UiT

THE ARCTIC UNIVERSITY OF NORWAY

Obligatory Project 3

Mat-3200 - Mathematical Methods
Institute for mathematics and statistics
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Problem 1

In the first part of this project we are going to solve the heat equation numerically using the finite difference method and the finite Fourier transform. The problem is given as

$$\begin{aligned} u_t - u_{xx} &= \rho(x,t), \quad 0 < x < l, \quad t > 0 \\ u(0,t) &= f(t) \\ u(l,t) &= g(t) \\ u(x,0) &= \varphi(x) \end{aligned}$$

a)

We are here asked to implement a finite difference method for the given problem, where the code is flexible enough to handle arbitrary choices for $f(t)$, $g(t)$, $\rho(x,t)$ and $\varphi(x,t)$.

We will use forward difference in time, which is given by

$$(u_t)_i^n \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

and center difference in space, given by

$$(u_{xx})_i^n \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Using these two relations, we can rewrite the differential equation.

$$\begin{aligned} (u_t)_i^n - (u_{xx})_i^n &= (\rho)_i^n \\ \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} &= \rho_i^n \end{aligned}$$

which can be written as

$$\boxed{u_i^{n+1} = s(u_{i+1}^n + u_{i-1}^n) + (1-2s)u_i^n + \rho_i^n \Delta t}$$

where we have introduced the parameter $s = \frac{\Delta t}{\Delta x^2}$. From earlier in the course notes, we know that this numerical scheme is Von Neumann stable for

$$s < \frac{1}{2}, \text{ when } \rho = 0$$

For the boundary conditions and initial condition, we have

$$\begin{aligned} &u_0^n = f_n, \ u_l^n = g_n \\ &u_i^0 = \varphi_i \end{aligned}$$

This can be implemented in code, for arbitrary f_n, g_n, ρ_i^n and φ_i.

b)

We are given the function

$$u_e(x,t) = \frac{1}{l} h(x,t) \left(x\frac{g(t)}{h(l,t)} + (l-x)\frac{f(t)}{h(0,t)} \right)$$

that satisfies the boundary conditions. We are asked to calculate the corresponding artificial source $\rho(x,t)$ and initial condition $\varphi(x)$. We are then asked to run the finite difference code with the calculated source term and initial condition and compare the numerical solution with the exact solution $u_e(x,t)$. We are to do this for a couple of different choices of $f(t)$, $g(t)$ and $h(x,t)$ and several values of parameters. For each choice we are to plot the exact solution and numerical solution in the same plot, and thereby verify that they coincide.

We can find the source term by computing

$$\rho(x,t) = (u_e)_t - (u_e)_{xx}$$

We can first compute the time derivative of u_e

$$(u_e)_t = \frac{1}{l} \left(h(x,t) \left(x\frac{g(t)}{h(l,t)} + (l-x)\frac{f(t)}{h(0,t)} \right) \right)_t$$

we can define

$$v(x,t) = x\frac{g(t)}{h(l,t)} + (l-x)\frac{f(t)}{h(0,t)}$$

so

$$(u_e)_t = \frac{1}{l}(h(x,t)v(x,t))_t = \frac{1}{l}(h_t(x,t)v(x,t) + v_t(x,t)h(x,t))$$

So we need to compute the time derivative of $v(x,t)$, thus

$$\begin{aligned}
v_t(x,t) &= \left(x\frac{g(t)}{h(l,t)} + (l-x)\frac{f(t)}{h(0,t)} \right)_t \\
&= x\left(\frac{g(t)}{h(l,t)} \right)_t + (l-x)\left(\frac{f(t)}{h(0,t)} \right)_t \\
&= \underline{x\frac{g_t(t)h(l,t) - h_t(l,t)g(t)}{h^2(l,t)} + (l-x)\frac{f_t(t)h(0,t) - h_t(0,t)f(t)}{h^2(0,t)}}
\end{aligned}$$

