MODULE 2 LESSON 6

AN ALTERNATIVE TO THE EKF: THE UNSCENTED KALMAN FILTER

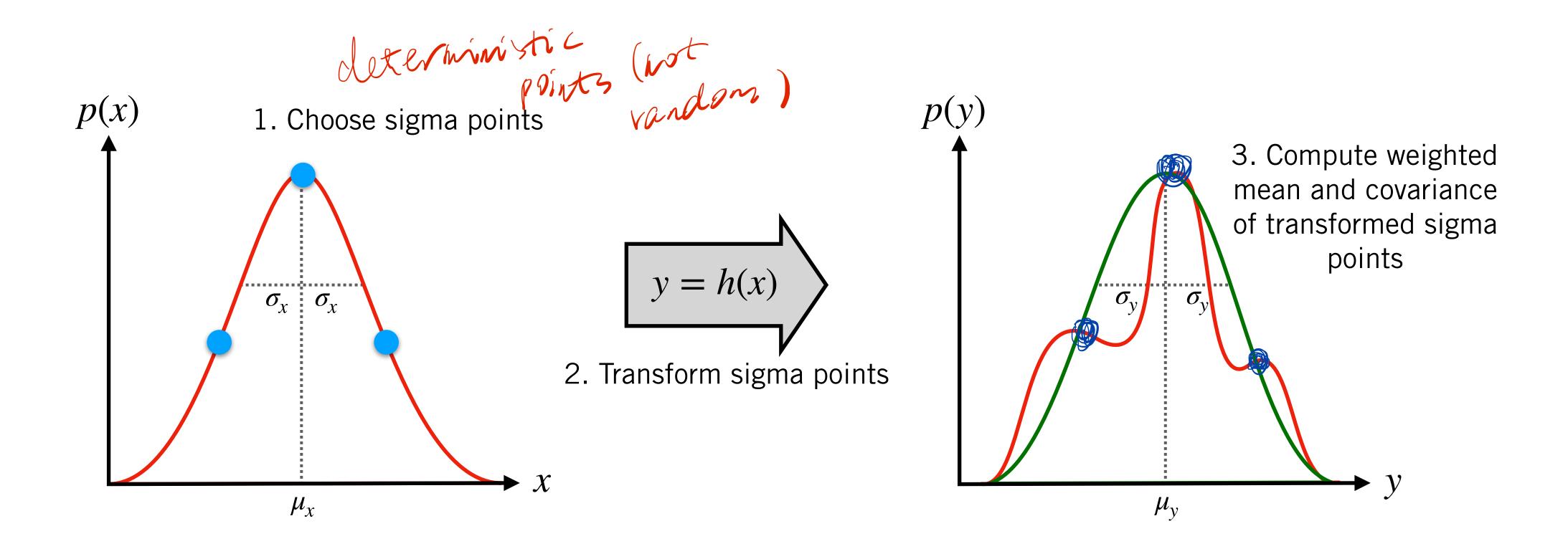
The Unscented Kalman Filter (UKF)

By the end of this video, you will be able to...

- Use the Unscented Transform to pass a probability distribution through a nonlinear function
- Describe how the Unscented Kalman Filter (UKF) uses the Unscented Transform in the prediction and correction steps
- Explain the advantages of the UKF over the EKF
- Apply the UKF to a simple nonlinear tracking problem

