Kinematic Modeling in 2D

Course 1, Module 4, Lesson 1

UNIVERSITY OF TORONTO
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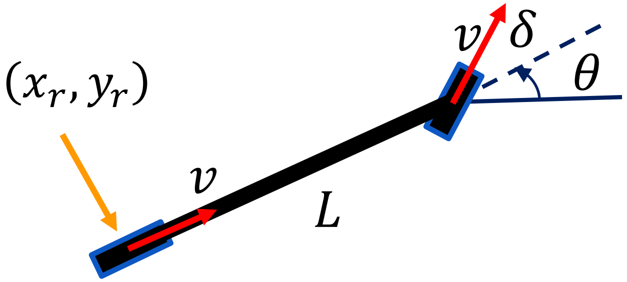
Overview of Module 4

- Basics of kinematic and coordinates
- Kinematic model development of a bicycle
- Basics of dynamic modeling
- Vehicle longitudinal dynamics and modeling
- Vehicle lateral dynamics and modeling
- Vehicle actuation system
- Tire slips and modeling

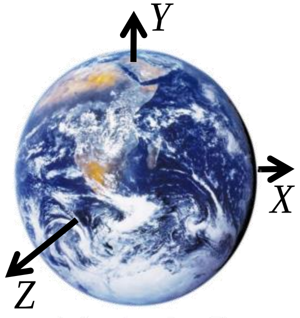
Kinematic Vs Dynamic Modeling

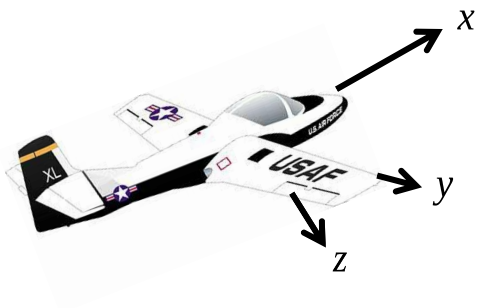
- At low speeds, it is often sufficient to look only at kinematic models of vehicles
  - Examples: Two wheeled robot, Bicycle model
- Dynamic modeling is more involved, but captures vehicle behavior more precisely over a wide operating range
  - Examples: Dynamic vehicle model

