Game Theory

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Contents

1 Nash Equilibrium in Pure Actions

A game is triple (I,(A_i), (u_i)) where:

- I is a set of players
- A_i is the set of actions available to player i
- u_i is the payoff function for player i

For the moment, the functions u_i represent ordinal preferences over $\prod_{i \in I} A_i$.

Note that we are allowing arbitrary externalities.

Consider a game with N players. A strategy profile

$$\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$$

is a **Nash equilibrium** of the game if, for every player i,

$$u_i(s_i^*, \mathbf{s}_{-i}^*) \geq u_i(s_i', \mathbf{s}_{-i}^*)$$

Given a game $(I, (A_i)_{i \in I}, (u_i)_{i \in I})$, a Nash equilibrium of this game is an n-tuple of actions $(a_i^*)_{i \in I}$ such that for every player $i \in I$ and for every action $a_i \in A_i$:

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*),$$

where (a_i, a_{-i}^*) is the list of actions that is identical to a^* except that we have replaced a_i^* by a_i.

Interpretation: A Nash equilibrium represents a rest point of a learning process in which each player chooses the optimal action assuming that all other players choose the same actions as in the previous period.

Best Response: A strategy s_i is a best response to a strategy profile s_{-i} if it maximizes the payoff of player i given the strategies of the other players.

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i})$$

Best Response Correspondence: The best response correspondence of player i is the correspondence that assigns to each strategy profile s_{-i} the set of best responses of player i to s_{-i}.

1.1 Key points

1. Nash equilibria are fixed points of the best reply correspondence.

For every player $i \in I$ define a best reply correspondence $BR_i : \prod_{i \in I} A_i \rightrightarrows A_i$ by setting for every $a \in \prod_{i \in I} A_i$:

$$BR_i(a) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every } a_i' \in A_i\}.$$

The best reply correspondence $BR : \prod_{i \in I} A_i \rightrightarrows \prod_{i \in I} A_i$ is defined by setting for every $a \in \prod_{i \in I} A_i$:

$$BR(a) = \prod_{i \in I} BR_i(a).$$

By definition: a^* is a Nash equilibrium if and only if $a^* \in BR(a^*)$, i.e., if and only if a^* is a fixed point of BR.

2 Static Games

2.1 Case 1: Betrend Competition

In betrend competition, firm face a total cost curve for producing their goods and simply choose the price for their respective goods. Whichever firm has the lower price will get all the customers and if the prices are the same, the customers will be split evenly. We will assume that identical firms face a constant marginal cost c. The firms simultaneously choose their prices, p1 and p2. let's look at it from firm 1's perspective (it will be the same for firm firm 2).

Firm 1 has three strategies:

- $p_1 < p_2$: Firm 1 gets all the customers and makes a profit of $p_1 - c$.
- $p_1 = p_2$: Firm 1 gets half the customers and makes a profit of $\frac{p_1 - c}{2}$.
- $p_1 > p_2$: Firm 1 gets no customers and makes a profit of 0.

Naturally, depending on what firm 2's price is, we could have any of these situations. Let s look at some different conditions.

- Starting with one extreme, suppose $p_2 > p^m$, the monopoly price.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.
- Naturally, firm 1 would actually want to set $p_1 = p^m$ in this case since that's the price that maximizes their profit level.
- Now, let's suppose firm 2 chose some price that was at most the monopoly price, i.e., $p_2 \leq p^m$. The same results hold.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.
- It should be obvious that firm 1 wants to set a price lower than firm 2, $p_1 < p_2$. In fact, to maximize profit, firm 1 wants to undercut firm 2 by as little as possible (a single penny). $p_1 = p_2 - \varepsilon$ where $\varepsilon > 0$ is the smallest possible number that firm 1 can pick.
  - That way they get all of the customers while lowering their profit margin by as little as possible.
- Lastly, suppose firm 2's price were at or below marginal cost, $p_2 \leq c$.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.
- Pricing below marginal cost would not be an optimal strategy for firm 1. If $p_1 < c$, firm 1 would actually lose money on each unit sold.

  - Thus, the lowest (and only possible) price firm 1 is willing to charge is $p_1 = c$.

- The analysis for firm 2 is identical.
  - If firm 1 prices above the monopoly price, $p_2 = p^m$.
  - If firm 1 prices between marginal cost and the monopoly price, $p_2 = p_1 - \varepsilon$, where $\varepsilon > 0$ is the smallest possible number greater than zero.
  - If firm 1 prices at or below marginal cost, $p_2 = c$.

- Returning to our Nash equilibrium solution concept, we know that our equilibrium is when neither firm has any incentive to deviate from their chosen strategy.
  - This occurs where the best response functions intersect.

- There is exactly one intersection point in the previous figure, where $p_1 = p_2 = c$. Thus, both firms price at marginal cost in equilibrium.
  - This should make sense. Each firm wants to undercut the other to claim the whole market. Yet they can't undercut anymore once the price is at marginal cost or they'll suffer losses.
  - Bertrand competition implies that with just two firms, we reach the perfectly competitive equilibrium.

- Since price is set at marginal cost for both firms, economic profit under Bertrand competition equals zero.