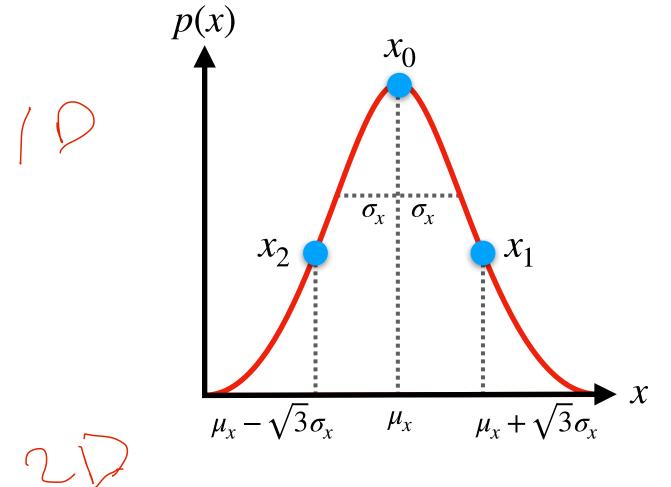
The Unscented Transform

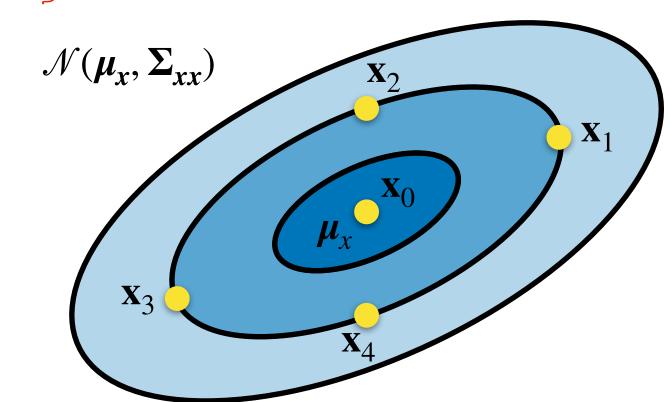
"It is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function" — S. Julier, J. Uhlmann, and H. Durrant-Whyte (2000)



The Unscented Transform | Choosing Sigma Points

For an *N*-dimensional PDF $\mathcal{N}(\mu_x, \Sigma_{xx})$, we need 2N + 1 sigma points:





1. Compute the Cholesky Decomposition of the covariance matrix

$$\mathbf{L}\mathbf{L}^T = \mathbf{\Sigma}_{xx}$$
 (L lower triangular)

2. Calculate the sigma points

$$\mathbf{x}_{0} = \boldsymbol{\mu}_{x}$$

$$\mathbf{x}_{i} = \boldsymbol{\mu}_{x} + \sqrt{N + \kappa} \operatorname{col}_{i} \mathbf{L} \qquad i = 1, ..., N$$

$$\mathbf{x}_{i+N} = \boldsymbol{\mu}_{x} - \sqrt{N + \kappa} \operatorname{col}_{i} \mathbf{L} \qquad i = 1, ..., N$$

$$\kappa = 3 - N$$
for Gaussian PDFs

The Unscented Transform | Transforming and Recombining

Next we pass each of our 2N + 1 sigma points through the nonlinear function h(x)