So the derivative is given by

$$(u_e)_t = \frac{1}{l}(h_t(x,t)v(x,t) + h(x,t)v_t(x,t))$$

where $v_t(x,t)$ is given above. Further we can find the second space derivative, using the same logic as the time derivative

$$(u_e)_x = \frac{1}{l}(h_x(x,t)v(x,t) + v_x(x,t)h(x,t))$$

So we need the derivative with respect to x of $v(x,t)$, so

$$v_x(x,t) = \frac{g(t)}{h(l,t)} - \frac{f(t)}{h(0,t)} = A(t)$$

And we can take the derivative with respect to x of $(u_e)_x$, thus

$$
\begin{aligned}
(u_e)_{xx} &= ((u_e)_x)_x \\
&= \frac{1}{l}(h_x(x,t)v(x,t) + h(x,t)A(t))_x \\
&= \frac{1}{l}(h_{xx}(x,t)v(x,t) + h_x(x,t)v_x(x,t) + h_x(x,t)A(t)) \\
&= \underline{\frac{1}{l}(h_{xx}(x,t)v(x,t) + 2h_x(x,t)A(t))}
\end{aligned}
$$

So now we can compute the source term, thus

$$
\begin{aligned}
\rho(x,t) &= (u_e)_t - (u_e)_{xx} \\
&= \boxed{\frac{1}{l}(h_t(x.t)v(x.t) + h(x,t)v_t(x,t)) - \frac{1}{l}(h_{xx}(x,t)v(x,t) + 2h_x(x,t)A(t))}
\end{aligned}
$$

with

$$
\begin{aligned}
v(x,t) &= x\frac{g(t)}{h(l,t)} + (l-x)\frac{f(t)}{h(0,t)} \\
v_t(x,t) &= x\frac{g_t(t)h(l,t) - h_t(l,t)g(t)}{h^2(l,t)} + (l-x)\frac{f_t(t)h(0,t) - h_t(0,t)f(t)}{h^2(0,t)} \\
A(t) &= \frac{g(t)}{h(l,t)} - \frac{f(t)}{h(0,t)}
\end{aligned}
$$

For the initial condition, we can simply input $t = 0$, thus

$$
\begin{aligned}
\varphi(x) &= u_e(x,0) \\
&= \boxed{\frac{1}{l}h(x,0)\left(x\frac{g(0)}{h(l,0)} + (l-x)\frac{f(0)}{h(0,0)}\right)}
\end{aligned}
$$

c)

We are here asked to solve the problem using the finite Fourier transform. We are to use the same choices for $f(t)$ and $g(t)$ that is used in problem *b*) and we are asked to choose our own source. We are asked to compare the finite Fourier transform solution with the finite solution for some choice of initial condition, where the initial condition is consistent with the boundary conditions.

Since this is a boundary value problem with non-homogeneous boundary values, we can transform the problem by defining

$$w(x,t) = u(x,t) - v(x,t)$$

where $v(x,t)$ is an arbitrary function which satisfy the non-homogeneous boundary conditions

$$v(0,t) = f(t), \; v(l,t) = g(t)$$

Using linear interpolation, we get that one such function is given by

$$v(x,t) = \frac{1}{l}(xg(t) + (l-x)f(t))$$

where $f(t)$ and $g(t)$ are the non-homogeneous boundary conditions. We can now put in the transform

$$u(x,t) = w(x,t) + v(x,t)$$

into the original problem, so

$$\begin{aligned} \Rightarrow (w(x,t)+v(x,t))_t - (w(x,t)+v(x,t))_{xx} &= w_t + v_t - w_{xx} - \cancelto{0}{v_{xx}} \\ w_t + v_t - w_{xx} &= \rho(x,t) \end{aligned}$$

Using the expression for $v(x,t)$, we get

$$w_t - w_{xx} = \rho(x,t) + \frac{1}{l}(xg'(t) + (l-x)f'(t)) = \alpha(x,t)$$

The problem will now have new boundary conditions, which we can find by

$$\begin{aligned} w(0,t) &= u(0,t) - v(0,t) \\ &= f(t) - \frac{1}{l}((0)g(t) + (l-0)f(t)) \\ &= f(t) - f(t) = \underline{0} \end{aligned}$$

and

$$\begin{aligned} w(l,t) &= u(l,t) - v(l,t) \\ &= g(t) - \frac{1}{l}((l)g(t) + (l-l)f(t)) \\ &= g(t) - g(t) = \underline{0} \end{aligned}$$

