

Econometrics and Applications

Kirby CHEN

Academic Year 2024-2025

Contents

1	Lecture 3: Endogeneity and Instrumental Variables	1
1.1	Motivation and Overlook	1
1.2	Math Section	4
1.2.1	Assumption	4
1.2.2	Property of OLS	5
1.2.3	WTS: Consistency	7
1.2.4	The Property of 2SLS	9
1.2.5	Efficiency	11
1.2.6	Test	11
1.3	Problem Set	14

1 Lecture 3: Endogeneity and Instrumental Variables

1.1 Motivation and Overlook

Example:

- Omitted variables bias
- Measurement error
- Simultaneous equations bias (reverse causality)

Our Goal

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

The endogenous variable x has a real impact on Y , and we aim to find the true value of β_1 .

1. Using an Instrumental Variable to Derive the Model's Covariance

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

Taking the covariance of both sides with the instrumental variable z :

$$\text{cov}(Y, z) = \text{cov}(\beta_0 + \beta_1 X + \varepsilon, z)$$

Expanding the covariance expression:

$$\text{cov}(Y, z) = \text{cov}(\beta_0, z) + \beta_1 \times \text{cov}(X, z) + \text{cov}(\varepsilon, z)$$

Since the instrumental variable z is uncorrelated with both β_0 and the error term ε , these covariance terms disappear:

$$\text{cov}(Y, z) = \beta_1 \times \text{cov}(X, z)$$

Solving for β_1 :

$$\beta_1 = \frac{\text{cov}(Y, z)}{\text{cov}(X, z)}$$

Instrumental Variables (IV) estimator of β_1 , β_{IV} .

2. Reduced-form Equation: Indirect Least Square, ILS

$$x = \delta_0 + \delta_1 \times z + u$$

$$Y = \pi_0 + \pi_1 \times z + v$$

Reduced-form equation: Writing an endogenous variable in terms of exogenous variables.

$$x = \delta_0 + \delta_1 \times z + u$$

$$Y = \pi_0 + \pi_1 \times z + v$$

$$\delta_1 = \frac{\text{cov}(x, z)}{\text{var}(z)}$$

$$\pi_1 = \frac{\text{cov}(Y, z)}{\text{var}(z)}$$

We know:

$$Y = \beta_0 + \beta_1 \times x + \varepsilon$$

Regression coefficient:

$$\beta_1 = \frac{\text{cov}(Y, x)}{\text{var}(x)}$$

Using the instrumental variable:

$$\frac{\pi_1}{\delta_1} = \frac{\frac{\text{cov}(Y,z)}{\text{var}(z)}}{\frac{\text{cov}(x,z)}{\text{var}(z)}} = \frac{\text{cov}(Y,z)}{\text{cov}(x,z)} = \beta_{IV} = \beta_1$$

$$x = \delta_0 + \delta_1 \times z + u$$

$$Y = \pi_0 + \pi_1 \times z + v$$

$$\delta_1 = \frac{\text{cov}(x,z)}{\text{var}(z)}$$

$$\pi_1 = \frac{\text{cov}(Y,z)}{\text{var}(z)}$$

*Reduced-form Equation

$$x = \delta_0 + \delta_1 \times z + u$$

$$Y = \pi_0 + \pi_1 \times z + v$$

$$\delta_1 = \frac{\text{cov}(x,z)}{\text{var}(z)}$$

$$\pi_1 = \frac{\text{cov}(Y,z)}{\text{var}(z)}$$

*Indirect Least Squares (ILS) Method

$$Y = \beta_0 + \beta_1 \times x + \varepsilon$$

$$= \beta_0 + \beta_1 \times (\delta_0 + \delta_1 \times z + u) + \varepsilon$$

$$= \beta_0 + \beta_1 \times \delta_0 + \beta_1 \times \delta_1 \times z + \beta_1 \times u + \varepsilon$$

$$= (\beta_0 + \beta_1 \times \delta_0) + \beta_1 \times \delta_1 \times z + (\beta_1 \times u + \varepsilon)$$

$$\pi_0 = \beta_0 + \beta_1 \times \delta_0, \quad \pi_1 = \beta_1 \times \delta_1, \quad v = \beta_1 \times u + \varepsilon$$

Question: when IVs more than endogenous variables, the above two method fails.

3. Two Stage Least Squares (2SLS/TSLS)

*First Stage

$$x = \delta_0 + \delta_1 \times z + u$$

$$x = \hat{\delta}_0 + \hat{\delta}_1 \times z + \hat{u}$$

$$\hat{x} = \delta_0 + \delta_1 \times z$$

*Second Stage

$$Y = \beta_{0,2SLS} + \beta_{1,2SLS} \times \hat{x} + \varepsilon_{2SLS}$$

*Does the Model Have Endogeneity?