$$\mathbf{y}_i = \mathbf{h}(\mathbf{x}_i) \qquad i = 0, ..., 2N$$

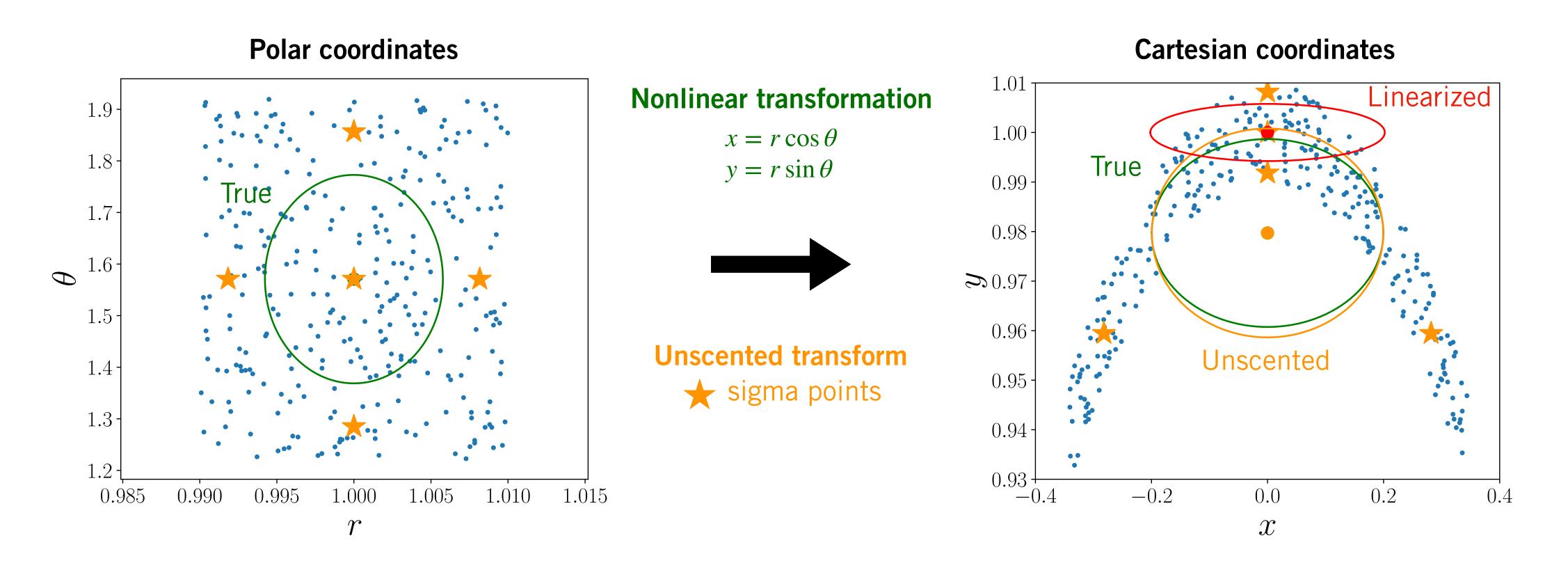
And finally compute the mean and covariance of the output PDF

Mean:
$$\mu_y = \sum_{i=0}^{2N} \alpha_i \mathbf{y}_i$$

Weights:
$$\alpha_i = \begin{cases} \frac{\kappa}{N+\kappa} & i=0\\ \frac{1}{2}\frac{1}{N+\kappa} & \text{otherwise} \end{cases}$$

The Unscented Transform vs. Linearization

Let's revisit our nonlinear transformation example from the previous video.



The Unscented Transform gives a much better approximation for similar work!

The Unscented Kalman Filter (UKF)

We can easily use the Unscented Transform in our Kalman Filtering framework with nonlinear models:

Nonlinear motion model

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$$

Nonlinear measurement model

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{v}_k)$$

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

Instead of approximating the system equations by linearizing, we will calculate sigma points and use the Unscented Transform to approximate the PDFs directly!

UKF | Prediction step

To propagate the state from time (k-1) to time k, apply the Unscented Transform using the current best guess for the mean and covariance

1. Compute sigma points

$$\hat{\mathbf{L}}_{k-1}\hat{\mathbf{L}}_{k-1}^{T} = \hat{\mathbf{P}}_{k-1}$$

$$\hat{\mathbf{x}}_{k-1}^{(0)} = \hat{\mathbf{x}}_{k-1}$$

$$\hat{\mathbf{x}}_{k-1}^{(i)} = \hat{\mathbf{x}}_{k-1} + \sqrt{N+\kappa} \operatorname{col}_{i}\hat{\mathbf{L}}_{k-1} \qquad i = 1...N$$

$$\hat{\mathbf{x}}_{k-1}^{(i+N)} = \hat{\mathbf{x}}_{k-1} - \sqrt{N+\kappa} \operatorname{col}_{i}\hat{\mathbf{L}}_{k-1} \qquad i = 1...N$$

2. Propagate sigma points

$$\check{\mathbf{x}}_{k}^{(i)} = \mathbf{f}_{k-1}(\hat{\mathbf{x}}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{0}) \qquad i = 0...2N$$

