Part I
Physics

1 Equality constraints

Legyen C a ket test kozott ertelmezett constraint fuggveny

$$C(p(t)_1, \alpha(t)_1, p(t)_2, \alpha(t)_2)$$

Amennyiben a parameterrekkel C eppen nulla, a constraint kielegitett. Ha a pozicio parameterekkel nem teljeseul a constraint, az azt jelenti, hogy a testek illegalis allapotba kerultek. A constraint gyokei adjak a constraint hiperfeluletet, a legallis alapotvektorok halmazat. A hiperfelulet s dimenzios, ahol s a C fuggveny parametereinek a szama. A hiperfelulet koruli gradiensek azok az iranyok az allapotvektorok halmazaban, amik torik a constraintet. Ha a testek ezen gradiensek menten mozognak, az allapotuk illegalis, ekkor ezen gradiensekkel parhuzamosan mozgatva a testet ujra kielegitheto a constraint.

A C pozicio constraint ido szerinti dervilatja adja a $C^{'}$ velocity constraitet

$$C^{'} = \frac{\partial C}{\partial t}$$

Mivel C a pozicio es az orientacio fuggvenye, amik viszont t ido szerinti fuggvenyek, a lanc szabaly szerint derivalni kell oket, igy kapjuk a linearis es angularis sebesseget.

$$p^{'} = \frac{\partial p}{\partial t} = v(t) \qquad \alpha^{'} = \frac{\partial \alpha}{\partial t} = \omega(t)$$

A C derivalasa a Jacobian matrixhoz vezet, aminek az egyutthatoja a v sebesseg vektor. Ahol v vektor az erintett testek linearis es angularis vektorai.

$$C^{'} = Jv = J \begin{pmatrix} v_1 \\ \omega_1 \\ v_2 \\ \omega_2 \end{pmatrix}$$

asdf Constraint

1.1 Distance Constraint

1.1.1 Position constraint

Let be L the initial distance between q_1 and q_2 where q_1 and q_2 are global points on the two bodies, and u the difference vector from p_1 to p_2, and n is the unit vector of u. $n = \frac{u}{\|u\|}$

$$C = \|(p_1 + r_1) - (p_2 + r_2)\| - L = \|q_1 - q_2\| - L \tag{1}$$

$$C = \|u\| - L \tag{2}$$

1.1.2 Velocity constraint

$$C^{'} = (\|u\| - L)^{'} = \|u\|^{'} \tag{3}$$

$$C^{'} = \frac{1}{2\sqrt{u^2}}(u \cdot u) = \frac{1}{2\sqrt{u^2}}(u \cdot u^{'} + u^{'} \cdot u) \tag{4}$$

$$C^{'} = \frac{1}{2\sqrt{u^2}}2(u \cdot u^{'}) = \frac{1}{\sqrt{u^2}}(u \cdot u^{'}) \tag{5}$$

$$C^{'} = \frac{u}{u\sqrt{u^2}} \cdot u^{'} = \frac{u}{\|u\|} \cdot u^{'} = n \cdot u^{'} \tag{6}$$

$$C^{'} = n \cdot (-v_1 - (\omega_1 \times r_1) - v_2 + (\omega_2 \times r_2)) \tag{7}$$

$$C^{'} = -n \cdot v_1 - n \cdot (\omega_1 \times r_1) + v_2 + n \cdot (\omega_2 \times r_2) \tag{8}$$

$$C^{'} = \boxed{(-n)} \cdot v_1 + \omega_1 \cdot \boxed{(r_1 \times -n)} + \boxed{(n)} \cdot v_2 + \omega_2 \cdot \boxed{(r_2 \times n)} \tag{9}$$

$$\tag{10}$$

1.1.3 Jacobian matrix

Based on the coeficients of the velocity components in the $C^{'}$, the J matrix is

$$J = \begin{bmatrix} -n & -(r_1 \times n) & n & (r_2 \times n) \end{bmatrix}$$

1.1.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -n \\ -(r_1 \times n) \\ n \\ (r_2 \times n) \end{bmatrix} = \begin{bmatrix} -nM_1^{-1} \\ -I_1^{-1}(r_1 \times n) \\ nM_2^{-1} \\ I_2^{-1}(r_2 \times n) \end{bmatrix}$$

1.1.5 Effective mass

$$M_{eff} = JM^{-1}J^T = \begin{bmatrix} -n & -(r_1 \times n) & n & (r_2 \times n) \end{bmatrix} \begin{bmatrix} -nM_1^{-1} \\ -I_1^{-1}(r_1 \times n) \\ nM_2^{-1} \\ I_2^{-1}(r_2 \times n) \end{bmatrix} \tag{11}$$

$$M_{eff} = M_1^{-1} + M_2^{-1} + (r_1 \times n)^2 I_1^{-1} + (r_2 \times n)^2 I_2^{-1} \tag{12}$$