2.2 Case 2: Cournet Competition

2.3 Case 3: Stackelberg Competition

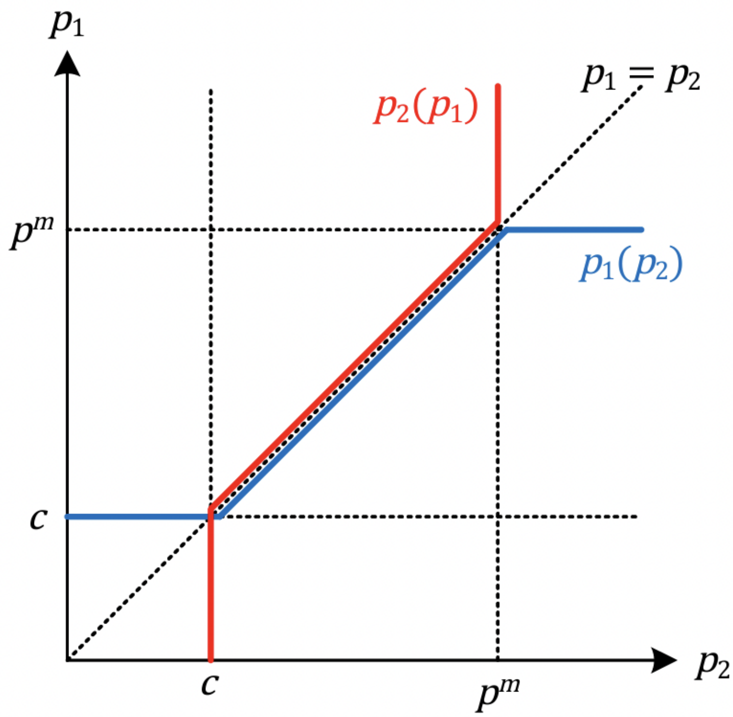


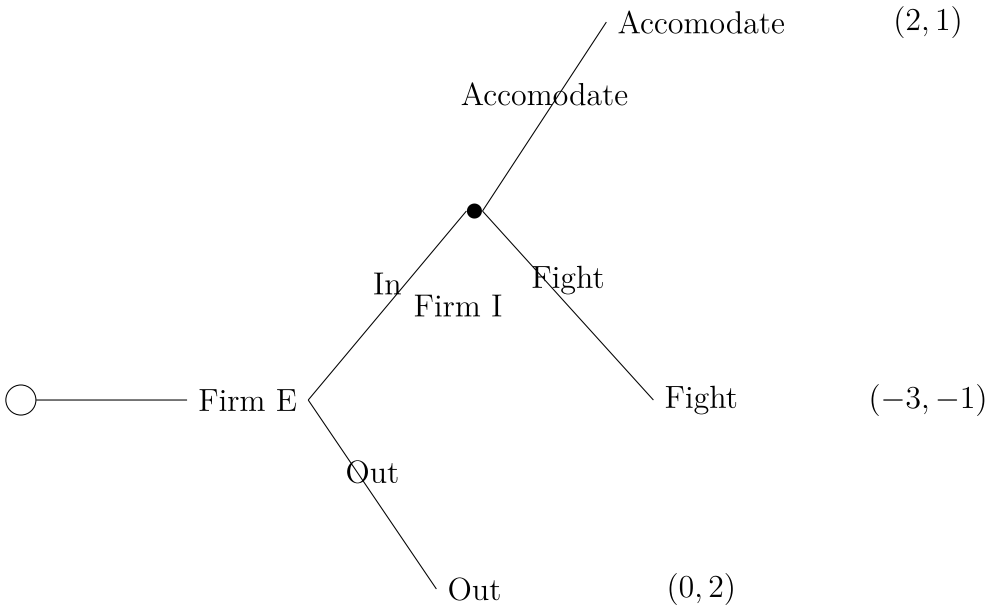
Figure 1: EQ

3 Nash Equilibria in Mixed Actions

4 Commplete Information Dynamic Games

4.1 Entry Game

A firm called Firm E is considering entering a market that currently has a single incumbent Firm I. If it chooses "in", the incumbent can respond in one of the two ways: It can either accomodate the entrant, of fight the entrant. We use the following tree to represent the whole process.



5 Incomplete Information

A game has (is of) incomplete information when at least one player does not know the payoff that some player receives from some strategy profile (or terminal nodes).

In this section, we will consider simultaneously-move games with incomplete information.

As an example, consider the following game. This is a modified Prisoner's dilemma. Prisoner 1 is not clear about the crime she committed so she doesn't know which of the following games is played.

Prisoner 2, on the other, has more knowledge about law so he knows which game is played.

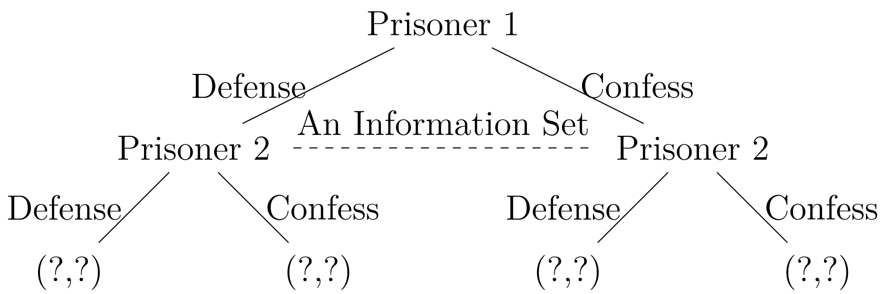
		Prisoner 2	
		Defense	Confess
Prisoner 1	Defense	-2,-2	-10,-1
	Confess	-1,-10	-5,-5

Good

		Prisoner 2	
		Defense	Confess
Prisoner 1	Defense	-5,-5	-13,-4
	Confess	-4,-13	-8,-8

Bad

Let's use a game tree to represent this. In the eyes of Prisoner 1, this game is like this:



5.1 Types and strategies

Dealing with games with incomplete information is difficult. But Harsany came up with a solution to address this problem. His idea is to transform any game of incomplete information into a game with complete but imperfect information.

When doing so, we suppose the nature randomly chooses a "type" for each player. Each player knows only her own type, but not other players' type. The types of all players influence the payoff of each player.

For example, after introducing the nature and type, then we can say Prison 2 has 2 types: Good or Bad. Prisoner 2's type not only influences his own payoff, it also influences the payoff of Prisoner 1. The game tree now looks like the following.

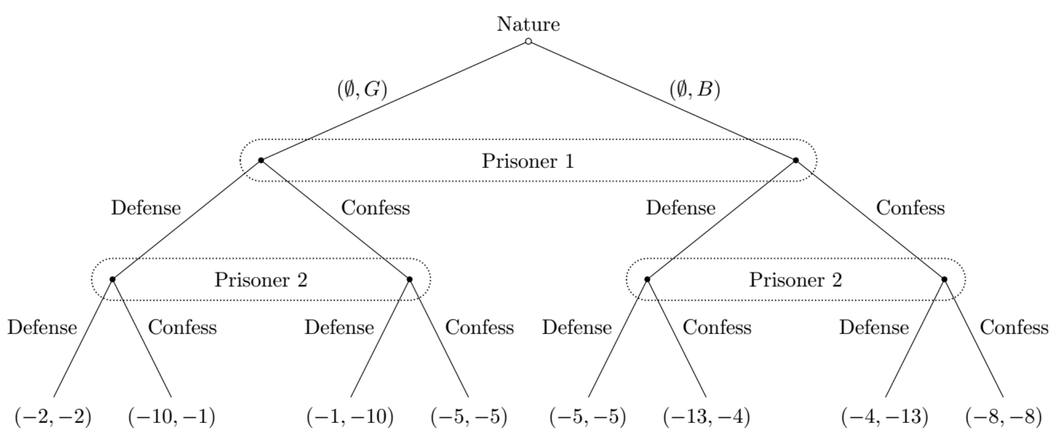


Figure 2: Incomplete Information Game Tree

Prisoner 1 cannot distinguish the two decision nodes because she cannot observe the type of Prisoner 2. Prisoner 2 has 2 types, and he can observe his own type, so he has 2 information sets.