$$Y = \beta_0 + \beta_1 \times x + \varepsilon$$

$$= \beta_0 + \beta_1 \times (\hat{x} + \hat{u}) + \varepsilon$$

$$= \beta_0 + \beta_1 \times \hat{x} + \beta_1 \times \hat{u} + \varepsilon$$

$$\text{cov}(\hat{x}, \varepsilon_{2SLS}) = \text{cov}(\hat{x}, \beta_1 \times \hat{u} + \varepsilon)$$

$$= \beta_1 \times \text{cov}(\hat{x}, \hat{u}) + \text{cov}(\hat{x}, \varepsilon) = 0$$

When there exists many IVs:

*First Stage

$$x = \delta_0 + \delta_1 \times z_1 + \delta_2 \times z_2 + u$$

$$\hat{x} = \hat{\delta}_0 + \hat{\delta}_1 \times z_1 + \hat{\delta}_2 \times z_2$$

*Second Stage

$$Y = \beta_{0,2SLS} + \beta_{1,2SLS} \times \hat{x} + \varepsilon_{2SLS}$$

1.2 Math Section

1.2.1 Assumption

1. **Linearity:** $Y = X\beta + \epsilon$.
2. **Full rank:** $\text{rank}(X) = k$.
3. **Exogeneity:** $\mathbb{E}[\epsilon|X] = 0$.

Law of iterated expectations:

$$\mathbb{E}[\epsilon] = \mathbb{E}[\mathbb{E}[\epsilon|X]] = \mathbb{E}[0] = 0.$$

4. **Homoscedasticity and nonautocorrelation:**

$$\text{Var}(\epsilon_i|X) = \sigma^2, \quad i = 1, 2, \dots, n.$$

$$\text{Var}(\epsilon_i, \epsilon_j|X) = 0, \quad i \neq j, \quad \text{Var}(\epsilon_i \epsilon) = \sigma^2 I.$$

5. X may be fixed and random.

We assume that there is an additional vector of variables z_i , with $L \geq k$.

- (1) **Exogeneity**: z_i is uncorrelated with disturbance ϵ_i .
- (2) **Relevance**: z_i is correlated with explanatory variable x_i .
- (3) **Homoscedasticity**: $\mathbb{E}[\epsilon_i^2|z_i] = \sigma^2$.
- (4) **Random Sampling** $(x_i, z_i, \epsilon_i) \stackrel{iid}{\sim}$.
- (5) **Moments of x_i and z_i** :

$$\mathbb{E}[x_i x_i'] = Q_{XX} < \infty, \quad \text{rank}(Q_{XX}) = k.$$

$$\mathbb{E}[z_i z_i'] = Q_{ZZ} < \infty, \quad \text{rank}(Q_{ZZ}) = L.$$

$$\mathbb{E}[z_i x_i'] = Q_{ZX} < \infty, \quad \text{rank}(Q_{ZX}) = k.$$

$(L \times k) \quad (\text{since } L \geq k).$

- (6) **Exogeneity of Instruments**:

$$\mathbb{E}[\epsilon_i | b_i] = 0.$$

1.2.2 Property of OLS

1. OLS is biased.

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon.$$

$$\mathbb{E}[\hat{\beta}|X] = \beta + \mathbb{E}[(X'X)^{-1}X'\epsilon|X].$$

$$= \beta + (X'X)^{-1}X'\mathbb{E}[\epsilon|X].$$

$$= \beta + (X'X)^{-1}X'\eta \neq \beta$$

(biased).

2. OLS is inconsistent in big sample.

Recall: $\mathbb{E}[\epsilon|X] = 0, \quad \mathbb{E}[\epsilon_i x_i]$

$$= \mathbb{E}[\mathbb{E}[\epsilon_i x_i|X]] = \mathbb{E}[x_i \mathbb{E}[\epsilon_i|X]] = 0.$$

2. OLS is inconsistent.

$$\mathbb{E}[x_i \epsilon_i] = \mathbb{E}[x_i \eta] \neq 0.$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon = \beta + \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n x_i \epsilon_i \right).$$

$$\frac{1}{n} \sum_{i=1}^n x_i x_i' \xrightarrow{p} Q_{XX}$$

$$\frac{1}{n} \sum_{i=1}^n x_i \epsilon_i \xrightarrow{p} \eta \neq 0.$$

$$\Rightarrow \hat{\beta} \xrightarrow{p} \neq \beta.$$

moment non.

$$\mathbb{E}[x_i \epsilon_i] = \mathbb{E}[x_i (y_i - x_i' \beta)] = 0.$$

OLS? .