1.1.6 Linear equation solved for λ

$$\lambda = -\frac{C^{'}}{M_{eff}}$$

1.1.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \lambda \begin{bmatrix} -nM_1^{-1} \\ -I_1^{-1}(r_1 \times n) \\ nM_2^{-1} \\ I_2^{-1}(r_2 \times n) \end{bmatrix} = \begin{bmatrix} v_1 - \lambda n M_1^{-1} \\ \omega_1 - \lambda I_1^{-1}(r_1 \times n) \\ v_2 + \lambda n M_2^{-1} \\ \omega_2 + \lambda I_2^{-1}(r_2 \times n) \end{bmatrix}.$$

1.2 Revolute Constraint

1.2.1 Position constraint

Let be r a global point and r_1 and r_2 the difference vectors from p_1 and p_2.

$$C = (p_2 + r_2) - (p_1 + r_1).$$

1.2.2 Velocity constraint

$$C^{'} = ((p_2 + r_2) - (p_1 + r_1))^{'} = (p_2 + r_2)^{'} - (p_1 + r_1)^{'} \tag{13}$$

$$C^{'} = (v_2 + (\omega_2 \times r_2)) - (v_1 + (\omega_1 \times r_1)) \tag{14}$$

$$C^{'} = \boxed{(-1)}v_1 \boxed{-[r_1]_\times} \omega_1 + \boxed{(1)} v_2 + \boxed{[r_2]_\times} \omega_2. \tag{15}$$

1.2.3 Jacobian matrix

Since the dimensions of the coeficent of the components are different we use $\boldsymbol{I_2}$ identity matrix instead of 1.

$$J = \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \end{bmatrix}.$$

1.2.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -\boldsymbol{I_2} \\ -[r_1]_\times^T \\ \boldsymbol{I_2} \\ [r_2]_\times^T \end{bmatrix} = \begin{bmatrix} -M_1^{-1}\boldsymbol{I_2} \\ -I_1^{-1}[r_1]_\times^T \\ M_2^{-1}\boldsymbol{I_2} \\ I_2^{-1}[r_2]_\times^T \end{bmatrix}$$

1.2.5 Effective mass

$$M_{eff} = JM^{-1}J^T = \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \end{bmatrix} \begin{bmatrix} -M_1^{-1}\boldsymbol{I_2} \\ -I_1^{-1}[r_1]_\times^T \\ M_2^{-1}\boldsymbol{I_2} \\ I_2^{-1}[r_2]_\times^T \end{bmatrix} \tag{16}$$

$$= \boldsymbol{I_2}(M_1^{-1} + M_2^{-1}) + I_1^{-1}[r_1]_\times[r_1]_\times^T + I_2^{-1}[r_2]_\times[r_2]_\times^T = \tag{17}$$

$$= \boldsymbol{I_2}(M_1^{-1} + M_2^{-1}) + \begin{bmatrix} I_1^{-1}r_{1_y}^2 & -I_1^{-1}r_{1_x}r_{1_y} \\ -I_1^{-1}r_{1_x}r_{1_y} & I_1^{-1}r_{1_x}^2 \end{bmatrix} + \begin{bmatrix} I_2^{-1}r_{2_y}^2 & -I_2^{-1}r_{2_x}r_{2_y} \\ -I_2^{-1}r_{2_x}r_{2_y} & I_2^{-1}r_{2_x}^2 \end{bmatrix} = \tag{18}$$

$$= \begin{bmatrix} M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_y}^2 + I_2^{-1}r_{2_y}^2 & -I_1^{-1}r_{1_x}r_{1_y} - I_2^{-1}r_{2_x}r_{2_y} \\ -I_1^{-1}r_{1_x}r_{1_y} - I_2^{-1}r_{2_x}r_{2_y} & M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_x}^2 + I_2^{-1}r_{2_x}^2 \end{bmatrix} \tag{19}$$

1.2.6 Linear equation solved for λ

$$\lambda = -\frac{C^{'}}{M_{eff}}$$

1.2.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -M_1^{-1}\lambda\boldsymbol{I_2} \\ -I_1^{-1}\lambda[r_1]_\times^T \\ M_2^{-1}\lambda\boldsymbol{I_2} \\ I_2^{-1}\lambda[r_2]_\times^T \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -\lambda M_1^{-1} \\ -I_1^{-1}(r_1 \times \lambda) \\ \lambda M_2^{-1} \\ I_2^{-1}(r_2 \times \lambda)) \end{bmatrix}.$$

