Test-2

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1) Give an example of a set P in $\mathbb{R}^2$ and a value of the parameter 't' for which $C_t(P)$ and $R_t(P)$ are not the same complex.

Ans:- Let us consider

$$P = \{P_0 = (0,0), P_1 = (1,0), P_2 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\} \text{ and}$$

Let $t = \frac{1}{\sqrt{3}}$

Then the 3-simplex $\{P_0, P_1, P_2\}$ is not in $C_t(P)$ because the condition $B(P_0,t) \cap B(P_1,t) \cap B(P_2,t) \neq \phi$. is not satisifed.

This emptyness can be seen as follows. consider the in-center of the equilateral triangle $P_0 P_1 P_2 P_0$.

$$P = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$$

Then

$$d(P,P_1) = d(P,P_2) = d(P,P_0) = \frac{1}{\sqrt{3}}$$

Hence

$$P \notin B(P_0, 0.55) \cap B(P_1, 0.55) \cap B(P_2, 0.55)$$

But 'p' being the in-centre, the common points of the in-circles centred at the vertices of the triangle is Just P, hence, $B(P_0,t) \cap B(P_1,t) \cap B(P_2,t) = \phi$.

on the other hand, the 3-simples

$\{P_0, P_1, P_2\}$ is not in. $R_t(P)$ because the condition.

$$B(P_i,t) \cap B(P_j,t) \neq \phi, \forall\, 0 \leq i \neq j \leq 2$$

$d(P_i, P_j) = 0.5 \leq \frac{1}{\sqrt{3}}$ for all is satisfied because

$$0 \leq i \neq j \leq 2$$

$\therefore$ Hence $C_t(P) \neq R_t(P)$.

2) Prove that the Hamming distance gives a metric on binary sequence of fixed length. you need to verify all the aximos for metric spaces in this case.

Ans:- Let 'S' be set of all binary sequences of length n.

Claim:- Hamming distance is a metric on S.

we know that Hamming distance b/w two strings of equal length is the number of positions at which corresponding symbols are different.

Let $d(x,y)$ be Hamming distance b/w string x & y, where $x=(x_1,x_2,\dots x_n)$ $y=(y_1,y_2,\dots y_n)$, $x_i, y_i = 0$ (or) 1.

(1) if $x=y$, then there is no symbol difference in x & y, so $d(x,y)=0$

if $d(x,y)=0$, it means corresponding symblos are same at all positions.

$\Rightarrow x=y$

Hence $\boxed{d(x,y)=0 \iff x=y}$

(2) clearly, no. of positions at which corresponding symbols are different b/w x & y is same as b/w y & x

$\boxed{d(x,y)=d(y,x)}$

(3) Let $x=(x_1,x_2,\dots x_n)$

$y=(y_1,y_2,\dots y_n)$

$z=(z_1,z_2,\dots z_n)$

suppose $d(x,y)=\alpha$ i.e, at α positions symblos are different

Let $d(x,z)=\beta$ & $d(y,z)=\gamma$

Claim:- $\alpha \le \beta+\gamma$

SUPPOSE $x_m \neq y_m$

Then, it can't happen both at same that $x_m = z_m$ & $z_m = y_m$

So, either $x_m \neq z_m$ (or) $z_m \neq y_m$

=) For, each symbol difference b/w x & y, there is alteeyt one symbol difference b/w x & z (or) b/w y & z.

$$\therefore d(x,y) \leq d(x,z) + d(z,y)$$

Hence, d is metric on S.

$\therefore$ Hence Proved.

5) Prove that the vertices $V(G)$ in a graph G is a metric space where the distance is measured by lengths of shortest paths b/w the vertices. You need to verify all of the axioms for metric spaces in this case.

Ans:- Let X be a non-empty set.

A function $d: X \times X \to \mathbb{R}$ is said to be a metric on a distance function if 'd' satifies the following properties.

(1) $d(x, y) \geq 0$, for all $x, y \in X$ & $d(x, y) = 0$ if $x = y$.

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$

The pair (X, d) is then called a metric space. Here $X = V(G)$, the set of vertices in a graph G.

The distance is measured by lengths of shortest paths b/w the vertices. Since G is a graph so $V(G)$ is non-empty.

(1) Let $x, y \in V(G)$. Then $d(x, y) \geq 0$ [$\because$ length of shortest path b/w two vertices is always non-negative]

$\therefore$ Since x, y are arbitrary, so

$d(x, y) \geq 0, \forall x, y \in V(G)$

Let $x = y$, then the length of shortest path b/w x & y is zero.