We can still define a strategy as a function that maps an information set to a distribution over actions. Now an information set just represents a type of a player. Here, Prisoner 1 has only one type, so she has only one information set.

Thus, we can also define a strategy as a function from a type to a distribution over action.

More formally, a static game of incomplete information can be described by:

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}]$$

such that:

1. I: a set of players. We assume there are n players.
2. $\Delta(A_i)$: the set of mixed action of player i.
3. Θ_i: a set of types of player i. We define $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$ and (θ_i, θ_{-i}) to define a type profile, where $\theta_{-i} = (\theta_j)_{j\neq i}$.
4. $u_i : \prod_{j=1}^n \Delta(A_i) \times \prod_{j=1}^n \Theta_j \to \mathbb{R}$: the utility function of player i, which depends on the action profile and players' types. We use $u_i(\alpha_1, \alpha_2, \ldots, \alpha_n, \theta_1, \theta_2, \ldots, \theta_n)$ to denote the expected payoff of player i when the action profile is $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and the type profile is $(\theta_1, \theta_2, \ldots, \theta_n)$.

In the above example, $I = \{1, 2\}$, $A_i = \{D, C\}$, $\Theta_1 = \{\emptyset\}$ and $\Theta_2 = \{G, B\}$. The utility function of P1 is such that $u_1(D, D, \emptyset, G) = -2$ and so on.

A pure strategy of player i is a function $s_i : \Theta_i \to A_i$ such that $s_i(\theta)$ is the action chosen by player i of type $\theta \in \Theta_i$.

A mixed strategy of player i is a function $\sigma_i : \Theta_i \to \Delta(A_i)$ such that $\sigma_i(a|\theta)$ is the probability that action $a \in A_i$ is chosen by player i of type $\theta \in \Theta_i$. If A_i has m elements, we can also use $\sigma_i(\theta) = (p_1, p_2, \ldots, p_m)$ to represent a strategy of player i of type θ.

So for Player 1, $s_1(\emptyset) = D$ is a pure strategy. For Player 2, $\sigma_2(G) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\sigma_2(B) = (1, 0)$ is a mixed strategy.

5.2 Dominant and Dominated Strategies

The idea of dominant and dominated strategies can be extended to an incomplete information game. We give the following definitions.

Definition 1. In an incomplete information game

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}],$$

a strategy σ_i is said to strictly dominate σ_i' if for all $(\theta_i, \theta_{-i}) \in \Theta$ and all $a_{-i} \in \prod_{j\neq i} A_j$,

$$u_i(\sigma_i(\theta_i), a_{-i}, \theta_i, \theta_{-i}) > u_i(\sigma_i'(\theta_i), a_{-i}, \theta_i, \theta_{-i}).$$

The strategy σ_i is strictly dominant if it strictly dominates every other strategy α_i'.

A strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a strictly dominant strategy equilibrium if σ_i is a strictly dominant strategy for all i.

In the previous example, regardless of the type of Player 2, for both players, Confess is a strictly dominant strategy so we can predict that both players of any type choose Confess.

The above condition is too strong. We also state a weaker one as below.

Definition 2. In an incomplete information game

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}],$$

a strategy σ_i is said to weakly dominate σ_i' if for all $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$ and all $a_{-i} \in \prod_{j\neq i} A_j$,

$$u_i(\sigma_i(\theta_i), a_{-i}, \theta_i, \theta_{-i}) \geq u_i(\sigma_i'(\theta_i), a_{-i}, \theta_i, \theta_{-i})$$

with strictly inequality for some a_{-i} and (θ_i, θ_{-i}).

The strategy σ_i is weakly dominant if it weakly dominates every other strategy α_i'.

A strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a weakly dominant strategy equilibrium if σ_i is a weakly dominant strategy for all i.

5.3 Beliefs and Bayesian Games

Now we change the payoff in the previous example a little bit. We still assume Player 1 does not know which game is played and thus is of one type, while Player 2 knows which game is played and has 2 types.

		Prisoner 2	
		Defense	Confess
Prisoner 1	Defense	-2,-2	-10,-1
	Confess	-1,-10	-5,-5

Good

		Prisoner 2	
		Defense	Confess
Prisoner 1	Defense	-4,-5	-8,-4
	Confess	-5,-13	-13,-8

Bad

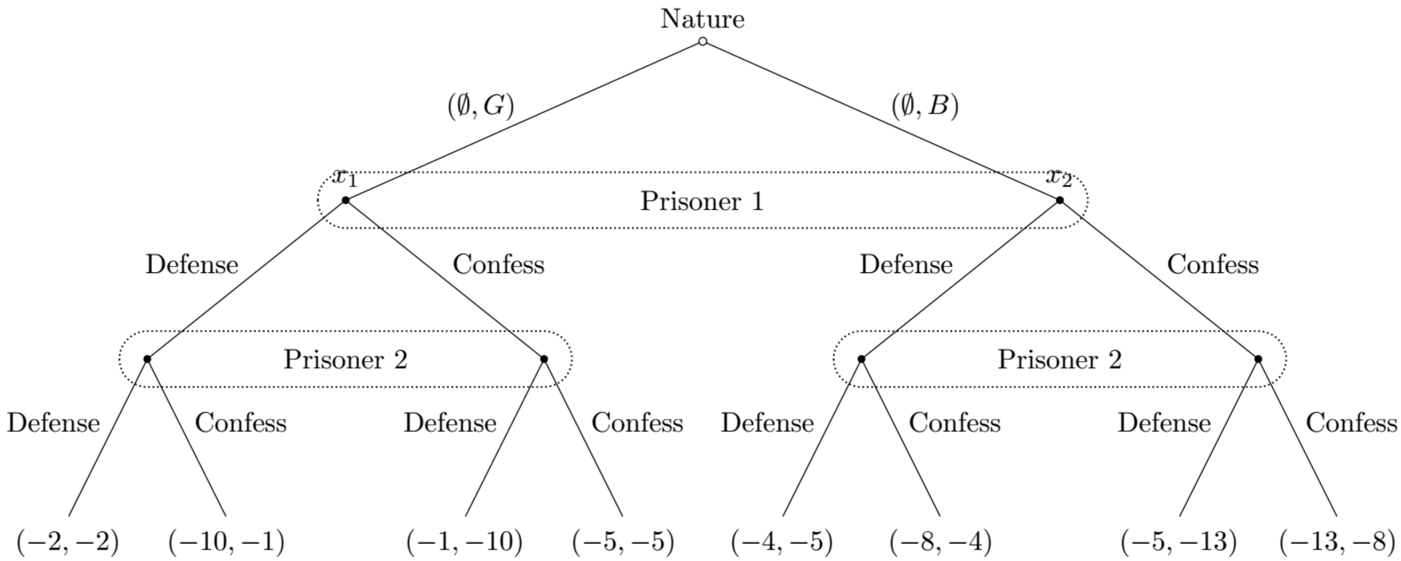
Prisoner 1 has no dominant strategy now. If Prisoner 2's type is Good, it is optimal for Player 1 to choose Confess. If his type is Bad, it is optimal to choose Defense.