So, now we have a problem with boundary conditions

$$w(0,t) = w(l,t) = 0$$

The initial condition is now given as

$$\begin{aligned} w(x,0) &= u(x,0) - v(x,0) \\ &= \varphi(x) - \frac{1}{l}(xg(0) + (l-x)f(0)) \end{aligned}$$

We now how a problem that can be solved using the finite Fourier transform, and is of the form

$$\begin{aligned} w_t - w_{xx} &= \alpha(x,t) \\ w(0,t) = w(l,t) &= 0 \\ w(x,0) &= \varphi(x) - \frac{1}{l}(xg(0) + (l-x)f(0)) \end{aligned}$$

To solve the differential equation, we start by assuming that the solution of $w(x,t)$ can be written as a Fourier sine series in space, which has the form

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

where we allow the coefficients to evolve in time. Taking the derivatives in time and space gives

$$\begin{aligned} w_t(x,t) &= \sum_{n=1}^{\infty} w_n'(t) \sin\left(\frac{n\pi}{l}x\right) \\ w_{xx}(x,t) &= \sum_{n=1}^{\infty} w_n(t) \left[-\frac{n^2\pi^2}{l^2}\right] \sin\left(\frac{n\pi}{l}x\right) \end{aligned}$$

Which we can put this into the equation for the transformed problem.

$$
\begin{aligned}
w_t - w_{xx} &= \alpha(x,t) \\
\sum_{n=1}^{\infty} w_n'(t)\sin\left(\frac{n\pi}{l}x\right) - \sum_{n=1}^{\infty} w_n(t)\left[-\frac{n^2\pi^2}{l^2}\right]\sin\left(\frac{n\pi}{l}x\right) &= \alpha(x,t)
\end{aligned}
$$

or

$$
\sum_{n=1}^{\infty}\left[w_n'(t) + \frac{n^2\pi^2}{l^2}w_n(t)\right]\sin\left(\frac{n\pi}{l}x\right) = \alpha(x,t)
$$

Since the left side can be written as a Fourier series in x, then this must also be true for the right side. We have

$$
\alpha(x,t) = \rho(x.t) + \frac{1}{l}(xg'(t) + (l-x)f'(t))
$$

We can for simplicity use $\rho(x,t) = 0$ in the first case, and use the boundary conditions

$$
\begin{aligned}
g(t) = 0,\ f(t) = \sin t \\
g'(t) = 0,\ f'(t) = \cos t
\end{aligned}
$$

So $\alpha(x,t)$ has the form

$$
\alpha(x,t) = \frac{1}{l}(x(0) + (l-x)\cos t) = \frac{l-x}{l}\cos t
$$

Further we can express $\alpha(x,t)$ as a Fourier series, given by

$$
\alpha(x,t) = \sum_{n=1}^{\infty}\alpha_n(t)\sin\left(\frac{n\pi}{l}x\right)
$$

The coefficients is found by

$$
\begin{aligned}
\alpha_n(t) &= \frac{2}{l}\int_0^l \alpha(x,t)\sin\left(\frac{n\pi}{l}x\right)dx \\
&= \frac{2}{l}\int_0^l \left(\frac{l-x}{l}\cos t\right)\sin\left(\frac{n\pi}{l}x\right)dx \\
&= \frac{2}{\cancel{l}}\left(\frac{\cancel{l}\cos t(n\pi - \sin n\pi)}{n^2\pi^2}\right) \\
&= \underline{\frac{2}{n\pi}\cos t}
\end{aligned}
$$

Since $\sin n\pi = 0,\ n = 1, 2, ...$
So we have

$$
\sum_{n=1}^{\infty}\left[w_n'(t) + \frac{n^2\pi^2}{l^2}w_n(t)\right]\sin\left(\frac{n\pi}{l}x\right) = \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\cos t\right]\sin\left(\frac{n\pi}{l}x\right)
$$

Since they are equal, then the coefficients must be equal and we get

$$
w_n'(t) - \frac{n^2\pi^2}{l^2}w_n = \frac{2}{n\pi}\cos t
$$

which can be solved using integrating factor, thus

$$
e^{\int \frac{n^2\pi^2}{l^2}dt} = e^{\frac{n^2\pi^2}{l^2}t}
$$

Which leads to

$$\left(e^{\frac{n^2\pi^2}{l^2}t}w_n\right)' = \frac{2}{n\pi}e^{\frac{n^2\pi^2}{l^2}t}\cos t$$

