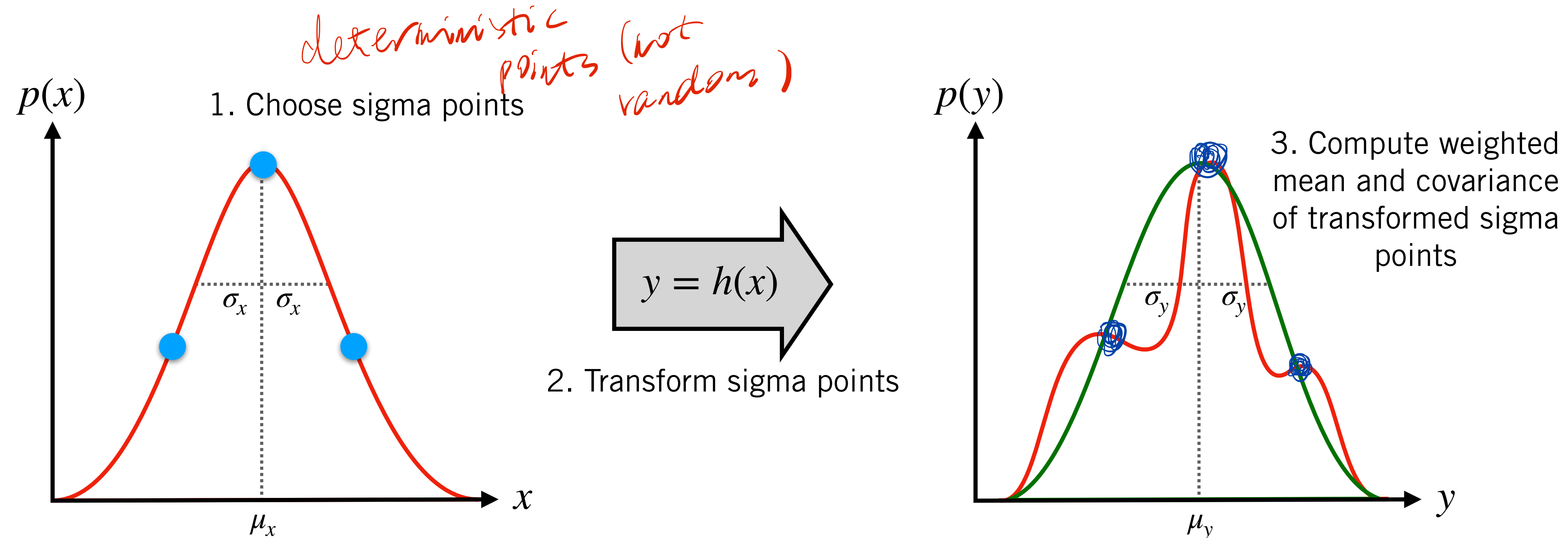


**MODULE 2 LESSON 6**

# **AN ALTERNATIVE TO THE EKF: THE UNSCENTED KALMAN FILTER**

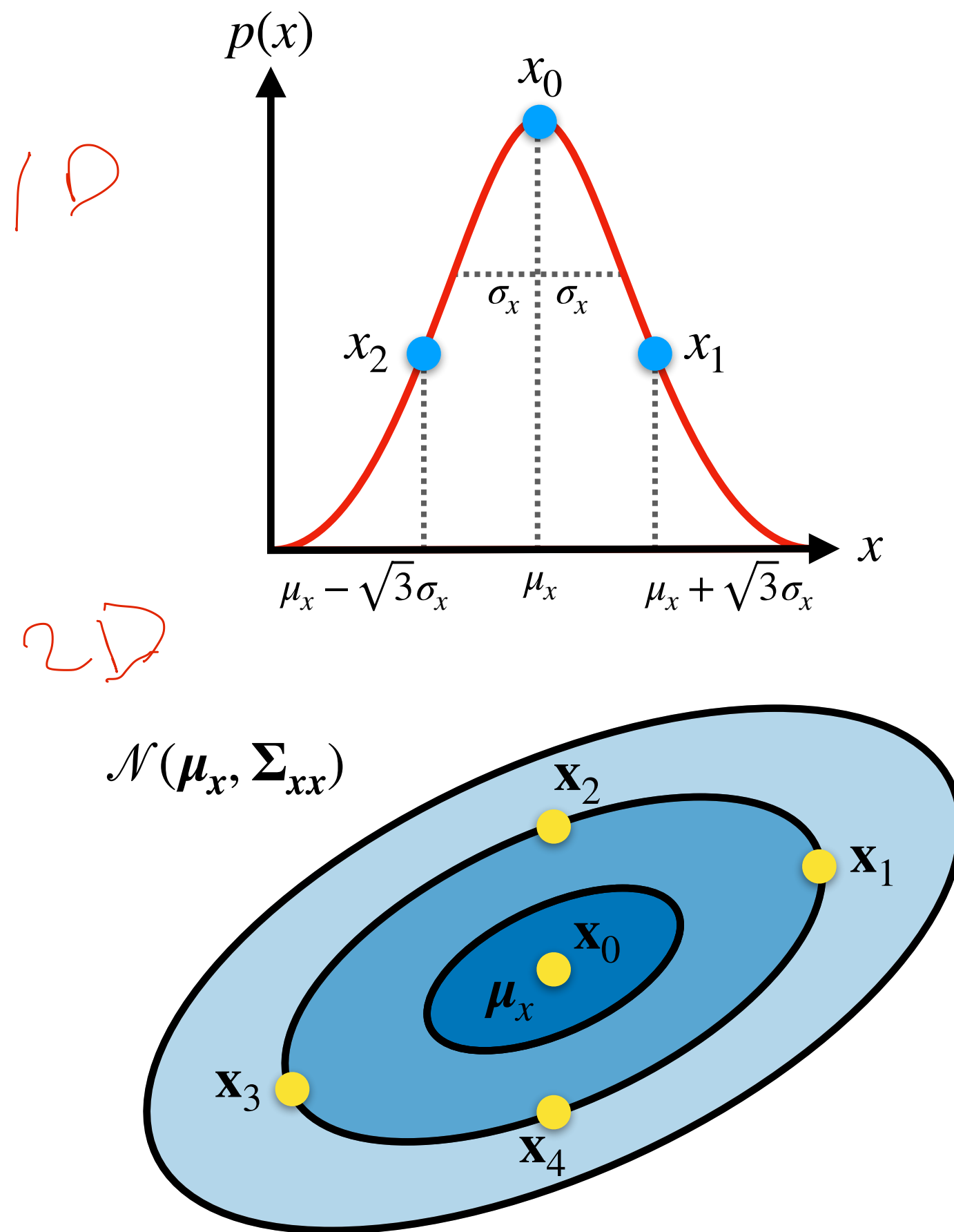
# The Unscented Transform

“It is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function” — S. Julier, J. Uhlmann, and H. Durrant-Whyte (2000)



# The Unscented Transform | Choosing Sigma Points

For an  $N$ -dimensional PDF  $\mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ , we need  $2N + 1$  sigma points:



1. Compute the Cholesky Decomposition of the covariance matrix  $\mathbf{L}$

$$\mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}_{xx} \quad (\mathbf{L} \text{ lower triangular})$$

2. Calculate the sigma points

$$\mathbf{x}_0 = \boldsymbol{\mu}_x$$

$$\mathbf{x}_i = \boldsymbol{\mu}_x + \sqrt{N + \kappa} \text{col}_i \mathbf{L} \quad i = 1, \dots, N$$

$$\mathbf{x}_{i+N} = \boldsymbol{\mu}_x - \sqrt{N + \kappa} \text{col}_i \mathbf{L} \quad i = 1, \dots, N$$

