ISOMORPHISMS

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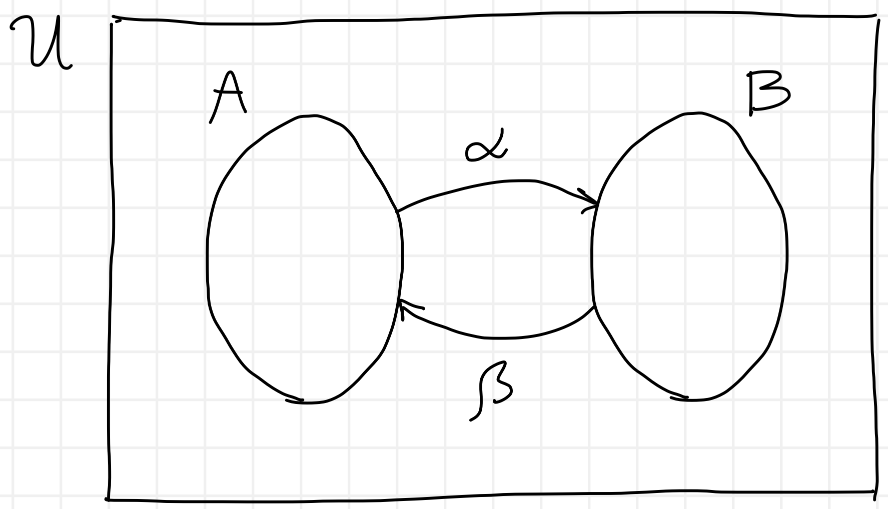
Isomorphic Mapping

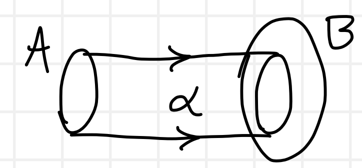


$$\left.\begin{array}{l} A \subseteq \mathcal{U} \\ B \subseteq \mathcal{U} \end{array}\right\} \text{domains (unary predicates)}$$

$$\left.\begin{array}{l} \alpha : \mathcal{U} \to \mathcal{U} \\ \beta : \mathcal{U} \to \mathcal{U} \end{array}\right\} \text{conversions (unary functions)}$$

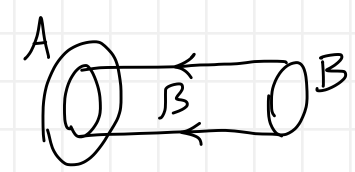
conditions

$\boxed{\alpha A}$ $\quad \forall a \in A.\ \alpha(a) \in B$ — α maps A to B



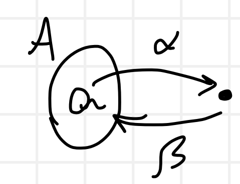
$\alpha(A) \subseteq B$

$\boxed{\beta B}$ $\quad \forall b \in B.\ \beta(b) \in A$ — β maps B to A

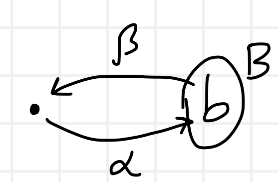


$\beta(B) \subseteq A$

$\boxed{\beta\alpha}$ $\quad \forall a \in A.\ \beta(\alpha(a)) = a$ — $\begin{cases} \beta \text{ is left inverse of } \alpha \text{ over } A \\ \alpha \text{ is right inverse of } \beta \text{ over } A \end{cases}$



$\boxed{\alpha\beta}$ $\quad \forall b \in B.\ \alpha(\beta(b)) = b$ — $\begin{cases} \alpha \text{ is left inverse of } \beta \text{ over } B \\ \beta \text{ is right inverse of } \alpha \text{ over } B \end{cases}$



$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \triangleq \alpha A \wedge \beta B \wedge \beta\alpha \wedge \alpha\beta$ — α and β are mutually inverse isomorphisms between A and B

$\vdash \boxed{\alpha i}$ $\forall a, a' \in A.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — α injective on A

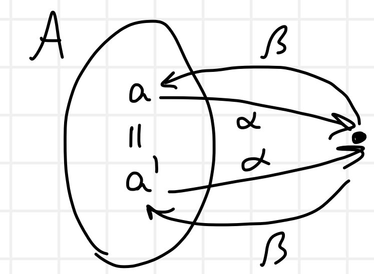
$a \in A \qquad \alpha(a) = \alpha(a') \qquad a' \in A$

$\alpha(a) = \alpha(a') \downarrow$

$a \xleftarrow{\beta\alpha}$ $\beta(\alpha(a)) = \beta(\alpha(a'))$ $\xrightarrow{\beta\alpha} a'$