We want to have an incomplete information game version of Nash equilibrium, which we can guarantee its existence and it is an extension of the previous concepts. To do so, we will have to specify how Prisoner 1 thinks of Prisoner 2's type. Or how Prisoner 1 assigns probabilities to the two decision nodes in her information set. We introduce the following concept.

Definition 3. In an extensive form game Γ_E, a belief system $\mu : N \to [0, 1]$ assigns probability to each decision node such that for any information set $h \in H$,

$$\sum_{x\in h} \mu(x) = 1$$

In our example, we need to specify $\mu(x_1)$ and $\mu(x_2)$, the probabilities that Prisoner 1 assigns to the 2 decision nodes in her information set.



To specify the beliefs of players, we usually assume the incomplete information games we consider are Bayesian games. In addition to what constitutes an incomplete information game, in a Bayesian game, nature assigns probabilities to type profiles of the players.

A Bayesian game is an incomplete information game with a probability distribution p over type profiles $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. We assume this distribution p is a common knowledge.

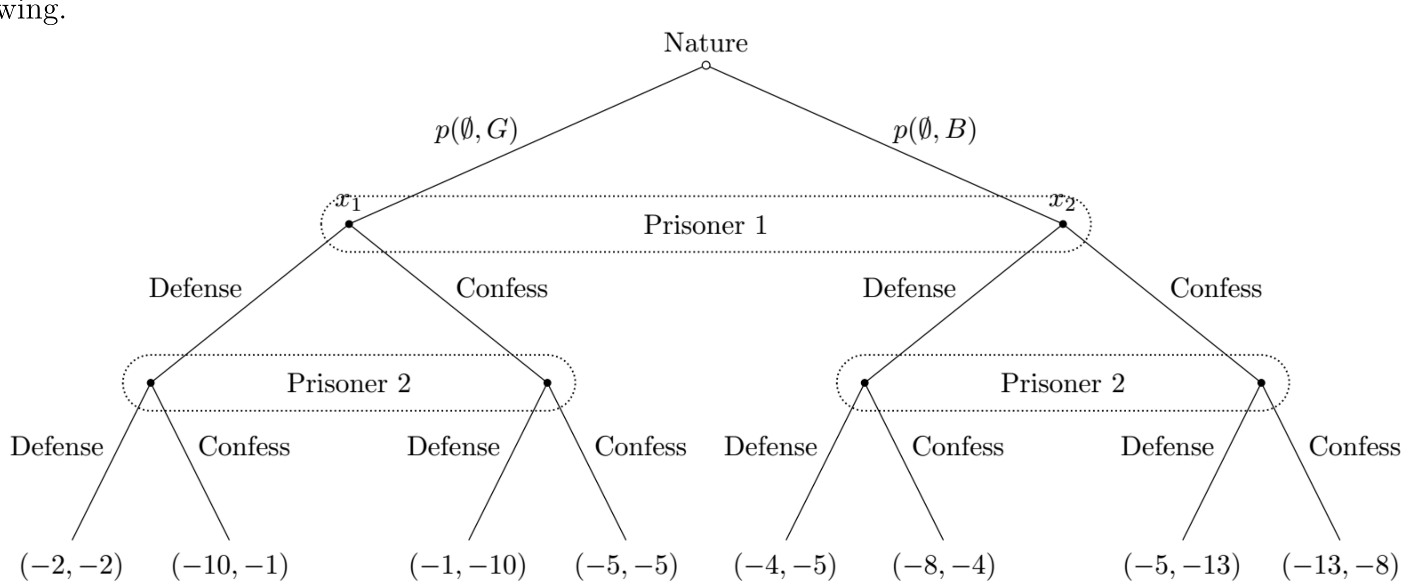
When each player has finite types, we use a probability mass function p such that $p(\theta_1, \theta_2, \ldots, \theta_n)$ is the probability the nature assigns to type profile $(\theta_1, \theta_2, \ldots, \theta_n)$. When types are real numbers, p is the probability density function.

We use

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}, p(\cdot)]$$

to represent a Bayesian game.

In the previous example, we can assume the nature assigns probability $p(\emptyset, G)$ to type profile $(\emptyset, G)$, and assigns probability $p(\emptyset, B)$ to type profile $(\emptyset, B)$. The game tree thus looks like the following.



p is called the prior belief in the sense that players know this even without knowing their own types. But after observing θ_i, player i will update her belief using Bayes rule. She will assign belief to (θ_i, θ_{-i}) conditional on θ_i. In notation, that is

$$p(\theta_i, \theta_{-i}|\theta_i) = \frac{p(\theta_i, \theta_{-i})}{\sum_{\theta'_{-i}} p(\theta_i, \theta'_{-i})}$$

This is called the posterior belief of player i of type θ_i.

Now we update the belief of Prisoner 1. This is trivial because Prisoner 1 has only one type.

$$p(\emptyset, B|\emptyset) = \frac{p(\emptyset, B)}{p(\emptyset, B) + p(\emptyset, G)} = p(\emptyset, B)$$

This is the probability that Prisoner 1 assigns to x_2 in her information set.

$$\mu(x_2) = p(\emptyset, B)$$

We call $p(\cdot)$ the prior belief, which is the belief of a player when she has not observed her type yet. We call $p(\cdot|\theta_i)$ the posterior belief of player i, which is the belief of her after she observed her type.

5.4 Bayesian Nash Equilibrium

5.5 Ex-Ante Optimization

We know what a strategy is in a Bayesian game. If we define the utility properly, we can directly use the definition of Nash equilibrium here by just changing some notations. Recall the definition of Nash equilibrium.

Definition 4. A strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ constitutes a Nash equilibrium of

$$\Gamma_N = [I, \{\Delta(S_i)\}_{i \in I}, \{u_i(\cdot)\}_{i \in I}]$$

if for every $i = 1, \ldots, n$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$$

for all $\sigma'_i \in \Delta(S_i)$.