$\kappa = 3 - N$   
for Gaussian PDFs

# The Unscented Transform I

## Transforming and Recombining

Next we pass each of our  $2N + 1$  sigma points through the nonlinear function  $\mathbf{h}(\mathbf{x})$

$$\mathbf{y}_i = \mathbf{h}(\mathbf{x}_i) \quad i = 0, \dots, 2N$$

And finally compute the mean and covariance of the output PDF

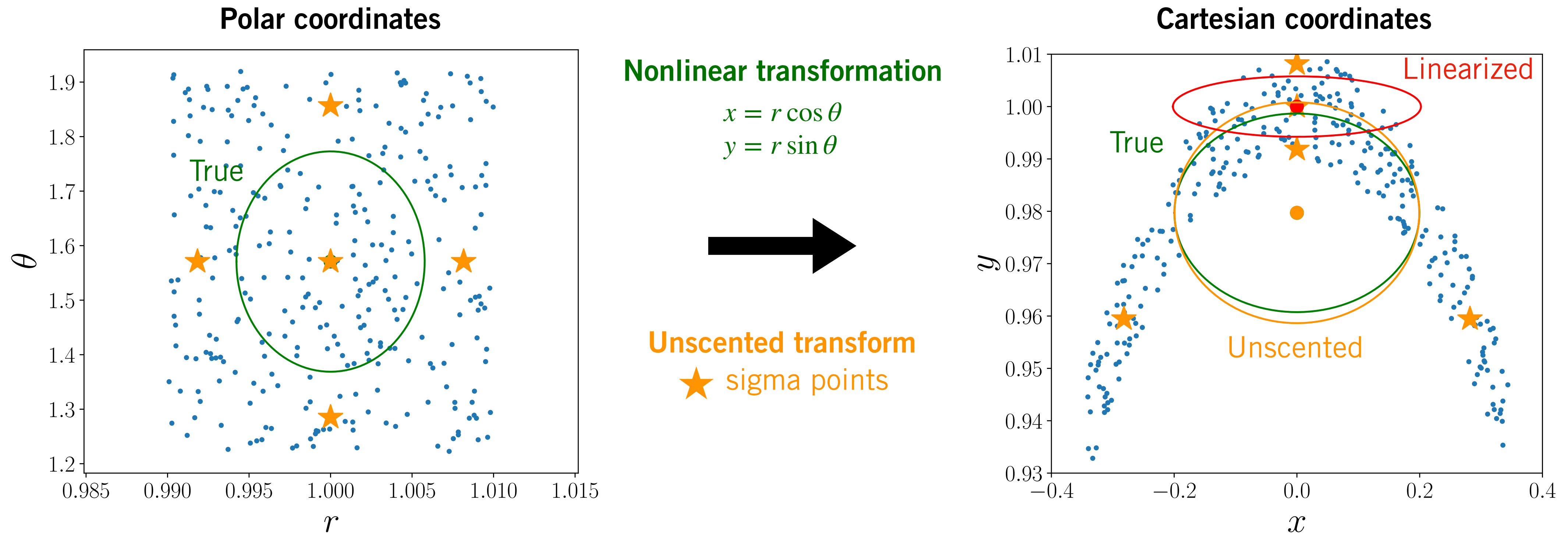
$$\text{Mean: } \boldsymbol{\mu}_y = \sum_{i=0}^{2N} \alpha_i \mathbf{y}_i$$

$$\text{Covariance: } \boldsymbol{\Sigma}_{yy} = \sum_{i=0}^{2N} \alpha_i \left( \mathbf{y}_i - \boldsymbol{\mu}_y \right) \left( \mathbf{y}_i - \boldsymbol{\mu}_y \right)^T$$

$$\text{Weights: } \alpha_i = \begin{cases} \frac{\kappa}{N + \kappa} & i = 0 \\ \frac{1}{2} \frac{1}{N + \kappa} & \text{otherwise} \end{cases}$$

# The Unscented Transform vs. Linearization

Let's revisit our nonlinear transformation example from the previous video.



**The Unscented Transform gives a much better approximation for similar work!**

# The Unscented Kalman Filter (UKF)

We can easily use the Unscented Transform in our Kalman Filtering framework with nonlinear models:

Nonlinear motion model

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$$

Nonlinear measurement model

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{v}_k)$$

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

Instead of approximating the system equations by linearizing, we will calculate sigma points and use the Unscented Transform to approximate the PDFs directly!

