EXERCISE 6.5

BASIC

1** A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk(v+v^3)$ Newtons, its speed is v ms^{-1} and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line. Prove $x=-\frac{1}{k}\int\frac{1}{1+v^2}dv$

2 A body is moving in a horizontal straight line. At time t seconds, its displacement is x metres from a fixed point O on the line, and its acceleration is $-\frac{1}{10}\sqrt{v}(1+\sqrt{v})$ where $v\geq 0$ is its velocity. The body is initially at O with velocity $V>0$.

Show that $t=20\log_e\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$

MEDIUM

3 A high speed train of mass m starts from rest and moves along a straight track. At time t hours, the distance travelled by the train from its starting point is x km, and its velocity is v km/h. The train is driven by a constant force F in the forward direction. The resistive force in the opposite direction is Kv^2, where K is a positive constant. The terminal velocity of the train is 300 km/h.

i Show that the equation of motion for the train is $m\ddot{x}=F\left(1-\left(\frac{v}{300}\right)^2\right)$

ii Find, in terms of F and m, the time it takes the train to reach a velocity of 200 km/h.

4 A 20 kg trolley is pushed with a force of 100 N. Friction causes a resistive force which is proportional to the square of the trolley's velocity.

i Show that $\ddot{x}=5-\frac{kv^2}{20}$ where k is a positive constant.

ii If the trolley is initially stationary at the origin, show that the distance travelled when its speed V is given by

$$x=\frac{10}{k}\ln\left(\frac{100}{100-kV^2}\right)$$

5** A landing aeroplane of mass m kg is brought to rest by the action of two retarding forces: a force of $4m$ Newtons due to the reverse thrust of the engines; and a force due to the brakes of $\frac{mv^2}{40\,000}$ Newtons.

i Show that the aeroplane's equation of motion for its speed v at time t seconds after landing is

$$\dot{v}=-\frac{v^2+400^2}{40\,000}$$

ii Assuming the aeroplane lands at a speed of U m/s, find an expression for the time it takes to come to rest.

iii Show that, given a sufficiently long runway, no matter how fast its landing speed, it will always come to rest within approximately 2.6 minutes of landing.

** Resultant force is given as a function of mass which makes our calculations easier but is not reflective of real life, as resistance is unrelated to mass.

6 A car, starting from rest, moves along a straight horizontal road. The car's engine produces a constant horizontal force of magnitude 4000 Newtons. At time t seconds, the speed of the car is v ms^{-1} and a resistance force of magnitude $40v$ Newtons acts upon the car. The mass of the car is 1600 kg.

i Show that $\dfrac{dv}{dt} = \dfrac{100 - v}{40}$

ii Find the velocity of the car at time t.

7 A supercar has a mass of 1000 kilograms and its engine generates a force of $1125\ \mathrm{N}$. Its motion is opposed by a resistive force of $\frac{v^2}{20}$ N.

a What is the maximum possible speed (terminal velocity) of the car on flat ground?

b If the car starts from rest, prove that the time taken to reach a speed of v, where $v < 150$, is given by $t = \dfrac{200}{3}\ln\left(\dfrac{150 + v}{150 - v}\right)$

c How does this formula help support the idea that the car can never reach the terminal velocity?

CHALLENGING

8 A fishing boat drifts with a current in a straight line across a fishing ground. The boat's velocity v, at time t after the start of this drift is given by $v = \mathrm{b} - (\mathrm{b} - v_0)e^{-\alpha t}$, where v_0, α and b are positive constants, and $v_0 < b$.

i Show that $\dfrac{dv}{dt} = \alpha(b - v)$

ii The physical significance of v_0 is that it represents the initial velocity of the boat. What is the physical significance of b?

iii Let x be the distance travelled by the boat from the start of the drift. Find x as a function of t. Hence show that

$$x = \frac{b}{\alpha}\log_{\mathrm{e}}\left(\frac{b - v_0}{b - v}\right) + \frac{v^0 - v}{\alpha}$$

iv The initial velocity of the boat is $\dfrac{b}{10}$. How far has the boat drifted when $v = \dfrac{b}{2}$?

9 A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is traveling with velocity Q, where $Q > 0$.

i Explain why $\ddot{x} = -(v + v^3)$

ii Show that v is related to the displacement x by the formula $x = \tan^{-1}\left[\dfrac{Q - v}{1 + Qv}\right]$

iii Show that the time t which has elapsed when the particle is traveling with velocity V is given by $t = \dfrac{1}{2}\log_e\left[\dfrac{Q^2(1 + V^2)}{V^2(1 + Q^2)}\right]$

iv Find V^2 as a function of t.

