Integration

From u-Substitution to Residue Calculus

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Dedicated to those who read it.

Contents

Preface

Integration is usually taught in first semester calculus, but can take years to master and understand nuances. This book tries to capture different techniques involved in calculating integrals and the intricacies of choosing a method of attack.

Prerequisites

The content of this book assumes basic knowledge of algebra, differential/integral calculus, and real/complex analysis.

Structure of book

Each method discussed will be broken into four parts:

- Intuition
- Definition
- Example
- Practice

Acknowledgements

- I'd like to thank Mike the Capybara.

1

Substitution

"The aim of science is to make difficult things understandable in a simpler way; the aim of poetry is to state simple things in an incomprehensible way. The two are incompatible."

– Paul Dirac

1.1 *u*-Substitution

The well known, u-substitution, is often one of the first integration techniques taught to calculus students. It gets its name from the variable often chosen to represent the substitution.

Definition

Substitutions can lead to trouble when you're not careful. Consider $\int_a^b f(x)\mathrm{d}x$ and the substitution $u \mapsto (x-a)(x-b)$. This leads to the solution,

$$\int_0^0 h(u)\mathrm{d}u = 0.$$

This is because our choice of substitution needs to be bijective on $[a, b]$. Substitution has many little nuances, which shows the need for a more formal definition. Let I be a real interval where $\phi : [a, b] \to I$ is differentiable and $f : I \to \mathbb{R}$ is continuous, then

$$\int_{\phi(a)}^{\phi(b)} f(x)\mathrm{d}x = \int_a^b f(\phi(t))\phi'(t)\mathrm{d}t$$

Now if we let $x \mapsto \phi(t)$, then the differential for x is $\mathrm{d}x = \phi'(t)\mathrm{d}t$. To go from a to $\phi(a)$ requires an inverse function of ϕ, meaning *phi* must be bijective at $x = a$ and likewise for b.

Intuition

The single variable chain rule states

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x),$$

which means any integral in the form of

$$\int f'(g(x))g'(x)\mathrm{d}x = f(g(x)) + c.$$

Setting $u \mapsto g(x)$,

$$\begin{aligned}\int f'(g(x))g'(x)\mathrm{d}x &= \int f'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ &= \int f'(u)\mathrm{d}u \\ &= f(u) + c \\ &= f(g(x)) + c\end{aligned}$$

When evaluating definite integrals, one must remember that the endpoints of integration are in terms of x, i.e.

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x.$$

When we make the integral with respect to $u(x)$, then we have to change the endpoints as well, i.e.

$$\int_{u(a)}^{u(b)} h(u)\mathrm{d}u.$$

Examples

1. Integrate $\displaystyle\int \sin(x^2)x\,\mathrm{d}x$.

   Substitution is handy when some function f and some constant multiple of its derivative f' lie within the integrand. Here it is trivial that $\frac{\mathrm{d}}{\mathrm{d}x}[x^2] = 2x$, so we make the substitution $u \mapsto x^2, \quad \mathrm{d}u = 2x\mathrm{d}x \implies (1/2)\mathrm{d}u = x\mathrm{d}x$.

$$\begin{aligned}\int \sin(x^2)x\mathrm{d}x &= \frac{1}{2}\int \sin u\mathrm{d}u && (1.1) \\ &= -\frac{1}{2}\cos u + c \\ &= -\frac{1}{2}\cos x^2 + c && (1.2)\end{aligned}$$

In step (1.1) we substitute u for x^2 and likewise $(1/2)\mathrm{d}u$ for $x\mathrm{d}x$. Next we integrate with respect to u. It is important to substitute x^2 back in for u as shown in (1.2).

2. Integrate $\int_0^4 \sqrt{5-\sqrt{x}}\mathrm{d}x$.

   Let $u \mapsto 5-\sqrt{x} \implies \sqrt{x} = 5-u \implies x = (5-u)^2 = 25 - 10u + u^2$ and therefore $\mathrm{d}x = -10 + 2u\,\mathrm{d}u$. (1.3) below shows this substitution.

