Game Theory

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1 Nash Equilibrium in Pure Actions

A game is triple (I,(A_i), (u_i)) where:

- I is a set of players
- A_i is the set of actions available to player i
- u_i is the payoff function for player i

For the moment, the functions u_i represent ordinal preferences over $\prod_{i \in I} A_i$.

Note that we are allowing arbitrary externalities.

Consider a game with N players. A strategy profile

$$\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$$

is a **Nash equilibrium** of the game if, for every player i,

$$u_i(s_i^*, \mathbf{s}_{-i}^*) \geq u_i(s_i', \mathbf{s}_{-i}^*)$$

Given a game $(I, (A_i)_{i \in I}, (u_i)_{i \in I})$, a Nash equilibrium of this game is an n-tuple of actions $(a_i^*)_{i \in I}$ such that for every player $i \in I$ and for every action $a_i \in A_i$:

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*),$$

where (a_i, a_{-i}^*) is the list of actions that is identical to a^* except that we have replaced a_i^* by a_i.

Interpretation: A Nash equilibrium represents a rest point of a learning process in which each player chooses the optimal action assuming that all other players choose the same actions as in the previous period.

Best Response: A strategy s_i is a best response to a strategy profile s_{-i} if it maximizes the payoff of player i given the strategies of the other players.

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i})$$

Best Response Correspondence: The best response correspondence of player i is the correspondence that assigns to each strategy profile s_{-i} the set of best responses of player i to s_{-i}.

1.1 Key points

1. Nash equilibria are fixed points of the best reply correspondence.

For every player $i \in I$ define a best reply correspondence $BR_i : \prod_{i \in I} A_i \rightrightarrows A_i$ by setting for every $a \in \prod_{i \in I} A_i$:

$$BR_i(a) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every } a_i' \in A_i\}.$$

The best reply correspondence $BR : \prod_{i\in I} A_i \rightrightarrows \prod_{i\in I} A_i$ is defined by setting for every $a \in \prod_{i\in I} A_i$:

$$BR(a) = \prod_{i\in I} BR_i(a).$$

By definition: a^* is a Nash equilibrium if and only if $a^* \in BR(a^*)$, i.e., if and only if a^* is a fixed point of BR.

2 Static Games

2.1 Case 1: Betrend Competition

In betrend competition, firm face a total cost curve for producing their goods and simply choose the price for their respective goods. Whichever firm has the lower price will get all the customers and if the prices are the same, the customers will be split evenly. We will assume that identical firms face a constant marginal cost c. The firms simultaneously choose their prices, p1 and p2. let's look at it from firm 1's perspective (it will be the same for firm firm 2).

Firm 1 has three strategies:

- $p_1 < p_2$: Firm 1 gets all the customers and makes a profit of $p_1 - c$.
- $p_1 = p_2$: Firm 1 gets half the customers and makes a profit of $\frac{p_1-c}{2}$.
- $p_1 > p_2$: Firm 1 gets no customers and makes a profit of 0.

Naturally, depending on what firm 2's price is, we could have any of these situations. Let s look at some different conditions.

- Starting with one extreme, suppose $p_2 > p^m$, the monopoly price.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.

- Naturally, firm 1 would actually want to set $p_1 = p^m$ in this case since that's the price that maximizes their profit level.
- Now, let's suppose firm 2 chose some price that was at most the monopoly price, i.e., $p_2 \leq p^m$. The same results hold.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.
- It should be obvious that firm 1 wants to set a price lower than firm 2, $p_1 < p_2$. In fact, to maximize profit, firm 1 wants to undercut firm 2 by as little as possible (a single penny). $p_1 = p_2 - \varepsilon$ where $\varepsilon > 0$ is the smallest possible number that firm 1 can pick.
  - That way they get all of the customers while lowering their profit margin by as little as possible.
- Lastly, suppose firm 2's price were at or below marginal cost, $p_2 \leq c$.
  - If firm 1 chose a price above this, firm 2 would get all of the consumers since their price is lower.
  - If firm 1 matched this price, they would get half of the consumers.
  - If firm 1 set a price lower than this, they would get all of the consumers.
- Pricing below marginal cost would not be an optimal strategy for firm 1. If $p_1 < c$, firm 1 would actually lose money on each unit sold.
  - Thus, the lowest (and only possible) price firm 1 is willing to charge is $p_1 = c$.
- The analysis for firm 2 is identical.
  - If firm 1 prices above the monopoly price, $p_2 = p^m$.
  - If firm 1 prices between marginal cost and the monopoly price, $p_2 = p_1 - \varepsilon$, where $\varepsilon > 0$ is the smallest possible number greater than zero.
  - If firm 1 prices at or below marginal cost, $p_2 = c$.