10 A particle of mass one kg is moving in a straight line. It is initially at the origin and is travelling with velocity $\sqrt{3}$ ms^{-1}. The particle is moving against a resisting force $v + v^3$, where v is the velocity.

i Briefly explain why the acceleration of the particle is given by $\dfrac{dv}{dt} = -(v + v^3)$

ii Show that the displacement x of the particle from the origin is given by $x = \tan^{-1}\left(\dfrac{\sqrt{3} - v}{1 + v\sqrt{3}}\right)$

iii Show that the time t which has elapsed when the particle is travelling with velocity V is given by $t = \dfrac{1}{2}\log_e\left[\dfrac{3(1 + V^2)}{4V^2}\right]$

iv Find V^2 as a function of t.

v Hence find the limiting position of the particle as $t \to \infty$.

1

$$mv\frac{dv}{dx} = -mk(v+v^3)$$
$$\frac{dv}{dx} = -k(1+v^2)$$
$$\frac{dx}{dv} = -\frac{1}{k}\times\frac{1}{1+v^2}$$
$$x = -\frac{1}{k}\int\frac{1}{1+v^2}\,dv$$

2

$$\frac{dv}{dt} = -\frac{1}{10}\sqrt{v}(1+\sqrt{v})$$
$$\frac{dt}{dv} = -\frac{10}{\sqrt{v}(1+\sqrt{v})}$$
$$t = -\int_V^v \frac{10}{\sqrt{v}(1+\sqrt{v})}\,dv$$
$$= 20\int_v^V \frac{\frac{1}{2}v^{-\frac{1}{2}}}{1+\sqrt{v}}\,dv$$
$$= 20\left[\ln(1+\sqrt{v})\right]_v^V$$
$$= 20\left(\ln\left(1+\sqrt{V}\right)-\ln(1+\sqrt{v})\right)$$
$$= 20\ln\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$$

3

i

The equation of motion is given by

$m\ddot{x} = F - kv^2$

At terminal velocity of 300 km/h $\ddot{x} = 0$

$\therefore 0 = F - k\times 300^2$

$$k = \frac{F}{300^2}$$

The equation of motion is:

$$m\ddot{x} = F - \frac{F}{300^2}v^2$$
$$= F\left[1-\left(\frac{v}{300}\right)^2\right]$$

ii

$$m\ddot{x} = F\left[1-\left(\frac{v}{300}\right)^2\right]$$
$$\frac{dv}{dt} = \frac{F}{m}\left[\frac{300^2-v^2}{300^2}\right]$$
$$\frac{dt}{dv} = \frac{m}{F}\left(\frac{300^2}{300^2-v^2}\right)$$
$$t = \frac{m}{F}\int_0^{200}\frac{300}{300^2-v^2}\,dv$$
$$= \frac{300^2 m}{600F}\int_0^{200}\left(\frac{1}{300+v}+\frac{1}{300-v}\right)dv$$
$$= \frac{150m}{F}\left[\ln(300+v)-\ln(300-v)\right]_0^{200}$$
$$= \frac{150m}{F}\left[\ln\frac{300+v}{300-v}\right]_0^{200}$$
$$= \frac{150m}{F}\left(\ln\frac{500}{100}-\ln\frac{300}{300}\right)$$
$$= \frac{150m}{F}\ln 5 \text{ hours}$$

4

i

$$20\ddot{x} = 100 - kv^2$$
$$\ddot{x} = 5 - \frac{kv^2}{20}$$

ii

$$v\frac{dv}{dx} = 5 - \frac{kv^2}{20}$$
$$\frac{dv}{dx} = \frac{100-kv^2}{20v}$$
$$\frac{dx}{dv} = \frac{20v}{100-kv^2}$$
$$x = \int_0^V \frac{20v}{100-kv^2}\,dv$$
$$= -\frac{10}{k}\left[\ln(100-kv^2)\right]_0^V$$
$$= -\frac{10}{k}\left(\ln(100-kV^2)-\ln 100\right)$$
$$= \frac{10}{k}\ln\left(\frac{100}{100-kV^2}\right)$$