3. A method of moment estimator β_{mom} sets the sample analogue to 0:

$$\frac{1}{n} \sum_{i=1}^n x_i (y_i - x_i' \beta_{\text{mom}}) = 0.$$

$$\sum_{i=1}^n x_i y_i - \left(\sum_{i=1}^n x_i x_i' \right) \beta_{\text{mom}} = 0.$$

.

$$\left(\sum_{i=1}^n x_i x_i' \right) \beta_{\text{mom}} = \sum_{i=1}^n x_i y_i.$$

$$\beta_{\text{mom}} = \left(\sum_{i=1}^n x_i x_i' \right)^{-1} \left(\sum_{i=1}^n x_i y_i \right).$$

$$= (X'X)^{-1}X'y = \beta_{\text{ols}}.$$

IV Model Assumptions

- (1), (2), (3) were replaced with (7).

$$\mathbb{E}[x_i | z_i] = 0.$$

$$\mathbb{E}[z_i \epsilon_i] = \mathbb{E}[\mathbb{E}[z_i \epsilon_i | z_i]] = \mathbb{E}[z_i \mathbb{E}[\epsilon_i]] = 0.$$

$$\mathbb{E}[z_i(y_i - x_i' \beta)] = 0.$$

(In sample),

$$\frac{1}{n} \sum_{i=1}^n z_i'(y_i - x_i' \beta_{IV}) = 0.$$

$$\sum_{i=1}^n z_i y_i - \left(\sum_{i=1}^n z_i x_i' \right) \beta_{IV} = 0.$$

$$\left[\sum_{i=1}^n z_i x_i' \right] \beta_{IV} = \sum_{i=1}^n z_i y_i.$$

If $L = k$, then

$$\beta_{IV} = \left(\sum_{i=1}^n z_i x_i' \right)^{-1} \left(\sum_{i=1}^n z_i y_i \right).$$

$$\beta_{IV} = (Z'X)^{-1}Z'y.$$

$$\beta_{OLS} = (X'X)^{-1}X'y.$$

1.2.3 WTS: Consistency

When $L = k$, $\mathbb{E}[z_i x_i'] = Q_{ZX}$, and:

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y.$$

$$= (Z'X)^{-1}Z'(X\beta + \epsilon).$$

$$= \beta + (Z'X)^{-1}Z'\epsilon.$$

$$= \beta + \left(\frac{1}{n} \sum_{i=1}^n z_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n z_i \epsilon_i \right).$$

$$\xrightarrow{p} \mathbb{E}[z_i x_i'] = Q_{ZX}, \quad \text{for using WLLN.}$$

$$\Rightarrow \hat{\beta}_{IV} \xrightarrow{p} \beta + (\mathbb{E}[z_i x_i'])^{-1} \mathbb{E}[z_i \epsilon_i].$$

$$\mathbb{E}[z_i \epsilon_i] = \mathbb{E}[\mathbb{E}[z_i \epsilon_i | z_i]] = \mathbb{E}[z_i \mathbb{E}[\epsilon_i | z_i]].$$

$$= \mathbb{E}[z_i \cdot 0] = 0.$$

$$\Rightarrow \hat{\beta}_{IV} \xrightarrow{p} \beta.$$

IV estimator is consistent.

WTS: Asymptotic normality proof

$$\hat{\beta}_{IV} - \beta = \left(\frac{1}{n} \sum_{i=1}^n z_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n z_i \epsilon_i \right).$$

By CLT,

$$\begin{aligned} \sqrt{n}(\hat{\beta}_{IV} - \beta) &= \left[\frac{1}{n} \sum_{i=1}^n z_i x_i' \right]^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \epsilon_i \right) \\ &\xrightarrow{p} Q_{ZX} \end{aligned}$$

$$\begin{aligned} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n z_i \epsilon_i \right) &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n z_i \epsilon_i - \mathbb{E}[z_i \epsilon_i] \right) \\ &\xrightarrow{d} N(0, \sigma^2 Q_{ZZ}). \end{aligned}$$

$$\text{Var}(z_i \epsilon_i) = \mathbb{E}z_i \epsilon_i - 0'$$

$$= \mathbb{E}[z_i \epsilon_i \epsilon_i' z_i'] = \mathbb{E}[\epsilon_i^2 z_i z_i'].$$

.

$$= \mathbb{E}[\mathbb{E}[\epsilon_i^2 | z_i] z_i z_i'].$$

$$= \sigma^2 \mathbb{E}[z_i z_i'] = \sigma^2 Q_{ZZ}.$$

By Slutsky's theorem,

$$\sqrt{n}(\hat{\beta}_{IV} - \beta) \rightarrow dN(0, \sigma^2 Q_{ZX}^{-1} Q_{ZZ} Q_{ZX}^{-1}).$$

Consistency.

But IV is biased:

$$\hat{\beta}_{IV} = \beta + (Z'X)^{-1} Z' \epsilon.$$

$$\mathbb{E}[\hat{\beta}_{IV} | X, Z] = \beta + (Z'X)^{-1} Z' \mathbb{E}[\epsilon | X, Z] \neq \beta.$$

$$\hat{\beta}_{IV} = (Z'X)^{-1} Z' y.$$

Matrix dimensions:

$$Z : n \times L, \quad Z' : L \times n, \quad X : n \times k.$$

$$L > k.$$

When $L > k$:

$X \rightarrow Z$ projection.

$$P_Z = Z(Z'Z)^{-1}Z'.$$

$$= ZCZ'Z'.$$

$$\begin{matrix} L & L & Z & & Z & . & . \\ & & & & & & \end{matrix}$$

$$\hat{X} = P_Z X.$$

$$\hat{X} = Z(Z'Z)^{-1}Z'X.$$

$$L \times L, \quad L \times n, \quad L \times k.$$

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1}\hat{X}'y.$$

$$= (X'P_Z X)^{-1}X'P_Z y.$$

Replaced the Z .