# UKF | Prediction step

To propagate the state from time  $(k - 1)$  to time  $k$ , apply the Unscented Transform using the current best guess for the mean and covariance

1. Compute sigma points

$$\hat{\mathbf{L}}_{k-1} \hat{\mathbf{L}}_{k-1}^T = \hat{\mathbf{P}}_{k-1}$$

$$\hat{\mathbf{x}}_{k-1}^{(0)} = \hat{\mathbf{x}}_{k-1}$$

$$\hat{\mathbf{x}}_{k-1}^{(i)} = \hat{\mathbf{x}}_{k-1} + \sqrt{N + \kappa} \text{col}_i \hat{\mathbf{L}}_{k-1} \quad i = 1 \dots N$$

$$\hat{\mathbf{x}}_{k-1}^{(i+N)} = \hat{\mathbf{x}}_{k-1} - \sqrt{N + \kappa} \text{col}_i \hat{\mathbf{L}}_{k-1} \quad i = 1 \dots N$$

2. Propagate sigma points

$$\check{\mathbf{x}}_k^{(i)} = \mathbf{f}_{k-1}(\hat{\mathbf{x}}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{0}) \quad i = 0 \dots 2N$$

3. Compute predicted mean and covariance

$$\alpha^{(i)} = \begin{cases} \frac{\kappa}{N + \kappa} & i = 0 \\ \frac{1}{2} \frac{1}{N + \kappa} & \text{otherwise} \end{cases}$$

$$\check{\mathbf{x}}_k = \sum_{i=0}^{2N} \alpha^{(i)} \check{\mathbf{x}}_k^{(i)}$$

$$\check{\mathbf{P}}_k = \sum_{i=0}^{2N} \alpha^{(i)} \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right)^T + \mathbf{Q}_{k-1}$$

Additive  
process noise!

# UKF | Correction step

To correct the state estimate using measurements at time  $k$ , use the nonlinear measurement model and the sigma points from the prediction step to predict the measurements

1. Predict measurements from (redrawn) propagated sigma points

$$\hat{\mathbf{y}}_k^{(i)} = \mathbf{h}_k(\check{\mathbf{x}}_k^{(i)}, \mathbf{0}) \quad i = 0, \dots, 2N$$

*redrawn*

With process noise considered!

2. Estimate mean and covariance of predicted measurements

$$\hat{\mathbf{y}}_k = \sum_{i=0}^{2N} \alpha^{(i)} \hat{\mathbf{y}}_k^{(i)}$$

$$\mathbf{P}_y = \sum_{i=0}^{2N} \alpha^{(i)} \left( \hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right) \left( \hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right)^T + \mathbf{R}_k$$

Additive  
measurement  
noise!

3. Compute cross-covariance and Kalman gain

$$\mathbf{P}_{xy} = \sum_{i=0}^{2N} \alpha^{(i)} \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left( \hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k \right)^T$$

