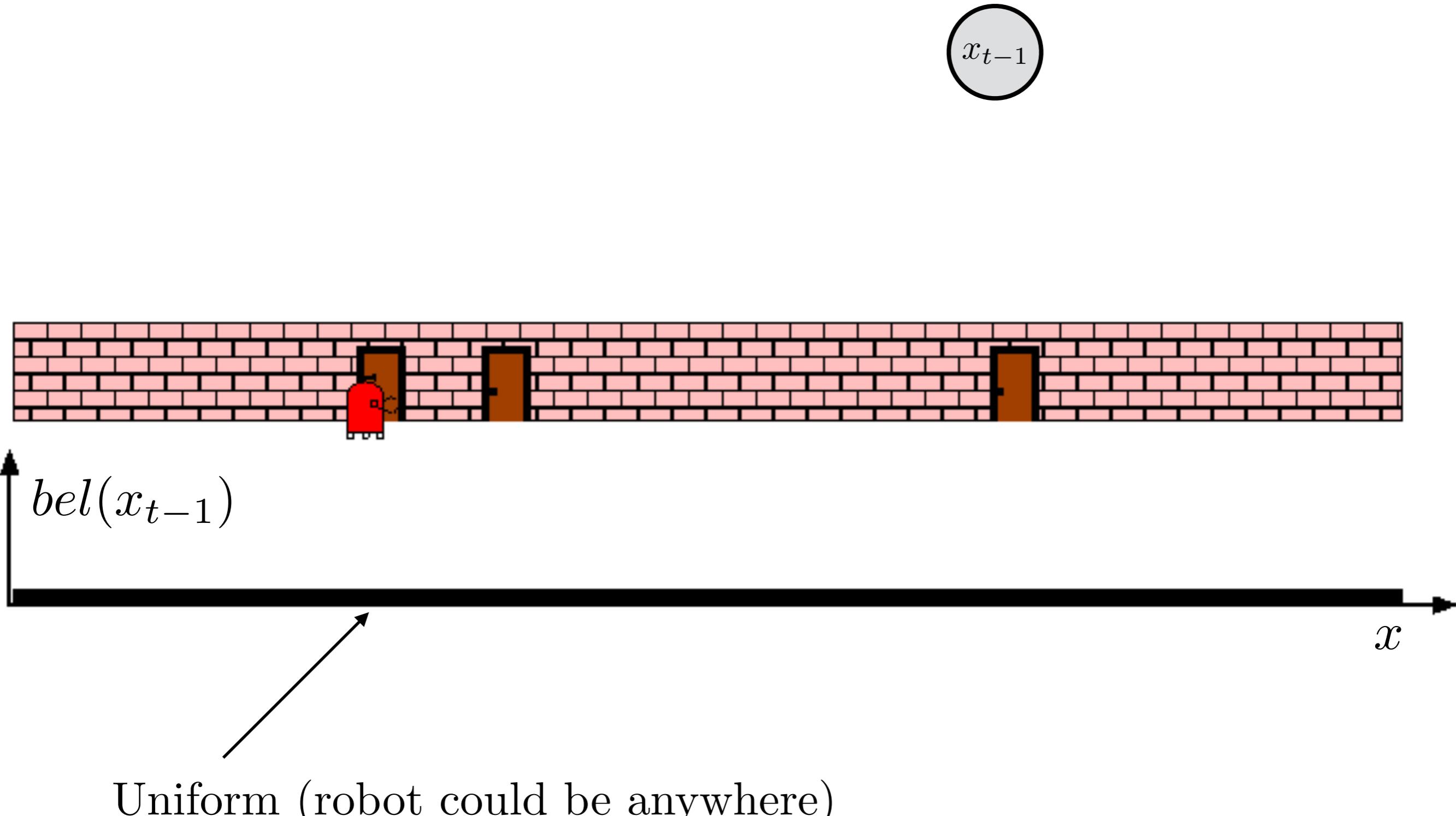
Step 1: Prediction - push belief through dynamics given **action**

$$\overline{bel}(x_t) = \int P(x_t | \textcolor{blue}{u}_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