1.3 Line Constraint

1.3.1 Position constraint

Let be r_1 and r_2 two points from two bodies, and n unit vector. The t vector is tangent of the n.

$$\begin{aligned}
C &= t \cdot ((p_2 + r_2) - (p_1 + r_1)) \\
C &= t \cdot (q_2 - q_1) \\
C &= t \cdot u
\end{aligned}$$

1.3.2 Velocity constraint

$$\begin{aligned}
C &= (t \cdot u)^{'} = t \cdot u^{'} + t^{'} \cdot u \\
C &= t \cdot (v_2 + (\omega_2 \times r_2) - v_1 - (\omega_1 \times r_1)) + (\omega_1 \times t) \\
C &= t \cdot v_2 + t \cdot (\omega_2 \times r_2) - t \cdot v_1 - t \cdot (\omega_1 \times r_1) + u \cdot (\omega_1 \times t) \\
C &= t \cdot v_2 + \omega_2 \cdot (r_2 \times t) - t \cdot v_1 - \omega_1 \cdot (r_1 \times t) + \omega_1 \cdot (t \times u) \\
C &= \boxed{(-t)} \cdot v_1 + \omega_1 \cdot \boxed{(t \times (r_1 + u))} + \boxed{(t)} \cdot v_2 + \omega_2 \cdot \boxed{(r_2 \times t)}
\end{aligned}$$

1.3.3 Jacobian matrix

$$J = \begin{bmatrix} -t & (t \times (r_1 + u)) & t & (r_2 \times t) \end{bmatrix}$$

1.3.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -t \\ (t \times (r_1 + u)) \\ t \\ (r_2 \times t) \end{bmatrix} = \begin{bmatrix} -M_1^{-1}t \\ I_1^{-1}(t \times (r_1 + u)) \\ M_2^{-1}t \\ I_2^{-1}(r_2 \times t) \end{bmatrix}$$

1.3.5 Effective mass

$$M_{eff} = JM^{-1}J^T = \begin{bmatrix} -t & (t \times (r_1 + u)) & t & (r_2 \times t) \end{bmatrix} \begin{bmatrix} -M_1^{-1}t \\ I_1^{-1}(t \times (r_1 + u)) \\ M_2^{-1}t \\ I_2^{-1}(r_2 \times t) \end{bmatrix} \tag{20}$$

$$= M_1^{-1} + (t \times (r_1 + u))^2 I_1^{-1} + M_2^{-1} + (r_2 \times t)^2 I_2^{-1} \tag{21}$$

1.3.6 Linear equation solved for λ

$$\lambda = -\frac{C^{'}}{M_{eff}}$$

1.3.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -\lambda M_1^{-1}t \\ \lambda I_1^{-1}(t \times (r_1 + u)) \\ \lambda M_2^{-1}t \\ \lambda I_2^{-1}(r_2 \times t) \end{bmatrix}$$

1.4 Linear Motor Constraint

1.4.1 Position constraint

Let be r_1 and r_2 two points from two bodies, and n unit vector.

$$
\begin{aligned}
C &= n \cdot ((p_2 + r_2) - (p_1 + r_1)) \\
C &= n \cdot (q_2 - q_1) \\
C &= n \cdot u
\end{aligned}
$$

1.4.2 Velocity constraint

$$
\begin{aligned}
C &= (n \cdot u)^{'} = n \cdot u^{'} + n^{'} \cdot u \\
C &= n \cdot (v_2 + (\omega_2 \times r_2) - v_1 - (\omega_1 \times r_1)) + (\omega_1 \times n) \\
C &= n \cdot v_2 + n \cdot (\omega_2 \times r_2) - n \cdot v_1 - n \cdot (\omega_1 \times r_1) + u \cdot (\omega_1 \times n) \\
C &= n \cdot v_2 + \omega_2 \cdot (r_2 \times n) - n \cdot v_1 - \omega_1 \cdot (r_1 \times n) + \omega_1 \cdot (n \times u) \\
C &= \boxed{(-n)} \cdot v_1 + \omega_1 \cdot \boxed{(n \times (r_1 + u))} + \boxed{(n)} \cdot v_2 + \omega_2 \cdot \boxed{(r_2 \times n)}
\end{aligned}
$$

1.4.3 Jacobian matrix

$$
J = \begin{bmatrix} -n & (n \times (r_1 + u)) & n & (r_2 \times n) \end{bmatrix}
$$

1.4.4 Coeficent of λ

$$
M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -n \\ (n \times (r_1 + u)) \\ n \\ (r_2 \times n) \end{bmatrix} = \begin{bmatrix} -M_1^{-1}n \\ I_1^{-1}(n \times (r_1 + u)) \\ M_2^{-1}n \\ I_2^{-1}(r_2 \times n) \end{bmatrix}
$$