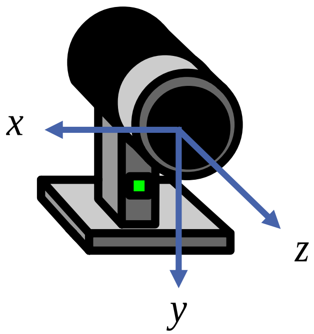


COORDINATE FRAMES

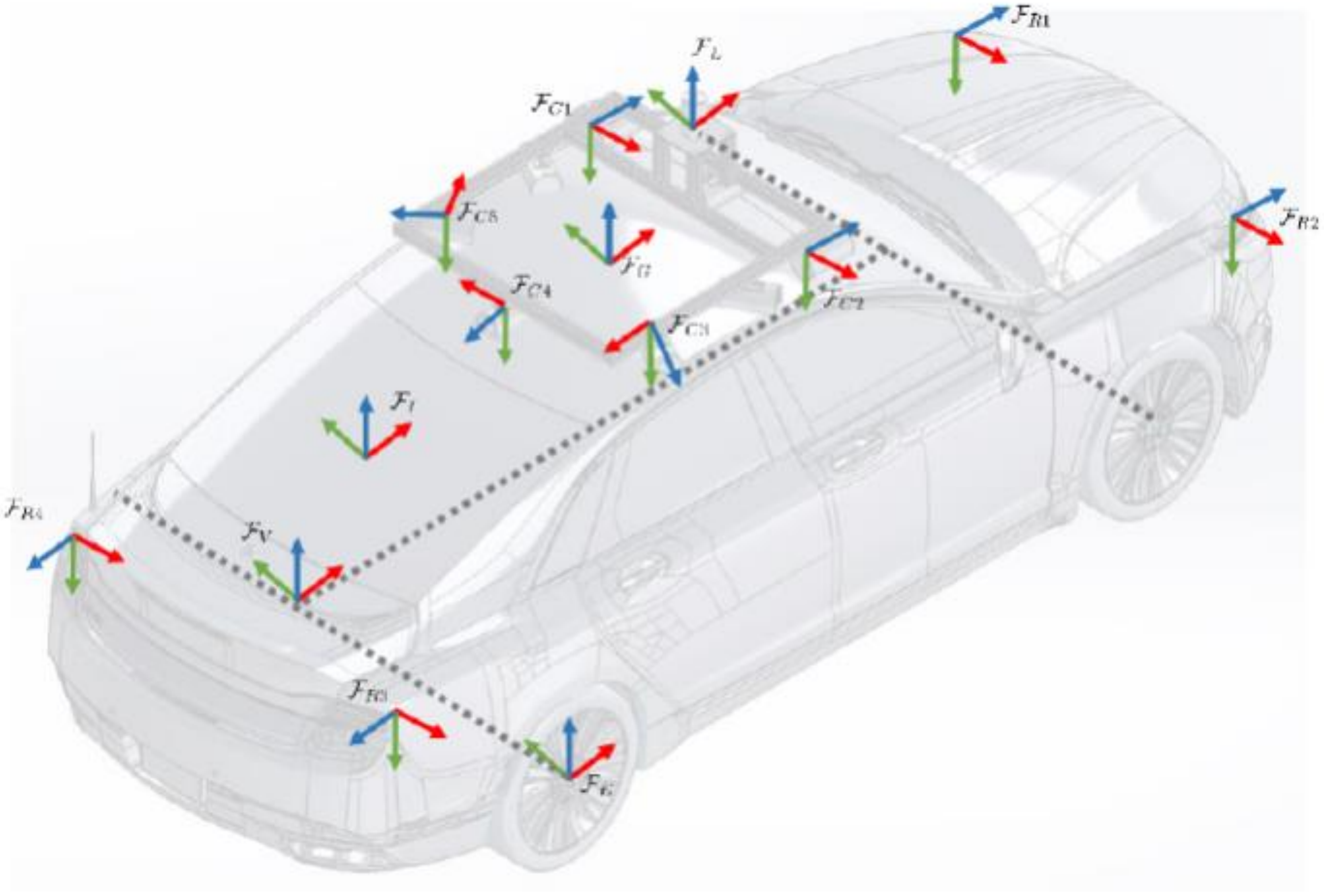
- Right handed by convention
- Inertial frame
  - Fixed, usually relative to earth
- Body frame
  - Attached to vehicle, origin at vehicle center of gravity, or center of rotation
- Sensor frame
  - Attached to sensor, convenient for expressing sensor measurements