   Since this is a definite integral we must re-evaluate the endpoints ($x = 0$ and $x = 4$), which leaves us with $u = 5 - \sqrt{0} = 5$ and $u = 5 - \sqrt{4} = 3$. Now we're left with,

$$\begin{aligned} \int_0^4 \sqrt{5-\sqrt{x}}\mathrm{d}x &= \int_5^3 \sqrt{u}(-10+2u)\,\mathrm{d}u \qquad (1.3) \\ &= -\int_3^5 \left(-10u^{1/2} + 2u^{3/2}\right)\,\mathrm{d}u \\ &= -\left[-\frac{20}{3}u^{3/2} + \frac{4}{5}u^{5/2}\right]_{u=3}^{u=5} \qquad (1.4) \\ &= -\frac{64\sqrt{3}}{5} + \frac{40\sqrt{5}}{3} \end{aligned}$$

   Since we changed changed the endpoints from x to u values, in step (1.4) we don't need to substitute $5-\sqrt{x}$ back in for u.

Exercises

For the following exercises integrate using u-substitution:

1.1.1. $\displaystyle\int \frac{(5+\ln^2 x)(2-\ln x)}{7x}\,\mathrm{d}x$.

1.1.2. $\displaystyle\int \sqrt{1+5x}\,\mathrm{d}x$.

1.1.3. $\displaystyle\int \sin x \cos x\,\mathrm{d}x$.

1.1.4. $\displaystyle\int \sqrt{4-x^2}\,\mathrm{d}x$ [*Hint:* substitute $x \mapsto 2\sin\theta$].

1.1.5. $\displaystyle\int e^{e^{kx}+3kx}\,\mathrm{d}x$.

1.2 Reflection Substitution

Reflection substitution is just a special case of u-substitution, but it's remarkably useful. We simply make the substitution $x \mapsto a+b-x$,

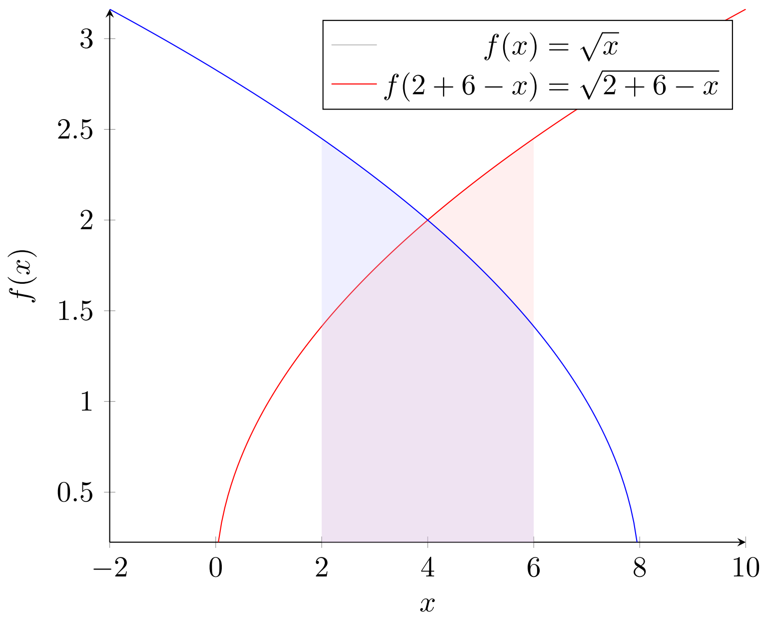

Figure 1.1: Graphical intuition behind $x \mapsto a + b - x$.

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x.$$

Definition

The definition is rather easy to state as we are only making a simple substitution. Consider, $\int_a^b f(x)\,\mathrm{d}x$. Let $x \mapsto a + b - x$ and $\mathrm{d}x \mapsto -\mathrm{d}x$.

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= -\int_b^a f(a+b-x)\,\mathrm{d}x \\ &= \int_a^b f(a+b-x)\,\mathrm{d}x\end{aligned}$$

Intuition

Graphically the method represents a *reflection* about the line $x = \frac{b+a}{2}$, depicted in figure (1.1). We see here that the area of the shaded region is the same for both functions. Consider the integral

$$\int_2^6 \sqrt{x}\,\mathrm{d}x.$$

Using our substitution rule we know that

$$\begin{aligned}\int_2^6 \sqrt{x}\,\mathrm{d}x &= \int_2^6 \sqrt{2+6-x}\,\mathrm{d}x \\ &= \left[-\frac{2}{3}(8-x)^{3/2}\right]_2^6 \\ &= -\frac{2}{3}\left(2\sqrt{2}-6\sqrt{6}\right).\end{aligned}$$

Now we can validate this by checking the original integral, which indeed is the same (*you can check for yourself*).