$$\mathbf{K}_k = \mathbf{P}_{xy} \mathbf{P}_y^{-1}$$

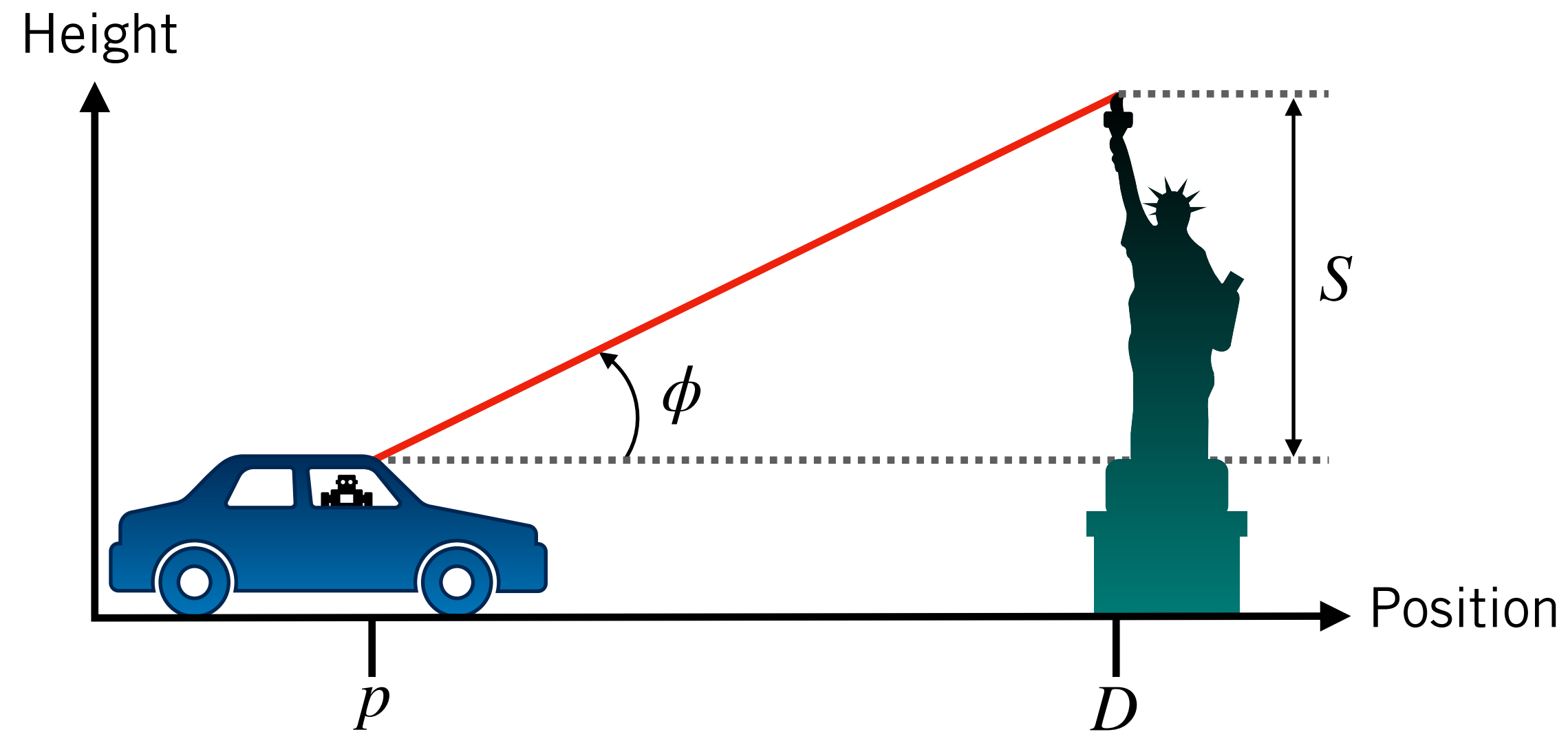
4. Compute corrected mean and covariance

$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k)$$

$$\hat{\mathbf{P}}_k = \check{\mathbf{P}}_k - \mathbf{K}_k \mathbf{P}_y \mathbf{K}_k^T$$



# UKF | Short example



$$\mathbf{x} = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \quad \mathbf{u} = \ddot{p}$$

$S$  and  $D$  are known in advance

## Motion/Process model

$$\begin{aligned} \mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}) \\ &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} \mathbf{u}_{k-1} + \mathbf{w}_{k-1} \end{aligned}$$