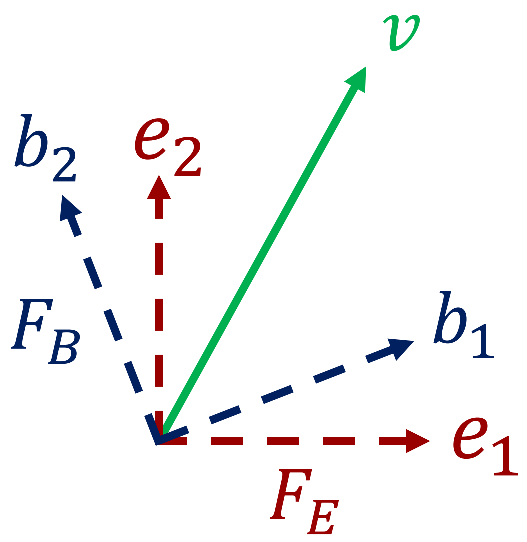


Why We Need Coordinate Transformation



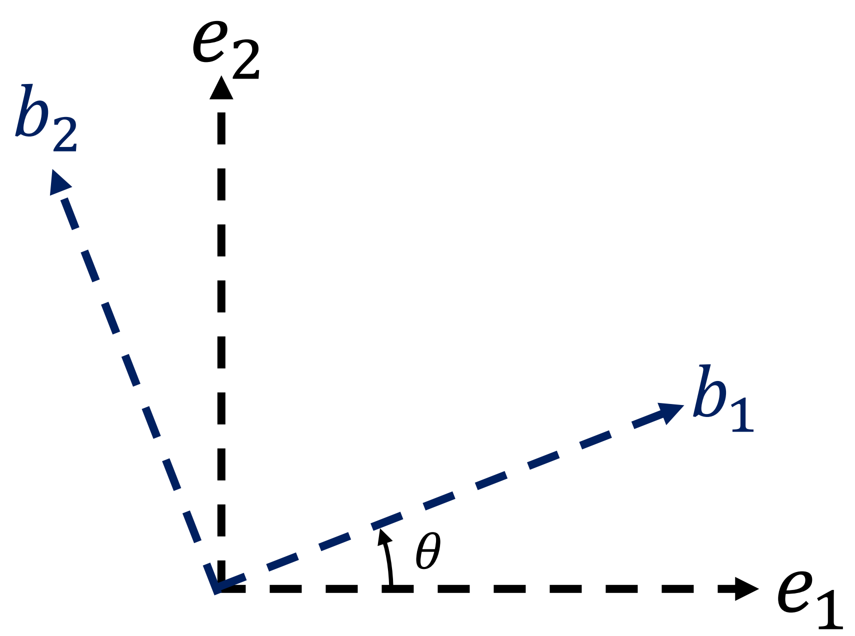
Vectors

- Vectors are variables with both magnitude and direction
- In this figure, v is a vector
- The vectors $\{b_1, b_2\}$, $\{e_1, e_2\}$ define two different coordinate frames, F_B and F_E



Rotation Matrices in 2D

$$C_{EB} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad C_{BE} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Coordinate Transformation

- Conversion between Inertial frame and Body coordinates is done with a translation vector and a rotation matrix
  - Location of point (P) in Body Frame (B)

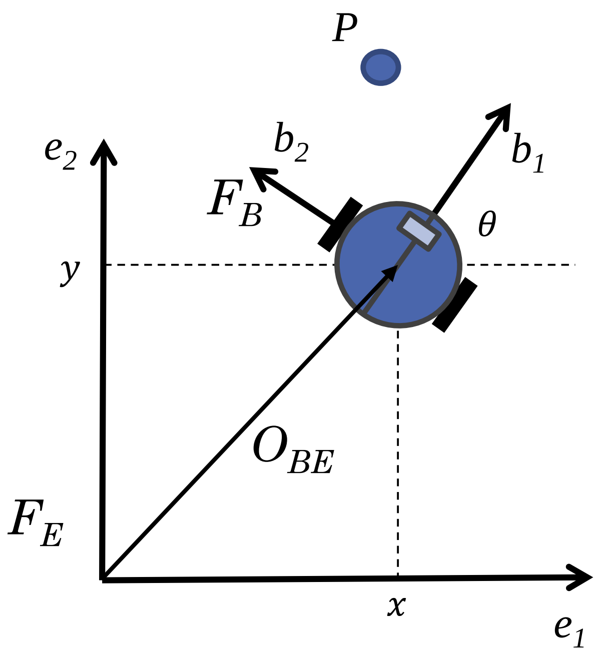
$$P_B = C_{EB}(\theta)\, P_E + O_{EB}$$

Translation term, expressed in body frame

  - Location of point (P) in Inertial Frame (E)

$$P_E = C_{BE}(\theta)\, P_B + O_{BE}$$

Translation term, expressed in inertial frame



Homogeneous Coordinate Form

- A 2D vector in homogeneous form

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \bar{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Transforming a point from body to inertial coordinates with homogeneous coordinates

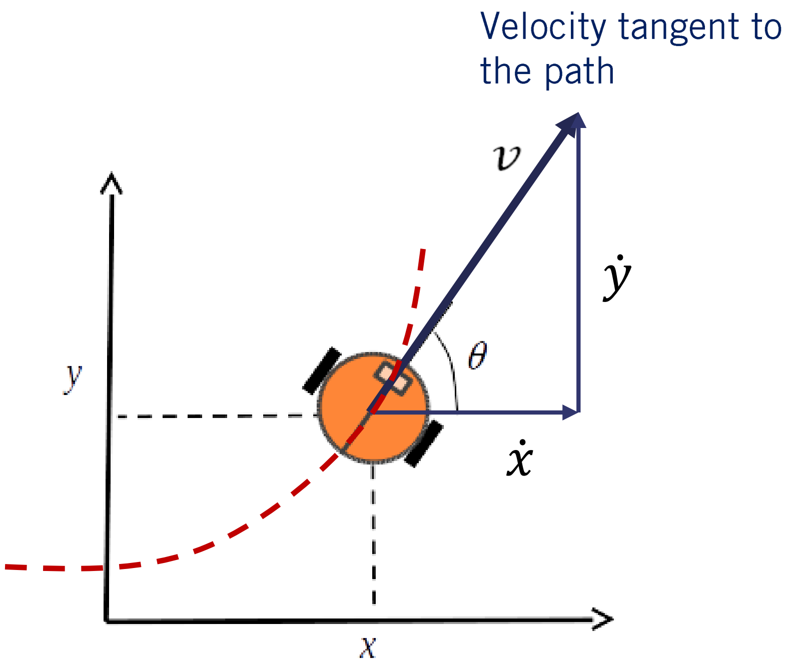
$$\overline{P_E} = [C_{EB}(\theta) \quad | \; O_{EB}]\overline{P_B}$$