1.4.5 Effective mass

$$
\begin{aligned}
M_{eff} = JM^{-1}J^T &= \begin{bmatrix} -n & (n \times (r_1 + u)) & n & (r_2 \times n) \end{bmatrix} \begin{bmatrix} -M_1^{-1}n \\ I_1^{-1}(n \times (r_1 + u)) \\ M_2^{-1}n \\ I_2^{-1}(r_2 \times n) \end{bmatrix} && (22) \\
&= M_1^{-1} + (n \times (r_1 + u))^2 I_1^{-1} + M_2^{-1} + (r_2 \times n)^2 I_2^{-1} && (23)
\end{aligned}
$$

1.4.6 Linear equation solved for λ

$$
\lambda = -\frac{C^{'}}{M_{eff}}
$$

1.4.7 Solved velocity

$$
v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -\lambda M_1^{-1}n \\ \lambda I_1^{-1}(n \times (r_1 + u)) \\ \lambda M_2^{-1}n \\ \lambda I_2^{-1}(r_2 \times n) \end{bmatrix}
$$

1.5 Angle Constraint

1.5.1 Position constraint

Let be α_{ref} the initial angle between two bodies

$$C = \alpha_2 - \alpha_1 - \alpha_{ref}$$

1.5.2 Velocity constraint

$$C' = \omega_2 - \omega_1$$

1.5.3 Jacobian matrix

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$$J = \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}$$

1.5.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -I_1^{-1} \\ 0 \\ I_2^{-1} \end{bmatrix}$$

1.5.5 Effective mass

$$M_{eff} = JM^{-1}J^T = \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -I_1^{-1} \\ 0 \\ I_2^{-1} \end{bmatrix} = I_2^{-1} + I_1^{-1}$$

1.5.6 Linear equation solved for λ

$$\lambda = -\frac{C'}{M_{eff}}$$

1.5.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\lambda I_1^{-1} \\ 0 \\ \lambda I_2^{-1} \end{bmatrix}$$

1.6 Prismatic Constraint

1.6.1 Position constraint

The prismatic contstrat is the combination of the angle and line joint

$$C = \begin{bmatrix} C_{line} \\ C_{angle} \end{bmatrix} = \begin{bmatrix} t \cdot u \\ \alpha_1 - \alpha_2 - \alpha_{ref} \end{bmatrix}$$

1.6.2 Velocity constraint

$$C^{'} = \begin{bmatrix} C^{'}_{line} \\ C^{'}_{angle} \end{bmatrix} = \begin{bmatrix} t^T \cdot v_2 + \omega_2 \cdot (r_2 \times t) - t^T \cdot v_1 - \omega_1 \cdot (r_1 \times t) + \omega_1 \cdot (t \times u) \\ \omega_1 - \omega_2 \end{bmatrix}$$

1.6.3 Jacobian matrix

$$J = \begin{bmatrix} J_{line} \\ J_{angle} \end{bmatrix} = \begin{bmatrix} -t & (t \times (r_1 + u)) & t & (r_2 \times t) \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

1.6.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -t & 0 \\ (t \times (r_1 + u)) & 1 \\ t & 0 \\ (r_2 \times t) & -1 \end{bmatrix} = \begin{bmatrix} -tM_1^{-1} & 0 \\ I_1^{-1}(t \times (r_1 + u)) & I_1^{-1} \\ tM_2^{-1} & 0 \\ I_2^{-1}(r_2 \times t) & -I_2^{-1} \end{bmatrix}$$

1.6.5 Effective mass

$$\begin{aligned} M_{eff} = JM^{-1}J^T &= \begin{bmatrix} -t & (t \times (r_1 + u)) & t & (r_2 \times t) \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -tM_1^{-1} & 0 \\ I_1^{-1}(t \times (r_1 + u)) & I_1^{-1} \\ tM_2^{-1} & 0 \\ I_2^{-1}(r_2 \times t) & -I_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (M_1^{-1} + I_1^{-1}(t \times (r_1 + u))^2 + M_2^{-1} + I_2^{-1}(r_2 \times t)^2 & (t \times (r_1 + u))I_1^{-1} - I_2^{-1}(r_2 \times t) \\ I_1^{-1}(t \times (r_1 + u)) - I_2^{-1}(r_2 \times t) & I_1^{-1} + I_2^{-1} \end{bmatrix} \end{aligned}$$