We now extend this to include types. Firstly, the strategy and the payoff should now be relevant to the type profile. Thus, we need to replace $u_i(\sigma_i, \sigma_{-i})$ by

$$u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

*Note: my payoff is related to my strategies, type and others' types. Other players' payoffs are similarly defined.

If we use this in the definition, it will become the ex-post equilibrium. And we know that ex-post equilibrium may not exist. Let's now think about the probability of the above type profile, that will be

$$p(\theta_i, \theta_{-i})$$

when there are finite or countably infinite types, or

$$p(\theta_i, \theta_{-i})d\theta$$

when there are uncountably infinite types.

With the probability, we can now define the expected payoff of a player i when players choose strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

$$\sum_{\theta \in \Theta} p(\theta_i, \theta_{-i}) u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

or

$$\int_{\theta \in \Theta} p(\theta_i, \theta_{-i}) u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) d\theta$$

Both of them will be denoted as

$$E[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})]$$

Then, we have done all our preparations. We can now define the Nash equilibrium in a Bayesian game, which is called the Bayesian Nash equilibrium.

Definition 5. A strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ constitutes a Bayesian Nash equilibrium of game

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i \in I}, \{u_i(\cdot)\}_{i \in I}, \{\Theta_i\}_{i \in I}, p(\cdot)]$$

if for every $i = 1, \ldots, n$,

$$E[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})] \geq E[u_i(\sigma_i'(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})]$$

for all σ_i'.

*NOTE: I optimize my strategy given my different type, others' strategies and types. Other players do the same. I don't know my type, but I know the probability of my type. I optimize my strategy given this probability.

This is called ex-ante because in the above definition, a player has not observed her own type. You can imagine that she is choosing what (mixed) action she is going to take when she knows her type. And she makes the plans for each possible type.

From the definition, we can see that a weakly dominant strategy equilibrium is a Bayesian Nash equilibrium. One can see that by multiplying both sides of the following inequality by a probability and then adding over all possible strategy profiles.

$$u_i(\sigma_i(\theta_i), a_{-i}, \theta_i, \theta_{-i}) \geq u_i(\sigma_i'(\theta_i), a_{-i}, \theta_i, \theta_{-i})$$

Now come back to the discussion of our example, suppose $p(\emptyset, G) = \frac{1}{3}$ and $p(\emptyset, B) = \frac{2}{3}$.

We first find out whether there is a pure-strategy Bayesian Nash equilibrium such that each player of each type always chooses a pure action. P1 has two pure strategies: D and C. P2 has 4 pure strategies: (D,D), (D,C), (C,D), and (D,D). In each tuple, the first element represents his action when his type is G and the second element represents his action when his type is B. Or we can explain them using the game tree.

We can now rewrite the game in the following normal form, where each row represents a pure strategy of P1 and each column represents a pure strategy of P2.

Now we need to decide the payoffs. Notice that even though we are just considering the pure strategies of players, we have to consider expected payoff because we don't know what types the nature choose.

As an example, consider the case where P1 chooses D, and P2 chooses (C,D). With probability $p(\emptyset, G) = \frac{1}{3}$, the outcome will be (D,C), which yields payoffs (-10,-1). With probability $p(\emptyset, B) = \frac{2}{3}$,

		P2		
	(D,D)	(D,C)	(C,D)	(C,C)
P1				
D				
C				

the outcome will be (D,D), which yields payoffs (-4,-5). Thus, the expected payoffs should be

$$\left(-6, -\frac{11}{3}\right).$$

Let's do this for all other pure strategy profiles.

For example, P1 : D , P2 : (C,D)

$\frac{1}{3}$: G : (D,C) $\rightarrow$ (-10,-1)

$\frac{2}{3}$: B : (D,D) $\rightarrow$ (-4,-5)

(D, (C,D)) $\rightarrow$ $\frac{1}{3}$(-10,-1) + $\frac{2}{3}$(-4,-5) = (-6, -

P1	**P2**	$(\emptyset, G)$	$p(\emptyset, G)$	$(\emptyset, B)$	$p(\emptyset, B)$		**Payoffs**
D	(D,D)	(-2,-2)	$\frac{1}{3}$	$\left(-\frac{2}{3}, -\frac{2}{3}\right)$	(-4,-5)	$\left(-\frac{8}{3}, -\frac{10}{3}\right)$	$\left(-\frac{10}{3}, -4\right)$
	(D,C)	(-2,-2)	$\frac{1}{3}$	$\left(-\frac{2}{3}, -\frac{2}{3}\right)$	(-8,-4)	$\left(-\frac{16}{3}, -\frac{8}{3}\right)$	$\left(-6, -\frac{10}{3}\right)$
	(C,D)	(-10,-1)	$\frac{1}{3}$	$\left(-\frac{10}{3}, -\frac{1}{3}\right)$	(-4,-5)	$\left(-\frac{8}{3}, -\frac{10}{3}\right)$	$\left(-6, -\frac{11}{3}\right)$
	(C,C)	(-10,-1)	$\frac{1}{3}$	$\left(-\frac{10}{3}, -\frac{1}{3}\right)$	(-8,-4)	$\left(-\frac{16}{3}, -\frac{8}{3}\right)$	$\left(-\frac{26}{3}, -3\right)$
C	(D,D)	(-1,-10)	$\frac{1}{3}$	$\left(-\frac{1}{3}, -\frac{10}{3}\right)$	(-5,-13)	$\left(-\frac{10}{3}, -\frac{26}{3}\right)$	$\left(-\frac{11}{3}, -12\right)$
	(D,C)	(-1,-10)	$\frac{1}{3}$	$\left(-\frac{1}{3}, -\frac{10}{3}\right)$	(-13,-8)	$\left(-\frac{26}{3}, -\frac{16}{3}\right)$	$\left(-9, -\frac{26}{3}\right)$
	(C,D)	(-5,-5)	$\frac{1}{3}$	$\left(-\frac{5}{3}, -\frac{5}{3}\right)$	(-5,-13)	$\left(-\frac{10}{3}, -\frac{26}{3}\right)$	$\left(-5, \frac{31}{3}\right)$
	(C,C)	(-5,-5)	$\frac{1}{3}$	$\left(-\frac{5}{3}, -\frac{5}{3}\right)$	(-13,-8)	$\left(-\frac{26}{3}, -\frac{16}{3}\right)$	$\left(-\frac{31}{3}, -7\right)$

Then we can fill in the table.

