SP25 8732: Homework 2

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1 Question 1

Solution:

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. /// 1.A
> reg price sqrft bdrms

      Source |       SS           df       MS      Number of obs   =        88
-------------+----------------------------------   F(2, 85)        =     72.96
       Model |  580009.152         2  290004.576   Prob > F        =    0.0000
    Residual |  337845.354        85  3974.65122   R-squared       =    0.6319
-------------+----------------------------------   Adj R-squared   =    0.6233
       Total |  917854.506        87  10550.0518   Root MSE        =    63.045

------------------------------------------------------------------------------
       price | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
       sqrft |   .1284362   .0138245     9.29   0.000     .1009495    .1559229
       bdrms |   15.19819   9.483517     1.60   0.113    -3.657582    34.05396
       _cons |    -19.315   31.04662    -0.62   0.536    -81.04399      42.414
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.
. reg sqrft bdrms

      Source |       SS           df       MS      Number of obs   =        88
-------------+----------------------------------   F(1, 86)        =     33.85
       Model |  8186960.43         1  8186960.43   Prob > F        =    0.0000
    Residual |  20797100.3        86  241826.748   R-squared       =    0.2825
-------------+----------------------------------   Adj R-squared   =    0.2741
       Total |  28984060.7        87  333150.123   Root MSE        =    491.76

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       sqrft | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
       bdrms |   364.5886    62.6605     5.82   0.000     240.0236    489.1535
       _cons |   712.7749   229.6473     3.10   0.003     256.2513    1169.299
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. predict e, resid

.
. reg price e

      Source |       SS           df       MS      Number of obs   =        88
-------------+----------------------------------   F(1, 86)        =     51.33
       Model |  343066.064         1  343066.064   Prob > F        =    0.0000
    Residual |  574788.441        86  6683.58653   R-squared       =    0.3738
-------------+----------------------------------   Adj R-squared   =    0.3665
       Total |  917854.506        87  10550.0518   Root MSE        =    81.753

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       price | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
           e |   .1284362   .0179268     7.16   0.000     .0927989    .1640736
       _cons |    293.546   8.714921    33.68   0.000     276.2213    310.8707
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2 Question 2

Solution:

P_2 :

(a) when $L \times k$, $\beta_{IV} = (Z'X)^{-1} Z y = (Z'X)^{-1} Z'(X\beta + \varepsilon) = (Z'X)^{-1} Z'X'\beta + (Z'X)^{-1} Z'\varepsilon$

$$= \beta + (Z'X)^{-1} Z'\varepsilon. = \beta + \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i'\varepsilon\right)$$

$$\Longrightarrow \beta + \underbrace{\left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i'\varepsilon\right)}_{\to_p E z_i x_i \quad E z_i \varepsilon_i)}$$

$$\xrightarrow{p} \beta$$

$\hat{\beta}_{IV} - \beta = \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i'\varepsilon\right)$, $\hat{\beta}_{2SLS} = (Z'X)^{-1} Z'y$, Let ~~$P_Z (Z'Z)$~~ $P_Z = Z(Z'Z)^{-1}Z'$

$$\hat{X} = P_Z X = Z(Z'Z)^{-1} Z' \cdot X$$

$$\hat{\beta}_{2SLS} = (X' P_Z X)^{-1} X' P_Z y = (X'Z(Z'Z)^{-1}Z'X)^{-1} X'Z(Z'Z)^{-1}Z'Xy$$

$$\hat{\beta}_{2SLS} - \beta = [X'Z(Z'Z)^{-1}Z'X)^{-1} X'Z(Z'Z)^{-1}Z'\varepsilon$$

(b) Moment of x_i and z_i, $E[x_i x_i'] = Q_{xx} < \infty$, rank $Q_{xx} = k$

$E[z_i z_i'] = Q_{zz} < \infty$. rank $Q_{zz} = L$

$E[z_i x_i'] = \underset{(L \times k)}{Q_{zx}} < \infty$ rank $Q_{zx} = k$.

By CLT and WLLN,

$$\sqrt{n}(\hat{\beta}_{2SLS} - \beta) = \sqrt{n} \underbrace{\left[\left(\frac{1}{n}\sum_{i=1}^{n} x_i z_i'\right)\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)\right]^{-1}\left(\frac{1}{n} x_i z_i'\right)\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}}\left(\sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n} z_i \varepsilon_i\right)$$

$$\xrightarrow{p} (Q_{xz} \cdot Q_{zz}^{-1} \cdot Q_{zx})^{-1} \cdot Q_{xz} \cdot Q_{zz}^{-1} .$$

$$\sqrt{n} \cdot \frac{Z'\varepsilon}{n} = \sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^{n} z_i \varepsilon_i \xrightarrow{d} N(0, \Omega)$$

