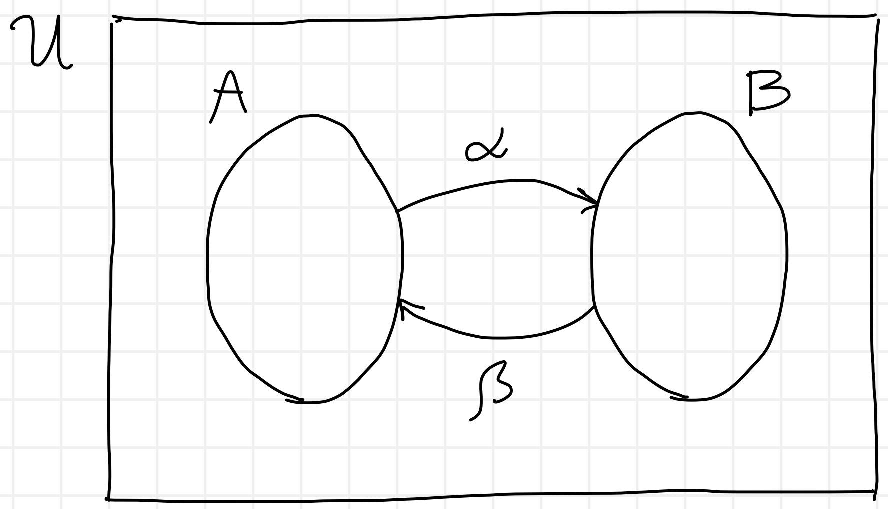
ISOMORPHISMS

Alessandro Coglio

Kestrel Institute

Isomorphic Domains

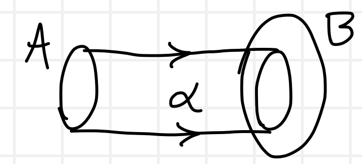


$$\left.\begin{array}{l} A \subseteq \mathcal{U} \\ B \subseteq \mathcal{U} \end{array}\right\} \text{domains (unary predicates)}$$

$$\left.\begin{array}{l} \alpha : \mathcal{U} \to \mathcal{U} \\ \beta : \mathcal{U} \to \mathcal{U} \end{array}\right\} \text{conversions (unary functions)}$$

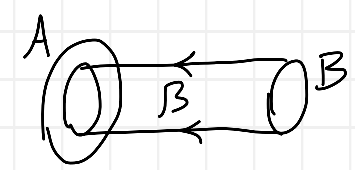
conditions

$\boxed{\alpha A}$ $\forall a \in A.\ \alpha(a) \in B$ — α maps A to B



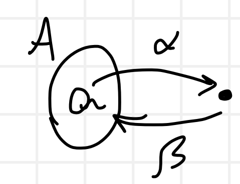
$\alpha(A) \subseteq B$

$\boxed{\beta B}$ $\forall b \in B.\ \beta(b) \in A$ — β maps B to A

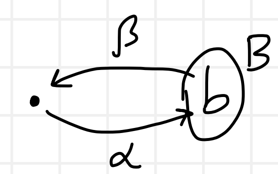


$\beta(B) \subseteq A$

$\boxed{\beta\alpha}$ $\forall a \in A.\ \beta(\alpha(a)) = a$ — β is left inverse of α over A



$\boxed{\alpha\beta}$ $\forall b \in B.\ \alpha(\beta(b)) = b$ — α is left inverse of β over B



$A \overset{\alpha}{\underset{\beta}{\rightleftarrows}} B \triangleq \alpha A \wedge \beta B \wedge \beta\alpha \wedge \alpha\beta$ — α and β are mutually inverse isomorphisms between A and B

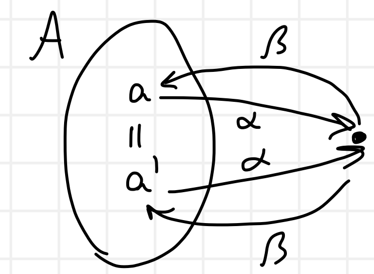
$\vdash \boxed{\alpha i}$ $\forall a, a' \in A.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — α injective on A

$a \in A \qquad \alpha(a) = \alpha(a') \qquad a' \in A$

$$\alpha(a) = \alpha(a') \;\Big\downarrow\; \beta(\alpha(a)) = \beta(\alpha(a'))$$

$$a \in A \xrightarrow{\beta\alpha} \underset{\displaystyle a}{\overset{\beta(\alpha(a))}{\parallel}} \qquad \underset{\displaystyle a'}{\overset{\beta(\alpha(a'))}{\parallel}} \xleftarrow{\beta\alpha} a' \in A$$