Step 2: Correction - apply Bayes rule given **measurement**

$$bel(x_t) = \eta P(\textcolor{pink}{z}_t | x_t) \overline{bel}(x_t)$$

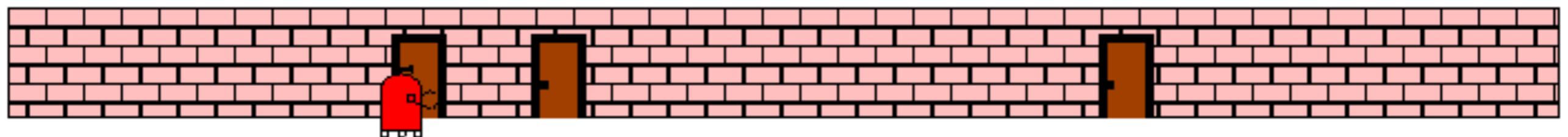
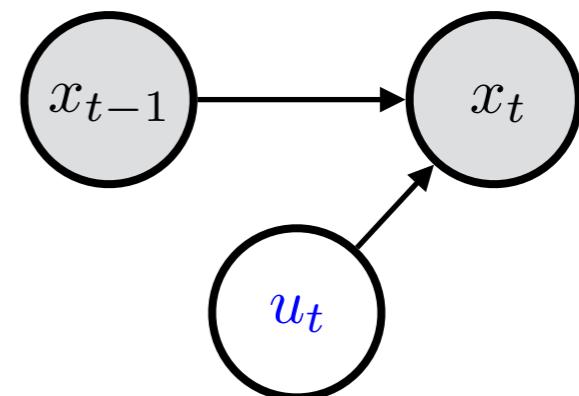
Robot lost in a 1-D hallway



Action at time t: NOP

$$u_t = \text{NOP}$$

$$P(x_t | u_t, x_{t-1}) = \delta(x_t = x_{t-1})$$



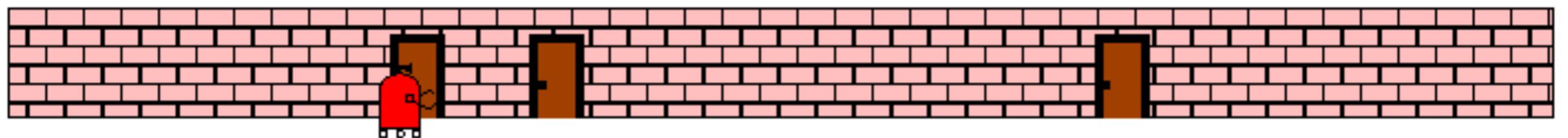
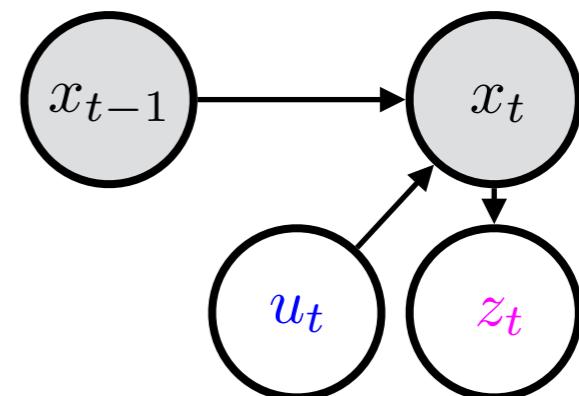
$$\overline{bel}(x_t) = \int P(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1} = bel(x_t)$$

NOP action implies belief remains the same!
(still uniform — no idea where I am)

Measurement at time t: “Door”

