Test-3

T. Dancetle Reddy
UAlbany20:00169169?

1) Prove that the vectors in the general solution to the homogeneous system $Ax=0$ are linearly independent. So they are a basis for $K(T)$, where A is the matrix from the respresentation $T(x)=Ax$. For the following matrix A over F_2, $[0\ 1\ 0\ 1\ 1\ 1]$ find a basis for $K(T)$

Sol:- $A=[0\ 1\ 0\ 1\ 1\ 1]$

$\therefore Ax=0$, where $x=[x_1, x_2, x_3, x_4, x_5, x_6]^T$

$\therefore 0x_1+1\cdot x_2+0\cdot x_3+1\cdot x_4+1\cdot x_5+1\cdot x_6=0$

$\therefore x_1, x_2, x_3, x_4, x_5, x_6$ are free variables i.e take any $x_1, x_2, x_3, x_4, x_5, x_6$ in F_2

$\therefore x_2=-x_4-x_5-x_6=x_4+x_5+x_6 \quad (\because (-1)=1)$

$\therefore x_2=x_4+x_5+x_6$

$$\therefore \begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix} x_1\\x_4+x_5+x_6\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix}$$

$$+x_5\begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}+x_6\begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix}$$

consider

$$a\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}+c\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}+e\begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

$\Rightarrow$ $\begin{bmatrix} a \\ c+d+e \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a=0,\ b=0,\ c=0,\ d=0,\ e=0$

$\therefore$ The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is linearly indepent & hence form basis for $Ker(T)$.

2) Prove that the Point columns in A (from the represen-tation $T(x) = Ax$) are linearly independent, so they are a basis for $R(T)$ for the following matrix A over F_2,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ find a basis for $R(T)$.

Sol :-

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To check the Pivot columns are independent (or) not, First we have to find pivot columns for A.

The matrix A is in echelon form then the pivot column is a column of A that contains a pivot position

Hence pivot column is $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Hence basis of range $R(T)$ is

$$R(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

8) a) Suppose W is the xy-plane inside $V = \mathbb{R}^3$. Compute a basis for the quotient space V/W.

b) Let $V = \text{span}\{e_1+2e_2\}$ in V from part (a). Compute a basis for the quotient space W/V.

Sol

a) Given W is the xy-plane inside $V = \mathbb{R}^3$

$\Rightarrow \{(1,0,0), (0,1,0)\}$ is a basis for W and clearly, we know that $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$

$\Rightarrow$ The vector $(0,0,1)$ is independent of the first two vectors.

$\Rightarrow \{(0,0,1)+W\}$ is a basis for $\frac{\mathbb{R}^3}{W}$

b) Given $V = \text{span}\{e+2e_2\}$

To find a basis for W/V

$\because V = \text{span}\{e_1+2e_2\} = \text{span}\{(1,2,0)\}$

Also $W = \text{span}\{(1,0,0), (0,1,0)\}$

Hence basis for W/V is given by

$\{\text{span}\{(1,0,0), (0,1,0)\}+V\}$

$\therefore$ Here $(1,2,0)$ is a linearly combination of vectors $(1,0,0)$ & $(0,1,0)$.

4) perform a homology computation with only three vector spaces non-zero:

$$\cdots \to 0 \xrightarrow{0} C_2 \xrightarrow{T_2} C_1 \xrightarrow{T_1} C_0 \xrightarrow{0} 0 \to \cdots$$

where $C_2 = (F_2)^3$, $C_1 = (1F_2)^3$. The non-zero linear transformation are $T_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ & $T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

you want to find the Betti numbers, then find a speific basis for each homology vector space.

Sol :- Given three vector spaces non-zero.

$$\cdots \to 0 \xrightarrow{0} C_2 \xrightarrow{T_1} C_1 \xrightarrow{T_1} C_0 \xrightarrow{0} 0 \to \cdots$$

where $C_2 = (1F_2)^3$, $C_1 = (1F_2)^4$, $C_0 = (1F_2)^3$

The non-zero transformer are

$$T_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\&\; T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we want to find the Betti numbers, then we should find a specific basis for each homology vector space.

For the given T_2, T_1 & the chain complex, the homology of $C_0 = H_0(C_0) = K(T_0) / R(T_1)$

Basis for $K(T_0)$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

pivot variables

↓ ↓

For $w_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$

$H_0(C)$ has basis

$$[v_1] + R[T_1] * [v_3] + R[T_1]$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + R(T_1) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + R(T_1)$$

Basis for $K(T_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

a basis for $R(T_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow\uparrow$ non-Pivot Variables

$H_1(C_0)$ has basis $v_3 + R(T_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + R(T_2)$

Basis for $K(T_2) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow$ Pivot variables

$H_2(C_0) = K(T_2) / R(T_3) \Rightarrow$ since $R(T_3) = 0$

we get, $H_2(C) = K(T_2)/0 = K(T_2)$

$$B_i(C_1) = \begin{cases} 0 & i < 0, i > 2 \\ 1, & i = 1 \\ 2, & i = 0, 2 \end{cases}$$

6) Show that the chain complex

$$\cdots \to 0 \to \mathbb{R}^2 \xrightarrow{\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}} \mathbb{R}^3 \xrightarrow{[1\ -1\ 1]} \mathbb{R}^1 \to 0 \to \cdots$$

is acyclic.

sol:- There is a useful characterization in Brown, This will help:-

set $C_n = \mathbb{Z}/8$ for $n \geq 0$ & $C_n = 0$ for $n < 0$;

for $n > 0$ let d_n send $x \pmod 8$ to $4x \pmod 8$.

→ we have to show that C is a chain complex $\mathbb{Z}$-8 modules & compute its homology modules.

→ There is category $C_n(\text{mod-}R)$ of a chain complex of (right) R-moduled, the object are chain complexes.

A morphism $v : C \to D$ is a chain complex map. that is family of R-module homomorphism.

$v_n : C_n \to D_n$ commuting with d in the sense that $v_{n-1} d_n = d_{n-1} v_n$ that is such that following diagram commutes

$$\cdots \xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} D_{n+1} \to D_n \xrightarrow{d} D_{n-1} \xrightarrow{d} \cdots$$

so $\cdots \xrightarrow{0} \mathbb{R}^2 \xrightarrow{\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}} \mathbb{R}^3 \xrightarrow{[1\ -1\ 1]} \mathbb{R}^1 \to 0 \to \cdots$

is acyclic