QED



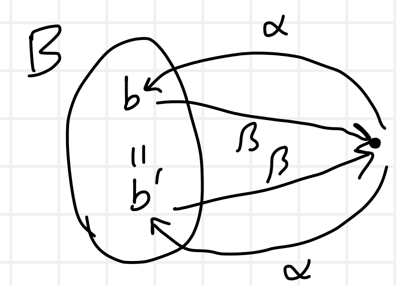
$\vdash \boxed{\beta i}$ $\forall b, b' \in B.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — β injective on B

$b \in B \qquad \beta(b) = \beta(b') \qquad b' \in B$

$$\beta(b) = \beta(b') \;\Big\downarrow\; \alpha(\beta(b)) = \alpha(\beta(b'))$$

$$b \in B \xrightarrow{\alpha\beta} \underset{\displaystyle b}{\overset{\alpha(\beta(b))}{\parallel}} \qquad \underset{\displaystyle b'}{\overset{\alpha(\beta(b'))}{\parallel}} \xleftarrow{\alpha\beta} b' \in B$$

QED



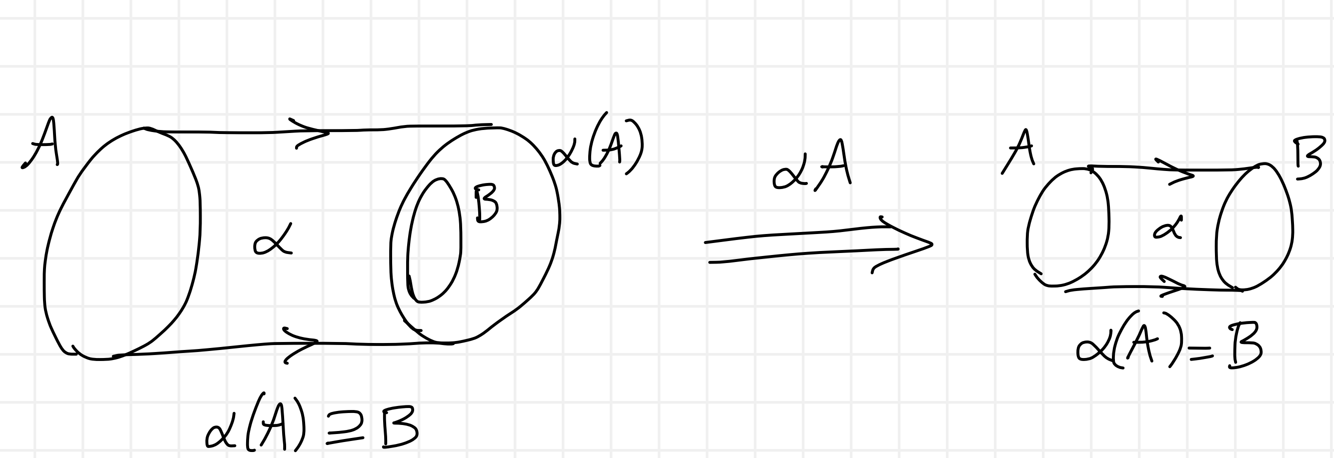
$\vdash \boxed{\alpha s}$ $\forall b \in B.\ \exists a \in A.\ b = \alpha(a)$ — α surjective on B from A

$$b \in B \xrightarrow{\alpha\beta} \alpha(\beta(b)) = b$$

$$\beta b \Big\downarrow \quad a \triangleq \beta(b) \nearrow \overset{\parallel}{\alpha(a)}$$

$$a \in A$$

QED



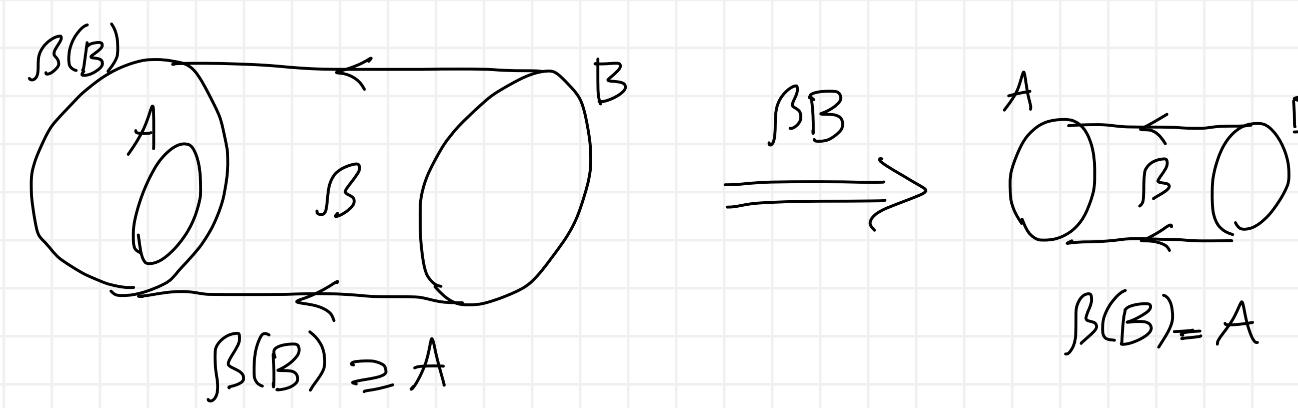
$\vdash \boxed{\beta s}$ $\forall a \in A.\ \exists b \in B.\ a = \beta(b)$ — β surjective on A from B