1.6.6 Linear equation solved for λ

$$\lambda = -\frac{C^{'}}{M_{eff}}$$

1.6.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -tM_1^{-1} & 0 \\ I_1^{-1}(t \times (r_1 + u)) & I_1^{-1} \\ tM_2^{-1} & 0 \\ I_2^{-1}(r_2 \times t) & -I_2^{-1} \end{bmatrix} \lambda = \begin{bmatrix} -tM_1^{-1}\lambda_1 \\ I_1^{-1}(t \times (r_1 + u))\lambda_1 + I_1^{-1}\lambda_2 \\ tM_2^{-1}\lambda_1 \\ I_2^{-1}(r_2 \times t)\lambda_1 - I_2^{-1}\lambda_2 \end{bmatrix}$$

1.7 Weld Constraint

1.7.1 Position constraint

The weld contstrat is the combination of the revolute and angle constraints

$$C = \begin{bmatrix} C_{revolute} \\ C_{angle} \end{bmatrix} = \begin{bmatrix} (p_1 + r_1) - (p_2 + r_2) \\ \alpha_1 - \alpha_2 - \alpha_{ref} \end{bmatrix}$$

1.7.2 Velocity constraint

$$C' = \begin{bmatrix} C'_{revolute} \\ C'_{angle} \end{bmatrix} = \begin{bmatrix} (v_1 + (\omega_1 \times r_1)) - (v_2 + (\omega_2 \times r_2)) \\ \omega_1 - \omega_2 \end{bmatrix}$$

1.7.3 Jacobian matrix

$$J = \begin{bmatrix} J_{revolute} \\ J_{angle} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

1.7.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -\boldsymbol{I_2} & 0 \\ -[r_1]_\times^T & 1 \\ \boldsymbol{I_2} & 0 \\ [r_2]_\times^T & -1 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I_2} M_1^{-1} & 0 \\ [r_{1_y} I_1^{-1} \quad -r_{1_x} I_1^{-1}] & I_1^{-1} \\ \boldsymbol{I_2} M_2^{-1} & 0 \\ [-r_{2_y} I_2^{-1} \quad r_{2_x} I_2^{-1}] & -I_2^{-1} \end{bmatrix}$$

1.7.5 Effective mass

$$M_{eff} = JM^{-1}J^T = \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\boldsymbol{I_2} M_1^{-1} & 0 \\ -[r_1]_\times^T I_1^{-1} & I_1^{-1} \\ \boldsymbol{I_2} M_2^{-1} & 0 \\ [r_2]_\times^T I_1^{-1} & -I_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I_2}(M_1^{-1} + M_2^{-1}) + [r_1]_\times [r_1]_\times^T I_1^{-1} + [r_2]_\times [r_2]_\times^T I_1^{-1} & -[r_1]_\times I_1^{-1} - [r_2]_\times I_2^{-1} \\ -[r_1]_\times^T I_1^{-1} - [r_2]_\times^T I_1^{-1} & I_1^{-1} + I_2^{-1} \end{bmatrix}$$

$$= E \begin{bmatrix} \begin{bmatrix} M_1^{-1} + M_2^{-1} + r_{1_y}^2 I_1^{-1} + r_{2_y}^2 I_2^{-1} & -r_{1_y} r_{1_x} I_1^{-1} - r_{2_y} r_{2_x} I_2^{-1} \\ -r_{1_y} r_{1_x} I_1^{-1} - r_{2_y} r_{2_x} I_2^{-1} & M_1^{-1} + M_2^{-1} + r_{1_x}^2 I_1^{-1} + r_{2_x}^2 I_2^{-1} \end{bmatrix} & \begin{bmatrix} r_{1_y} I_1^{-1} - r_{2_y} I_2^{-1} \\ r_{2_x} I_2^{-1} - r_{1_x} I_1^{-1} \end{bmatrix} \\ [r_{1_y} I_1^{-1} + r_{2_y} I_2^{-1} \quad -r_{1_x} I_1^{-1} - r_{2_x} I_2^{-1}] & I_1^{-1} + I_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} M_1^{-1} + M_2^{-1} + r_{1_y}^2 I_1^{-1} + r_{2_y}^2 I_2^{-1} & -r_{1_y} r_{1_x} I_1^{-1} - r_{2_y} r_{2_x} I_2^{-1} & r_{1_y} I_1^{-1} - r_{2_y} I_2^{-1} \\ -r_{1_y} r_{1_x} I_1^{-1} - r_{2_y} r_{2_x} I_2^{-1} & M_1^{-1} + M_2^{-1} + r_{1_x}^2 I_1^{-1} + r_{2_x}^2 I_2^{-1} & r_{2_x} I_2^{-1} - r_{1_x} I_1^{-1} \\ r_{1_y} I_1^{-1} + r_{2_y} I_2^{-1} & -r_{1_x} I_1^{-1} - r_{2_x} I_2^{-1} & I_1^{-1} + I_2^{-1} \end{bmatrix}$$