3. Compute predicted mean and covariance

$$\boldsymbol{\alpha}^{(i)} = \begin{cases} \frac{\kappa}{N+\kappa} & i = 0\\ \frac{1}{2} \frac{1}{N+\kappa} & \text{otherwise} \end{cases}$$

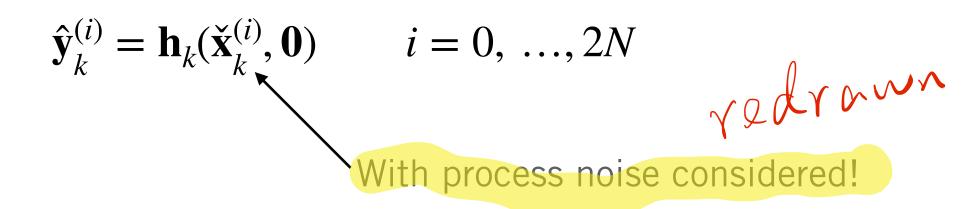
$$\check{\mathbf{x}}_k = \sum_{i=0}^{2N} \boldsymbol{\alpha}^{(i)} \check{\mathbf{x}}_k^{(i)}$$

$$\check{\mathbf{P}}_k = \sum_{i=0}^{2N} \boldsymbol{\alpha}^{(i)} \left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right)^T + \mathbf{Q}_{k-1}$$
Additive process noise!

UKF | Correction step

To correct the state estimate using measurements at time k, use the nonlinear measurement model and the sigma points from the prediction step to predict the measurements

1. Predict measurements from (redrawn) propagated sigma points



2. Estimate mean and covariance of predicted measurements

$$\hat{\mathbf{y}}_k = \sum_{i=0}^{2N} \alpha^{(i)} \hat{\mathbf{y}}_k^{(i)}$$

$$\mathbf{P}_y = \sum_{i=0}^{2N} \alpha^{(i)} \left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right) \left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right)^T + \mathbf{R}_k$$
Additive
measurement noise!

3. Compute cross-covariance and Kalman gain

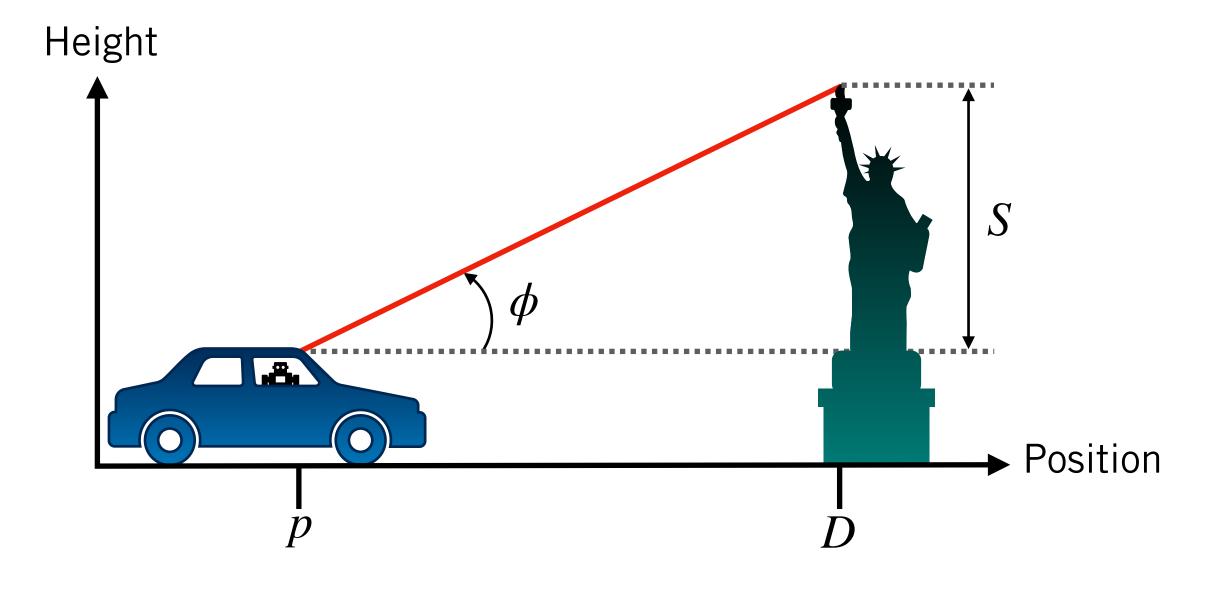
$$\mathbf{P}_{xy} = \sum_{i=0}^{2N} \alpha^{(i)} \left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right)^T$$
$$\mathbf{K}_k = \mathbf{P}_{xy} \mathbf{P}_y^{-1}$$