If we successfully find the above matrix, we can now forget about the fact that we are dealing

		P2		
	(D,D)	(D,C)	(C,D)	(C,C)
D	$\left(-\frac{10}{3}, -4\right)$	$(-6, -\frac{10}{3})$	$(-6, -\frac{11}{3})$	$\left(-\frac{26}{3}, -3\right)$
C	$\left(-\frac{11}{3}, -12\right)$	$(-9, -\frac{26}{3})$	$(-5, -\frac{31}{3})$	$\left(-\frac{31}{3}, -7\right)$

with a Bayesian game. We can treat the above game as a complete information game and find the equilibrium in the usual way.

5.6 Optimization by 'Type' Players

Another equivalent definition of Bayesian Nash equilibrium assumes players observe their types. Then they update their beliefs about others' types and then choose the optimal action given others' strategies.

After observing her own type θ_i, player i's expected payoff from the strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ can be written as

$$\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i, \theta_{-i}|\theta_i) u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

or

$$\int_{\theta_{-i} \in \Theta_{-i}} p(\theta_i, \theta_{-i}|\theta_i) u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) d\theta_{-i}$$

Both of them will be denoted as

$$E[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})|\theta_i]$$

Then we can define the Bayesian Nash equilibrium as the following.

Definition 6. A strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ constitutes a Bayesian Nash equilibrium of game

$$\Gamma_N = [I, \{\Delta(A_i)\}_{i \in I}, \{u_i(\cdot)\}_{i \in I}, \{\Theta_i\}_{i \in I}, p(\cdot)]$$

if for every type θ_i of player $i = 1, \ldots, n$,

$$E[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})|\theta_i] \geq E[u_i(a_i', \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})|\theta_i]$$

for all $a_i' \in A_i$.

It is easy to check that this definition is equivalent to the last definition.

The good thing about using this definition is that we can treat each type of a player as a "type" player. The reason is that the action of a type of a player has no influence on the payoff of a different type of the same player.

So, in our example, we can imagine there are 3 players. P1, P2 of type G, and P2 of type B. Then the game can be thought of as a three-player game, with P1 being the row player, P2 of type G being the column player, and P2 of type B being the matrix player.

5.7 Application

5.7.1 Cournot Competition with Incomplete Information

Consider the Cournot competition with two firms we have talked about. For simplicity, assume that the inverse demand function is given by the following.

$$p(q_1, q_2) = 2 - (q_1 + q_2)$$

That is, when the total output is too large, the price could be negative and the firms still have to produce and get a negative profit.

Suppose firm 1's marginal cost is 1. On the other hand, firm 2's marginal cost could be either $\frac{3}{4}$ or $\frac{5}{4}$, with equal probabilities.

This is a Bayesian game where firm 1 has one type and firm 2 has two types. We treat the two types of firm 2 as different players. The pure strategies of the players can be denoted as q_1, $q_2(L)$, and $q_2(H)$, respectively.

Firm 1 solves the following problem.

$$\max_{q_1} \frac{1}{2}\left[(2 - q_1 - q_2(L))q_1 - q_1\right] + \frac{1}{2}\left[(2 - q_1 - q_2(H))q_1 - q_1\right]$$

=

$$\max_{q_1} \left(1 - q_1 - \left(\frac{1}{2}q_2(L) + \frac{1}{2}q_2(H)\right)\right) q_1$$

Firm 2 with low cost solves the following problem.

$$\max_{q_2(L)} \left(\frac{5}{4} - q_1 - q_2(L)\right) q_2(L)$$

Firm 2 with high cost solves the following problem.

$$\max_{q_2(H)} \left(\frac{3}{4} - q_1 - q_2(H)\right) q_2(H)$$

We can have the following 3 FOCs:

$$1 - 2q_1 - \left(\frac{1}{2}q_2(L) + \frac{1}{2}q_2(H)\right) = 0$$

$$\frac{5}{4} - q_1 - 2q_2(L) = 0$$

$$\frac{3}{4} - q_1 - 2q_2(H) = 0$$

Solving the above, we have $q_1 = \frac{1}{3}$, $q_2(L) = \frac{11}{24}$, and $q_2(H) = \frac{5}{24}$.

5.7.2 Bertrand with Incomplete Information

Exercise: Betrend Competition

Consider the following asymmetric-information model of Bertrand duopoly with differentiated products. Demand for firm i is

$$q_i(p_i, p_j) = a - p_i - b_i \cdot p_j.$$

Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L, where $b_H > b_L > 0$. For each firm, $b_i = b_H$ with probability θ and $b_i = b_L$ with probability $1 - \theta$, independent of the realization of b_j.

Each firm knows its own b_i but not its competitor's. All of this is common knowledge. Solve for the Bayesian Nash equilibrium.

Solution: We look for a symmetric pure strategy Bayesian Nash equilibrium:

$$p^* : \{b_H, b_L\} \to \{p_H^*, p_L^*\}.$$

So that each firm chooses p_H^* if $b_i = b_H$ and p_L^* if $b_i = b_L$.

Given that firm 2 follows such BNE, firm 1 with b_i solves the following maximization problem:

$$\max_{p_i} \mathbb{E}_{b_j}[p_i(a - p_i - b_i p^*(b_j))]$$

$$= p_i[a - p_i - b_i(\theta p_H^* + (1-\theta)p_L^*)].$$

F.O.C is

$$a - p_i - b_i(\theta p_H^* + (1-\theta)p_L^*) - p_i = 0 \Rightarrow$$

$$p_i^* = \frac{1}{2}[a - b_i(\theta p_H^* + (1-\theta)p_L^*)].$$

For p^* to constitute a symmetric BNE, we must have $p_i^* = p_H^*$ if $b_i = b_H$ and $p_i^* = p_L^*$ if $b_i = b_L$. Hence the F.O.C. implies

$$p_H^* = \frac{1}{2}[a - b_H(\theta p_H^* + (1-\theta)p_L^*)]$$

$$p_L^* = \frac{1}{2}[a - b_L(\theta p_H^* + (1-\theta)p_L^*)].$$

$$a = (2 + \theta b_H)p_H^* + (1-\theta)b_H p_L^*$$

$$a = \theta b_L p_H^* + (2 + (1-\theta)b_L)p_L^*$$

The above two conditions characterize a symmetric pure strategy BNE.