2D Kinematic Modeling

- The kinematic constraint is nonholonomic
  - A constraint on rate of change of degrees of freedom
  - Vehicle velocity always tangent to current path

roll forward & turn while rolling
cannot move sideways directly

$$\frac{dy}{dx} = \tan\theta = \frac{\sin\theta}{\cos\theta}$$



- Nonholonomic constraint

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0$$

- Velocity components

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$

Simple Robot Motion Kinematics

Inputs → **Simple Model** → **States (Outputs)**

$$[v, \omega]$$

forward velocity

rotation rate

$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

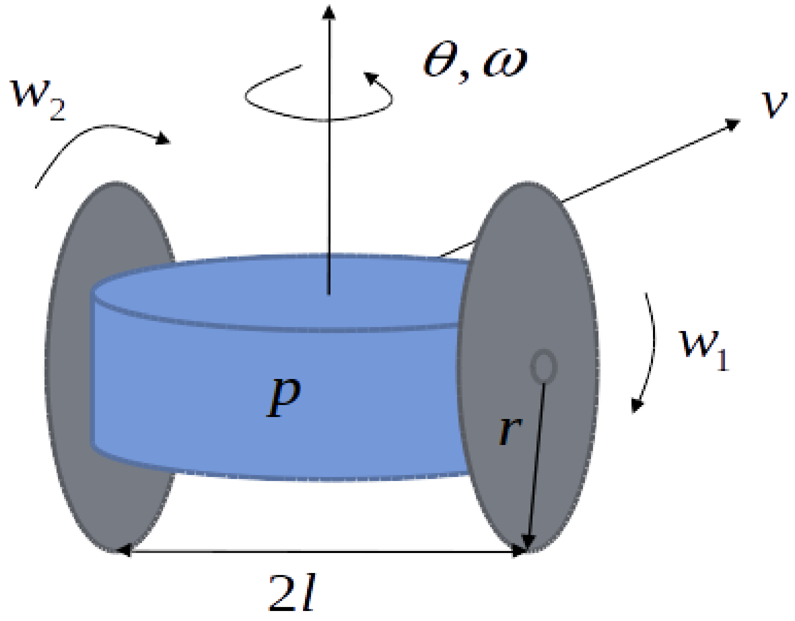
$$\dot{\theta} = \omega$$

$$[x, y, \theta]$$

The **state** is a set of variables (often arranged in the form of a vector) that fully describe the system at the current time.