Examples

1. Evaluate $\displaystyle\int_0^\pi \frac{\sin^2 x}{\pi - x}\,\mathrm{d}x$

   The Reflection Substitution often comes in handy when $a = 0$, $b = \pi$, and there's a sine function within the integrand. This is because $\sin(\pi - x) = \sin(x)$. This applies directly to the current problem.

$$\begin{aligned}\int_0^\pi \frac{\sin^2 x}{\pi - x}\,\mathrm{d}x &= \int_0^\pi \frac{\sin^2 x}{x}\,\mathrm{d}x && (1.5)\\ &\approx 1.21883 && (1.6)\end{aligned}$$

   In step (1.5) we make the substitution $x \mapsto \pi - x$. On the next line, (1.6), we use numeric approximations, because the resulting integral does not have an elementary anti-derivative.

2. Evaluate $\displaystyle\int_0^\pi \frac{x\tan x}{\sec x + \cos x}\,\mathrm{d}x$.

   Again, begin this by making the substitution $x \mapsto \pi - x$.

$$\begin{aligned}I &= \int_0^\pi \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \cos(\pi - x)}\,\mathrm{d}x \\ &= \int_0^\pi \frac{(x-\pi)(-\tan(x))}{-\sec(x) - \cos(x)}\,\mathrm{d}x \\ &= \pi \int_0^\pi \frac{\tan(x)}{\sec(x) + \cos(x)}\,\mathrm{d}x - I\end{aligned}$$

   Let $J = \int_0^\pi \frac{\tan(x)}{\sec(x)+\cos(x)}\,\mathrm{d}x$, such that $I = \frac{\pi}{2}J$. We can find J by substituting $u \mapsto \cos x$ and we get $J = \frac{\pi}{2}$. Substituting back in,

$$I = \frac{\pi}{2}J = \frac{\pi^2}{4}.$$

Exercises

1.2.1. Show that $\int_0^\pi x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x$. [*Hint:* $\sin(\pi - x) = \sin x$]

1.2.2. Show that $\int_0^{n\pi} f(\cos^2\theta)\,\mathrm{d}\theta = n\int_0^\pi f(\cos^2\theta)\,\mathrm{d}\theta$.

1.3 Euler Substitution

Leonhard Euler, a Swiss mathematician from the 1700s first introduced this method of integration, hence the name. Like reflection substitution, it's just a special case of a u-substitution.

Definition

Let $R(f, g)$ denote a rational function of f and g. Given the integral,

$$\int R(x, \sqrt{ax^2 + bx + c})\,\mathrm{d}x,$$

we make one of three substitutions:

1. If $a > 0$, then we make the substitution,

$$\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} + t$$

$$\Downarrow$$

$$x = \frac{t^2 - c}{b + 2t\sqrt{a}}$$

2. If $c > 0$, then we make the substitution,

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$$

$$\Downarrow$$

$$x = \frac{2t\sqrt{c} + b}{t^2 - a}$$

3. If $ax^2 + bx + c$ is has two real roots α and β, then we substitute,

$$\sqrt{ax^2 + bx + c} = (x - \alpha)t$$

$$\Downarrow$$

$$x = \frac{a\beta - \alpha t^2}{a - t^2}$$

Intuition

The goal of an Euler substitution is to eliminate a radical in a rational function. We see in the first two substitutions the ax^2 terms are canceled out so we don't have any $\sqrt{t}$ terms left over. In all three substitutions we are left with

$$\int R(P_1(x), P_2(x))\,\mathrm{d}x,$$

where $P(x)$ is a polynomial of x. This should be trivial to integrate using long division, partial fraction decomposition, etc.

Examples

1. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+x+1}}$.