5 **i**

$$m\dot{v} = -\frac{mv^2}{40\,000} - 4m$$
$$\dot{v} = -\frac{v^2 + 160\,000}{40\,000}$$
$$= -\frac{v^2 + 400^2}{40\,000}$$

ii

$$\frac{dv}{dt} = -\frac{v^2 + 400^2}{40\,000}$$
$$\frac{dt}{dv} = -\frac{40\,000}{v^2 + 400^2}$$
$$t = -\int_U^0 \frac{40\,000}{v^2 + 400^2}\,dv$$
$$= 40\,000\left[\frac{1}{400}\tan^{-1}\frac{v}{400}\right]_0^U$$
$$= 100\left(\tan^{-1}\frac{U}{400} - 0\right)$$
$$= 100\tan^{-1}\frac{U}{400}\text{ s}$$

iii

As $U \to \infty \quad \tan^{-1}\frac{U}{400} \to \frac{\pi}{2}$

$\therefore t \to 100 \times \frac{\pi}{2} \approx 157$ s ≈ 2.618 minutes

$\therefore$ The plane lands within approximately 2.6 minutes of landing regardless of speed.

6 **i**

$$1600\frac{dv}{dt} = 4000 - 40v$$
$$\frac{dv}{dt} = \frac{100 - v}{40}\text{ ms}^{-2}$$

ii

$$\frac{dt}{dv} = \frac{40}{100 - v}$$
$$t = \int_0^v \frac{40}{100 - v}\,dv$$
$$= -40\Big[\ln(100 - v)\Big]_0^v$$
$$-\frac{t}{40} = \ln(100 - v) - \ln 100$$
$$\ln 100 - \frac{t}{40} = \ln(100 - v)$$
$$100e^{-\frac{t}{40}} = 100 - v$$
$$v = 100\left(1 - e^{-\frac{t}{40}}\right)\text{ ms}^{-1}$$

7 **a**

$$1000\ddot{x} = 1125 - \frac{v^2}{20}$$
$$\ddot{x} = \frac{22500 - v^2}{20000}$$

Let $\ddot{x} = 0, \therefore 22500 - v_T^2 = 0 \;\; \to \;\; v_T = \sqrt{22500} = 150\text{ ms}^{-1}$

b

$$\frac{dv}{dt} = \frac{22500 - v^2}{20000}$$
$$\frac{dt}{dv} = \frac{20000}{22500 - v^2}$$
$$t = \frac{200}{3}\int_0^v \left(\frac{1}{150 - v} + \frac{1}{150 + v}\right)dv$$
$$= \frac{200}{3}\Big[-\ln(150 - v) + \ln(150 + v)\Big]_0^v$$
$$= \frac{200}{3}(-\ln(150 - v) + \ln(150 + v))$$
$$= \frac{200}{3}\ln\left(\frac{150 + v}{150 - v}\right)$$

c

The time taken to reach $v = 150$ would be $t = \frac{200}{3}\ln\left(\frac{150+150}{150-150}\right) = \frac{200}{3}\ln\left(\frac{300}{0}\right)$ which is undefined, which is an indicator that velocity can never reach terminal velocity (slope fields give a better explanation).

8

i

$$\frac{dv}{dt} = \alpha(b - v_0)e^{-\alpha t}$$
$$= \alpha\left(b - (b - (b - v_0)e^{-\alpha t})\right)$$
$$= \alpha(b - v)$$

ii

'b' is the speed of the current – the boat slowly approaches the speed of the current, as a limiting value.

iii

$$v\frac{dv}{dx} = \alpha(b - v)$$
$$\frac{dv}{dx} = \frac{\alpha(b - v)}{v}$$
$$\frac{dx}{dv} = \frac{1}{\alpha} \times \frac{v}{b - v}$$
$$x = \frac{1}{\alpha}\int_{v_0}^{v} \frac{v}{b - v}\,dv$$
$$= \frac{1}{\alpha}\int_{v_0}^{v} \frac{-(b - v) + b}{b - v}\,dv$$
$$= \frac{1}{\alpha}\int_{v_0}^{v} \left(-1 + \frac{b}{b - v}\right)dv$$
$$= \frac{1}{\alpha}\Big[-v - b\ln(b - v)\ \Big]_{v_0}^{v}$$
$$= \frac{1}{\alpha}\left((-v - b\ln(b - v)) - (-v_0 - b\ln(b - v_0))\right)$$
$$= \frac{1}{\alpha}\left(b\ln\left(\frac{b - v_0}{b - v}\right) + v_0 - v\right)$$
$$= \frac{b}{\alpha}\ln\left(\frac{b - v_0}{b - v}\right) + \frac{v_0 - v}{\alpha}$$

iv

$$x = \frac{b}{\alpha}\ln\left(\frac{b - \frac{b}{10}}{b - \frac{b}{2}}\right) + \frac{\frac{b}{10} - \frac{b}{2}}{\alpha}$$
$$= \frac{b}{\alpha}\ln\left(\frac{\frac{9}{10}}{\frac{1}{2}}\right) + \frac{b}{\alpha}\left(-\frac{2}{5}\right)$$
$$= \frac{b}{\alpha}\left(\ln\frac{9}{5} - \frac{2}{5}\right)$$

