SURJECTIONS

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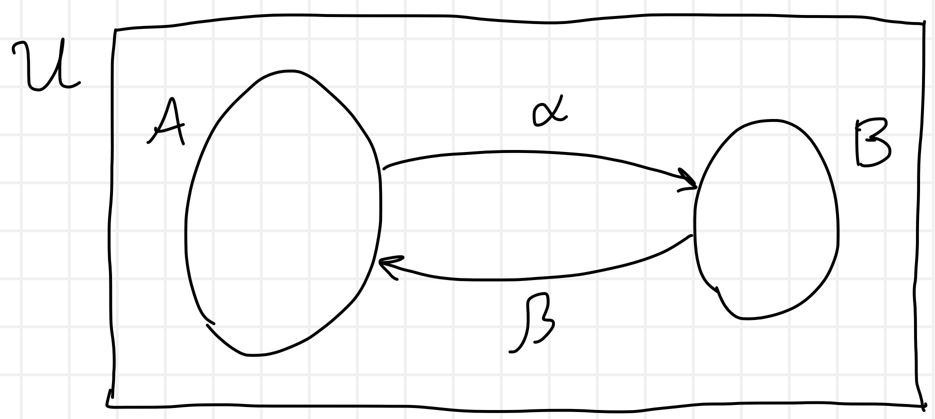
Surjective Mapping

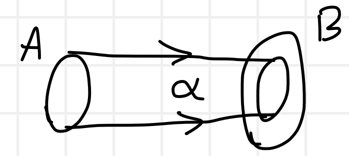


$$\left.\begin{array}{l} A \subseteq \mathcal{U} \\ B \subseteq \mathcal{U} \end{array}\right\} \text{ domains (unary predicates)}$$

$$\left.\begin{array}{l} \alpha : \mathcal{U} \to \mathcal{U} \\ \beta : \mathcal{U} \to \mathcal{U} \end{array}\right\} \text{ conversions (unary functions)}$$

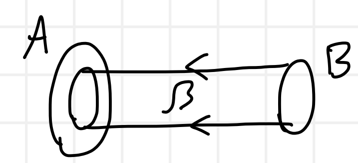
conditions:

- $\boxed{\alpha A}$ $\forall a \in A.\ \alpha(a) \in B$ — α maps A to B



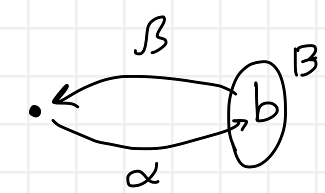
$\alpha(A) \subseteq B$

- $\boxed{\beta B}$ $\forall b \in B.\ \beta(b) \in A$ — β maps B to A

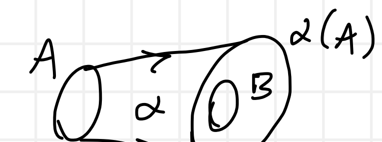


$\beta(B) \subseteq A$

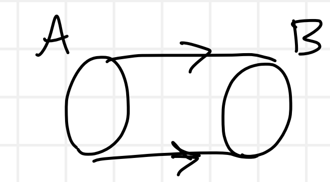
- $\boxed{\alpha\beta}$ $\forall b \in B.\ \alpha(\beta(b)) = b$ — $\left\{\begin{array}{l} \alpha \text{ is left inverse of } \beta \text{ over } B \\ \beta \text{ is right inverse of } \alpha \text{ over } B \end{array}\right.$



(βB, αβ) $\Rightarrow \alpha(A) \supseteq B$



(αA, $\alpha(A) \supseteq B$) $\Rightarrow \alpha(A) = B$



$$A \underset{\beta}{\overset{\alpha}{\twoheadrightarrow}} B \triangleq \alpha A \wedge \beta B \wedge \alpha\beta$$ — α is a surjection from A to B with right inverse β

$\vdash \forall b \in B.\ \exists a \in A.\ \alpha(a) = b$ — more typical definition of surjectivity — β is the witness function

$b \in B \xrightarrow{\beta B} \beta(b) \in A$

$b \in B \xrightarrow{\alpha\beta} \alpha(\beta(b)) = b$

$a \triangleq \beta(b)$

$\Rightarrow a \in A \wedge \alpha(a) = b$

QED

Guards

$$\text{conditions}\begin{cases} \boxed{GA} & \gamma_A = \mathcal{U} & \text{— } A \text{ well-defined everywhere} \\ \boxed{GB} & \gamma_B = \mathcal{U} & \text{— } B \text{ well-defined everywhere} \\ \boxed{G\alpha} & \gamma_\alpha \supseteq A & \text{— } \alpha \text{ well-defined at least over } A \\ \boxed{G\beta} & \gamma_\beta \supseteq B & \text{— } \beta \text{ well-defined at least over } B \end{cases}$$

$G[A \overset{\alpha}{\underset{\beta}{\twoheadrightarrow}} B] \triangleq GA \wedge GB \wedge G\alpha \wedge G\beta$ — the surjection $A \overset{\alpha}{\underset{\beta}{\twoheadrightarrow}} B$ satisfies the guard conditions

$\vdash \boxed{\sqrt{\alpha A}}$ $\omega_{\alpha A}(a)$

$$\omega_{\alpha A}(a) = [\overset{GA}{\cancel{\gamma_A(a)}} \wedge [a \in A \Rightarrow \overset{G\alpha}{\cancel{\gamma_\alpha(a)}} \wedge \overset{GB}{\cancel{\gamma_B(\alpha(a))}}]]$$

QED

$\vdash \boxed{\sqrt{\beta B}}$ $\omega_{\beta B}(b)$

$$\omega_{\beta B}(b) = [\overset{GB}{\cancel{\gamma_B(b)}} \wedge [b \in B \Rightarrow \overset{G\beta}{\cancel{\gamma_\beta(b)}} \wedge \overset{GA}{\cancel{\gamma_A(\beta(b))}}]]$$

QED

$\vdash \boxed{\sqrt{\alpha\beta}}$ $\omega_{\alpha\beta}(b)$

$$\omega_{\alpha\beta}(b) = [\overset{GB}{\cancel{\gamma_B(b)}} \wedge [b \in B \Rightarrow \overset{G\beta}{\cancel{\gamma_\beta(b)}} \wedge \overset{G\alpha}{\cancel{\gamma_\alpha(\beta(b))}}]]$$

$\beta B \quad \beta(b) \in A$

QED

Generalization to Tuples

$A \subseteq \mathcal{U}^n \qquad B \subseteq \mathcal{U}^m \qquad \alpha : \mathcal{U}^n \to \mathcal{U}^m \qquad \beta : \mathcal{U}^m \to \mathcal{U}^n$

everything works the same as in the unary case