1.2.4 The Property of 2SLS

$$\hat{\beta}_{2SLS} = (X'P_Z X)^{-1}X'P_Z Y$$

$$= (X'P_Z X)^{-1}X'P_Z (X\beta + \varepsilon)$$

$$= \beta + (X'P_Z X)^{-1}X'P_Z \varepsilon$$

We want to show that $\hat{\beta}_{2SLS}$ is a consistent estimator, which requires proving that:

$$(X'P_Z X)^{-1}X'P_Z \varepsilon \xrightarrow{p} 0.$$

*Step-by-Step Derivation

$$(X'P_Z X)^{-1}X'P_Z \varepsilon$$

$$= (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'\varepsilon$$

$$= \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'X}{n} \right) \right]^{-1} \left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'\varepsilon}{n} \right).$$

By the Weak Law of Large Numbers (WLLN):

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n X_i z'_i &\xrightarrow{p} E[X_i z'_i] \\ \frac{1}{n} \sum_{i=1}^n z_i z'_i &\xrightarrow{p} E[z_i z'_i] \\ \frac{1}{n} \sum_{i=1}^n z_i X'_i &\xrightarrow{p} E[z_i X'_i] \\ \frac{1}{n} \sum_{i=1}^n z_i \varepsilon_i &\xrightarrow{p} E[z_i \varepsilon_i] = 0. \end{aligned}$$

Thus,

$$(Q_{XZ} Q_{ZZ}^{-1} Q_{ZX})^{-1} (Q_{XZ} Q_{ZZ}^{-1} 0) = 0.$$

This shows that:

$$\hat{\beta}_{2SLS} \xrightarrow{p} \beta.$$

*Asymptotic Normality

$$\begin{aligned} &\sqrt{n}(\hat{\beta}_{2SLS} - \beta) \\ &= \left[\frac{X'Z}{n} \frac{Z'Z}{n}^{-1} \frac{Z'X}{n} \right]^{-1} \frac{X'Z}{n} \frac{Z'Z}{n}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i. \end{aligned}$$

By the Central Limit Theorem (CLT):

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i \xrightarrow{d} N(0, \sigma^2 Q_{ZZ}).$$

Since

$$\text{Var}(Z'\varepsilon) = E[z_i \varepsilon_i \varepsilon_i z'_i] = E[\varepsilon_i^2 z_i z'_i] = \sigma^2 E[z_i z'_i] = \sigma^2 Q_{ZZ},$$

we obtain:

$$\sqrt{n}(\hat{\beta}_{2SLS} - \beta) \xrightarrow{d} N(0, \sigma^2 (Q_{XZ} Q_{ZZ}^{-1} Q_{ZX})^{-1}).$$

Thus,

$$\hat{\beta}_{2SLS} \sim N\left(\beta, \frac{\sigma^2}{n} (Q_{XZ} Q_{ZZ}^{-1} Q_{ZX})^{-1}\right).$$

1.2.5 Efficiency

*Variance Comparison and Positive Semi-Definiteness

Statement: If β_{OLS} variance is smaller than β_{IV} ,

$$A - B > 0 \quad (\text{positive semi-definite})$$

then $B^{-1} - A^{-1}$ is also positive semi-definite.

*Derivation

$$\begin{aligned} & Q_{XX} - (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1} \\ &= Q_{XX} - Q_{XZ}Q_{ZZ}^{-1}Q_{ZX} \\ &= \text{plim}_{n \rightarrow \infty} \frac{X'X}{n} - \text{plim}_{n \rightarrow \infty} \frac{X'Z}{n} \left(\text{plim}_{n \rightarrow \infty} \frac{Z'Z}{n} \right)^{-1} \text{plim}_{n \rightarrow \infty} \frac{Z'X}{n} \\ &= \text{plim}_{n \rightarrow \infty} \left[\frac{X'X}{n} - \frac{X'Z}{n} (Z'Z/n)^{-1} Z'X/n \right] \\ &= \text{plim}_{n \rightarrow \infty} \left[\frac{X'(I - P_Z)X}{n} \right] = \text{plim}_{n \rightarrow \infty} \frac{X'M_Z X}{n} \end{aligned}$$

where $M_Z = I - P_Z$ and $P_Z = Z(Z'Z)^{-1}Z'$.

If A is positive semi-definite, then A is positive semi-definite, where $A_n \xrightarrow{p} A$.

*Important Observation

$$X'M_Z X = X'M_Z M_Z X = X'M_Z (X'M_Z)$$

For any $r \neq 0$, let $V = r'(X'M_Z)$,

$$r'X'M_Z X r' = \gamma'X'M_Z X M_Z X r' = \gamma' \mathbb{D} \gamma = \sum_{i=1}^p v_i^2$$

$$\sigma_{IV}^2 \geq \sigma_{OLS}^2.$$

Thus, the asymptotic variance satisfies:

$$\text{Asy. Var}(\beta_{OLS}) \leq \text{Asy. Var}(\beta_{IV}).$$

*Conclusion:

However, note that σ^2 is still useful, so further testing is needed.