Two-Wheeled Robot Kinematic Model

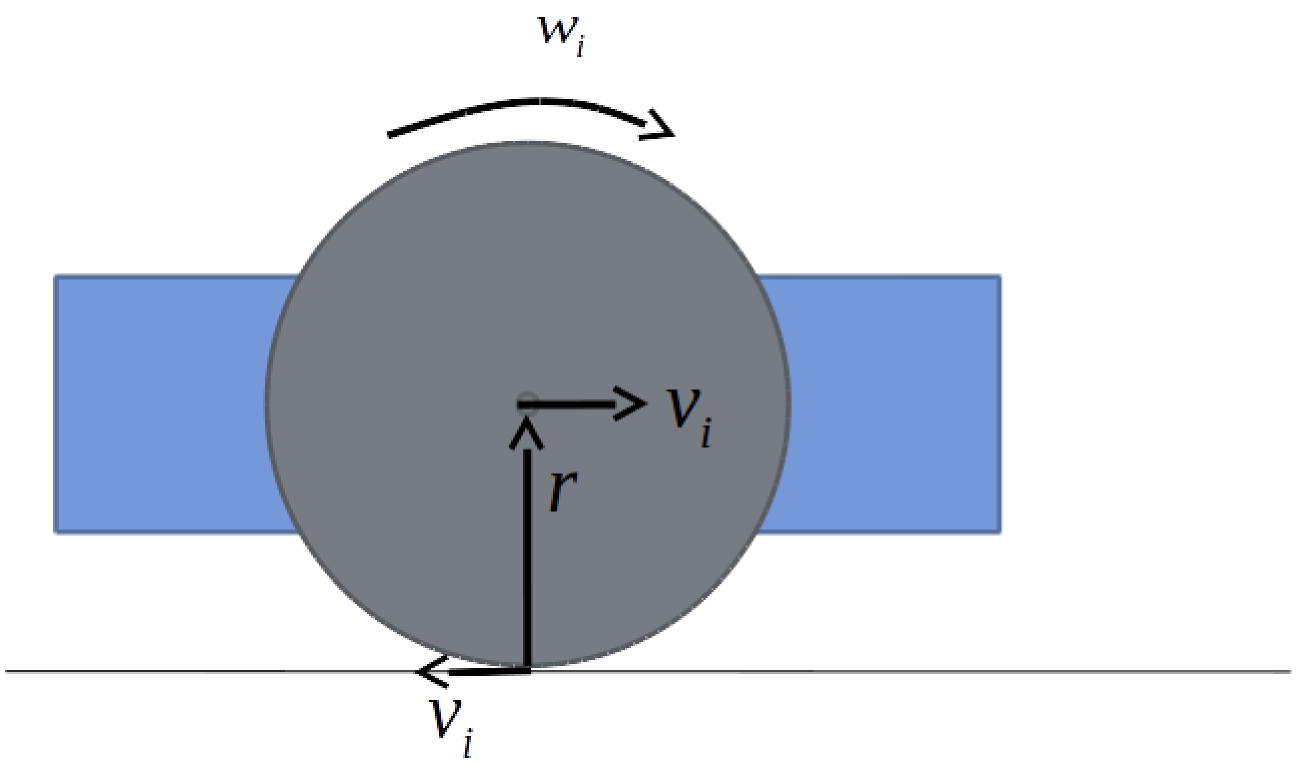
- Assume control inputs are wheel speeds
  - Center: p
  - Wheel to center: l
  - Wheel radius: r
  - Wheel rotation rates: w1, w2