z_t = Door

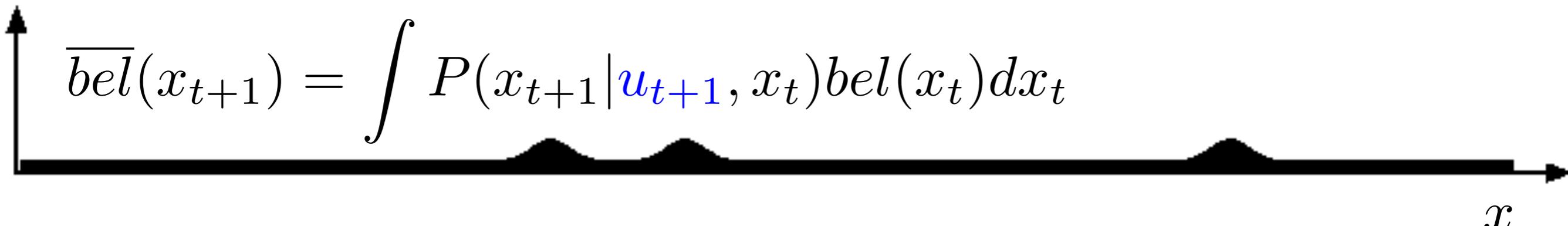
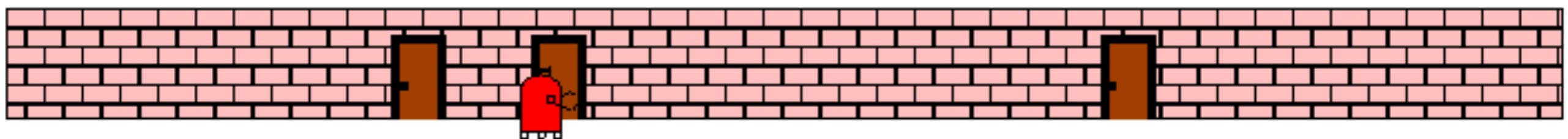
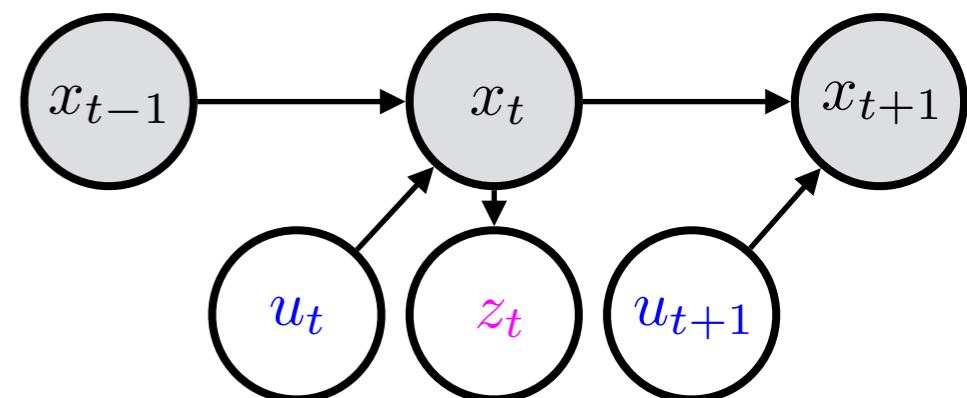
$P(z_t|x_t) = \mathcal{N}(\text{door centre}, 0.75m)$