9

i

$$m\ddot{x} = -m(v + v^3)$$
$$\ddot{x} = -(v + v^3)$$

ii

$$v\frac{dv}{dx} = -(v + v^3)$$
$$\frac{dv}{dx} = -(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1 + v^2}$$
$$x = -\int_Q^v \frac{1}{1 + v^2}\,dv$$
$$= \Big[\tan^{-1} v\Big]_v^Q$$
$$= \tan^{-1} Q - \tan^{-1} v$$
$$= \tan^{-1}\left(\frac{\tan(\tan^{-1} Q) - \tan(\tan^{-1} v)}{1 + (\tan(\tan^{-1} Q))(\tan(\tan^{-1} v))}\right)$$
$$= \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right)$$

iii

$$\frac{dv}{dt} = -(v + v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v + v^3}$$
$$t = -\int_Q^V \frac{1}{v + v^3}\,dv$$
$$= \int_V^Q \left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_V^Q$$
$$= \left(\ln Q - \frac{1}{2}\ln(1 + Q^2)\right)$$
$$-\left(\ln V - \frac{1}{2}\ln(1 + V^2)\right)$$
$$= \left(\frac{1}{2}\ln Q^2 - \frac{1}{2}\ln(1 + Q^2)\right)$$
$$-\left(\frac{1}{2}\ln V^2 - \frac{1}{2}\ln(1 + V^2)\right)$$
$$= \frac{1}{2}\ln\frac{Q^2}{1 + Q^2} - \frac{1}{2}\ln\frac{V^2}{1 + V^2}$$
$$= \frac{1}{2}\ln\left(\frac{Q^2(1 + V^2)}{V^2(1 + Q^2)}\right)$$

iv

$$e^{2t} = \frac{Q^2(1 + V^2)}{V^2(1 + Q^2)}$$
$$V^2e^{2t}(1 + Q^2) = Q^2 + Q^2V^2$$
$$V^2(e^{2t}(1 + Q^2) - Q^2) = Q^2$$
$$V^2 = \frac{Q^2}{e^{2t} + e^{2t}Q^2 - Q^2}$$

10

i

$$m\ddot{x} = -(v + v^3)$$
$$\ddot{x} = -(v + v^3)$$

ii

$$v\frac{dv}{dx} = -(v + v^3)$$
$$\frac{dv}{dx} = -(1 + v^2)$$
$$\frac{dx}{dv} = -\frac{1}{1 + v^2}$$
$$x = -\int_{\sqrt{3}}^v \frac{1}{1 + v^2}\,dv$$
$$= \Big[\tan^{-1} v\Big]_v^{\sqrt{3}}$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1} v$$
$$= \tan^{-1}\left(\tan\left(\tan^{-1}\sqrt{3} - \tan^{-1} v\right)\right)$$
$$= \tan^{-1}\left(\frac{\tan\left(\tan^{-1}\sqrt{3}\right) - \tan(\tan^{-1} v)}{1 + \tan\left(\tan^{-1}\sqrt{3}\right) \times \tan(\tan^{-1} v)}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{3} - v}{1 + \sqrt{3}v}\right)$$

iii

$$\frac{dv}{dt} = -(v + v^3)$$
$$\frac{dt}{dv} = -\frac{1}{v + v^3}$$
$$t = -\int_{\sqrt{3}}^V \frac{1}{v + v^3}\,dv$$
$$= \int_V^{\sqrt{3}} \left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$
$$= \left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_V^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \frac{1}{2}\ln 4 - \ln V + \frac{1}{2}\ln(1 + V^2)$$
$$= \frac{1}{2}\ln 3 + \frac{1}{2}\ln(1 + V^2) - \frac{1}{2}\ln 4 - \frac{1}{2}\ln V^2$$
$$= \frac{1}{2}\ln\left(\frac{3(1 + V^2)}{4V^2}\right)$$

iv

$$e^{2t} = \frac{3(1 + V^2)}{4V^2}$$
$$4V^2e^{2t} = 3 + 3V^2$$
$$V^2(4e^{2t} - 3) = 3$$
$$V^2 = \frac{3}{4e^{2t} - 3}$$

v

as $t \to \infty \quad e^{2t} \to \infty \quad \therefore V^2 \to 0 \quad \therefore V \to 0$

$$\therefore x \to \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$