   Since $a > 0$, we make the first Euler substitution and solve for x,

   $$\sqrt{x^2+x+1} = x+t \implies x = \frac{t^2-1}{1-2t}.$$

   Substitute the value for x back in,

   $$\sqrt{x^2+x+1} = \frac{t^2-1}{1-2t} + t = \frac{-t^2+t-1}{1-2t}$$

   Now find the differential,

   $$dx = \frac{2(-t^2+t-1)}{(1-2t)^2}dt.$$

   Finally we substitute these values into our integral,

   $$\begin{aligned}\int \frac{1-2t}{-t^2+t-1}\frac{2(-t^2+t-1)}{(1-2t)^2}\,\mathrm{d}t &= \int \frac{2}{1-2t}\,\mathrm{d}t \\ &= -\ln|1-2t| + c \\ &= -\ln\left|1-2\left(\sqrt{x^2+x+1}-x\right)\right| + c.\end{aligned}$$

2. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-3x-10}}$.

   First note that $x^2-3x-10 = (x-5)(x+2)$, which indicates we'll be using our third substitution. Substitute $\sqrt{x^2-3x-10} = (x-5)t$ and solve for x,

   $$x = \frac{2+5t^2}{t^2-1} \implies \sqrt{x^2-3x-10} = \left(\frac{2+5t^2}{t^2-1} - 5\right)t = \frac{7t}{t^2-1}$$

and the differential

$$dx = \frac{-14t}{(t^2-1)^2}dt.$$

Finally we substitute back in and integrate,

$$\begin{aligned}
\int \frac{t^2-1}{7t}\frac{-14t}{(t^2-1)^2}dt &= -2\int \frac{dt}{t^2-1} \\
&= \ln\left|\frac{t+1}{t-1}\right| + c \\
&= \ln\left|\frac{\sqrt{x^2-3x-10}+x-5}{\sqrt{x^2-3x-10}-x+5}\right| + c \qquad (1.7) \\
&= \ln\left|2x-3+2\sqrt{x^2-3x-10}\right| + c. \qquad (1.8)
\end{aligned}$$

From step (1.7) to (1.8) we rationalize the denominator and absorb the $-\ln 7$ term into the constant c. Euler's substitution has allowed us to remove the radical and x^2 term, making the integrand feasible.

Exercises

1.3.1. Evaluate $\int \frac{\mathrm{d}x}{\sqrt{x^2+c}}$ for arbitrary c. How does this compare to when $c = \pm 1$?

1.3.2. Evaluate $\int_1^2 \frac{\mathrm{d}x}{x+\sqrt{x^2+1}}$.

1.3.3. Evaluate $\int \frac{\mathrm{d}x}{x\sqrt{x^2+x+1}}$.

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Miscellaneous

"snoo snoo"

– Mike the Capybara

2.1 Integrals of Inverse Functions

First by Laisant in 1905, this formula is capable of allowing quick integration of inverse formulas.

Definition

Given a bijective map $f : A \to B$, where $A, B \in \mathbb{R}$, the integral of the inverse $f^{-1} : B \to A$ is given by,

$$\int f^{-1}(x)\,\mathrm{d}x = xf^{-1}(x) - (F \circ f^{-1})(x) + c,$$

where F is the anti-derivative of f.

Intuition

It is trivial to show the equality if we differentiate both sides,

$$\begin{aligned} f^{-1}(x) &= (f^{-1}(x) + x(f^{-1})'(x)) - F'(f^{-1}(x))(f^{-1})'(x) \\ &= f^{-1}(x) + x(f^{-1})'(x) - x(f^{-1})'(x) \\ &= f^{-1}(x). \end{aligned}$$

This is fine for showing why the identity is true, but there's an even better geometric representation. Assume that f^{-1} is differentiable. Let $f(a) = c$ and $f(b) = d$. Then we can rewrite the identity as,