1.7.6 Linear equation solved for λ

$$\lambda = -\frac{C'}{M_{eff}}$$

1.7.7 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{I_2} M_1^{-1} & 0 \\ [r_{1_y} I_1^{-1} \quad -r_{1_x} I_1^{-1}] & I_1^{-1} \\ \boldsymbol{I_2} M_2^{-1} & 0 \\ [-r_{2_y} I_2^{-1} \quad r_{2_x} I_2^{-1}] & -I_2^{-1} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} M_1^{-1} \\ \lambda_1 r_{1_y} I_1^{-1} - \lambda_2 r_{1_x} I_1^{-1} + I_1^{-1}\lambda_3 \\ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} M_2^{-1} \\ \lambda_2 r_{2_x} I_2^{-1} - \lambda_1 r_{2_y} I_2^{-1} - I_2^{-1}\lambda_3 \end{bmatrix}$$

1.8 Pulley Constraint

Let be r_1 and r_2 points on $body_1$ and $body_2$, a_1 and a_2 global anchor points for the bodies. The difference vector between a_i and r_1 is u_i and the lengh of the u_i is l_i. r is the ratio between the two anchor points.

$$u_i = r_i - a_i \qquad l_i = \|u_i\| = \sqrt{u_i \cdot u_i} \qquad n_i = \frac{u_i}{\|u_i\|}$$

The initial distances saved in C_i and it helps to obtain the distance of the bodies

$$\begin{aligned} C_i &= r\|r_1 - a_1\| + \|r_2 + a_2\| = r\|u_1\| + \|u_2\| \\ C_i &= rl_1 + l_2 \end{aligned}$$

1.8.1 Position constraint

$$C = C_i - (rl_1 + l_2)$$

1.8.2 Velocity constraint

First find $l_i^{'}$ and use it in $C^{'}$ because of the chain rule

$$\begin{aligned} l_i^{'} &= \|u_i\|^{'} = (\sqrt{u_i \cdot u_i})^{'} = \frac{1}{2\sqrt{u_i \cdot u_i}}(u_i \cdot u_i)^{'} \\ l_i^{'} &= \frac{1}{2\sqrt{u_i \cdot u_i}}(u_i \cdot u_i^{'} + u_i^{'} \cdot u_i) = \frac{1}{2\sqrt{u_i \cdot u_i}}2(u_i \cdot u_i^{'}) \\ l_i^{'} &= \frac{1}{\sqrt{u_i \cdot u_i}}(u_i \cdot u_i^{'}) = \frac{u_i}{\|u_i\|}u_i^{'} = n_i \cdot u_i^{'} = n_i \cdot (v_i + (\omega_i \times r_i)). \\ C^{'} &= (C_i - (rl_1 + l_2))^{'} = (0 - (rl_1 + l_2))^{'} = -rl_1^{'} - l_2^{'} \\ C^{'} &= -rn_1 \cdot (v_1 + (\omega_1 \times r_1)) - n_2 \cdot (v_2 + (\omega_2 \times r_2)) \\ C^{'} &= \boxed{(-rn_1)} \cdot v_1 + \omega_1 \cdot \boxed{r(r_1 \times -n_1)} + \boxed{(-n_2)} \cdot v_2 + \omega_2 \cdot \boxed{(r_2 \times -n_2)} \end{aligned}$$

1.8.3 Jacobian matrix

$$J = \begin{bmatrix} -rn_1 & -r(r_1 \times n_1) & -n_2 & -(r_2 \times n_2) \end{bmatrix}$$

1.8.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -rn_1 \\ -r(r_1 \times n_1) \\ -n_2 \\ -(r_2 \times n_2) \end{bmatrix} = \begin{bmatrix} -M_1^{-1}rn_1 \\ -I_1^{-1}r(r_1 \times n_1) \\ -M_2^{-1}n_2 \\ -I_2^{-1}(r_2 \times n_2) \end{bmatrix}$$

1.8.5 Effective mass

$$\begin{aligned} M_{eff} = JM^{-1}J^T &= \begin{bmatrix} -rn_1 & -r(r_1 \times n_1) & -n_2 & -(r_2 \times n_2) \end{bmatrix} \begin{bmatrix} -M_1^{-1}rn_1 \\ -I_1^{-1}r(r_1 \times n_1) \\ -M_2^{-1}n_2 \\ -I_2^{-1}(r_2 \times n_2) \end{bmatrix} \\ &= M_1^{-1}r^2 + M_2^{-1} + r^2 I_1^{-1}(r_1 \times n_1)^2 + I_2^{-1}(r_2 \times n_2)^2 \end{aligned}$$