Action at time t+1: Move 3m right

$$u_{t+1} = 3\text{m right}$$

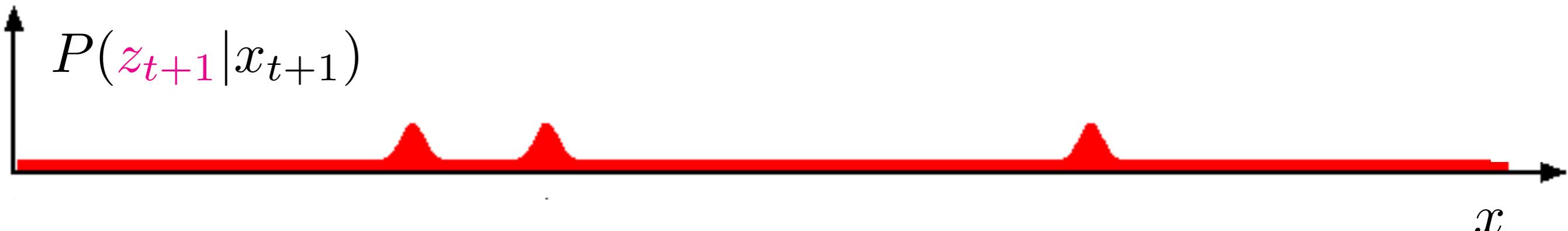
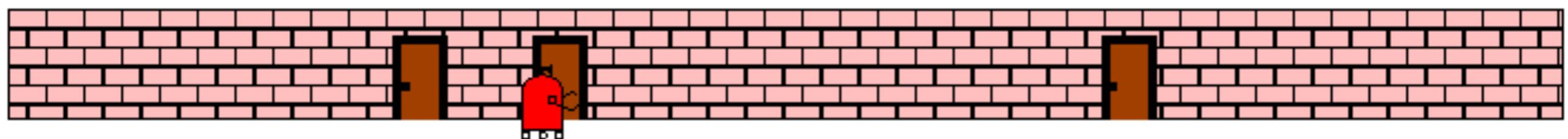
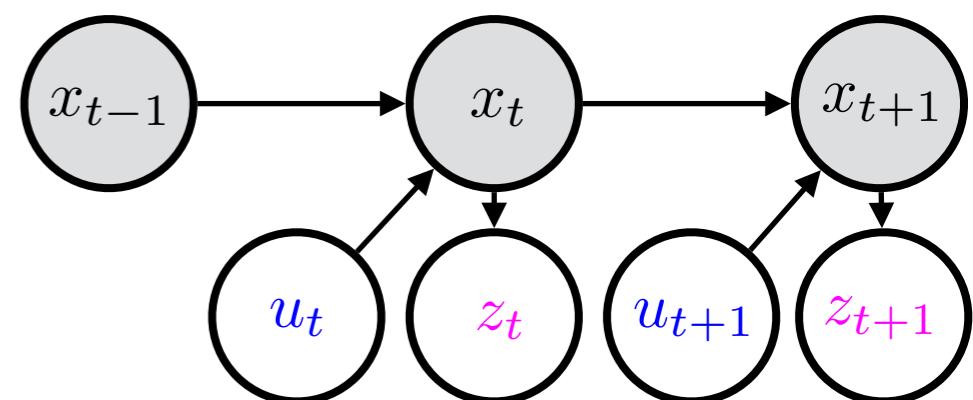
$$P(x_{t+1} | \mathbf{u}_{t+1}, x_t) = \mathcal{N}(x_t + u_{t+1}, 0.25m)$$



Measurement at time $t+1$: “Door”

z_{t+1} = Door

$P(z_{t+1}|x_{t+1}) = \mathcal{N}(\text{door centre}, 0.75m)$



Exercise: Discrete bayes filter

Step 1: Prediction - push belief through dynamics given **action**

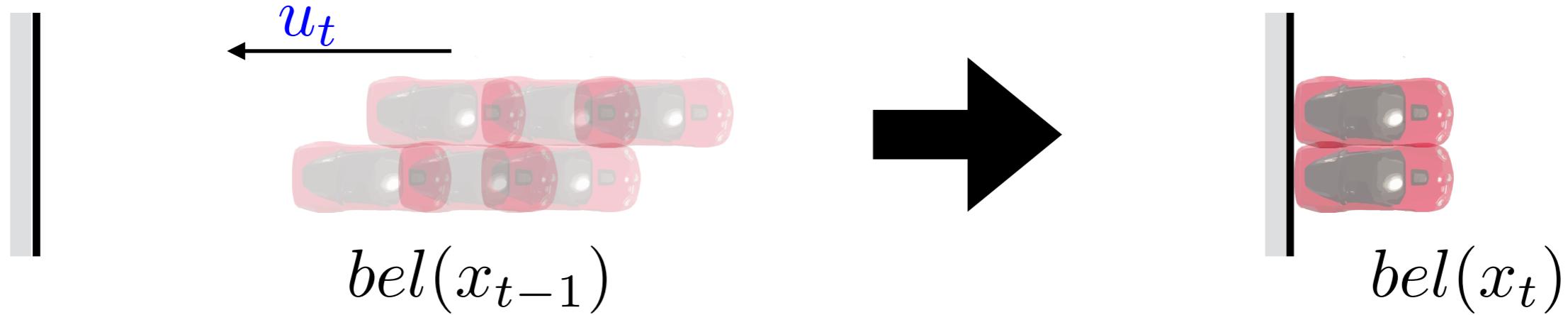
$$\begin{bmatrix} P(\bar{x}_t = 1) \\ \vdots \\ P(\bar{x}_t = n) \end{bmatrix} = \begin{bmatrix} P(x_t = 1 | \mathbf{u}_t, x_{t-1} = 1) & \cdots & P(x_t = 1 | \mathbf{u}_t, x_{t-1} = n) \\ \vdots & \ddots & \vdots \\ P(x_t = n | \mathbf{u}_t, x_{t-1} = 1) & \cdots & P(x_t = n | \mathbf{u}_t, x_{t-1} = n) \end{bmatrix} \begin{bmatrix} P(x_{t-1} = 1) \\ \vdots \\ P(x_{t-1} = n) \end{bmatrix}$$