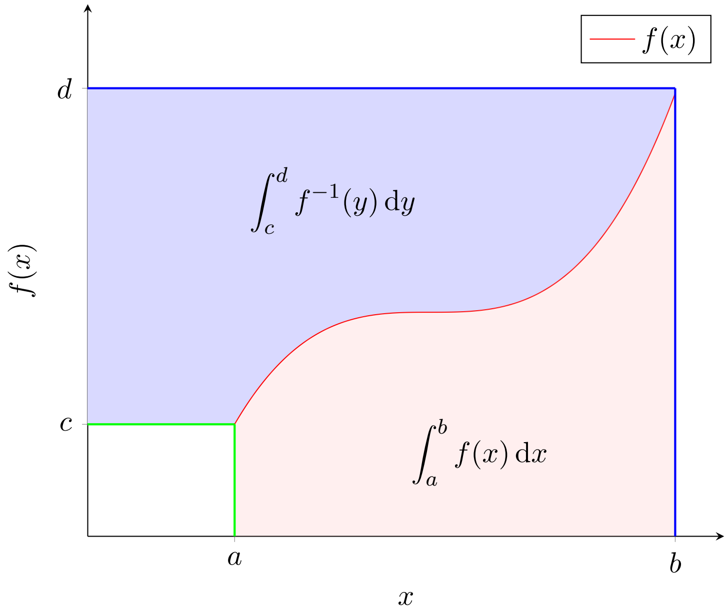

Figure 2.1: Graphical intuition behind inverse derivative.

$$\int_c^d f^{-1}(x)\,\mathrm{d}x + \int_a^b f(x)\,\mathrm{d}x = bd - ac.$$

We can now trivially see that this holds from Figure (2.1).

Examples

1. $\int \arcsin x \,\mathrm{d}x$

   First, notice that the function of the integrand is an inverse function, where $f^{-1}(x) = \arcsin x$ and $f(x) = \sin x$. According to the identity, we need to also find F, the anti-derivative of f, which in this case $F(x) = -\cos x$.

$$\begin{aligned}\int \arcsin x\,\mathrm{d}x &= x\arcsin x - (-\cos\circ\arcsin)(x) + c\\ &= x\arcsin x + \cos\arcsin x + c\\ &= x\arcsin x + \sqrt{1-x^2} + c \qquad (2.1)\end{aligned}$$

   In step (2.1) we use the identity that $\cos\arcsin x = \sqrt{1-x^2}$ (think of the geometric meanings of the trig functions).

2. $\int \ln x \,\mathrm{d}x$

Again, we can simply plug into the identity,

$$\begin{aligned}\int \ln x \,\mathrm{d}x &= x \ln x - (\exp \circ \ln)(x) + c \\ &= x(\ln x - 1) + c\end{aligned}$$

Exercises

Evaluate the following integrals using the inverse identity of integration.

2.1.1. $\displaystyle\int W(z)\,\mathrm{d}z$ where $W(z)$, the W-Lambert function, is the inverse of ze^z.

2.1.2. $\displaystyle\int \arctan x \,\mathrm{d}x$.

2.1.3. $\displaystyle\int \arccos x \,\mathrm{d}x$.

3

Complex Integrals

"The shortest path between two truths in the real domain passes through the complex domain."

– Jacques Hadamard

3.1 Cauchy's Integral Formula

Cauchy revolutionized the study of complex functions with his integral and residue formulas. However, his work is applicable to many real integrals as well via extension.

Definition

Suppose $U \subseteq \mathbb{C}$ is open, the disk $D = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is contained within U, $f : U \to \mathbb{C}$ is a holomorphic function, and $\gamma = \partial D$, then Cauchy's integral formula states, $\forall a \in D \setminus \gamma$

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - a} \, \mathrm{d}z.$$

Intuition

Consider the theorem formula,

$$f(a) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z - a} \, \mathrm{d}z.$$

Since we assumed f was a holomorphic map, we know it has a power series $\sum c_n (z - a)^n$. Substituting this into the integral,

$$
\begin{aligned}
2\pi i f(a) &= \oint_{\partial D} \frac{\sum c_n (z-a)^n}{z-a} \,\mathrm{d}z \\
&= \sum c_n \oint_{\partial D} (z-a)^{n-1} \,\mathrm{d}z
\end{aligned}
$$

Let us assume that D is an arbitrary disk centered around a. It can be parameterized by $z = a + e^{it}$. Now consider just the integral within the sum. For $n \geq 1$,

$$
\begin{aligned}
\oint_{\partial D} (z-a)^{n-1} \,\mathrm{d}z &= \int_{-\pi}^{\pi} (a + e^{it} - a)^m (a + e^{it}) \,\mathrm{d}t \quad (m \geq 0) \\
&= \int_{-\pi}^{-\pi} \left(a e^{itm} + e^{it(m+1)} \right) \,\mathrm{d}t \\
&= 0 \quad (\forall m \geq 0).
\end{aligned}
$$