Integrating this gives

$$\begin{aligned}
e^{\frac{n^2\pi^2}{l^2}t}w_n(t) &= \frac{2l^2e^{\frac{n^2\pi^2}{l^2}t}(l^2\sin t + n^2\pi^2\cos t)}{n\pi(l^4+n^4\pi^4)} + c_n \\
\Rightarrow w_n(t) &= \frac{2l^2(l^2\sin t + n^2\pi^2\cos t)}{n\pi(l^4+n^4\pi^4)} + c_ne^{-\frac{n^2\pi^2}{l^2}t}
\end{aligned}$$

So the Fourier series is given as

$$w(x,t) = \sum_{n=1}^{\infty}\left[2\frac{l^2(l^2\sin t + n^2\pi^2\cos t)}{n\pi(l^4+n^4\pi^4)} + c_ne^{-\frac{n^2\pi^2}{l^2}t}\right]\sin\left(\frac{n\pi}{l}x\right)$$

Now we need to use the initial condition to find c_n. We can use $\varphi(x) = x(l-x)$, thus we have

$$\begin{aligned}
w(x,0) &= \varphi(x) - \frac{1}{l}(xg(0) + (l-x)f(0)) \\
&= x(l-x) - \frac{1}{l}(x(0) + (l-x)(0)) = \underline{x(l-x)}
\end{aligned}$$

Using the Fourier series solution found above and again using the argument of Fourier series on the left and right side of equality sign, we get

$$w(x,0) = \sum_{n=1}^{\infty}\left[2\frac{l^2n\pi}{l^4+n^4\pi^4} + c_n\right]\sin\left(\frac{n\pi}{l}x\right) = \sum_{n=1}^{\infty}b_n(t)\sin\left(\frac{n\pi}{l}x\right) = x(l-x)$$

Thus we assume that

$$\sum_{n=1}^{\infty}b_n(t)\sin\left(\frac{n\pi}{l}x\right) = x(l-x)$$

is a Fourier series in x for the initial condition. And we can now find the coefficients the same way as before

$$\begin{aligned}
b_n(t) &= \frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi}{l}x\right)dx \\
&= -\frac{2l^2(2(-1)^n-2)}{n^3\pi^3} \\
&= \begin{cases} 8\frac{l^2}{n^3\pi^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}$$

So we have

$$\sum_{n=1}^{\infty}\left[2\frac{l^2n\pi}{l^4+n^4\pi^4} + c_n\right]\sin\left(\frac{n\pi}{l}x\right) = \sum_{n \text{ odd}}\frac{8l^2}{n^3\pi^3}\sin\left(\frac{n\pi}{l}x\right)$$

So the coefficients must be equal, but we only have for odd n. So

$$\begin{aligned}
2\frac{l^2n\pi}{l^4+n^4\pi^4} + c_n &= \frac{8l^2}{n^3\pi^3} \\
\underline{c_n = \frac{8l^2}{n^3\pi^3} - 2\frac{l^2n\pi}{l^4+n^4\pi^4}}
\end{aligned}$$

We can put this into the solution for $w(x,t)$ and by using $l=1$ simplify to

$$w(x,t)=\sum_{n=1}^{\infty}\frac{\sin t+n^2\pi^2\cos t}{n\pi(1+n^4\pi^4)}\sin(n\pi x)+\sum_{n\text{ odd}}\left[\frac{8}{n^3\pi^3}-2\frac{n\pi}{1+n^4\pi^4}\right]e^{-n^2\pi^2t}\sin(n\pi x)$$

And so a solution to the original problem is given as

$$u(x,t)=w(x,t)+v(x,t)=w(x,t)+(1-x)\sin t$$

where $w(x,t)$ is given in the expression above.
We can now compare the numerical solution to the same problem we have solved above and compare using RMSE, or the root-mean-square error for each time step.