$\beta(\alpha(a)) = a, \quad \beta(\alpha(a')) = a'$

QED



$\vdash \boxed{\beta i}$ $\forall b, b' \in B.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — β injective on B

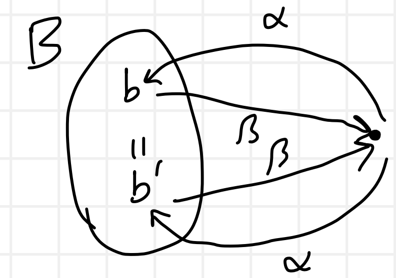
$b \in B \qquad \beta(b) = \beta(b') \qquad b' \in B$

$\beta(b) = \beta(b') \downarrow$

$b \xleftarrow{\alpha\beta}$ $\alpha(\beta(b)) = \alpha(\beta(b'))$ $\xrightarrow{\alpha\beta} b'$

$\alpha(\beta(b)) = b, \quad \alpha(\beta(b')) = b'$

QED



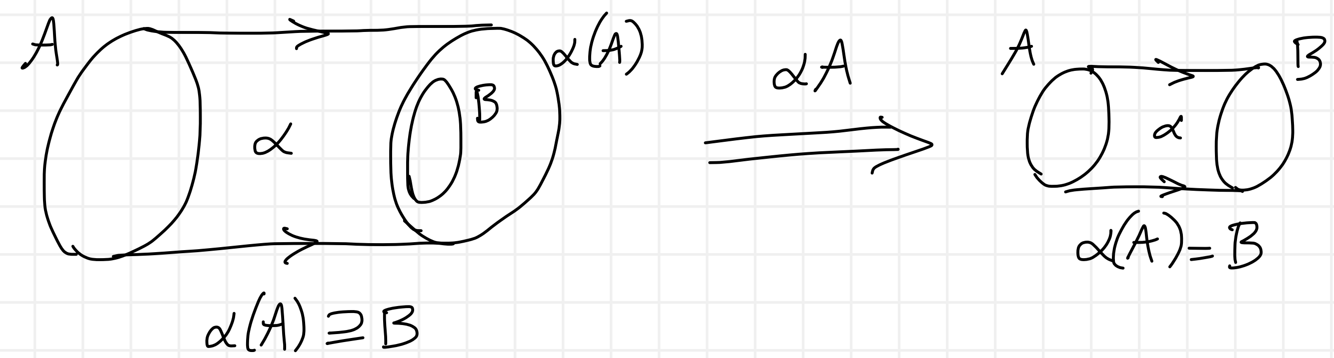
$\vdash \boxed{\alpha s}$ $\forall b \in B.\ \exists a \in A.\ b = \alpha(a)$ — α surjective on B from A

$b \in B \xrightarrow{\alpha\beta} \alpha(\beta(b)) = b$

$\beta b \downarrow \quad a \triangleq \beta(b) \nearrow \alpha(a)$

$a \in A$

QED



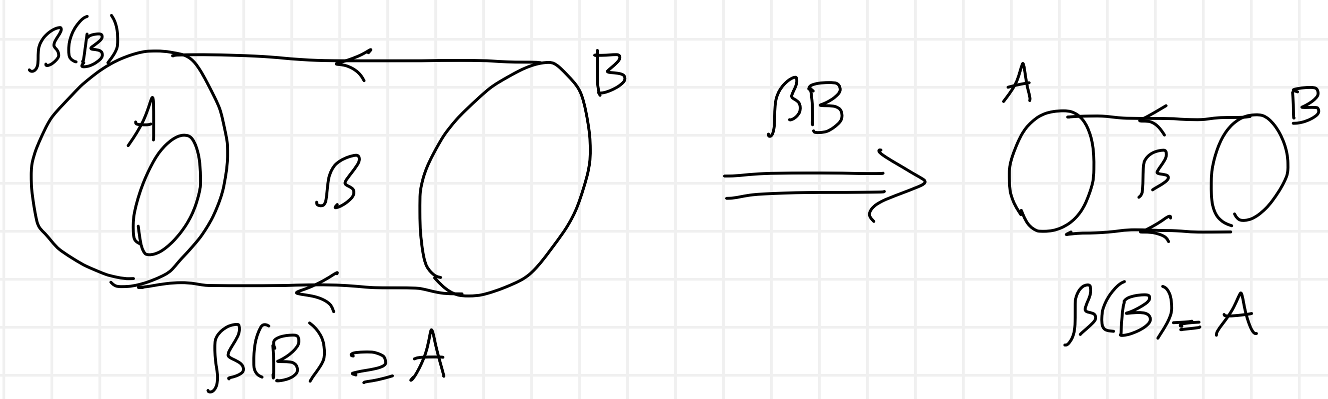
$\vdash \boxed{\beta s}$ $\forall a \in A.\ \exists b \in B.\ a = \beta(b)$ — β surjective on A from B