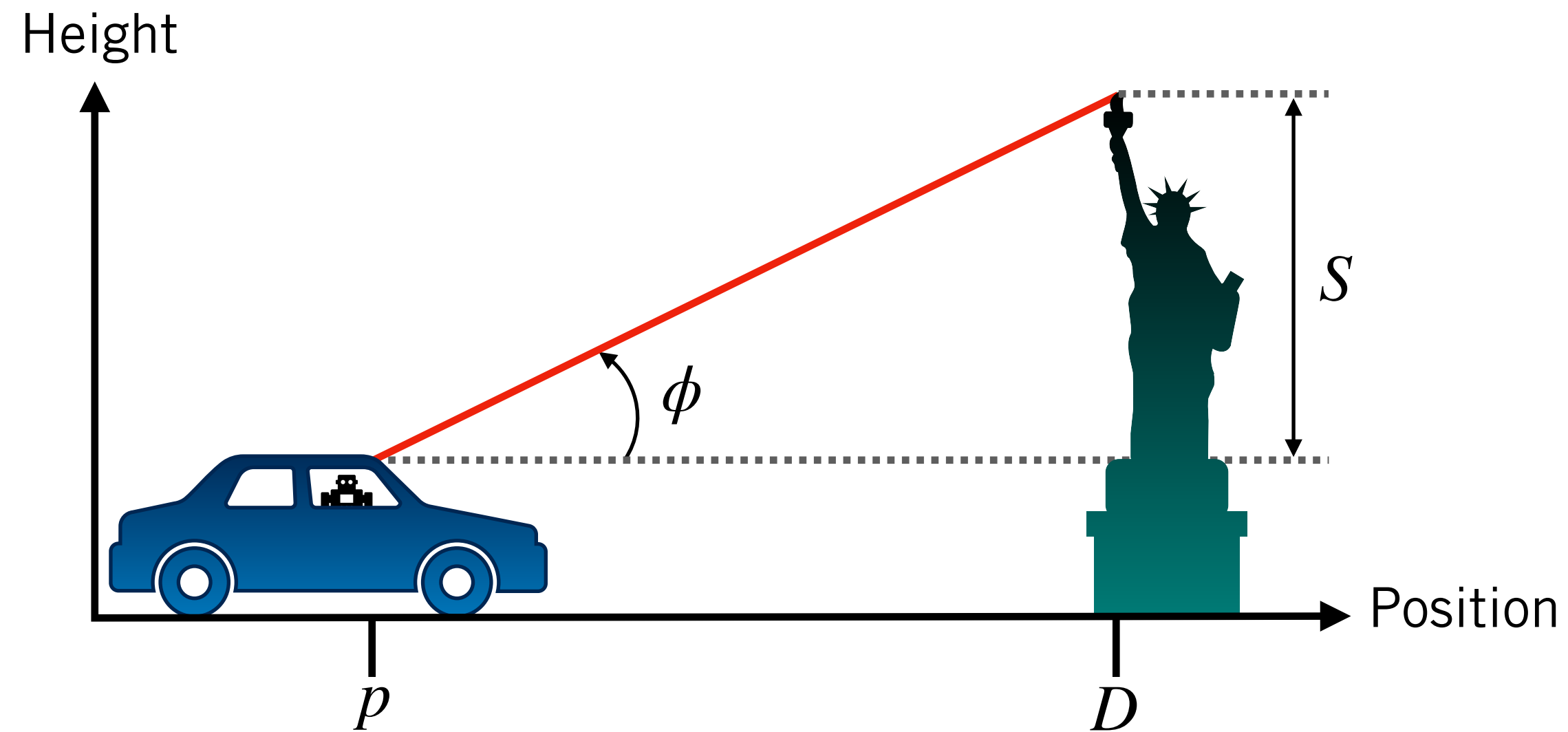
## Landmark measurement model

$$\begin{aligned} y_k &= \phi_k = h(p_k, v_k) \\ &= \tan^{-1} \left( \frac{S}{D - p_k} \right) + v_k \end{aligned}$$

## Noise densities

$$v_k \sim \mathcal{N}(0, 0.01) \quad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, (0.1)\mathbf{1}_{2 \times 2})$$

# UKF | Short example



## Data

$$\hat{\mathbf{x}}_0 \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Delta t = 0.5 \text{ s}$$

$$u_0 = -2 \text{ [m/s}^2\text{]} \quad y_1 = \pi/6 \text{ [rad]}$$

$$S = 20 \text{ [m]} \quad D = 40 \text{ [m]}$$

Using the Unscented Kalman Filter equations, what is our updated position?