We can easily show that

$$p_H^* = \frac{[2 - (1-\theta)(b_H - b_L)]}{[2 + \theta(b_H - b_L)]} p_L^*.$$

Substitute this into the equilibrium conditions, then get,

$$p_H^* = \frac{a[2 - (1-\theta)(b_H - b_L)]}{2[2 + \theta b_H + (1-\theta)b_L]}$$

$$p_L^* = \frac{a[2+\theta(b_H - b_L)]}{2[2+\theta b_H + (1-\theta)b_L]}$$

(which are non-negative if $2-(1-\theta)(b_H - b_L) \geq 0$ or equivalently $b_L + \frac{2}{1-\theta} \geq b_H$).

5.7.3 Auction with Incomplete Information

We will consider an important application of incomplete information static games.

(1) Independent Private Values

A seller is going to sell a good. There are n potential buyers. Let θ_i be the valuation of buyer i, which is the maximum amount buyer i is willing to pay for the object. Each buyer knows her own valuation, but not other buyers' valuations.

Suppose $\theta_1, \theta_2, \ldots, \theta_n$ are independently and identically distributed on interval $[0, \bar{\theta}]$ according to a distribution function F with probability density function $f = F'$. This is a common knowledge among all buyers.

This is called the independent private values (IPV) setting, where each buyer's value of the good is not related to other buyers' values. We will focus on this setting.

We will consider two types of auctions through which the seller sells the good. We will call the buyers bidders from now on.

(2) Second-Price Sealed-Bid Auction

In a second-price sealed-bid auction, each bidder writes down his bid and places it in an envelope. Then the envelopes are opened simultaneously. The highest bidder wins the auction and pays a price equal to the second-highest bid or the highest-losing bid. Assume that if there is a tie, no one wins.

Let b_i denote the bid of bidder i. Then we can write down the payoff of bidder i as the following.

$$u_i(b_i, b_{-i}, \theta_i) = \begin{cases} \theta_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i \leq \max_{j \neq i} b_j \end{cases}$$

We don't include other bidders' valuations in the payoff of bidder i because they do not influence bidder i's payoff.

A pure strategy of bidder i is a function $s_i : [0, \bar{\theta}] \to [0, \infty)$, such that $s_i(\theta_i)$ is the bid of bidder i when his valuation is θ_i.

A weakly dominant strategy of bidder i is $s_i(\theta_i) = \theta_i$, i.e., each bidder bids her own valuation. To see this, first consider the payoff of bidder i when she bids $b_i > \theta_i$ instead. Let b_j be the highest bid of other players.

1. If $b_j < b_i$, buyer i still wins and pays the same amount b_j. Her payoff does not change.
2. If $b_i > b_j < \theta_i$, buyer i loses. Her payoff decreases from $\theta_i - b_j$ to 0.
3. If $b_j \geq \theta_i$, buyer i still loses. Her payoff does not change.

Next, consider the payoff of bidder i when she bids $b_i > \theta_i$ instead. Let b_j be the highest bid of other players.

1. If $b_j < \theta_i$, buyer i still wins and pays the same amount b_j. Her payoff does not change.
2. If $\theta_i \leq b_j < b_i$, buyer i wins. Her payoff decreases from 0 to $\theta_i - b_j$.

3. If $b_j \geq b_i$, buyer i still loses. Her payoff does not change.

Then, we can conclude that there is a weakly dominant strategy equilibrium in the second-price sealed-bid auction, where each bidder bids her own valuation.

(3) First Price Sealed-Bid Auction

In a second-price sealed-bid auction, each bidder writes down his bid and places it in an envelope. Then the envelopes are opened simultaneously. The highest bidder wins the auction and pays her bid. Assume that if there is a tie, no one wins.

Let b_i denote the bid of bidder i. Then we can write down the payoff of bidder i as the following:

$$u_i(b_i, b_{-i}, \theta_i) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i \leq \max_{j \neq i} b_j \end{cases}$$

Let's find a Nash equilibrium of this Bayesian game. We focus on the strategy profile $(s_1, s_2, \ldots, s_n)$ such that $s_1 = s_2 = \cdots = s_n$ and $s_i(\cdot)$ is a strictly increasing function with $s_i(0) = 0$ for all i.

$s_i(\cdot)$ is an increasing function because s_i is increasing with θ_i

This is to say that we focus on symmetric and increasing pure strategies of bidders.

Without loss, consider bidder 1 of type θ_i. If she bids b_1, her expected payoff is:

$$Pr(b_1 > s_2(\theta_2)) \times Pr(b_1 > s_3(\theta_3)) \times \cdots \times Pr(b_1 > s_n(\theta_n)) \times (\theta_1 - b_1)$$

Here, $Pr(b_1 > s_i(\theta_i))$ is the probability that b_1 is higher than what bidder i bids. Since bidders' valuations are independent, the probability that b_1 wins is

$$Pr(b_1 > s_2(\theta_2)) \times Pr(b_1 > s_3(\theta_3)) \times \cdots \times Pr(b_1 > s_n(\theta_n)).$$

Now we use the assumption that $s_2 = \cdots = s_n$ and $s_i(\cdot)$ is strictly increasing. Let's say $\beta = s_2 = \cdots = s_n$. Then there exists z such that $\beta(z) = b_i$. Rewrite the expected payoff:

$$Pr(\beta(z) > \beta(\theta_2)) \times Pr(\beta(z) > \beta(\theta_3)) \times \cdots \times Pr(\beta(z) > \beta(\theta_n)) \times (\theta_1 - \beta(z))$$

$$= Pr(z > \theta_2) \times Pr(z > \theta_3) \times \cdots \times Pr(z > \theta_n) \times (\theta_1 - \beta(z))$$

$$= [F(z)]^{n-1} \times (\theta_1 - \beta(z))$$

The first equality uses the assumption that β is strictly increasing. The second equality is the result of identical distribution.