4. Compute corrected mean and covariance

$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k \left(\mathbf{y}_k - \hat{\mathbf{y}}_k \right)$$

$$\hat{\mathbf{P}}_k = \check{\mathbf{P}}_k - \mathbf{K}_k \mathbf{P}_y \mathbf{K}_k^T$$

UKF | Short example



$$\mathbf{x} = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \qquad \mathbf{u} = \ddot{p}$$

S and D are known in advance

Motion/Process model

$$\mathbf{x}_{k} = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} \mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

Landmark measurement model

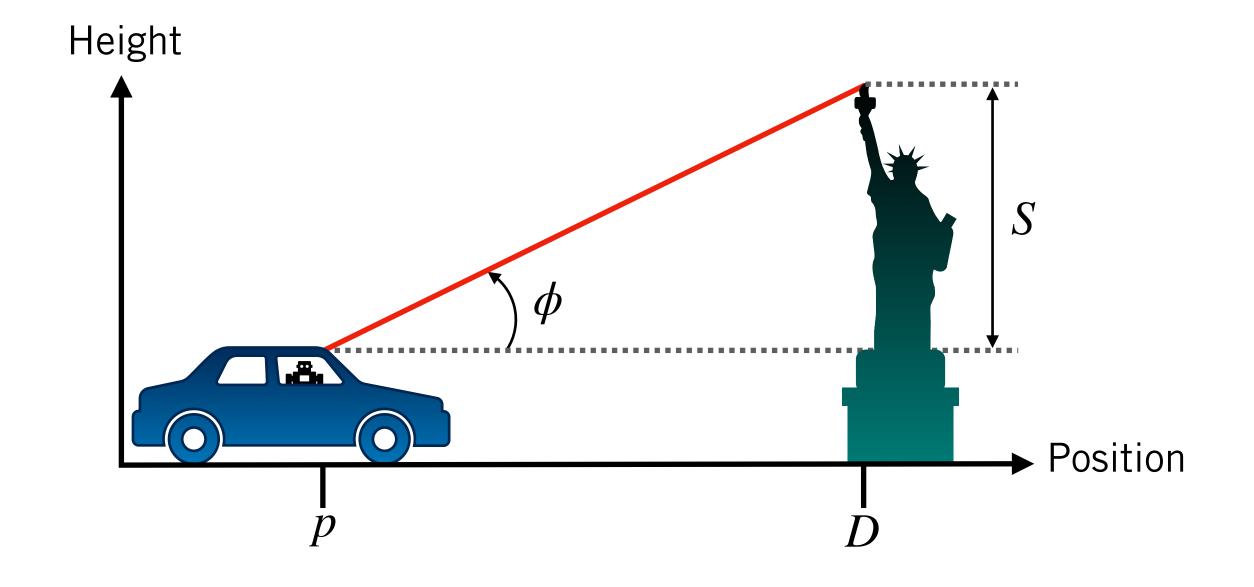
$$y_k = \phi_k = h(p_k, v_k)$$

$$= \tan^{-1} \left(\frac{S}{D - p_k} \right) + v_k$$

Noise densities

$$v_k \sim \mathcal{N}(0, 0.01)$$
 $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, (0.1)\mathbf{1}_{2\times 2})$

UKF | Short example



<u>Data</u>

$$\hat{\mathbf{x}}_0 \sim \mathcal{N}\left(\begin{bmatrix}0\\5\end{bmatrix}, \begin{bmatrix}0.01 & 0\\0 & 1\end{bmatrix}\right)$$

$$\Delta t = 0.5$$
s

$$u_0 = -2 [m/s^2]$$
 $y_1 = \pi/6 [rad]$

$$S = 20 \ [m]$$
 $D = 40 \ [m]$

Using the Unscented Kalman Filter equations, what is our updated position?