Two-wheeled kinematic model

- Kinematic constraint

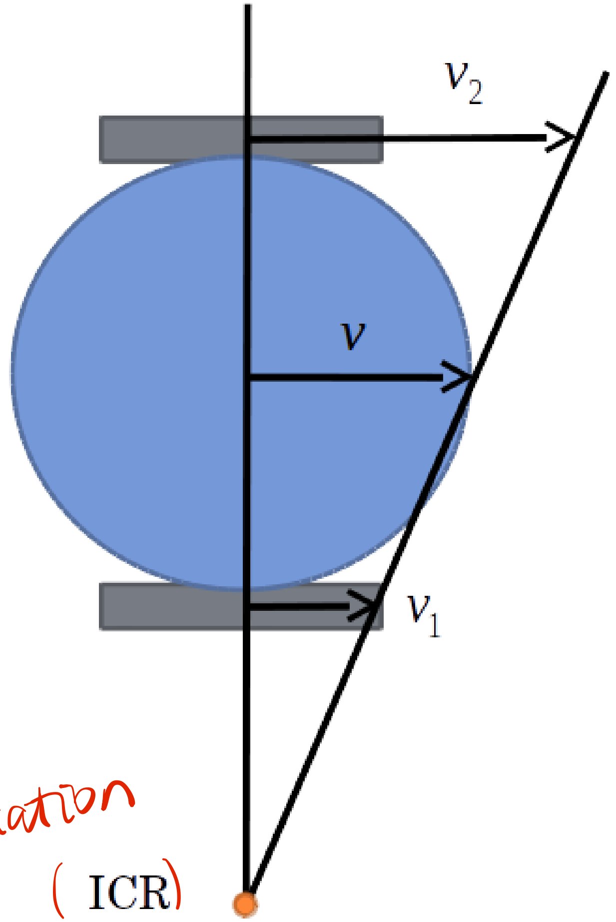
$$v_i = r w_i$$



Two-wheeled Kinematic Model

- Velocity is the average of the two wheel velocities

$$v = \frac{v_1 + v_2}{2} = \frac{rw_1 + rw_2}{2}$$



Instantaneous center of rotation

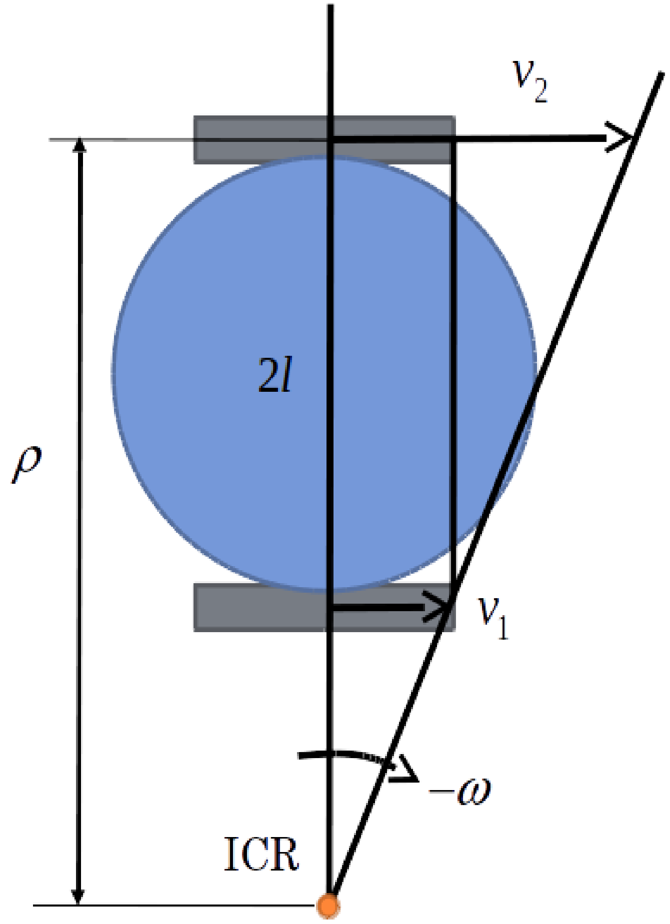
Two-wheeled Kinematic Model

- Use the instantaneous center of rotation (ICR)
- Equivalent triangles give the angular rate of rotation

$$\omega = \frac{-v_2}{\rho} = \frac{-(v_2 - v_1)}{2l}$$

$$\omega = \frac{rw_1 - rw_2}{2l}$$

wheel rotation speed



Kinematic Model of a Simple 2D Robot

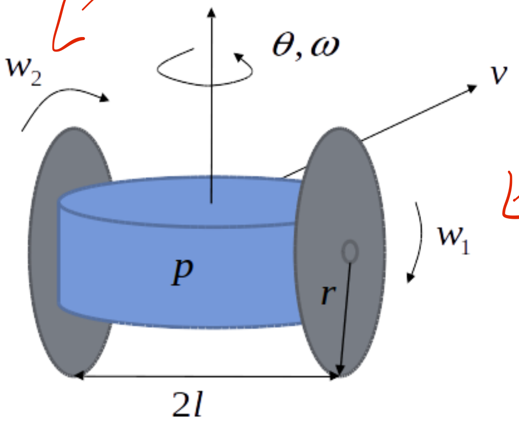
- Continuous time model:

$$\dot{x} = \left[\left(\frac{rw_1 + rw_2}{2}\right)\cos\theta\right]$$

$$\dot{y} = \left[\left(\frac{rw_1 + rw_2}{2}\right)\sin\theta\right]$$

$$\dot{\theta} = \left(\frac{rw_1 - rw_2}{2l}\right)$$

inputs



- Discrete time model:

next current

$$x_{k+1} = x_k + \left[\left(\frac{rw_{1,k} + rw_{2,k}}{2}\right)\cos\theta_k\right]\Delta t$$

$$y_{k+1} = y_k + \left[\left(\frac{rw_{1,k}+rw_{2,k}}{2}\right)\sin\theta_k\right]\Delta t$$

$$\theta_{k+1} = \theta_k + \left(\frac{rw_{1,k} - rw_{2,k}}{2l}\right)\Delta t$$

Summary

What we have learned from this lesson:

- Basics of 2D kinematics
- Coordinate frames and transformations
- Continuous and discrete kinematic model of a two wheeled robot

What is next?

- Going through the kinematics formulation of a bicycle model