SCHEMATIC ALGORITHM TRANSFORMATION

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Generic Schema

old specification : $S(f) \triangleq \Phi[f]$, $S \subseteq \mathcal{U}^n \to \mathcal{U}^m$

schematic algorithm $\begin{cases} A(f_1, \ldots, f_p) \triangleq \ldots , \quad A \in (\mathcal{U}^{n_1} \to \mathcal{U}^{m_1}) \times \ldots \times (\mathcal{U}^{n_p} \to \mathcal{U}^{m_p}) \to \mathcal{U}^n \to \mathcal{U}^m & \text{— algorithm function} \\ \boxed{\text{COR}} \vdash \underbrace{\Psi_1[f_1, \ldots, f_p] \wedge \cdots \wedge \Psi_q[f_1, \ldots, f_p]}_{\text{each of these may actually depend on a strict subset of } \{f_1, \ldots, f_p\}} \Rightarrow \Psi[A(f_1, \ldots, f_p)] & \text{— correctness theorem} \end{cases}$ $\Big\}$ 2^{nd}-order

condition : $\boxed{\text{MTC}}$ $\Phi[f]$ matches $\Psi[f]$, i.e. $\exists$ substitution σ. $\Phi[f] = \sigma(\Psi[f])$

new specifications $\begin{cases} S_1(f_1, \ldots, f_p) \triangleq \sigma(\Psi_1[f_1, \ldots, f_p]) \\ \vdots \\ S_q(f_1, \ldots, f_p) \triangleq \sigma(\Psi_q[f_1, \ldots, f_p]) \\ S'(f, f_1, \ldots, f_p) \triangleq [f = A(f_1, \ldots, f_p) \wedge S_1(f_1, \ldots, f_p) \wedge \cdots \wedge S_q(f_1, \ldots, f_p)] \end{cases}$

these may be easier to solve when they depend on strict subsets of $\{f_1, \ldots, f_p\}$

$\boxed{SS'}$ $\vdash S'(f, f_1, \ldots, f_p) \Rightarrow S(f)$

$S'(f, f_1, \ldots, f_p) \xrightarrow{\delta_{S'}} f = A(f_1, \ldots, f_p) \longrightarrow \sigma(\Psi[f]) \overset{\text{MTC}}{=} \Phi[f] \overset{\delta_S}{=} S(f)$

$S_1(f_1, \ldots, f_p) \xrightarrow{\delta_{S_1}} \sigma(\Psi_1[f_1, \ldots, f_p]) \xrightarrow{\text{COR}} \sigma(\Psi[A(f_1, \ldots, f_p)]) \longrightarrow \sigma(\Psi[f])$

$\vdots$

$S_q(f_1, \ldots, f_p) \xrightarrow{\delta_{S_q}} \sigma(\Psi_q[f_1, \ldots, f_p]) \longrightarrow \sigma(\Psi[A(f_1, \ldots, f_p)])$

QED

$\underbrace{\hat{f}_1, \ldots, \hat{f}_p \text{ solutions for } S_1, \ldots, S_q}_{\vdash S_1(\hat{f}_1, \ldots, \hat{f}_p) \wedge \cdots \wedge S_q(\hat{f}_1, \ldots, \hat{f}_p)} \Longrightarrow \underbrace{A(\hat{f}_1, \ldots, \hat{f}_p) \text{ solution for } S}_{\vdash S(A(\hat{f}_1, \ldots, \hat{f}_p))}$ — final solution from sub-solutions

see 'Specifications & Refinements' notes for background on S and its forms

Divide & Conquer List 0 – 1 Schema

$A(g,h)(x,\bar{z}) \triangleq$ <u>if</u> $\mathrm{atom}(x)$ <u>then</u> $g(x,\bar{z})$ <u>else</u> $h(\mathrm{car}(x), \bar{z}, A(g,h)(\mathrm{cdr}(x),\bar{z}))$ $\qquad z = z_1, \dots, z_p \quad p \geq 0$