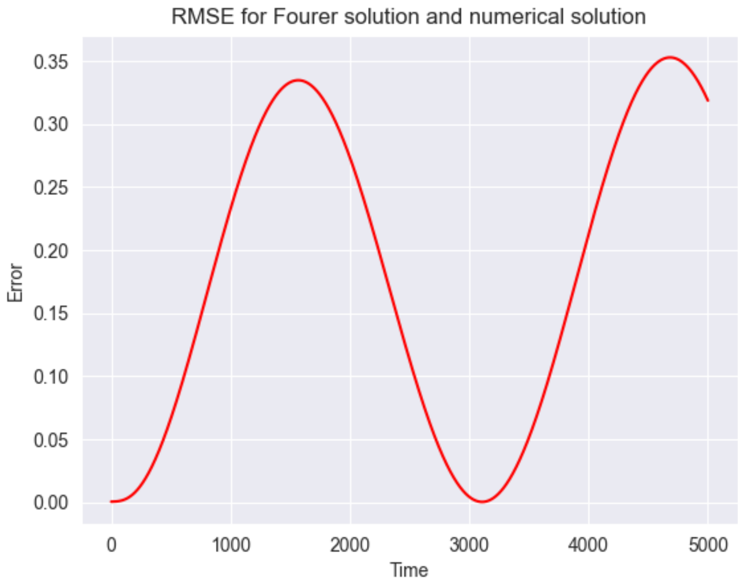

Figure 1: *RMSE for the first Fourier series and the finite difference method solution of the same problem.* $s=0.10$ *for this plot.*

As we can see there is some alternating error over time, so we can see that at some time steps we high error rate and other time steps, we have small error rate. So the solutions only similar for some time steps.
As a second example, we can consider the problem given as

$$\begin{aligned}\rho(t)&=e^{-t}\\ f(t)&=0\\ g(t)&=0\\ \varphi(x)&=x(1-x)\end{aligned}$$

We will also continue to use $l=1$ for simplicity. So we now have the modified problem

$$\begin{aligned}u_t-u_{xx}&=e^{-t},\ 0<x<1,\ t>0\\ u(0,t)&=u(1,t)=0\\ u(x,0)&=x(1-x)\end{aligned}$$

We now have a problem with homogeneous boundary conditions. And so we can do the same as in the last problem, and we seek a solution on the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

which is a Fourier series in x, to get rid of the source term $\rho(x,t)$. We can further put this into the differential equation

$$\begin{aligned} u_t - u_{xx} &= e^{-t} \\ \sum_{n=1}^{\infty} a_n'(t)\sin(n\pi x) + \sum_{n=1}^{\infty} n^2\pi^2 a_n(t)\sin(n\pi x) &= e^{-t} \end{aligned}$$

Since we have a Fourier series in x as a solution on the left, then the same must be true on the right side. So further we can expand e^{-t} as a Fourier series

$$e^{-t} = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$$

and we can find the coefficients by

$$\begin{aligned} b-n(t) &= 2\int_0^1 e^{-t}\sin(n\pi x) \\ &= \frac{2e^{-t}}{n\pi}(1-(-1)^n) \\ &= \begin{cases} \frac{4e^{-t}}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

Since the coefficients must be equal, we get the differential equations

$$a_n' + n^2\pi^2 a_n = 0$$

and

$$a_n' + n^2\pi^2 a_n = \frac{4e^{-t}}{n\pi}$$

The first equation, for even n, gives

$$\begin{aligned} a_n' + n^2\pi^2 a_n &= 0 \\ \Rightarrow a_n(t) &= c_n e^{-n^2\pi^2 t} \end{aligned}$$

Which is the complementary solution for odd n. For odd n, we get

$$a_n' + n^2\pi^2 a_n = \frac{4}{n\pi}e^{-t}$$

We can solve this equation using the method of undetermined coefficients. We guess a particular solution of the form

$$a_n = Ce^{-t}$$

which we can insert into the equation

$$\begin{aligned} -Ce^{-t} + n^2\pi^2 Ce^{-t} &= \frac{4}{n\pi}e^{-t} \\ \Rightarrow (-1+n^2\pi^2)Ce^{-t} &= \frac{4}{n\pi}e^{-t} \end{aligned}$$