Prediction

$$N = 2$$
, $\kappa = 3 - N = 1$

$$\hat{\mathbf{L}}_0 \hat{\mathbf{L}}_0^T = \hat{\mathbf{P}}_0$$

$$\begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(0)} = \hat{\mathbf{x}}_0$$

$$\hat{\mathbf{x}}_0^{(i)} = \hat{\mathbf{x}}_0 + \sqrt{N + \kappa} \operatorname{col}_i \hat{\mathbf{L}}_0 \qquad i = 1 \dots, N$$

$$\hat{\mathbf{x}}_0^{(i+N)} = \hat{\mathbf{x}}_0 - \sqrt{N + \kappa} \operatorname{col}_i \hat{\mathbf{L}}_0 \qquad i = 1, \dots, N$$

$$\hat{\mathbf{x}}_{0}^{(0)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_{0}^{(1)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_{0}^{(2)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.73 \end{bmatrix}$$

$$\hat{\mathbf{x}}_{0}^{(3)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.17 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_{0}^{(4)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3.27 \end{bmatrix}$$

Prediction

$$\check{\mathbf{x}}_{1}^{(i)} = \mathbf{f}_{0}(\hat{\mathbf{x}}_{0}^{(i)}, \mathbf{u}_{0}, \mathbf{0})$$
 $i = 0, ..., 2N$

$$\check{\mathbf{x}}_{1}^{(0)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(1)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.17 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.67 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(2)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6.73 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(3)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.17 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.33 \\ 4 \end{bmatrix}$$

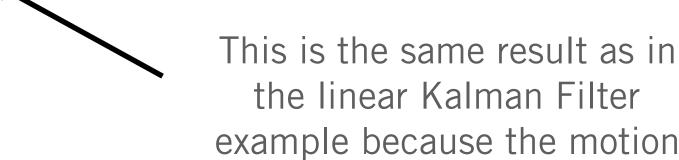
$$\check{\mathbf{x}}_{1}^{(4)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3.27 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix}$$

$$\alpha^{(i)} = \begin{cases} \frac{\kappa}{N+\kappa} = \frac{1}{2+1} = \frac{1}{3} & i = 0\\ \frac{1}{2N+\kappa} = \frac{1}{2} \frac{1}{2+1} = \frac{1}{6} & \text{otherwise} \end{cases}$$

$$\check{\mathbf{x}}_{k} = \sum_{i=0}^{2N} \alpha^{(i)} \check{\mathbf{x}}_{k}^{(i)}
= \frac{1}{3} \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2.67 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2.33 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

Prediction

$$\begin{split} \check{\mathbf{P}}_{k} &= \sum_{i=0}^{2N} \alpha^{(i)} \left(\check{\mathbf{x}}_{k}^{(i)} - \check{\mathbf{x}}_{k} \right) \left(\check{\mathbf{x}}_{k}^{(i)} - \check{\mathbf{x}}_{k} \right)^{T} + \mathbf{Q}_{k-1} \\ &= \frac{1}{3} \left(\begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left(\begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^{T} + \\ &\frac{1}{6} \left(\begin{bmatrix} 2.67 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left(\begin{bmatrix} 2.67 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^{T} + \frac{1}{6} \left(\begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left(\begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^{T} + \\ &\frac{1}{6} \left(\begin{bmatrix} 2.33 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left(\begin{bmatrix} 2.33 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^{T} + \frac{1}{6} \left(\begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left(\begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^{T} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \\ &= \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \end{split}$$



the linear Kalman Filter example because the motion model is already linear!