1.2.6 Test

1. Hausman Test

- Null Hypothesis

$$H_0 : E[\varepsilon_i | x_i] = 0 \quad \Rightarrow \quad \text{Exogeneity}$$

– Under H_0 , IV and OLS are consistent.

- Define the difference:

$$d = \hat{\beta}_{IV} - \hat{\beta}_{OLS}$$

(similar to a linear restriction). Under H_0 ,

$$d \xrightarrow{p} 0.$$

- **Test Statistic** If we can derive:

$$\sqrt{n}d \xrightarrow{d} N(0, V),$$

and estimate V by \hat{V} , then we can test H_0 using the Wald statistic:

$$W = \sqrt{n}d'\hat{V}^{-1}\sqrt{n}d = nd'\hat{V}^{-1}d \xrightarrow{d} \chi^2(r).$$

- **Variance of d**

$$\text{Var}(\hat{\beta}_{IV} - \hat{\beta}_{OLS})$$

$$= \text{Var}(\hat{\beta}_{IV}) + \text{Var}(\hat{\beta}_{OLS}) - 2\text{Cov}(\hat{\beta}_{IV}, \hat{\beta}_{OLS}).$$

- **Hausman's Principle**

Let b_E be an estimator of β such that:

$$\sqrt{n}(b_E - \beta) \xrightarrow{d} N(0, V_E).$$

Suppose b_E is efficient in the sense that for any other estimator b of β such that:

$$\sqrt{n}(b - \beta) \xrightarrow{d} N(0, V),$$

we have:

$$V \geq V_E.$$

Let b_I be an inefficient estimator of β , namely:

$$\sqrt{n}(b_I - \beta) \xrightarrow{d} N(0, \Sigma), \quad \text{where } \Sigma \geq V_E.$$

Then the asymptotic variance satisfies:

$$\text{Asy. Var}(b_E, b_I) = \text{Asy. Var}(b_E).$$

- **Proof of a Scalar Case**

Let β be a scalar.

Consider an estimator:

$$\hat{\beta} = \alpha b_I + (1 - \alpha)b_E = b_E + d(b_I - b_E)$$

for a constant α .

Then,

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \Omega).$$

Asymptotic Variance

$$\begin{aligned}\Omega &= \text{Asy. Var}[b_E + d(b_I - b_E)] \\ &= \text{Asy. Var}[b_E] + d^2 \text{Asy. Var}[b_I - b_E] + 2d \text{Asy. Cov}(b_E, b_I - b_E) \\ &= \text{Asy. Var}[b_E] + 2d \text{Asy. Cov}(b_E, b_I - b_E) + d^2 \text{Asy. Var}(b_I - b_E).\end{aligned}$$

Minimization Condition

$$\Omega \text{ is minimized when } d = -\frac{\text{Asy. Cov}(b_E, b_I - b_E)}{\text{Asy. Var}(b_I - b_E)}.$$

Efficiency Argument If $\alpha^* \neq 0$, then $\hat{\beta}$ with $\alpha = \alpha^*$ will have a smaller asymptotic variance than $\hat{\beta}$ with $\alpha = 0$, which contradicts the efficiency of b_E .

Thus, we conclude:

$$\alpha^* = 0 \quad \Rightarrow \quad \text{Asy. Cov}(b_E, b_I - b_E) = 0.$$

Final Covariance Expression Using the identity:

$$\text{Cov}(A + B, C) = \text{Cov}(A, C) + \text{Cov}(B, C),$$

we obtain:

$$\text{Asy. Cov}(b_I, b_E) - \underbrace{\text{Asy. Cov}(b_E, b_E)}_{\text{Asy. Var}(b_E)} = 0.$$

- **Final Test Statistic**

$$\sqrt{nd} \xrightarrow{d} N(0, V),$$

where:

$$\begin{aligned}V &= \text{Asy. Var}(\hat{\beta}_{IV} - \hat{\beta}_{OLS}) \\ &= \text{Asy. Var}(\hat{\beta}_{IV}) - \text{Asy. Var}(\hat{\beta}_{OLS}) - 2\text{Asy. Cov}(\hat{\beta}_{IV}, \hat{\beta}_{OLS}) \\ &= \text{Asy. Var}(\hat{\beta}_{IV}) - \text{Asy. Var}(\hat{\beta}_{OLS}).\end{aligned}$$

Let:

$$\hat{V}_{IV} \xrightarrow{p} \text{Asy. Var}(\hat{\beta}_{IV}),$$

$$\hat{V}_{OLS} \xrightarrow{p} \text{Asy. Var}(\hat{\beta}_{OLS}).$$

Then the final test statistic is:

$$W = nd'(\hat{V}_{IV} - \hat{V}_{OLS})^{-1}d \xrightarrow{d} \chi^2(r).$$

Question: Does the instrumental variable z need to be uncorrelated with the dependent variable y ?