$a \in A \xrightarrow{A\iota} \beta(\alpha(a)) = a$

$\alpha A \downarrow \quad b \triangleq \alpha(a) \nearrow \beta(b)$

$b \in B$

QED



Guards

conditions

$\boxed{GA}$	$\gamma_A = \mathcal{U}$	— A well-defined everywhere
$\boxed{GB}$	$\gamma_B = \mathcal{U}$	— B well-defined everywhere
$\boxed{G\alpha}$	$\gamma_\alpha \supseteq A$	— α well-defined at least over A
$\boxed{G\beta}$	$\gamma_\beta \supseteq B$	— β well-defined at least over B

$G[A \underset{\beta}{\overset{\alpha}{\leftrightarrows}} B] \triangleq GA \wedge GB \wedge G\alpha \wedge G\beta$ — the constituents of $A \underset{\beta}{\overset{\alpha}{\leftrightarrows}} B$ satisfy the guard conditions

$\vdash \boxed{\surd \alpha A}$ $\omega_{\alpha A}(a)$

$$\omega_{\alpha A}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \Big[a \in A \underset{G\alpha}{\Rightarrow} \cancel{\gamma_\alpha(a)} \wedge \underbrace{\cancel{\gamma_B(\alpha(a))}}_{GB}\Big]\Big]$$

QED

$\vdash \boxed{\surd \beta B}$ $\omega_{\beta B}(b)$

$$\omega_{\beta B}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \Big[b \in B \underset{G\beta}{\Rightarrow} \cancel{\gamma_\beta(b)} \wedge \underbrace{\cancel{\gamma_A(\beta(b))}}_{GA}\Big]\Big]$$

QED

$\vdash \boxed{\surd \beta\alpha}$ $\omega_{\beta\alpha}(a)$

$$\omega_{\beta\alpha}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \Big[a \in A \overset{G\alpha}{\Rightarrow} \cancel{\gamma_\alpha(a)} \wedge \cancel{\gamma_\beta(\alpha(a))}\Big]\Big]$$

$$a \in A \xrightarrow{\alpha A} \alpha(a) \in B \xrightarrow{G\beta}$$

QED

$\vdash \boxed{\surd \alpha\beta}$ $\omega_{\alpha\beta}(b)$

$$\omega_{\alpha\beta}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \Big[b \in B \overset{G\beta}{\Rightarrow} \cancel{\gamma_\beta(b)} \wedge \cancel{\gamma_\alpha(\beta(b))}\Big]\Big]$$

$$b \in B \xrightarrow{\beta B} B(b) \in A \xrightarrow{G\alpha}$$

QED

Generalization to Tuples

$A \subseteq \mathcal{U}^n$ $\quad B \subseteq \mathcal{U}^m$ $\quad \alpha : \mathcal{U}^n \to \mathcal{U}^m$ $\quad \beta : \mathcal{U}^m \to \mathcal{U}^n$

everything works the same as in the unary case

Variant: Unconditional Theorems

$\boxed{\beta\alpha'}$ $\forall a.\ \beta(\alpha(a)) = a$ — holds for $a \notin A$ too

$\boxed{\alpha\beta'}$ $\forall b.\ \alpha(\beta(b)) = b$ — holds for $b \notin B$ too

$\vdash \boxed{\alpha i'}$ $\forall a, a'.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — holds for $a \notin A$ or $a' \notin A$ too

$$\alpha(a) = \alpha(a')$$
$$\downarrow$$
$$a \overset{\beta\alpha'}{=} \beta(\alpha(a)) = \beta(\alpha(a')) \overset{\beta\alpha'}{=} a'$$

QED

$\vdash \boxed{\beta i'}$ $\forall b, b'.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — holds for $b \notin B$ or $b' \notin B$ too

$$\beta(b) = \beta(b')$$
$$\downarrow$$
$$b \overset{\alpha\beta'}{=} \alpha(\beta(b)) = \alpha(\beta(b')) \overset{\alpha\beta'}{=} b'$$

QED

making also αA and βB unconditional seems unnecessery : just have $A = B = \mathcal{U}$ instead