Step 2: Correction - apply Bayes rule given **measurement**

$$\begin{bmatrix} P(x_t = 1) \\ P(x_t = 2) \\ \vdots \\ P(x_t = n) \end{bmatrix} = \begin{bmatrix} P(\mathbf{z}_t | x_t = 1) & 0 & \cdots & 0 \\ 0 & P(\mathbf{z}_t | x_t = 2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(\mathbf{z}_t | x_t = n) \end{bmatrix} \begin{bmatrix} P(x_{t-1} = 1) \\ P(x_{t-1} = 2) \\ \vdots \\ P(x_{t-1} = n) \end{bmatrix}$$

Questions

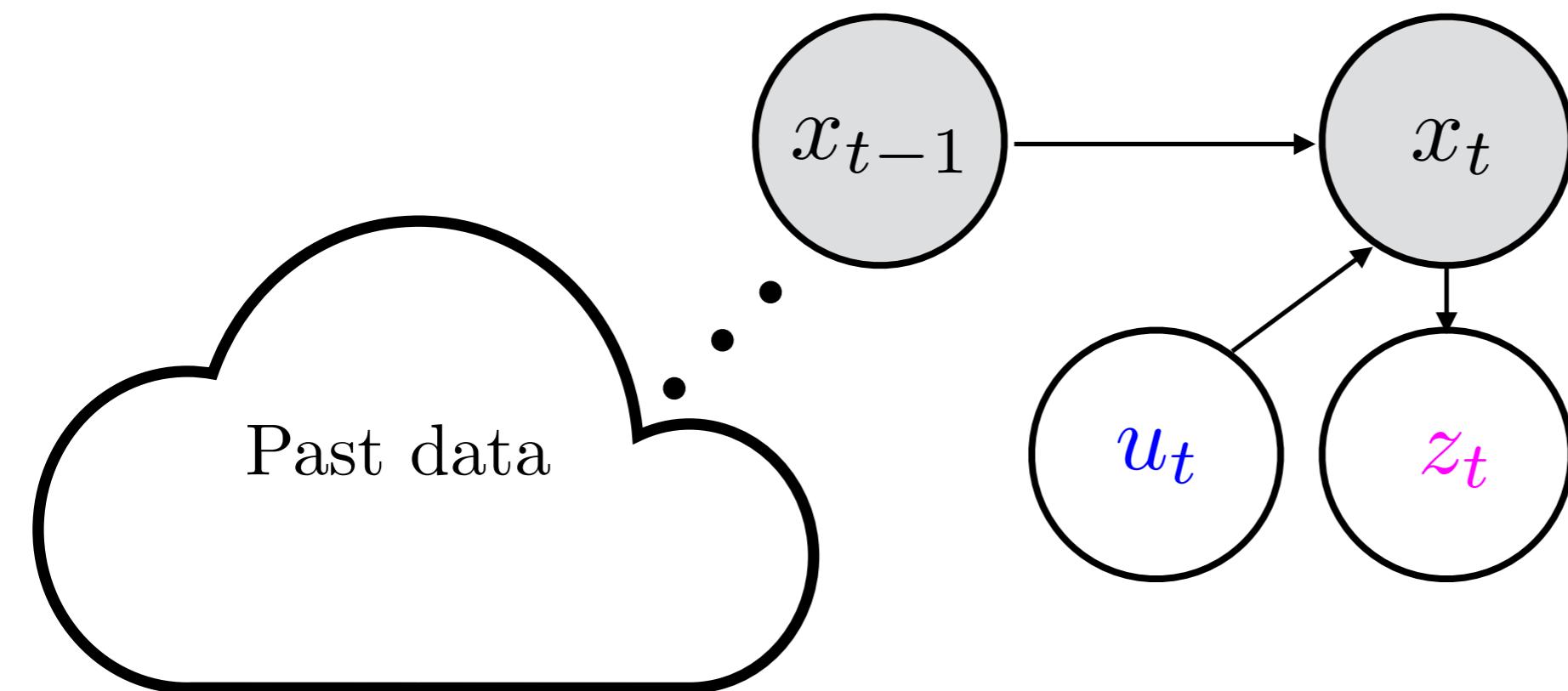
Do actions always increase uncertainty?



Do measurements always reduce uncertainty?

(What happens when you reach into your bag and don't find your keys?
Example of a negative measurement)

Bayes derivation



Bayes derivation

$$bel(x_t) = P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{magenta}{z_t}, \textcolor{blue}{u_t}) \xleftarrow{\text{Incorporate new action, measurement}}$$

(Bayes) $= \eta P(\textcolor{magenta}{z_t} | x_t, z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u_t}) \frac{P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u_t})}{P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u_t})}$

$P(A|B) = \eta P(B|A)P(A)$

Get rid of this

(Markov) $= \eta P(\textcolor{magenta}{z_t} | x_t) \frac{P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u_t})}{P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u_t})}$

$= \eta P(\textcolor{magenta}{z_t} | x_t) \overline{bel}(x_t)$

Bayes derivation

$$\overline{bel}(x_t) = P(x_t | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u}_t)$$

$$\begin{aligned} \text{(Total prob.)} &= \int P(x_t | x_{t-1}, \underbrace{z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u}_t}_{\text{Get rid of this}}) P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, \textcolor{blue}{u}_t) dx_{t-1} \end{aligned}$$

$$\text{(Markov)} = \int P(x_t | x_{t-1}, \textcolor{blue}{u}_t) P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, \cancel{\textcolor{blue}{u}_t}) dx_{t-1}$$

$$\begin{aligned} \text{(Cond. indep)} &= \int P(x_t | x_{t-1}, \textcolor{blue}{u}_t) \boxed{P(x_{t-1} | z_{1:t-1}, u_{1:t-1})} dx_{t-1} \\ &\quad \text{Previous Belief!} \end{aligned}$$

$$= \int P(x_t | x_{t-1}, \textcolor{blue}{u}_t) bel(x_{t-1}) dx_{t-1}$$

After thoughts ...

Question: When is cond. independence not true?

$$= \int P(x_t | x_{t-1}, \mathbf{u}_t) P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, \mathbf{u}_t) dx_{t-1}$$

(Cond.
indep) $= \int P(x_t | x_{t-1}, \mathbf{u}_t) P(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$

i.e. when can you tell something about the past
based on future data?

E.g. Motion capture data of a human.

Human knows the true state and generates control actions accordingly.

Bayes filter in a single line

$$P(x_t | x_{t-1}, \textcolor{blue}{u}_t)$$

Motion model

$$P(\textcolor{magenta}{z}_t | x_t)$$

Measurement model

$$bel(x_t) = \eta P(\textcolor{magenta}{z}_t | x_t) \int P(x_t | x_{t-1}, \textcolor{blue}{u}_t) bel(x_{t-1}) dx_{t-1}$$

Note that order does not really matter -
we can flip measurement and control.

Bayes Filter Pseudocode (Asynchronous)

$$Bel(x_t) = \eta \int P(z_t | x_t) P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

1. Algorithm **Bayes_filter**($Bel(x), d$):
2. $\eta=0$
3. If d is a **perceptual** data item z then
4. For all x do
5. $Bel'(x) = P(z | x)Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all x do
8. $Bel'(x) = \eta^{-1}Bel'(x)$
9. Else if d is an **action** data item u then
10. For all x do
11. $Bel'(x) = \int P(x | u, x') Bel(x') dx'$
12. Return $Bel'(x)$

Slide by Prof. Dieter Fox

Practical Issues

1. Bayes filter can be overconfident

Once belief collapses to 0/1 only motion model can shake it loose

2. Too many measurements will collapse belief

3. Correlated incorrect measurements are dangerous

Bayes filter is a powerful tool



Localization



Mapping



SLAM



POMDP