$$\hat{p}_1$$

# UKF | Short Example Solution

## Prediction

$$N = 2, \quad \kappa = 3 - N = 1$$

$$\hat{\mathbf{L}}_0 \hat{\mathbf{L}}_0^T = \hat{\mathbf{P}}_0$$

$$\begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(0)} = \hat{\mathbf{x}}_0$$

$$\hat{\mathbf{x}}_0^{(i)} = \hat{\mathbf{x}}_0 + \sqrt{N + \kappa} \text{col}_i \hat{\mathbf{L}}_0 \quad i = 1 \dots, N$$

$$\hat{\mathbf{x}}_0^{(i+N)} = \hat{\mathbf{x}}_0 - \sqrt{N + \kappa} \text{col}_i \hat{\mathbf{L}}_0 \quad i = 1, \dots, N$$

$$\hat{\mathbf{x}}_0^{(0)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(1)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(2)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.73 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(3)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.17 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0^{(4)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3.27 \end{bmatrix}$$

# UKF | Short Example Solution

## Prediction

$$\check{\mathbf{x}}_1^{(i)} = \mathbf{f}_0(\hat{\mathbf{x}}_0^{(i)}, \mathbf{u}_0, \mathbf{0}) \quad i = 0, \dots, 2N$$

$$\check{\mathbf{x}}_1^{(0)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(1)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.17 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.67 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(2)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6.73 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(3)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.17 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.33 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(4)} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3.27 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix}$$

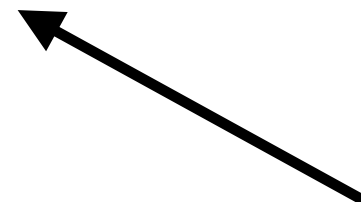
$$\alpha^{(i)} = \begin{cases} \frac{\kappa}{N + \kappa} = \frac{1}{2 + 1} = \frac{1}{3} & i = 0 \\ \frac{1}{2} \frac{1}{N + \kappa} = \frac{1}{2} \frac{1}{2 + 1} = \frac{1}{6} & \text{otherwise} \end{cases}$$

$$\begin{aligned} \check{\mathbf{x}}_k &= \sum_{i=0}^{2N} \alpha^{(i)} \check{\mathbf{x}}_k^{(i)} \\ &= \frac{1}{3} \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2.67 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2.33 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \end{aligned}$$

# UKF | Short Example Solution

## Prediction

$$\begin{aligned}\check{\mathbf{P}}_k &= \sum_{i=0}^{2N} \alpha^{(i)} \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right)^T + \mathbf{Q}_{k-1} \\&= \frac{1}{3} \left( \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left( \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^T + \\&\quad \frac{1}{6} \left( \begin{bmatrix} 2.67 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left( \begin{bmatrix} 2.67 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^T + \frac{1}{6} \left( \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left( \begin{bmatrix} 3.37 \\ 5.73 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^T + \\&\quad \frac{1}{6} \left( \begin{bmatrix} 2.33 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left( \begin{bmatrix} 2.33 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^T + \frac{1}{6} \left( \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) \left( \begin{bmatrix} 1.63 \\ 2.27 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right)^T + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \\&= \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}\end{aligned}$$



This is the same result as in  
the linear Kalman Filter  
example because the motion  
model is already linear!

# UKF | Short Example Solution

## Correction

$$\check{\mathbf{L}}_k \check{\mathbf{L}}_k^T = \check{\mathbf{P}}_k$$

$$\begin{bmatrix} 0.60 & 0 \\ 0.83 & 0.64 \end{bmatrix} \begin{bmatrix} 0.60 & 0 \\ 0.83 & 0.64 \end{bmatrix}^T = \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(0)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

Redrawn  
sigma points!