1.8.6 Solved velocity

$$v_{res} = v_{imp} + M^{-1}J^T\lambda = \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -M_1^{-1}rn_1\lambda \\ -I_1^{-1}r(r_1 \times n_1)\lambda \\ -M_2^{-1}n_2\lambda \\ -I_2^{-1}(r_2 \times n_2)\lambda \end{bmatrix}$$

1.9 Motored Revolute Constraint

The motored revolute constraint is a hybrid constraint, which is a combination of revolute and a angle joint

1.9.1 Position constraint

$$C = \begin{bmatrix} C_{revolute} \\ C_{angle} \end{bmatrix} = \begin{bmatrix} (p_2 + r_2) - (p_1 + r_1) \\ \alpha_2 - \alpha_1 - \alpha_{target} \end{bmatrix}$$

1.9.2 Velocity constraint

$$C^{'} = \begin{bmatrix} C^{'}_{revolute} \\ C^{'}_{angle} \end{bmatrix} = \begin{bmatrix} (v_2 + (\omega_2 \times r_2)) - (v_1 + (\omega_1 \times r_1)) \\ \omega_2 - \omega_1 \end{bmatrix}$$

1.9.3 Jacobian matrix

$$J = \begin{bmatrix} J_{revolute} \\ J_{angle} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

1.9.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{bmatrix} \begin{bmatrix} -\boldsymbol{I_2} & 0 \\ -[r_1]_\times^T & -1 \\ \boldsymbol{I_2} & 0 \\ [r_2]_\times^T & 1 \end{bmatrix} = \begin{bmatrix} -M_1^{-1}\boldsymbol{I_2} & 0 \\ -I_1^{-1}[r_1]_\times^T & -I_1^{-1} \\ M_2^{-1}\boldsymbol{I_2} & 0 \\ I_2^{-1}[r_2]_\times^T & I_2^{-1} \end{bmatrix}$$

1.9.5 Effective mass

$$\begin{aligned}
M_{eff} = JM^{-1}J^T &= \begin{bmatrix} -\boldsymbol{I_2} & -[r_1]_\times & \boldsymbol{I_2} & [r_2]_\times \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -M_1^{-1}\boldsymbol{I_2} & 0 \\ -I_1^{-1}[r_1]_\times^T & -I_1^{-1} \\ M_2^{-1}\boldsymbol{I_2} & 0 \\ I_2^{-1}[r_2]_\times^T & I_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I_2}(M_1^{-1} + M_2^{-1}) + I_1^{-1}[r_1]_\times[r_1]_\times^T + I_2^{-1}[r_2]_\times[r_2]_\times^T & I_1^{-1}[r_1]_\times + I_2^{-1}[r_2]_\times \\ I_1^{-1}[r_1]_\times^T + I_2^{-1}[r_2]_\times^T & I_1^{-1} + I_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_y}^2 + I_2^{-1}r_{2_y}^2 & -I_1^{-1}r_{1_y}r_{1_x} - I_2^{-1}r_{2_y}r_{2_x} \\ -I_1^{-1}r_{1_x}r_{1_y} - I_2^{-1}r_{2_x}r_{2_y} & M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_x}^2 + I_2^{-1}r_{2_x}^2 \end{bmatrix} & \begin{bmatrix} -I_1^{-1}r_{1_y} - I_2^{-1}r_{2_y} \\ I_1^{-1}r_{1_x} + I_2^{-1}r_{2_x} \end{bmatrix} \\ \begin{bmatrix} -I_1^{-1}r_{1_y} - I_2^{-1}r_{2_y} & I_1^{-1}r_{1_x} + I_2^{-1}r_{2_x} \end{bmatrix} & I_1^{-1} + I_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_y}^2 + I_2^{-1}r_{2_y}^2 & -I_1^{-1}r_{1_y}r_{1_x} - I_2^{-1}r_{2_y}r_{2_x} & -I_1^{-1}r_{1_y} - I_2^{-1}r_{2_y} \\ -I_1^{-1}r_{1_x}r_{1_y} - I_2^{-1}r_{2_x}r_{2_y} & M_1^{-1} + M_2^{-1} + I_1^{-1}r_{1_x}^2 + I_2^{-1}r_{2_x}^2 & I_1^{-1}r_{1_x} + I_2^{-1}r_{2_x} \\ -I_1^{-1}r_{1_y} - I_2^{-1}r_{2_y} & I_1^{-1}r_{1_x} + I_2^{-1}r_{2_x} & I_1^{-1} + I_2^{-1} \end{bmatrix}
\end{aligned}$$