Correction

$$\check{\mathbf{L}}_k \check{\mathbf{L}}_k^T = \check{\mathbf{P}}_k$$

$$\begin{bmatrix} 0.60 & 0 \\ 0.83 & 0.64 \end{bmatrix} \begin{bmatrix} 0.60 & 0 \\ 0.83 & 0.64 \end{bmatrix}^{T} = \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(0)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$
 Redrawn sigma points!

$$\check{\mathbf{x}}_{1}^{(1)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.60 \\ 0.83 \end{bmatrix} = \begin{bmatrix} 3.54 \\ 5.44 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(2)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0 \\ 0.64 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5.10 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(3)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0.60 \\ 0.83 \end{bmatrix} = \begin{bmatrix} 1.46 \\ 2.56 \end{bmatrix}$$

$$\check{\mathbf{x}}_{1}^{(4)} = \begin{bmatrix} 2.5\\4 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0\\0.64 \end{bmatrix} = \begin{bmatrix} 2.5\\2.90 \end{bmatrix}$$

$$\hat{y}_1^{(i)} = h_1(\check{\mathbf{x}}_1^{(i)}, 0)$$
 $i = 0, ..., 2N$

$$\hat{y}_1^{(0)} = \tan^{-1}\left(\frac{20}{40 - 2.5}\right) = 0.49$$

$$\hat{y}_1^{(1)} = \tan^{-1}\left(\frac{20}{40 - 3.54}\right) = 0.50$$

$$\hat{y}_1^{(2)} = \tan^{-1}\left(\frac{20}{40 - 2.5}\right) = 0.49$$

$$\hat{y}_1^{(3)} = \tan^{-1}\left(\frac{20}{40 - 1.46}\right) = 0.48$$

$$\hat{y}_1^{(4)} = \tan^{-1}\left(\frac{20}{40 - 2.5}\right) = 0.49$$

Correction

$$\hat{y}_1 = \sum_{i=0}^{2N} \alpha^{(i)} \hat{y}_1^{(i)}$$

$$= \frac{1}{3} (0.49) + \frac{1}{6} (0.50) + \frac{1}{6} (0.49) + \frac{1}{6} (0.48) + \frac{1}{6} (0.49)$$

$$= 0.49$$

$$P_{y} = \sum_{i=0}^{2N} \alpha^{(i)} \left(\hat{y}_{1}^{(i)} - \hat{y}_{k} \right) \left(\hat{y}_{1}^{(i)} - \hat{y}_{k} \right)^{T} + R_{k}$$

$$= \frac{1}{3} (0.49 - 0.49)^{2} + \frac{1}{6} (0.50 - 0.49)^{2} + \frac{1}{6} (0.49 - 0.49)^{2} + \frac{1}{6} (0.48 - 0.49)^{2} + \frac{1}{6} (0.49 - 0.49)^{2} + 0.01$$

$$= 0.01$$

Correction

$$\begin{aligned} \mathbf{P}_{xy} &= \sum_{i=0}^{2N} \alpha^{(i)} \left(\check{\mathbf{x}}_{k}^{(i)} - \check{\mathbf{x}}_{k} \right) \left(\hat{y}_{1}^{(i)} - \hat{y}_{k} \right)^{T} \\ &= \frac{1}{3} \left(\begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) + \frac{1}{6} \left(\begin{bmatrix} 3.54 \\ 5.44 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.50 - 0.49) + \frac{1}{6} \left(\begin{bmatrix} 2.5 \\ 5.10 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) \\ &+ \frac{1}{6} \left(\begin{bmatrix} 1.46 \\ 2.56 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.48 - 0.49) + \frac{1}{6} \left(\begin{bmatrix} 2.5 \\ 2.90 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) \\ &= \begin{bmatrix} 0.004 \\ 0.006 \end{bmatrix} \end{aligned}$$