No!

- The instrumental variable z affects the dependent variable y through the endogenous variable x :

$$z \rightarrow x \rightarrow y$$

- The instrumental variable z does not directly affect the dependent variable y :

$$\text{cov}(z, y|x) = 0$$

- The instrumental variable z **can and must** influence the dependent variable y **only through** the endogenous variable x .

Suppose that there is a set of instrumental variables $Z = (Z_0 \ Z_1 \ \dots Z_K)$ that meet the following condition:

1. $\text{plim } n^{-1}Z'X = Q_{ZX}$ (non-singular)
2. $\text{plim } n^{-1}Z'Z = Q_{ZZ}$ (positive definite)
3. $\text{plim } n^{-1}Z'u = 0$

$$Y = X\beta + u \Rightarrow Z'Y = Z'X\beta + Z'u$$

Let $\tilde{\beta}$ be an estimator of β . Then we have:

$$Z'Y = Z'X\tilde{\beta} + Z'\tilde{u} \Rightarrow Z'\tilde{u} =$$

$$Z'(Y - X\tilde{\beta}) \Rightarrow \tilde{u} = Y - X\tilde{\beta}$$

$$(Z'\tilde{u})(Z'\tilde{u}) = (Z'Y - Z'X\tilde{\beta})'(Z'Y - Z'X\tilde{\beta})$$

$$= Y'Z'ZY - 2\tilde{\beta}'X'Z'ZY + \tilde{\beta}'X'Z'Z'X\tilde{\beta}$$

$$\frac{\partial(Z'\tilde{u})(Z'\tilde{u})}{\partial\tilde{\beta}} = -2X'Z'ZY + 2X'Z'Z'X\tilde{\beta} = 0$$

hence $X'Z'ZY = X'Z'Z'X\tilde{\beta}$. Then premultiplying by $(X'Z)^{-1}$ leads to

$$\tilde{\beta}^{IV} = (Z'X)^{-1}Z'Y$$

We further have:

$$\tilde{\beta}^{IV} = (Z'X)^{-1}Z'(X\beta + u)$$

$$= \beta + (Z'X)^{-1}Z'u$$

$$\text{plim } \tilde{\beta}^{IV} = \beta + \left[\text{plim } \left(\frac{Z'X}{n} \right) \right]^{-1} \cdot \text{plim } \frac{Z'u}{n}$$

$$= \beta + Q_{ZX}^{-1} \cdot 0 = \beta$$

Therefore $\tilde{\beta}^{IV}$ is consistent.

1.3 Problem Set

*Problem 2 Derive the limiting distribution of the two-stage least squares estimator (2SLS) and consistency of the estimator for the variance-covariance matrix. For each step make exactly clear which assumptions are needed. You may assume homoskedasticity of the errors, or not, but if so state it as an assumption.

(a). **Verify that**

$$\hat{\beta}_{2SLS} - \beta = [X'Z(Z'Z)^{-1}Z'X]^{-1} X'Z(Z'Z)^{-1}Z'\varepsilon.$$

1. **Solution**

$$\begin{aligned} \hat{\beta}_{2SLS} - \beta &= [X'Z(Z'Z)^{-1}Z'X]^{-1} X'Z(Z'Z)^{-1}Z'\varepsilon \\ &= \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'X}{n} \right) \right]^{-1} \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'\varepsilon}{n} \right) \right]. \end{aligned}$$

To use the Weak Law of Large Numbers (WLLN) in Hansen chapter 6, P164, the following assumptions are needed:

2. *Assumptions

- **A1:** (y_i, x_i, z_i) are i.i.d.
- **A2:** $E|y_i|^2 < \infty$, $E||x_i||^2 < \infty$, $E||z_i||^2 < \infty$.

3. *Detour:

- The WLLN in Hansen only needs the first moment, as in A2: $E|y_i| < \infty$, $E||x_i|| < \infty$, $E||z_i|| < \infty$; but in A2, we ask for the second moment to exist. The reason is that the cross product behaves like a degree-2 term. By the **Cauchy-Schwarz inequality**, one can prove that the expectation of the cross product exists and is finite using A2.

- For example, using the inequality:

$$E(|x_{ik}z_{i\ell}|) \leq \sqrt{E|x_{ik}^2|E|z_{i\ell}^2|}$$

where x_{ik} is the k -th element, and $z_{i\ell}$ is the ℓ -th element. Since A2 ensures the second moment of x_i and z_i exists and is finite, it follows that $E[x_i z_i']$ exists and is finite.

4. *By the WLLN, we obtain:

$$\frac{X'Z}{n} \xrightarrow{p} Q_{XZ}, \quad \frac{Z'Z}{n} \xrightarrow{p} Q_{ZZ}, \quad \frac{Z'X}{n} \xrightarrow{p} Q_{ZX}.$$

5. *By the Continuous Mapping Theorem, and the additional assumptions:

- **A3:** $E[z_i \varepsilon_i] = 0$ (the **exogeneity condition**).
- **A4:** $E[z_i z_i'] = Q_{ZZ}$ is full rank/invertible/positive definite.
- **A5:** $E[z_i x_i']$ has full column rank K (the **relevance condition**).