$$\sqrt{n}(\hat{\beta}_{2SLS} - \beta) \xrightarrow{d} (Q_{xz} \cdot Q_{zz}^{-1} Q_{zx})^{-1} Q_{xz} Q_{zz}^{-1} \cdot N(0, \Omega) . = N(0, V)$$

$$V = (Q_{xz} Q_{zz}^{-1} Q_{zx})^{-1} Q_{xz}(Q_{zz}^{-1} \Omega Q_{zz}^{-1} Q_{zx})(Q_{xz} Q_{zz}^{-1} Q_{zx})^{-1}$$

c. (i) If under Homoskedasticity, $E(\varepsilon_i^2) = \sigma^2$

Then $\left(\sqrt{n}\,\frac{1}{n}\sum z_i \varepsilon_i\right) \xrightarrow{d} N(0, \sigma^2 Q_{zz})$

~~V can read~~ $\Rightarrow V = \sigma^2 (Q_{xz} Q_{zz}^{-1} Q_{zx})^{-1}$

A sample analog $\hat{V} = \hat{\sigma}^2 (\hat{Q}_{xz} \hat{Q}_{zz} \hat{Q}_{zx})^{-1}$ where

$\hat{Q}_{xz} = \frac{1}{n} X'Z$; $\hat{Q}_{zz} = \frac{1}{n} Z'Z$; $\hat{Q}_{zx} = \frac{1}{n} Z'X$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n} (y_i - x_i'\hat{\beta}_{2SLS})^2$

(ii) If under heteroskedasticity, $\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} z_i z_i' \hat{\varepsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n} z_i z_i' (y_i - x_i'\hat{\beta}_{2SLS})^2$

$\neq \frac{1}{n}\sum \hat{\varepsilon}_i^2$

d. (i) Under Homoskedasticity, $\hat{V} = \hat{\sigma}^2 (\hat{Q}_{xz} \hat{Q}_{zz} \hat{Q}_{zx})^{-1}$

$$\hat{\varepsilon}_i = y_i - x'\hat{\beta} = x_i'\beta + \varepsilon_i - x_i'\hat{\beta} = x_i'(\beta - \hat{\beta}) + \varepsilon_i$$

$$\hat{\varepsilon}_i^2 = (y_i - x_i'\hat{\beta})^2 = (x_i'(\beta - \hat{\beta}) + \varepsilon_i)^2 = (\beta - \hat{\beta})' x_i x_i' (\beta - \hat{\beta}) + 2(\beta - \hat{\beta})' x_i \varepsilon_i + \varepsilon_i^2$$

$$\hat{\sigma}_i^2 = \frac{1}{n}\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 + 2(\beta - \hat{\beta})' \frac{1}{n}\sum_{i=1}^{n} x_i \varepsilon_i + (\beta - \hat{\beta})' \frac{1}{n}\sum_{i=1}^{n} x_i x_i' (\beta - \hat{\beta})$$

By WLLN, $\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 \xrightarrow{p} \sigma^2$;

Since $E X_{ik}^2 < \infty$ and $E\varepsilon_i^2 < \infty$, $E X_{ik}\varepsilon_i < \infty$, we can get

$\frac{1}{n}\sum_{i=1}^{n} x_i \varepsilon_i \xrightarrow{p} 0$ (By WLLN)

By WLLN, $\frac{1}{n}\sum_{i=1}^{n} x_i' x_i \xrightarrow{p} E x_i' x_i = 0$ as $n \to \infty$

Thus: $\hat{\sigma} \xrightarrow{p} \sigma$. $\hat{V} \xrightarrow{p} V$.

(ii) If under heteroskedasticity,

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} z_i z_i' (y_i - x_i'\hat{\beta}_{2SLS})^2$$

$$= \frac{1}{n}\sum z_i z_i' \varepsilon_i^2 + 2 \cdot \frac{1}{n}\sum_{i=1}^{n} z_i z_i' [(\beta - \hat{\beta})' x_i \varepsilon_i] + \frac{1}{n}\sum_{i=1}^{n} z_i z_i' [(\beta - \hat{\beta})' x_i x_i' (\beta - \hat{\beta})$$

$\frac{1}{n}\sum_{i=1}^{n} z_i z_i' \varepsilon_i^2 \xrightarrow{p} E[z_i z_i' \varepsilon_i^2] < \infty$

$$\Omega_{kl} = \frac{1}{n}\sum z_{ik} z_{il} [(\beta - \hat{\beta})' x_i \varepsilon_i] = (\beta - \hat{\beta})' \frac{1}{n}\sum z_{ik} z_{il} x_i \varepsilon_i$$

$$\xrightarrow{p} E[z_{ik} z_{il} x_i \varepsilon_i] < \infty .$$

3 Question 3

Solution:

4 Question 4

Solution:

5 Question 5

Solution: