EXERCISE 5.3

BASIC

1 Sketch $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{-1}$

2 Sketch $\underset{\sim}{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$

3 Sketch the interval $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{-1}$ for $-1 \le \lambda \le 1$

4 Sketch the interval $\underset{\sim}{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ for $-1 \le \lambda \le 2$

5 **a** Find a vector equation of the line through $\binom{1}{1}$ and $\binom{2}{3}$.
b Find a vector equation for the interval from $\binom{1}{1}$ to $\binom{2}{3}$.

6 Consider the points $A\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $B\begin{pmatrix}1\\2\\1\end{pmatrix}$.
a Find a vector equation of the line through A and B.
b Find a vector equation for the interval from A to B.

7 Prove the following lines are parallel: $\underset{\sim}{r} = \binom{1}{-1} + \lambda\binom{1}{2}$ and $\underset{\sim}{q} = \binom{3}{1} + \lambda\binom{-2}{-4}$.

8 Prove the following lines are parallel: $\underset{\sim}{r} = \begin{pmatrix}2\\1\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\-9\end{pmatrix}$

9 Prove the lines $\underset{\sim}{r} = \binom{2}{1} + \lambda\binom{3}{2}$ and $\underset{\sim}{q} = \binom{4}{-2} + \lambda\binom{-2}{3}$ are perpendicular.

10 Prove the lines $\underset{\sim}{r} = \begin{pmatrix}-1\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}-2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ are perpendicular.

MEDIUM

11 Find the vector equation of the line through $\mathrm{A}\binom{-1}{2}$ parallel to $\overrightarrow{BC}$ with $\mathrm{B}\binom{2}{1}$ and $\mathrm{C}\binom{1}{2}$

12 Find the vector equation of the line through $\mathrm{A}\begin{pmatrix}1\\2\\1\end{pmatrix}$ parallel to $\overrightarrow{BC}$ with $\mathrm{B}\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathrm{C}\begin{pmatrix}3\\3\\1\end{pmatrix}$.

13 Find a vector equation for the line through $\binom{0}{1}$ with gradient $m = -2$

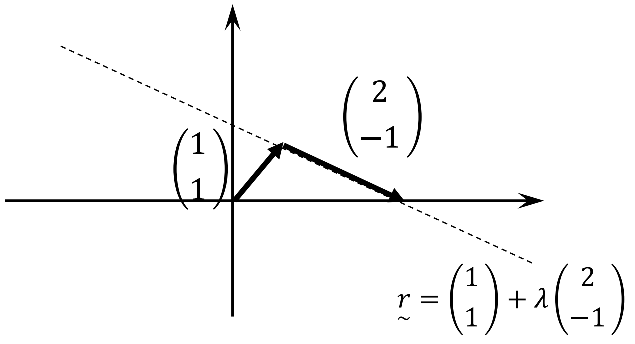
14 The lines $\underset{\sim}{r} = \lambda\binom{2}{3}$ and $\underset{\sim}{q} = \binom{4}{1} + \lambda\binom{p}{2}$ are perpendicular. Find p.

15 The lines $\underset{\sim}{r} = \lambda\begin{pmatrix}-2\\1\\-p\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ are perpendicular. Find p.

CHALLENGING

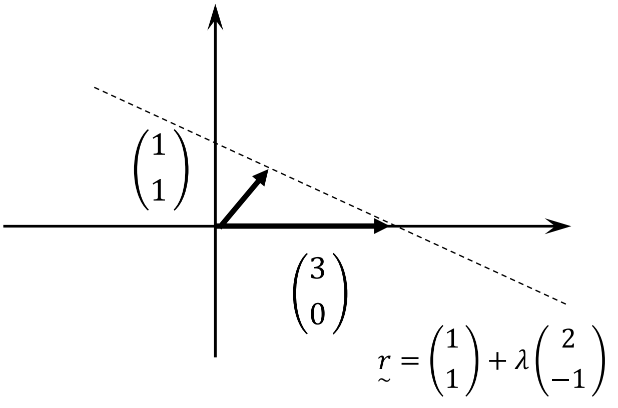
16 A cube has opposite vertices at the origin and $(2,2,2)$. State the equations of the four diagonals. Are the diagonals perpendicular?

1 Method 1

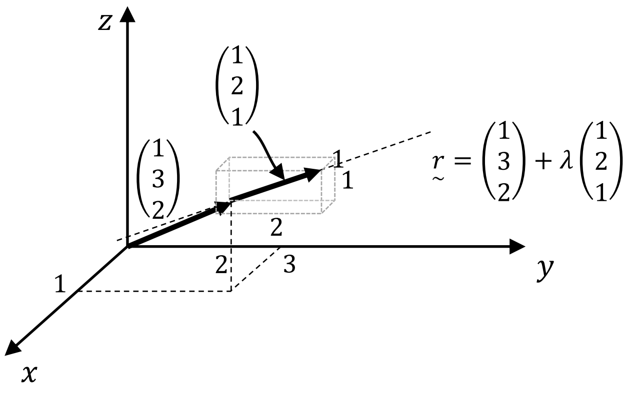


Method 2

Let $\lambda = 1$ to find a second point, in this case $\binom{3}{0}$, and plot this point and $\binom{1}{2}$, drawing a line through their tips.