And for $n = 0$,

$$
\begin{aligned}
\oint_{\partial D} \frac{1}{z-a} \,\mathrm{d}z &= \int_{-\pi}^{\pi} \frac{a + e^{it}}{e^{it}} \,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} \left(a e^{-it} + 1 \right) \,\mathrm{d}t \\
&= 2\pi i
\end{aligned}
$$

Remembering that $c_0 = f(a)$, our sum should result to

$$
\begin{aligned}
2\pi i f(a) &= c_0 2\pi i \\
&= f(a) 2\pi i
\end{aligned}
$$

Examples

1. Evaluate $\displaystyle\oint_C \frac{(z-2)^2}{z+i} \,\mathrm{d}z$ where $C : |z| = 2$.

   In this example we see $f(z) = (z-2)^2$ and $a = -i$. By Cauchy's formula,

$$
\begin{aligned}
\oint_C \frac{(z-2)^2}{z+i} \,\mathrm{d}z &= 2\pi i f(-i) \\
&= 2\pi i (i+2)^2 \\
&= \pi(6i - 8).
\end{aligned}
$$

Using Cauchy's method prevents us from having to expand the numerator, divide, and parametrize the integrand.

2. Evaluate $\oint_\gamma \frac{\exp(-z)}{z-\pi/2}\,\mathrm{d}z$ where $\gamma : |z| = 2$.

$$\begin{aligned}\oint_\gamma \frac{\exp(-z)}{z-\pi/2}\,\mathrm{d}z &= 2\pi i f(\pi/2)\\ &= 2\pi i e^{-\pi/2}\end{aligned}$$

Exercises

For the following exercises, integrate using Cauchy's Integral Formula

3.1.1. Evaluate $\oint_\gamma \frac{ze^z}{z-i}\,\mathrm{d}z$ where $\gamma : |z| = 2$.

3.1.2. Evaluate $\oint_D \frac{ze^z}{z^2+1}\,\mathrm{d}z$ where $D : |z| = 2$

3.2 Cauchy's Differentiation Formula

Cauchy's Differentiation Formula is very similar to his integral formula (3.1), but extends to higher derivatives of f. It's commonly called *Cauchy's Differentiation Formula*, because it is often used to find higher order derivatives, but it can also come in handy with integrating if higher order derivatives are known.

Definition

The definition is very similar to that of Cauchy's Integral Formula. If $U \subset \mathbb{C}$ is open, $D = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ s.t. $D \subseteq U$, $f : U \to \mathbb{C}$ is holomorphic, then we have, $\forall a \in D \setminus \partial D$, $n \in \mathbb{N}$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z.$$

Intuition

The intuition for Cauchy's Differentiation Formula is very similar to that of his integration formula if you think about the definition of a Taylor Series. Except when we find the power series of f at a, there ends up being an extra $n!$ that factors out of the series for the n-th derivative.

Examples

1. Compute $\oint_\Gamma \dfrac{\cos z}{(z-1/2)^3}\,\mathrm{d}z$ where $\Gamma : |z| = \pi$.

   Since $n = 2$ (from the denominator), we can evaluate the integral as follows,

$$\begin{aligned} \oint_\Gamma \frac{\cos z}{(z-1/2)^3}\,\mathrm{d}z &= \frac{2\pi i}{2!}\frac{d^2[\cos z]}{dz^2}(2) \\ &= -\pi i \cos 2 \\ &\approx 1.3073638i. \end{aligned} \tag{3.1}$$

   Similar to before, we move the constant factor of the integral to the RHS and $\cos z$ is easy to differentiate twice.

Exercises

Use Cauchy's Differentiation Formula to evaluate the following integrals

3.2.1. $\oint_C \dfrac{e^z}{(z-i)^4}\,\mathrm{d}z$ where $C : |z| = 2$.

3.2.2. $\oint_\gamma \dfrac{\sin z}{(z+1)^5}\,\mathrm{d}z$ where $\gamma : |z| = 3$.

Glossary

endpoints Represented by a and b in the integral $\int_a^b f \mathrm{d}x$. The start and end of the interval to integrate over. . 18

primitive For any function $f : A \to B$, F is called a primitive of f if $F' = f$. F is also called the anti-derivative of f.. 18