$\mu_A(x,\bar{z}) \triangleq \mathrm{len}(x) \qquad \prec_A \triangleq < \qquad \boxed{\tau_A} \vdash \neg\,\mathrm{atom}(x) \Rightarrow \mathrm{len}(\mathrm{cdr}(x)) < \mathrm{len}(x)$

$\boxed{\text{ZERO}}\ \forall x, \bar{x}, \bar{z}.\ \mathrm{atom}(x) \Rightarrow \rho(x, \bar{x}, g(x,\bar{z}))$

$\boxed{\text{ONE}}\ \forall x, \bar{x}, y, \bar{z}.\ \mathrm{consp}(x) \wedge \rho(\mathrm{cdr}(x), \bar{x}, y) \Rightarrow \rho(x, \bar{x}, h(\mathrm{car}(x), \bar{z}, y))$

$\boxed{\text{ALL}}\ \forall x, \bar{x}, \bar{z}.\ \rho(x, \bar{x}, A(g,h)(x,\bar{z}))$

$\boxed{\text{COR}} \vdash \boxed{\text{ZERO}} \wedge \boxed{\text{ONE}} \Rightarrow \boxed{\text{ALL}}$

induct A

- $\mathrm{atom}(x)$
  - $\xrightarrow{\delta_A} A(g,h)(x,\bar{z}) = g(x,\bar{z})$
  - $\xrightarrow{\text{ZERO}} \rho(x,\bar{x},g(x,\bar{z}))$
  - $\longrightarrow \rho(x,\bar{x},A(g,h)(x,\bar{z}))$
- $\mathrm{consp}(x)$
  - $\xrightarrow{\delta_A} A(g,h)(x,\bar{z}) = h(\mathrm{car}(x), \bar{z}, A(g,h)(\mathrm{cdr}(x),\bar{z}))$
  - $\xrightarrow{\text{IH}} \rho(\mathrm{cdr}(x), \bar{x}, A(g,h)(\mathrm{cdr}(x),\bar{z})) \xrightarrow[\text{ONE}]{} \rho(x,\bar{x},h(\mathrm{car}(x),\bar{z},A(g,h)(\mathrm{cdr}(x),\bar{z})))$
  - $y := A(g,h)(\mathrm{cdr}(x),\bar{z})$
  - $\longrightarrow \rho(x,\bar{x},A(g,h)(x,\bar{z}))$

QED

applicable to specification form $\boxed{Rf\alpha}$ $\quad S(f) = [\forall x, \bar{x}.\ R(x,\bar{x},f(x,\bar{\alpha}(\bar{x})))]$

$\left.\begin{array}{l} \bar{z} := \bar{\alpha}(\bar{x}) \\ \rho := R \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \boxed{\text{ZERO}\alpha}\ \forall x,\bar{x}.\ \mathrm{atom}(x) \Rightarrow R(x,\bar{x},g(x,\bar{\alpha}(\bar{x}))) \\ \boxed{\text{ONE}\alpha}\ \forall x,\bar{x},y.\ \mathrm{consp}(x) \wedge R(\mathrm{cdr}(x),\bar{x},y) \Rightarrow R(x,\bar{x},h(\mathrm{car}(x),\bar{\alpha}(\bar{x}),y)) \\ \boxed{\text{ALL}\alpha}\ \forall x,\bar{x}.\ R(x,\bar{x},A(g,h)(x,\bar{\alpha}(\bar{x}))) \end{array}\right.$

match if
$f = A(g,h)$

Divide & Conquer Set 0-1 Schema

analogous to divide & conquer list 0-1 schema, with:

atom $\longrightarrow$ empty

consp $\longrightarrow$ $\neg$ empty

car $\longrightarrow$ head

cdr $\longrightarrow$ tail

len $\longrightarrow$ cardinality