6. *Then, the 2SLS estimator is consistent as:

$$\hat{\beta}_{2SLS} - \beta \xrightarrow{P} (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}Q_{XZ}Q_{ZZ}^{-1} \frac{E[\cancel{z_i \varepsilon_i}]}{0} = 0$$

(Finite matrix).

b. Rescale the equation to converge to a random variable and establish the asymptotic distribution

*Solution

1. Use the usual scaling, multiply by \sqrt{n} , and

$$\begin{aligned} \sqrt{n}(\hat{\beta}_{2SLS} - \beta) &= [X'Z(Z'Z)^{-1}Z'X]^{-1}X'Z(Z'Z)^{-1}Z'\varepsilon \\ &= \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'X}{n} \right) \right]^{-1} \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'\varepsilon}{\sqrt{n}} \right) \right]. \end{aligned}$$

2. **Weak Law of Large Numbers (WLLN) and Central Limit Theorem (CLT)** WLLN and CLT are needed to obtain the distribution. To use the CLT as in Hansen chapter 6, P164, we need assumptions **A1**, **A2**, and:

- **A6:** $E||z_i z_i' \varepsilon_i^2|| < \infty$, since to use CLT for $z_i \varepsilon_i$, we need $z_i \varepsilon_i$ to have a **finite second moment**.
- **A7:** $\Omega = E[z_i z_i' \varepsilon_i^2]$ is positive definite, so it is a valid asymptotic variance matrix.

(Can have a different **A6'** as $E|y_i|^4 < \infty$, $E||z_i||^4 < \infty$, $E||x_i||^4 < \infty$, and then use the **Cauchy-Schwarz inequality** to prove $E||z_i z_i' \varepsilon_i^2|| < \infty$. Assumption **A6'** can replace both **A6** and **A2**, since a higher moment exists means a lower moment also exists.)

3. **Application of the Central Limit Theorem** By the CLT, we have:

$$\sqrt{n} \frac{Z' \varepsilon}{n} = \sqrt{n} \frac{1}{n} \sum_i z_i \varepsilon_i \xrightarrow{d} N(0, \Omega)$$

4. **Combining with WLLN**

$$\sqrt{n}(\hat{\beta}_{2SLS} - \beta) = \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'X}{n} \right) \right]^{-1} \left[\left(\frac{X'Z}{n} \right) \left(\frac{Z'Z}{n} \right)^{-1} \left(\frac{Z'\varepsilon}{\sqrt{n}} \right) \right].$$

$$\xrightarrow{d} (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}Q_{XZ}Q_{ZZ}^{-1}N(0, \Omega) = N(0, V)$$

where

$$V = (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}(Q_{XZ}Q_{ZZ}^{-1}\Omega Q_{ZZ}^{-1}Q_{ZX})(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}$$

(c). **Estimator \hat{V} for the Variance-Covariance Matrix**

Solution: Detour

1.
 - This V is the variance-covariance matrix in $\sqrt{n}(\hat{\beta}_{2SLS} - \beta) \xrightarrow{d} N(0, V)$, not the asymptotic variance of $\hat{\beta}_{2SLS}$.
 - The asymptotic variance of $\hat{\beta}_{2SLS}$ is $\frac{V}{n}$.
2. **Under A8: Homoskedasticity**, $E[\varepsilon_i^2] = \sigma^2 < \infty$

$$\Omega = E[z_i z_i' \varepsilon_i^2] = \sigma^2 E[z_i z_i'] = \sigma^2 Q_{ZZ}$$

Thus, V can be reduced to:

$$\begin{aligned} V &= (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}(Q_{XZ}Q_{ZZ}^{-1}\sigma^2 Q_{ZZ}Q_{ZZ}^{-1}Q_{ZX})(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1} \\ &= \sigma^2 (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1} \end{aligned}$$

3. **Sample Analog \hat{V}**

$$\hat{V} = \hat{\sigma}^2 (\hat{Q}_{XZ} \hat{Q}_{ZZ}^{-1} \hat{Q}_{ZX})^{-1}$$

where

$$\hat{Q}_{ZZ} = \frac{1}{n} \sum_{i=1}^n z_i z_i' = \frac{1}{n} Z'Z$$

$$\hat{Q}_{XZ} = \frac{1}{n} \sum_{i=1}^n x_i z_i' = \frac{1}{n} X'Z$$

$$\hat{Q}_{ZX} = \frac{1}{n} \sum_{i=1}^n z_i x_i' = \frac{1}{n} Z' X$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\varepsilon}_i^2 = \frac{1}{n} \sum_{i=1}^n (y_i - x_i' \hat{\beta}_{2SLS})^2$$

4. Heteroskedasticity Case

If heteroskedasticity is present, then V cannot be simplified. With the Q items the same as above, the Ω matrix can be estimated by:

$$\hat{\Omega} = \frac{1}{n} \sum_{i=1}^n z_i z_i' \hat{\varepsilon}_i^2 = \frac{1}{n} \sum_{i=1}^n z_i z_i' (y_i - x_i' \hat{\beta}_{2SLS})^2$$

d. Establish consistency of \hat{V} Solution

1. Under A8: Homoskedasticity,

$$\hat{V} = \hat{\sigma}^2 (\hat{Q}_{XZ} \hat{Q}_{ZZ}^{-1} \hat{Q}_{ZX})^{-1}$$

The convergence in probability of $(\hat{Q}_{XZ} \hat{Q}_{ZZ}^{-1} \hat{Q}_{ZX})^{-1}$ has been proven when establishing consistency, so the key is to show $\hat{\sigma}^2$ is a consistent estimator of σ^2 .

