MODULE 2 LESSON 2

THE KALMAN FILTER: BIAS BLUES

The Kalman Filter and The Bias BLUEs

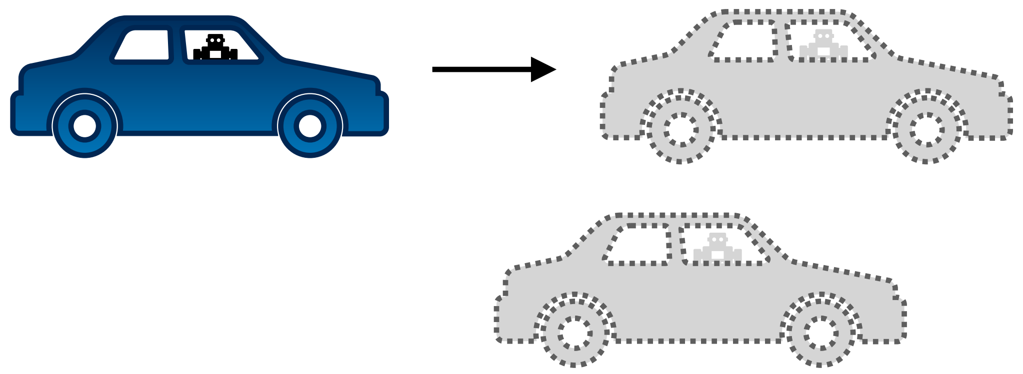
By the end of this video, you will be able to...

- Define *bias*
- Define *consistency*
- Explain why the Kalman filter is the Best Linear Unbiased Estimator (BLUE)

Bias in State Estimation

Process/Motion: $$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

Measurement: $$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k$$

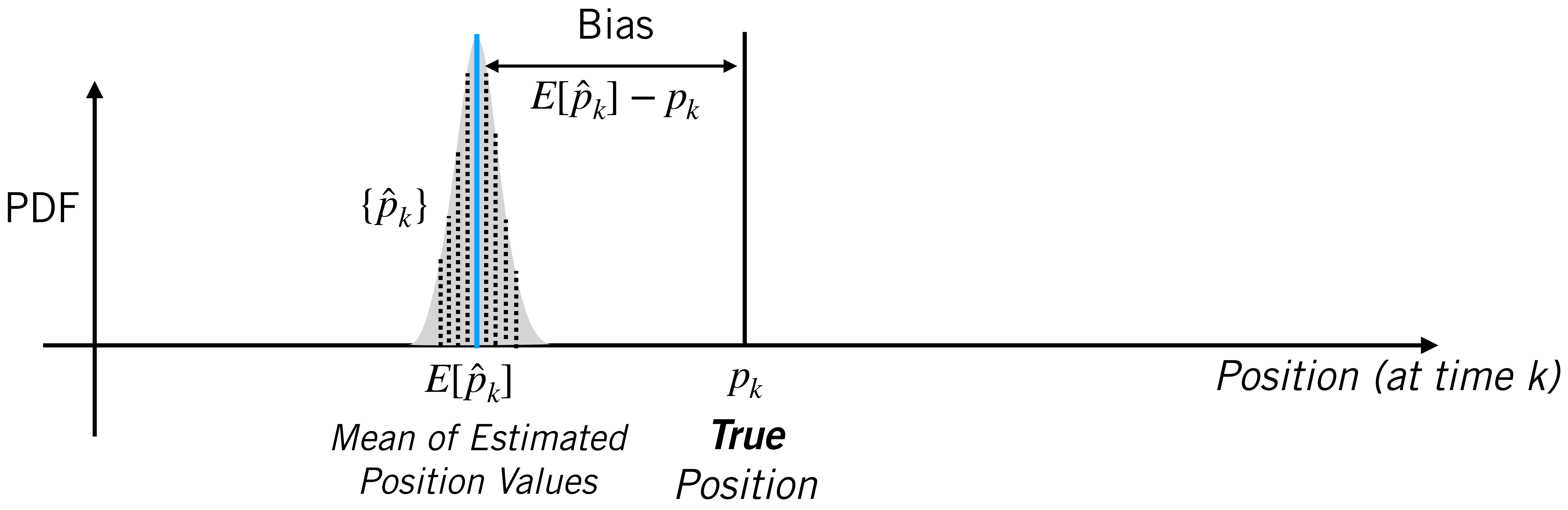


$k = 0, 1, \ldots. K$

Drive the car for K time steps, record estimation error, repeat.....

- We say an estimator or filter is unbiased if it produces an ‘average’ error of zero at a particular time step *k*, over many trials

Bias in State Estimation



This filter is an unbiased if **for all *k*,**

$$E[\hat{e}_k] = E[\hat{p}_k - p_k] = E[\hat{p}_k] - p_k = 0$$

Bias in State Estimation

- How can we compute this analytically for the Kalman filter?
- Consider the *error dynamics*:

$$\check{\mathbf{e}}_k = \check{\mathbf{x}}_k - \mathbf{x}_k$$

Predicted state error

$$\hat{\mathbf{e}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k$$

Corrected estimate error

- Using the Kalman Filter equations, we can derive:

$$\check{\mathbf{e}}_k = \mathbf{F}_{k-1}\check{\mathbf{e}}_{k-1} - \mathbf{w}_k$$

$$\hat{\mathbf{e}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{e}}_k + \mathbf{K}_k\mathbf{v}_k$$

Bias in State Estimation

- For the Kalman filter, for all k,

$$\begin{aligned} E[\check{\mathbf{e}}_k] &= E[\mathbf{F}_{k-1}\check{\mathbf{e}}_{k-1} - \mathbf{w}_k] \\ &= \mathbf{F}_{k-1}E[\check{\mathbf{e}}_{k-1}] - E[\mathbf{w}_k] \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} E[\hat{\mathbf{e}}_k] &= E[(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{e}}_k + \mathbf{K}_k\mathbf{v}_k] \\ &= (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)E[\check{\mathbf{e}}_k] + \mathbf{K}_kE[\mathbf{v}_k] \\ &= \mathbf{0} \end{aligned}$$

***Unbiased** predictions!*

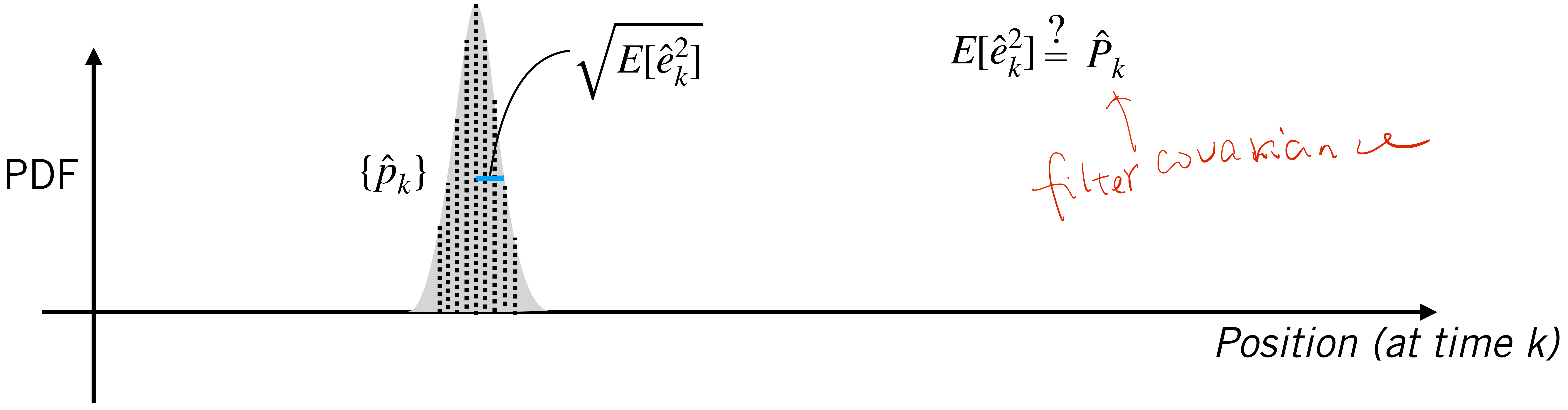
So long as $E[\hat{\mathbf{e}}_0] = \mathbf{0}$ $\quad E[\mathbf{v}] = \mathbf{0} \quad E[\mathbf{w}] = \mathbf{0}$

initial state estimate

+ white, uncorrelated noise

Note: this does *not* mean that the error on a *given* trial will be zero, but that, with enough trials, our expected error is zero!

Consistency in State Estimation



$$E[\hat{e}_k^2] \stackrel{?}{=} \hat{P}_k$$

filter covariance

This filter is consistent if **for all *k***,

$$E[\hat{e}_k^2] = E[(\hat{p}_k - p_k)^2] = \hat{P}_k$$

Empirical variance matches the variance reported by the filter.

Consistency in State Estimation

- One can also show (with more algebra!) that for all *k*,

$$E[\check{\mathbf{e}}_k \check{\mathbf{e}}_k^T] = \check{\mathbf{P}}_k \qquad E[\hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T] = \hat{\mathbf{P}}_k$$

Consistent *predictions!*

- Provided,

$$E[\hat{\mathbf{e}}_0 \hat{\mathbf{e}}_0^T] = \check{\mathbf{P}}_0 \qquad E[\mathbf{v}] = \mathbf{0} \qquad E[\mathbf{w}] = \mathbf{0}$$

Consistent initial estimate

+ white noise

The Kalman Filter is the BLUE
Best Linear Unbiased Estimator

- We've shown that given our linear formulation, and zero-mean, white noise the Kalman Filter is ***unbiased***
- We can also say that the filter is ***consistent:***

$$E[\hat{\mathbf{e}}_k] = \mathbf{0}$$

$$E[\hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T] = \hat{\mathbf{P}}_k$$

- In general, if we have white, uncorrelated zero-mean noise, the Kalman filter is the best (i.e., lowest variance) unbiased estimator that uses only a linear combination of measurements
- For this reason, we call it the **BLUE** (best linear unbiased estimator)

Summary | The Kalman Filter and The Bias BLUEs

The Kalman filter is

- unbiased
- consistent
- the lowest variance estimator that uses a linear combination of measurements: Best Linear Unbiased Estimator (**BLUE**)