So

$$(-1+n^2\pi^2)C = \frac{4}{n\pi} \Rightarrow C = \frac{4}{n\pi(n^2\pi^2-1)}$$

So the coefficients are given by

$$a_n(t) = \frac{4e^{-t}}{n\pi(n^2\pi^2-1)} + c_n e^{-n^2\pi^2 t}$$

And so we have

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x) + \sum_{n \text{ odd}} \frac{4e^{-t}}{n\pi(n^2\pi^2-1)}\sin(n\pi x)$$

We can now consider the initial condition, so

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) + \sum_{n \text{ odd}} \frac{4}{n\pi(n^2\pi^2-1)}\sin(n\pi x) = x(1-x)$$

We can now find the Fourier series for $x(1-x)$

$$x(1-x) = \sum_{n \text{ odd}} \frac{8}{n^3\pi^3}\sin(n\pi x)$$

So further we have

$$\begin{aligned} c_n + \frac{4}{n\pi(n^2\pi^2-1)} &= \frac{8}{n^3\pi^3} \\ c_n &= \frac{4(n^2\pi^2-2)}{n^3\pi^3(n^2\pi^2-1)} \end{aligned}$$

So the solution, using $2k+1$ for n since we only have odd terms.

$$\boxed{u(x,t) = \sum_{k=1}^{\infty}\left[\frac{4[(2k+1)^2\pi^2-2]}{(2k+1)^3\pi^3[(2k+1)^2\pi^2-1]}e^{-(2k+1)^2\pi^2 t} + \frac{4e^{-t}}{(2k+1)\pi[(2k+1)^2\pi^2-1]}\right]\sin([2k+1]\pi x)}$$

We can do the same here as we did in the first problem, and compare the Fourier solution and the finite difference solution to the same problem. We can again use RMSE for each time step and plot the evolution over time.

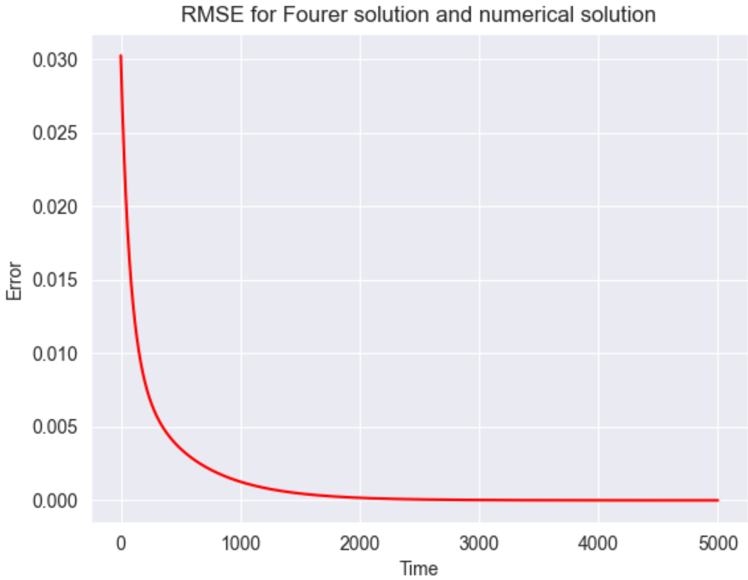

Figure 2: *RMSE for the second Fourier series and the finite difference method solution of the same problem.* $s = 0.10$ *for this plot*

From the figure above, we can see that the error starts relative low and has an exponential drop-off as the solutions evolve in time. After about 2000 time steps, then the error is approximately zero and the solutions are close to or in fact similar.

Problem 2

For the second part of this project, we are asked to consider the in-homogeneous wave equation analytically.

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L,\ t > 0 \\
u(x,0) = g(x),\ u_t(x,0) = h(x),\ 0 \le x \le L \\
u(0,t) = 0,\ u(L,t) = B,\ 0 \le \theta \le 2\pi,\ t > 0 \\
B \in \mathbb{R}
\end{aligned}
$$

a)

We are here asked to find the stationary (time independent) solution $u_0(x)$.