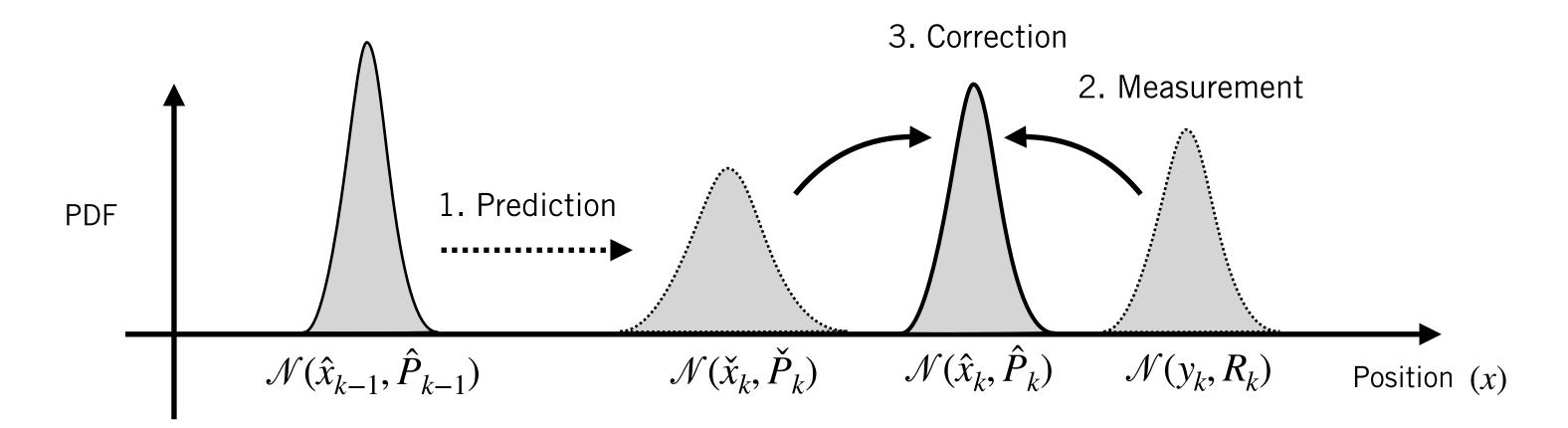
$$\mathbf{K}_1 = \mathbf{P}_{xy} P_y^{-1} = \begin{bmatrix} 0.004 \\ 0.006 \end{bmatrix} \frac{1}{0.01} = \begin{bmatrix} 0.40 \\ 0.55 \end{bmatrix}$$

$$\hat{\mathbf{x}}_1 = \check{\mathbf{x}}_1 + \mathbf{K}_1 \left(y_1 - \hat{y}_1 \right) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0.40 \\ 0.55 \end{bmatrix} (0.52 - 0.49) = \begin{bmatrix} 2.51 \\ 4.02 \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_1 \end{bmatrix}$$

Summary | Unscented Kalman Filter (UKF)

- The UKF uses the unscented transform to adapt the Kalman filter to nonlinear systems.
- The unscented transform works by passing a small set of carefully chosen samples through a nonlinear system, and computing the mean and covariance of the outputs.
- The unscented transform does a better job of approximating the output distribution than analytical local linearization, for similar computational cost.

Summary | Kalman Filtering



- The Kalman filter (KF) is a form of recursive least-squares estimation that allows us to combine information frsom a motion model and sensor measurements.
- The KF uses the motion model to make *predictions* of the state, and uses the measurements to make *corrections* to the predictions.
- The KF is the Best Linear Unbiased Estimator (BLUE).

Summary | Nonlinear Kalman Filtering

	EKF	ES-EKF	UKF
Operating Principle	Linearization (Full State)	Linearization (Error State)	Unscented Transform
Accuracy	Good	Better	Best
Jacobians	Required	Required	Not required
Speed	Slightly faster	Slightly faster	Slightly slower