$$a \in A \xrightarrow{A\iota} \beta(\alpha(a)) = a$$

$$\alpha A \Big\downarrow \quad b \triangleq \alpha(a) \nearrow \overset{\parallel}{\beta(b)}$$

$$b \in B$$

QED



Guards

conditions
$$\begin{cases} \boxed{GA} & \gamma_A = \mathcal{U} & \text{— } A \text{ well-defined everywhere} \\ \boxed{GB} & \gamma_B = \mathcal{U} & \text{— } B \text{ well-defined everywhere} \\ \boxed{G\alpha} & \gamma_\alpha \supseteq A & \text{— } \alpha \text{ well-defined at least over } A \\ \boxed{G\beta} & \gamma_\beta \supseteq B & \text{— } \beta \text{ well-defined at least over } B \end{cases}$$

$G[A \overset{\alpha}{\underset{\beta}{\leftrightarrows}} B] \triangleq GA \wedge GB \wedge G\alpha \wedge G\beta$ — the constituents of $A \overset{\alpha}{\underset{\beta}{\leftrightarrows}} B$ satisfy the guard conditions

$\vdash \boxed{\sqrt{\alpha A}}$ $\omega_{\alpha A}(a)$

$$\omega_{\alpha A}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \big[a \in A \xRightarrow{G\alpha} \cancel{\gamma_\alpha(a)} \wedge \underbrace{\cancel{\gamma_B(\alpha(a))}}_{GB}\big]\Big]$$

QED

$\vdash \boxed{\sqrt{\beta B}}$ $\omega_{\beta B}(b)$

$$\omega_{\beta B}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \big[b \in B \xRightarrow{G\beta} \cancel{\gamma_\beta(b)} \wedge \underbrace{\cancel{\gamma_A(\beta(b))}}_{GA}\big]\Big]$$

QED

$\vdash \boxed{\sqrt{\beta\alpha}}$ $\omega_{\beta\alpha}(a)$

$$\omega_{\beta\alpha}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \big[a \in A \xRightarrow{G\alpha} \cancel{\gamma_\alpha(a)} \wedge \cancel{\gamma_\beta(\alpha(a))}\big]\Big]$$

$$a \in A \xrightarrow{\alpha A} \alpha(a) \in B \xrightarrow{G\beta} \cancel{\gamma_\beta(\alpha(a))}$$

QED

$\vdash \boxed{\sqrt{\alpha\beta}}$ $\omega_{\alpha\beta}(b)$

$$\omega_{\alpha\beta}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \big[b \in B \xRightarrow{G\beta} \cancel{\gamma_\beta(b)} \wedge \cancel{\gamma_\alpha(\beta(b))}\big]\Big]$$

$$b \in B \xrightarrow{\beta B} B(b) \in A \xrightarrow{G\alpha} \cancel{\gamma_\alpha(\beta(b))}$$

QED

Generalization to Tuples

$A \subseteq \mathcal{U}^n$ $B \subseteq \mathcal{U}^m$ $\alpha : \mathcal{U}^n \to \mathcal{U}^m$ $\beta : \mathcal{U}^m \to \mathcal{U}^n$

everything works the same as in the unary case

Variant: Unconditional Theorems

$\boxed{\beta\alpha'}$ $\forall a.\ \beta(\alpha(a)) = a$ — holds for $a \notin A$ too

$\boxed{\alpha\beta'}$ $\forall b.\ \alpha(\beta(b)) = b$ — holds for $b \notin B$ too

$\vdash \boxed{\alpha i'}$ $\forall a, a'.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — holds for $a \notin A$ or $a' \notin A$ too

$$\alpha(a) = \alpha(a')$$
$$\downarrow$$
$$a \overset{\beta\alpha'}{=} \beta(\alpha(a)) = \beta(\alpha(a')) \overset{\beta\alpha'}{=} a'$$

QED

$\vdash \boxed{\beta i'}$ $\forall b, b'.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — holds for $b \notin B$ or $b' \notin B$ too

$$\beta(b) = \beta(b')$$
$$\downarrow$$
$$b \overset{\alpha\beta'}{=} \alpha(\beta(b)) = \alpha(\beta(b')) \overset{\alpha\beta'}{=} b'$$

QED

making also αA and βB unconditional seems unnecessery : just have $A = B = \mathcal{U}$ instead