Notice that bidding b_1 is equivalent to choosing z now. Because from b_1, we can use β to uniquely pin down z. Now take the first order condition of the above with respect to z:

$$(n-1)[F(z)]^{n-2} f(z) \times (\theta_1 - \beta(z)) - [F(z)]^{n-1} \beta'(z) = 0$$

What value of z satisfies the above FOC ? Since we are considering an equilibrium strategy, bidding $\beta(\theta_1)$ should be optimal. Therefore, $z = \theta_1$ satisfies the above FOC.

$$(n-1)[F(\theta_1)]^{n-2} f(\theta_1) \times (\theta_1 - \beta(\theta_1)) - [F(\theta_1)]^{n-1} \beta'(\theta_1) = 0$$

$$(n-1)[F(\theta_1)]^{n-2} f(\theta_1) \beta(\theta_1) + [F(\theta_1)]^{n-1} \beta'(\theta_1) = (n-1)[F(\theta_1)]^{n-2} f(\theta_1) \theta_1$$

$$\frac{d}{d\theta_1}([F(\theta_1)]^{n-1}\beta(\theta_1)) = (n-1)[F(\theta_1)]^{n-2}f(\theta_1)\theta_1$$

Let's first replace θ_1 by t.

$$\frac{d}{dt}([F(t)]^{n-1}\beta(t)) = (n-1)[F(t)]^{n-2}f(t)t$$

$$d([F(t)]^{n-1}\beta(t)) = (n-1)[F(t)]^{n-2}f(t)tdt$$

Then integrate t from 0 to θ_1, and we have

$$[F(\theta_1)]^{n-1}\beta(\theta_1) - [F(0)]^{n-1}\beta(0) = \int_0^{\theta_1} (n-1)[F(t)]^{n-2}f(t)tdt$$

Here, $\beta(0) = 0$ by our assumption. So we have

$$\beta(\theta_1) = \frac{1}{[F(\theta_1)]^{n-1}} \int_0^{\theta_1} (n-1)[F(t)]^{n-2}f(t)tdt$$

Let's deal with the integral a little bit.

$$\int_0^{\theta_1} (n-1)[F(t)]^{n-2}f(t)tdt = \int_0^{\theta_1} td[F(t)]^{n-1}$$

$$= \left\{[F(t)]^{n-1}t\right\}\Big|_{t=0}^{\theta_1} - \int_0^{\theta_1} [F(t)]^{n-1}dt$$

$$= [F(\theta_1)]^{n-1}\theta_1 - \int_0^{\theta_1} [F(t)]^{n-1}dt$$

Therefore, we have

$$\beta(\theta_1) = \theta_1 - \int_0^{\theta_1} \frac{[F(t)]^{n-1}}{[F(\theta_1)]^{n-1}}dt$$

Recall that we consider bidder 1 without loss. Therefore, we have found a Bayesian Nash equilibrium where bidder i uses a strategy s_i such that for any θ_i,

$$s_i(\theta_i) = \beta(\theta_i) = \theta_i - \int_0^{\theta_i} \frac{[F(t)]^{n-1}}{[F(\theta_i)]^{n-1}}dt$$

This is lower than θ_i. That makes sense because otherwise, the bidder's payoff will be 0 or negative.

Exercise: first-price sealed-bid auction

1. Consider the first-price sealed bid auction (introduced in the lecture) with two bidders, $i = 1, 2$. Each bidder's valuation is drawn independently from the cumulative distribution $F(v)$ on $[0, 1]$.

(a) Formulate bidder 1's maximization problem given that bidder 2 follows the equilibrium bid function, $\sigma(\cdot)$:

Solution: We assume that this equilibrium bid function is strictly increasing. Given that bidder 2 follows the equilibrium $\sigma(\cdot)$, bidder 1's problem is given by

$$\max_b (v-b)Pr(b > \sigma(v_2)) = (v-b)F(\sigma^{-1}(b))$$

■

(b) Derive the conditions that characterize the symmetric equilibrium (in which both bidders follow the same strategy σ):

Solution: We look for a symmetric equilibrium in which $\sigma_1 = \sigma_2 = \sigma$. The first-order condition with respect to b:

$$-F(\sigma^{-1}(b)) + (v-b)f(\sigma^{-1}(b))\frac{1}{\sigma'(\sigma^{-1}(b))}\Bigg|_{b=\sigma(v)} = 0$$

$$-F(v) + (v-\sigma(v))f(v)\frac{1}{\sigma'(v)} = 0$$

It is also easily seen that in equilibrium, $\sigma(0) = 0$. So the ODE system is given by

$$(1) \quad -F(v) + (v-\sigma(v))f(v)\frac{1}{\sigma'(v)} = 0$$

$$(2) \quad \sigma(0) = 0$$

The above two conditions characterize the symmetric pure strategy Bayesian Nash Equilibrium (BNE).

■

(c) Solve for the symmetric equilibrium bid function $\sigma(v)$.

Solution: *[Remark: if you don't know how to solve an ODE system derived in (c) above to get the solution, you may use the envelope theorem introduced in the lecture.]*

- **Method #1 (ODE)**

From equation (1),

$$-F(v) + (v-\sigma(v))f(v)\frac{1}{\sigma'(v)} = 0$$

$$\Leftrightarrow f(v)\sigma(v) + F(v)\sigma'(v) = vf(v)$$

$$\Leftrightarrow \frac{d}{dv}(F(v)\sigma(v)) = \frac{d}{dv}\left(\int_0^v \bar{v}f(\bar{v})d\bar{v} + C\right)$$

$$\Leftrightarrow F(v)\sigma(v) = \int_0^v \bar{v}f(\bar{v})d\bar{v} + C = \bar{v}F(\bar{v})\Big|_0^v - \int_0^v F(\bar{v})d\bar{v} + C$$

The last equality is followed by the integration by parts.

$$\Leftrightarrow F(v)\sigma(v) = vF(v) - \int_0^v F(\bar{v})d\bar{v} + C$$

From equation (2), $\sigma(0) = 0$, implying that $C = 0$. Thus,

$$\sigma(v) = v - \int_0^v \frac{F(\bar{v})}{F(v)}d\bar{v}.$$

- **Method #2 (Envelope Theorem)**

In equilibrium, the expected payoff for a bidder with valuation v is $\Pi(v) = \pi(v, \sigma(v))$.
Notice that

$$\frac{d\Pi(v)}{dv} = \frac{\partial \pi(v, \sigma(v))}{\partial v} + \frac{\partial \pi(v, \sigma(v))}{\partial \sigma} \times \frac{\partial \sigma(v)}{\partial v}.$$

By the Envelope Theorem, $\frac{\partial \pi(v,\sigma(v))}{\partial \sigma} = 0$.
Consequently,

$$\frac{d\Pi(v)}{dv} = F(\sigma^{-1}(\sigma(v))) = F(v).$$

Moreover, notice that

$$\Pi(v) = (v - \sigma(v))F(\sigma^{-1}(\sigma(v)))$$

$$= (v - \sigma(v))F(v).$$

Taking the previous two expressions and substituting them into the identity

$$\int_0^v \frac{d\Pi(w)}{dw} dw = \Pi(v) - \Pi(0),$$

we obtain

$$\int_0^v F(w)dw = (v - \sigma(v))F(v) - 0.$$

Rearranging provides an explicit solution $\sigma(v)$:

$$\sigma(v) = v - \int_0^v \frac{F(w)}{F(v)} dw,$$

which is the same as derived in Method #1.
■