- Returning to our Nash equilibrium solution concept, we know that our equilibrium is when neither firm has any incentive to deviate from their chosen strategy.
  - This occurs where the best response functions intersect.
- There is exactly one intersection point in the previous figure, where $p_1 = p_2 = c$. Thus, both firms price at marginal cost in equilibrium.
  - This should make sense. Each firm wants to undercut the other to claim the whole market. Yet they can't undercut anymore once the price is at marginal cost or they'll suffer losses.
  - Bertrand competition implies that with just two firms, we reach the perfectly competitive equilibrium.
- Since price is set at marginal cost for both firms, economic profit under Bertrand competition equals zero.

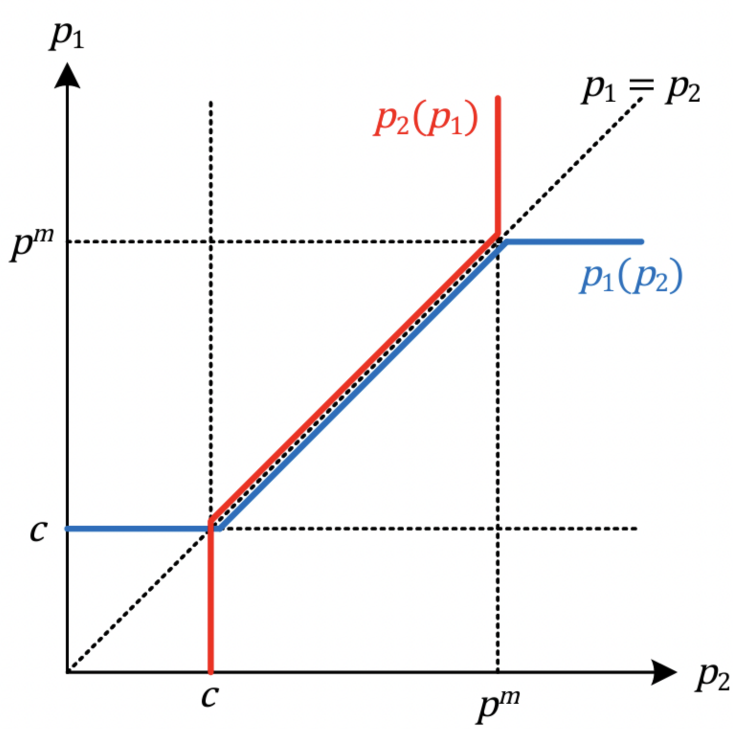

Figure 1: EQ

2.2 Case 2: Cournet Competition

2.3 Case 3: Stackelberg Competition

3 Topic 2: Nash Equilibria in Mixed Actions

3.1 Mon, Sept 9: Review of Newtonian Mechanics

- A *Newtonian trajectory* $\mathbf{x}(t)$ $(t \in \mathbb{R})$ is given by solutions of the second order ODE

$$m\,\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)),$$

  where $m > 0$ is a basic parameter associated with a given Newtonian particle, called its *mass*.

- The force field $\mathbf{F}(\mathbf{x})$ — which we take to be static (i.e., not intrinsically dependent on time) for simplicity — is said to be *conservative* if there is a *potential function* $V(\mathbf{x})$ such that

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}).$$

  Here, '∇' denotes the *gradient operator*,

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right).$$

- For a conservative force field, we can find a *conserved quantity* along the Newtonian trajectories, namely the *total mechanical energy*.

$$E = H(\mathbf{x}, \mathbf{p}) := \frac{1}{2m}\,\mathbf{p}^2 + V(\mathbf{x}).$$

  Here, $\mathbf{p}^2 := \mathbf{p} \cdot \mathbf{p} = \|\mathbf{p}\|^2$, and $\mathbf{p} := m\mathbf{v} := m\dot{\mathbf{x}}$ is the *momentum*.

3.2 Tue, Sept 10: Alternative Formulations of Newtonian Mechanics

- The **Hamiltonian formulation**:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}.$$

- The **Lagrangian formulation**:

$$\delta S[\mathbf{x}(t)] = 0,$$

where the *action* on the time interval $[t_a, t_b]$ is given by

$$S[\mathbf{x}(t)] := \int_{t_a}^{t_b} \left[\frac{m}{2}\, \dot{\mathbf{x}}(t)^2 - V(\mathbf{x}(t)) \right] dt.$$

- Etc.