$$\check{\mathbf{x}}_1^{(1)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.60 \\ 0.83 \end{bmatrix} = \begin{bmatrix} 3.54 \\ 5.44 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(2)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0 \\ 0.64 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5.10 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(3)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0.60 \\ 0.83 \end{bmatrix} = \begin{bmatrix} 1.46 \\ 2.56 \end{bmatrix}$$

$$\check{\mathbf{x}}_1^{(4)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0 \\ 0.64 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.90 \end{bmatrix}$$

$$\hat{y}_1^{(i)} = h_1(\check{\mathbf{x}}_1^{(i)}, 0) \quad i = 0, \dots, 2N$$

$$\hat{y}_1^{(0)} = \tan^{-1} \left( \frac{20}{40 - 2.5} \right) = 0.49$$

$$\hat{y}_1^{(1)} = \tan^{-1} \left( \frac{20}{40 - 3.54} \right) = 0.50$$

$$\hat{y}_1^{(2)} = \tan^{-1} \left( \frac{20}{40 - 2.5} \right) = 0.49$$

$$\hat{y}_1^{(3)} = \tan^{-1} \left( \frac{20}{40 - 1.46} \right) = 0.48$$

$$\hat{y}_1^{(4)} = \tan^{-1} \left( \frac{20}{40 - 2.5} \right) = 0.49$$

# UKF | Short Example Solution

## Correction

$$\begin{aligned}\hat{y}_1 &= \sum_{i=0}^{2N} \alpha^{(i)} \hat{y}_1^{(i)} \\ &= \frac{1}{3}(0.49) + \frac{1}{6}(0.50) + \frac{1}{6}(0.49) + \frac{1}{6}(0.48) + \frac{1}{6}(0.49) \\ &= 0.49\end{aligned}$$

$$\begin{aligned}P_y &= \sum_{i=0}^{2N} \alpha^{(i)} \left( \hat{y}_1^{(i)} - \hat{y}_k \right) \left( \hat{y}_1^{(i)} - \hat{y}_k \right)^T + R_k \\ &= \frac{1}{3}(0.49 - 0.49)^2 + \frac{1}{6}(0.50 - 0.49)^2 + \frac{1}{6}(0.49 - 0.49)^2 + \frac{1}{6}(0.48 - 0.49)^2 + \frac{1}{6}(0.49 - 0.49)^2 + 0.01 \\ &= 0.01\end{aligned}$$

# UKF | Short Example Solution

## Correction

$$\begin{aligned}\mathbf{P}_{xy} &= \sum_{i=0}^{2N} \alpha^{(i)} \left( \check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k \right) \left( \hat{y}_1^{(i)} - \hat{y}_k \right)^T \\ &= \frac{1}{3} \left( \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) + \frac{1}{6} \left( \begin{bmatrix} 3.54 \\ 5.44 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.50 - 0.49) + \frac{1}{6} \left( \begin{bmatrix} 2.5 \\ 5.10 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) \\ &\quad + \frac{1}{6} \left( \begin{bmatrix} 1.46 \\ 2.56 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.48 - 0.49) + \frac{1}{6} \left( \begin{bmatrix} 2.5 \\ 2.90 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} \right) (0.49 - 0.49) \\ &= \begin{bmatrix} 0.004 \\ 0.006 \end{bmatrix}\end{aligned}$$

$$\mathbf{K}_1 = \mathbf{P}_{xy} P_y^{-1} = \begin{bmatrix} 0.004 \\ 0.006 \end{bmatrix} \frac{1}{0.01} = \begin{bmatrix} 0.40 \\ 0.55 \end{bmatrix}$$

$$\hat{\mathbf{x}}_1 = \check{\mathbf{x}}_1 + \mathbf{K}_1 (y_1 - \hat{y}_1) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0.40 \\ 0.55 \end{bmatrix} (0.52 - 0.49) = \begin{bmatrix} 2.51 \\ 4.02 \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ \hat{\dot{p}}_1 \end{bmatrix}$$



# Summary | Unscented Kalman Filter (UKF)

- The UKF uses the unscented transform to adapt the Kalman filter to nonlinear systems.
- The unscented transform works by passing a small set of carefully chosen samples through a nonlinear system, and computing the mean and covariance of the outputs.
- The unscented transform does a better job of approximating the output distribution than analytical local linearization, for similar computational cost.

# Summary | Nonlinear Kalman Filtering

	EKF	ES-EKF	UKF
Operating Principle	Linearization (Full State)	Linearization (Error State)	Unscented Transform
Accuracy	Good	Better	Best
Jacobians	Required	Required	Not required
Speed	Slightly faster	Slightly faster	Slightly slower