$\therefore d(x, y) = 0$

$\therefore$ if $x = y$ then $d(x, y) = 0$ — ①

Again let $d(x, y) = 0$. Let us assume that $x \neq y$.

Then, the length of the shortest path b/w x & y is positive i.e., $d(x, y) > 0$, which is a contradiciton, since $d(x, y) = 0$

$\therefore$ our assumption is wrong

ii) Let $x, y \in V(G)$, Since the length of the shortest path b/w x & y is same as the length of the shortest path b/w y & x. So

$$d(x,y) = d(y,x)$$

Since x & y are arbitrary, so

$$d(x,y) = d(y,x), \forall x,y \in V(G)$$

(3) Let $x, y, z \in V(G)$

Then $d(x,z)$ = the length of the shortest path b/w x & z.

If 'y' lies in the shortest path b/w x & z, then the length of shortest path b/w x & z = (length of shortest path b/w x & y)

+

(length of shortest path b/w y & z)

i.e., $d(x,z) = d(x,y) + d(y,z)$ — (A)

If y doesn't lie in the shortest path b/w x & z. Then

$$d(x,z) < d(x,y) + d(y,z) \quad \text{— (B)}$$

Because $d(x,y) + d(y,z)$ is the length of a path b/w x & z and $d(x,z)$ is the length of the shortest path b/w x & z.

From (A) & (B), we get

$$d(x,z) \leq d(x,y) + d(y,z)$$

Since x, y, z are arbitrary elements of $V(G)$, so

$$d(x,z) \leq d(x,y) + d(y,z), \forall x,y,z \in V(G)$$

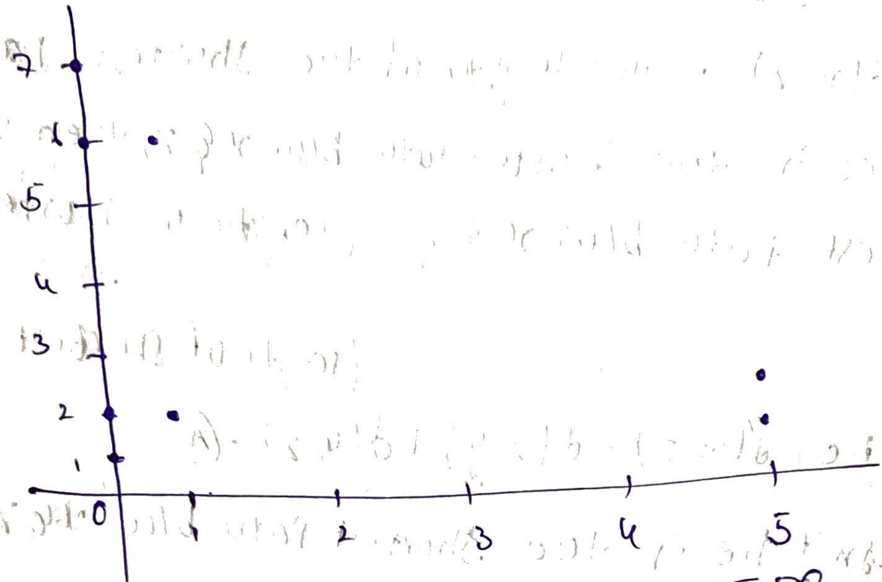
∴ The vertices $V(G)$ in a graph 'G' is a metric space.

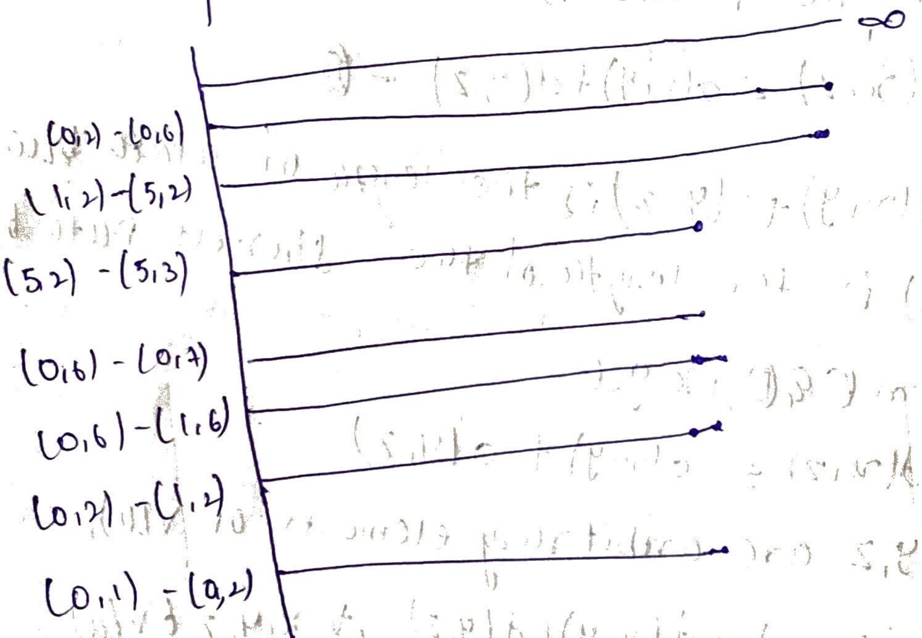
4) consider the following subset P of the 2-dimensional Plane:

$$\{(0,1),(0,2),(0,6),(0,7),(1,2),(1,6),(5,2),(5,3)\}$$

* construct the clustering bar code for this set 'P' with respect to the max distance.

Ans:





→ For given subset (P) the above constructed barcode gives cluster classification.

* ~~Because the~~

→ In, barcode the last five lines represent the points

$(0,1) \perp (0,2)$
$(0,2) \perp (1,2)$
$(0,6) \perp (0,7)$
$(0,6) \perp (1,6)$
$(5,2) \perp (5,3)$

These have unit distance among themselves.

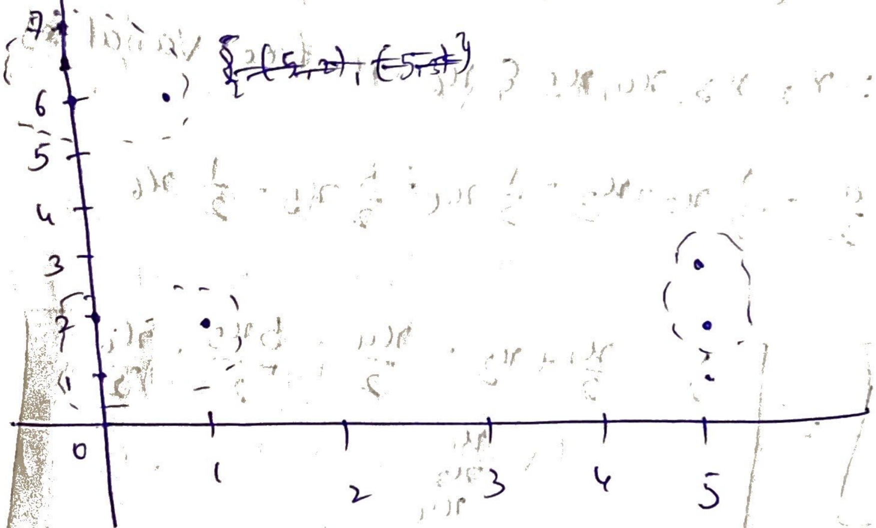
$(0,2) \overset{4}{-} (0,6)$
$(1,2) \overset{4}{-} (5,2)$

These two pairs of points have a distance '4' units.

→ max distance line represents entire subset 'P'.

→ The resulting classification cluster is

$\{(0,1), (0,2), (1,2)\}$; $\{(0,6)$ $(0,7)\}$; $\{(5,2)$ $(5,3)\}$



5) Find all real-valued solutions to the following system consisting of just one linear equation.

$$2x_1 + x_2 - 2x_3 + x_4 + 5x_5 + x_6 = 9.$$

Make sure to identify free variables & write the answers in terms of a particular solution plus a linear combination of homogenous solutions.

Ans:- Given system is

$$2x_1 + x_2 - 2x_3 + x_4 + 5x_5 + x_6 = 9$$

so augmented matrix can be

$$\left[\begin{array}{cccccc|c} 2 & 1 & -2 & 1 & 5 & 1 & 9 \end{array}\right]$$

$$R_1 \to \frac{R_1}{2} \sim \left[\begin{array}{cccccc|c} 1 & \frac{1}{2} & -1 & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{9}{2} \end{array}\right]$$

$\therefore x_2, x_3, x_4, x_5$ & x_6 are free variables

$$\text{so } x_1 = \frac{9}{2} - \frac{1}{2}x_2 + x_3 - \frac{1}{2}x_4 - \frac{5}{2}x_5 - \frac{1}{2}x_6$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} - \frac{x_2}{2} + x_3 - \frac{x_4}{2} - \frac{5x_5}{2} - \frac{x_6}{2} \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$+ x_5 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$