To show this, write:

$$\hat{\varepsilon}_i = y_i - x_i' \hat{\beta} = x_i' \beta + \varepsilon_i - x_i' \hat{\beta} = x_i' (\beta - \hat{\beta}) + \varepsilon_i$$

$$\hat{\varepsilon}_i^2 = \varepsilon_i^2 + 2(\beta - \hat{\beta})' x_i \varepsilon_i + (\beta - \hat{\beta})' x_i x_i' (\beta - \hat{\beta})$$

Summing up:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\varepsilon}_i^2 = \frac{1}{n} \sum_{i=1}^n \varepsilon_i^2 + 2(\beta - \hat{\beta})' \frac{1}{n} \sum_{i=1}^n x_i \varepsilon_i + (\beta - \hat{\beta})' \frac{1}{n} \sum_{i=1}^n x_i x_i' (\beta - \hat{\beta})$$

- (1) By WLLN, $\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2 \xrightarrow{p} E[\varepsilon_i^2] = \sigma^2$.
- (3) By A2, $E[x_i x_i'] < \infty$, and $\hat{\beta}_{2SLS}$ is a consistent estimator of β , using WLLN that $\frac{1}{n} \sum_{i=1}^n x_i x_i' \xrightarrow{p} E[x_i x_i'] < \infty$, thus part (3) vanishes as $n \rightarrow \infty$.
- (2) Under A2, both $E[x_{ik}^2] < \infty$ and $E[\varepsilon_i^2] < \infty$, and by the **Cauchy-Schwarz inequality**:

$$E(|x_{ik} \varepsilon_i|) \leq \sqrt{E[x_{ik}^2] E[\varepsilon_i^2]} < \infty.$$

Using WLLN that $\frac{1}{n} \sum_{i=1}^n x_i \varepsilon_i \xrightarrow{p} E[x_i \varepsilon_i] < \infty$, and again $\hat{\beta}_{2SLS}$ is a consistent estimator of β , part (2) vanishes as $n \rightarrow \infty$.

Hence, we obtain:

$$\hat{\sigma}^2 \xrightarrow{p} \sigma^2, \quad \text{and} \quad \hat{V} \xrightarrow{p} V.$$

2. Heteroskedasticity Case

$$V = (Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}(Q_{XZ}Q_{ZZ}^{-1}\Omega Q_{ZZ}^{-1}Q_{ZX})(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX})^{-1}$$

One needs to prove that:

$$\hat{\Omega} = \frac{1}{n} \sum_{i=1}^n z_i z_i' (y_i - x_i' \hat{\beta}_{2SLS})^2 \xrightarrow{p} E[z_i z_i' \varepsilon_i^2].$$

Inserting $\hat{\varepsilon}_i$ back into $\hat{\Omega}$:

$$\hat{\Omega} = \frac{1}{n} \sum_{i=1}^n z_i z_i' \varepsilon_i^2 + 2 \frac{1}{n} \sum_{i=1}^n z_i z_i' [(\beta - \hat{\beta})' x_i \varepsilon_i] + \frac{1}{n} \sum_{i=1}^n z_i z_i' [(\beta - \hat{\beta})' x_i x_i' (\beta - \hat{\beta})]$$

- (1) By WLLN, $\frac{1}{n} \sum_{i=1}^n z_i z_i' \varepsilon_i^2 \xrightarrow{p} E[z_i z_i' \varepsilon_i^2]$.
- (2) In homoskedasticity, we could take $(\beta - \hat{\beta})$ out of summation, but here we cannot directly because:

$$\frac{1}{n} \sum_{i=1}^n z_i z_i' [(\beta - \hat{\beta})' x_i \varepsilon_i]$$

Instead, consider the $k - \ell$ element in $\hat{\Omega}$:

$$\hat{\Omega}_{k\ell} = \frac{1}{n} \sum_{i=1}^n z_{ik} z_{i\ell} [(\beta - \hat{\beta})' x_i \varepsilon_i] = (\beta - \hat{\beta})' \frac{1}{n} \sum_{i=1}^n z_{ik} z_{i\ell} x_i \varepsilon_i$$

Then follow similar logic as in homoskedasticity and show that:

$$\hat{\Omega}_{k\ell} \xrightarrow{p} E[z_{ik} z_{i\ell} x_i \varepsilon_i] < \infty.$$