1.9.6 Solved velocity

$$\begin{aligned}
v_{res} = v_{imp} + M^{-1}J^T\lambda &= \begin{bmatrix} v_1 \\ \omega_1 \\ v2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -M_1^{-1} & 0 \\ -I_1^{-1}[r_1]_\times^T & -I_1^{-1} \\ M_2^{-1} & 0 \\ I_2^{-1}[r_2]_\times^T & I_2^{-1} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ \lambda_3 \end{bmatrix} \\
&= \begin{bmatrix} -M_1^{-1}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ -I_1^{-1}(r_{1_x}\lambda_2 - r_{1_y}\lambda_1) - I_1^{-1}\lambda_3 \\ M_2^{-1}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ I_2^{-1}(r_{2_x}\lambda_2 - r_{2_y}\lambda_1) + I_2^{-1}\lambda_3 \end{bmatrix}
\end{aligned}$$

1.10 Mouse Constraint

Let be r is an arbiraty point on a body, and m is the current position of the cursor.

1.10.1 Position constraint

$$C = p - m$$

1.10.2 Velocity constraint

$$C^{'} = (r - m)^{'} = r^{'} - m^{'} = r^{'} - 0 = r^{'} \tag{24}$$

$$C^{'} = v + (\omega \times r) \tag{25}$$

$$C^{'} = \boxed{(1)}\, v + \boxed{[r]_\times}\, \omega \tag{26}$$

1.10.3 Jacobian matrix

$$J = \begin{bmatrix} \boldsymbol{I_2} & [r]_\times \end{bmatrix}$$

1.10.4 Coeficent of λ

$$M^{-1}J^T = \begin{bmatrix} M_1^{-1} & 0 \\ 0 & I_1^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I_2} \\ [r]_\times^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{I_2} M_1^{-1} \\ I_1^{-1}[r]_\times^T \end{bmatrix}$$

1.10.5 Effective mass

$$\begin{aligned} M_{eff} = JM^{-1}J^T &= \begin{bmatrix} \boldsymbol{I_2} & [r]_\times \end{bmatrix} \begin{bmatrix} \boldsymbol{I_2} M_1^{-1} \\ I_1^{-1}[r]_\times^T \end{bmatrix} \\ &= \boldsymbol{I_2} M_1^{-1} + I_1^{-1}[r]_\times [r]_\times^T \\ &= \begin{bmatrix} M_1^{-1} & 0 \\ 0 & M_1^{-1} \end{bmatrix} + \begin{bmatrix} I_1^{-1} r_y^2 & -I_1^{-1} r_x r_y \\ -I_1^{-1} r_x r_y & I_1^{-1} r_x^2 \end{bmatrix} \\ &= \begin{bmatrix} M_1^{-1} + I_1^{-1} r_y^2 & -I_1^{-1} r_x r_y \\ -I_1^{-1} r_x r_y & M_1^{-1} + I_1^{-1} r_x^2 \end{bmatrix} \end{aligned}$$

1.10.6 Solved velocity

$$\begin{aligned} v_{res} = v_{imp} + M^{-1}J^T\lambda &= \begin{bmatrix} v \\ \omega \end{bmatrix} + \begin{bmatrix} \boldsymbol{I_2} M_1^{-1} \\ I_1^{-1}[r]_\times^T \end{bmatrix} \lambda \\ &= \begin{bmatrix} v \\ \omega \end{bmatrix} + \begin{bmatrix} \lambda M_1^{-1} \\ I^{-1}(\lambda_2 r_x - \lambda_1 r_y) \end{bmatrix} \end{aligned}$$

2 Soft constraints

From Newton's law

$$F = ma$$

The a acceleration is the velocity change in a given h time step

$$a = \Delta v = v_2 - v_1 \tag{27}$$

$$F = m(v_2 - v_1) \tag{28}$$

Velocity constraint with softness and Baumgarte

$$Jv_2 + softness\lambda + bias = 0$$

Where

$$bias = biasFactor * C^{'}$$

The only missing thins are v_2 and λ.

$$v_2 = v_1 + M^{-1}J^T\lambda$$

Substitue back this to the velocity constraint

$$J(v_1 + M^{-1}J^T\lambda) + softness\lambda + bias = 0$$

Coefficients of λ

$$(JM^{-1}J^T + softness)\lambda = -Jv_1 - bias$$

Effective mass

$$M_{eff} = \frac{1}{JM^{-1}J^T + softness}$$

Thus

$$M(v2_{new} - v2_{damaged}) = J^T P_d$$
$$Jv2_{new} + softness(P + P_d) + bias = 0$$
$$v2_{new} = v2_{damaged} + M^{-1}J^T P_d$$
$$J(v2_{damaged} + M^{-1}J^T P_d) + softness * P + softness * P_d + bias = 0$$
$$(JM^{-1}J^T + softness)P_d = -(Jv2_{damaged} + softness * P + bias)$$