SCHEMATIC ALGORITHM TRANSFORMATION

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Generic Schema

old specification : $S(f) \triangleq \Phi[f]$, $S \subseteq \mathcal{U}^n \to \mathcal{U}^m$

schematic algorithm $\begin{cases} A(f_1, \ldots, f_p) \triangleq \ldots , \quad A \in (\mathcal{U}^{n_1} \to \mathcal{U}^{m_1}) \times \ldots \times (\mathcal{U}^{n_p} \to \mathcal{U}^{m_p}) \to \mathcal{U}^n \to \mathcal{U}^m & \text{— algorithm function} \\ \boxed{COR} \vdash \underbrace{\Psi_1[f_1, \ldots, f_p] \wedge \cdots \wedge \Psi_q[f_1, \ldots, f_p]}_{\text{each of these may actually depend on a strict subset of } \{f_1, \ldots, f_p\}} \Rightarrow \Psi[A(f_1, \ldots, f_p)] & \text{— correctness theorem} \end{cases}$ } 2nd-order

condition : $\boxed{MTC}$ $\Phi[f]$ matches $\Psi[f]$, i.e. $\exists$ substitution σ. $\Phi[f] = \sigma(\Psi[f])$

new specifications $\begin{cases} S_1(f_1, \ldots, f_p) \triangleq \sigma(\Psi_1[f_1, \ldots, f_p]) \\ \vdots \\ S_q(f_1, \ldots, f_p) \triangleq \sigma(\Psi_q[f_1, \ldots, f_p]) \\ S'(f, f_1, \ldots, f_p) \triangleq [f = A(f_1, \ldots, f_p) \wedge S_1(f_1, \ldots, f_p) \wedge \cdots \wedge S_q(f_1, \ldots, f_p)] \end{cases}$

} these may be easier to solve when they depend on strict subsets of $\{f_1, \ldots, f_p\}$

$\boxed{SS'}$ $\vdash S'(f, f_1, \ldots, f_p) \Rightarrow S(f)$

$S'(f, f_1, \ldots, f_p) \xrightarrow{\delta_{S'}} f = A(f_1, \ldots, f_p) \longrightarrow \sigma(\Psi[f]) \overset{MTC}{=} \Phi[f] \overset{\delta_S}{=} S(f)$

$S_1(f_1, \ldots, f_p) \xrightarrow{\delta_{S_1}} \sigma(\Psi_1[f_1, \ldots, f_p]) \xrightarrow{COR} \sigma(\Psi[A(f_1, \ldots, f_p)])$

$\vdots$

$S_q(f_1, \ldots, f_p) \xrightarrow{\delta_{S_q}} \sigma(\Psi_q[f_1, \ldots, f_p])$

QED

$\underbrace{\hat{f}_1, \ldots, \hat{f}_p \text{ solutions for } S_1, \ldots, S_q}_{\vdash S_1(\hat{f}_1, \ldots, \hat{f}_p) \wedge \cdots \wedge S_q(\hat{f}_1, \ldots, \hat{f}_p)} \Longrightarrow \underbrace{A(\hat{f}_1, \ldots, \hat{f}_p) \text{ solution for } S}_{\vdash S(A(\hat{f}_1, \ldots, \hat{f}_p))}$ — final solution from sub-solutions

see 'Specifications & Refinements' notes for background on S and its forms

Divide & Conquer List 0 – 1 Schema

$A(g,h)(x,\bar z) \triangleq \underline{\text{if}}\ \text{atom}(x)\ \underline{\text{then}}\ g(x,\bar z)\ \underline{\text{else}}\ h(\text{car}(x), \bar z, A(g,h)(\text{cdr}(x), \bar z))$ $\qquad z = z_1, \dots, z_p \quad p \geq 0$

$\mu_A(x,\bar z) \triangleq \text{len}(x) \qquad \prec_A \triangleq < \qquad \boxed{\tau_A} \vdash \neg\,\text{atom}(x) \Rightarrow \text{len}(\text{cdr}(x)) < \text{len}(x)$

$\boxed{\text{ZERO}}\ \forall x, \bar x, \bar z.\ \text{atom}(x) \Rightarrow \rho(x, \bar x, g(x, \bar z))$

$\boxed{\text{ONE}}\ \forall x, \bar x, y, \bar z.\ \text{consp}(x) \wedge \rho(\text{cdr}(x), \bar x, y) \Rightarrow \rho(x, \bar x, h(\text{car}(x), \bar z, y))$

$\boxed{\text{ALL}}\ \forall x, \bar x, \bar z.\ \rho(x, \bar x, A(g,h)(x, \bar z))$

$\boxed{\text{COR}} \vdash \boxed{\text{ZERO}} \wedge \boxed{\text{ONE}} \Rightarrow \boxed{\text{ALL}}$

induct A

- $\text{atom}(x) \xrightarrow{\delta_A} A(g,h)(x,\bar z) = g(x,\bar z)$
- $\text{atom}(x) \xrightarrow{\text{ZERO}} \rho(x, \bar x, g(x, \bar z))$
- $\Rightarrow \rho(x, \bar x, A(g,h)(x, \bar z))$

- $\text{consp}(x) \xrightarrow{\delta_A} A(g,h)(x,\bar z) = h(\text{car}(x), \bar z, A(g,h)(\text{cdr}(x), \bar z))$
- $\text{consp}(x) \xrightarrow{\text{IH}} \rho(\text{cdr}(x), \bar x, A(g,h)(\text{cdr}(x), \bar z)) \xrightarrow[\text{ONE}]{} \rho(x, \bar x, h(\text{car}(x), \bar z, A(g,h)(\text{cdr}(x), \bar z)))$
- $y := A(g,h)(\text{cdr}(x), \bar z)$
- $\Rightarrow \rho(x, \bar x, A(g,h)(x, \bar z))$

QED

applicable to specification form $\boxed{Rf\alpha}$ $\quad S(f) = [\forall x, \bar x.\ R(x, \bar x, f(x, \bar\alpha(\bar x)))]$

$\left.\begin{array}{l} \bar z := \bar\alpha(\bar x) \\ \rho := R \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \boxed{\text{ZERO}\alpha}\ \forall x, \bar x.\ \text{atom}(x) \Rightarrow R(x, \bar x, g(x, \bar\alpha(\bar x))) \\ \boxed{\text{ONE}\alpha}\ \forall x, \bar x, y.\ \text{consp}(x) \wedge R(\text{cdr}(x), \bar x, y) \Rightarrow R(x, \bar x, h(\text{car}(x), \bar\alpha(\bar x), y)) \\ \boxed{\text{ALL}\alpha}\ \forall x, \bar x.\ R(x, \bar x, A(g,h)(x, \bar\alpha(\bar x))) \end{array}\right.$

match if $f = A(g,h)$