We can first define

$$w(x,t) = u(x,t) - u_0(x)$$

or

$$u(x,t) = w(x,t) + u_0(x)$$

$u_0(x)$ satisfy the boundary condition. So we can insert this into the differential equation

$$w_{tt} + \cancelto{0}{(u_0)_{tt}} - c^2(u_0)_{xx} = c^2 w_{xx}$$

Since we want

$$w_{tt} = c^2 w_{xx}$$

Then we need to have

$$-c^2(u_0)_{xx} = 0$$

or

$$(u_0)_{xx} = 0$$

with the boundary conditions $u_0(0) = 0$ and $u_0(L) = B,\ B \in \mathbb{R}$. The general solution for this ODE is

$$u_0(x) = C_1 x + C_2$$

Using the boundary conditions we can find the constants, thus

$$u_0(0) = C_1(0) + C_2 = \underline{C_2 = 0}$$

and

$$u_0(L) = C_1(L) = LC_1 = B \Rightarrow \underline{C_1 = \frac{B}{L}}$$

so the stationary solution is given by

$$\boxed{u_0(x) = \frac{B}{L}x}$$

b)

We are here asked to find the formal solution for $u(x,t)$.

In the first problem, we defined

$$w8x,t) = u(x,t) - u_0(x)$$

and we got

$$w_{tt} = c^2 w_{xx}$$

The boundary conditions for the transformed problem will now be

$$w(0,t) = w(L,t) = 0$$

with initial conditions

$$w(x,0) = u(x,0) - u_0(x) = \underline{g(x) - \frac{B}{L}x}$$

And

$$w_t(x,0) = u_t(x,0) - \cancel{(u_0)_t}^{\,0} = u_t(x,0) = \underline{h(x)}$$

We can solve this problem using separation of variables.

$$w(x,t) = N(t)M(x)$$

So we can put into the differential equation

$$N''(t)M(x) = c^2 N(t) M''(x)$$
$$\Rightarrow \frac{N''(t)}{c^2 N(t)} = \frac{M''(x)}{M(x)} = -\lambda$$

Which leads to two equations

$$N''(t) + c^2 \lambda N(t) = 0$$
$$M''(x) + \lambda M(x) = 0$$

The equation is of hyperbolic type, and thus the solution for the first equation is given by

$$N_n(t) = a_n \cos\left(c\frac{n\pi}{L}t\right) + b_n \sin\left(c\frac{n\pi}{L}t\right)$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ are the eigenvalues.
the second equation

$$M''(x) + \lambda M(x) = 0$$

wit boundary conditions

$$M(0) = M(L) = 0$$

is an eigenvalue problem with normalized solution

$$M_n(x) = \frac{2}{L}\sin\left(\frac{n\pi}{L}x\right)$$

So the solution is then

$$\begin{aligned} w_n &= N_n(t)M_n(x) \\ &= \left(a_n \cos\left(c\frac{n\pi}{L}t\right) + b_n \sin\left(c\frac{n\pi}{L}t\right)\right)\frac{2}{L}\sin\left(\frac{n\pi}{L}x\right) \end{aligned}$$

So a formal series solution is of the form

$$\underline{w(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(a_n\cos\left(c\frac{n\pi}{L}t\right) + b_n\sin\left(c\frac{n\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)}$$

where a_n and b_n are found by the initial conditions.

$$\begin{aligned} w(x,0) &= \sum_{n=1}^{\infty} a_n\frac{2}{L}\sin\left(\frac{n\pi}{L}x\right) \\ &= g(x) - \frac{B}{L}x \end{aligned}$$

and

$$\begin{aligned} w_t(x,0) &= \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)b_n\frac{2}{L}\sin\left(\frac{n\pi}{L}x\right) \\ &= h(x) \end{aligned}$$

So

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L\left(g(x) - \frac{B}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{2L}{n\pi c}\int_0^L h(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

We can now find the formal solution to the original problem. We have

$$u(x,t) = u_0(x) + w(x,t)$$

We can now insert the expression found for both the stationary solution and the transformed solution to obtain the solution to the in-homogeneous wave equation.

$$\boxed{u(x,t) = \frac{B}{L}x + \frac{2}{L}\sum_{n=1}^{\infty}\left(a_n\cos\left(c\frac{n\pi}{L}t\right) + b_n\sin\left(c\frac{n\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)}$$

c)

We are asked to establish whether $u(x,t) \to u_0$ as $t \to \infty$.