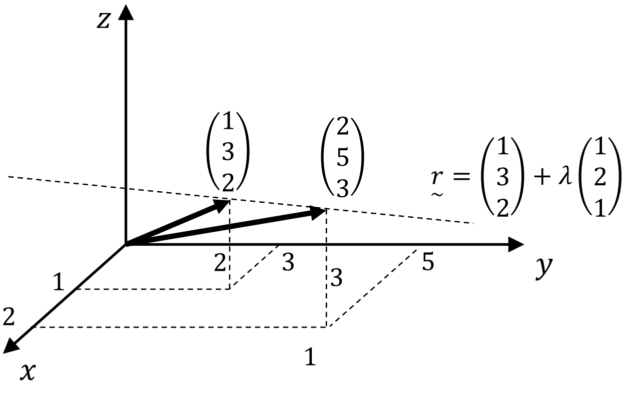


2 Method 1



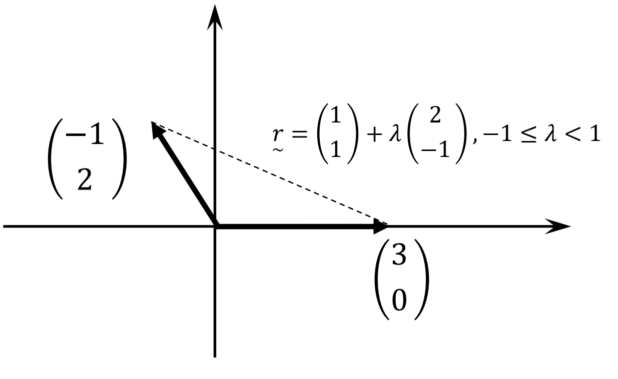
Method 2

Let $\lambda = 1$ to find a second point, in this case $\begin{pmatrix}2\\5\\3\end{pmatrix}$, and plot this point and $\begin{pmatrix}1\\3\\2\end{pmatrix}$, drawing a line through their tips.



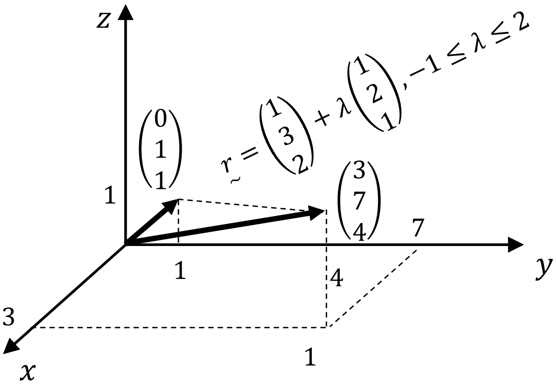
3 Substitute $\lambda = -1$ and $\lambda = 1$ to find the end points of the interval.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$
$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$



4 Substitute $\lambda = -1$ and $\lambda = 2$ to find the end points of the interval.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}$$



5 **a**

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore \underset{\sim}{r} = \binom{1}{1} + \lambda\binom{1}{2}$ is one correct answer.

b Letting $\lambda = 0$ in $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{1}{2}$ gives us $\binom{1}{1}$, and $\lambda = 1$ gives $\binom{2}{3}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad 0 \le \lambda \le 1$$

6 **a**

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is one correct answer.

b Letting $\lambda = 0$ in $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ gives us $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\lambda = 1$ gives $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, so one vector equation for the interval is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad 0 \le \lambda \le 1$$

7 $\begin{pmatrix} -2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel.

8 $\begin{pmatrix} -3 \\ 3 \\ -9 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are parallel

9 $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -6 + 6 = 0$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

10 $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 1(-1) + 2(1) - 1(1) = 0$

$\therefore \underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular

11 $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore \underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

12 $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

13 The gradient of -2 can be represented by the vector $\binom{1}{-2}$, or any vector where the y-value is minus two times the x-value.

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

14

$$\begin{aligned} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} p \\ 2 \end{pmatrix} &= 0 \\ 2p + 6 &= 0 \\ 2p &= -6 \\ p &= -3 \end{aligned}$$

15

$$\begin{aligned} \begin{pmatrix} -2 \\ 1 \\ -p \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} &= 0 \\ -2(2) + 1(1) - p(-2) &= 0 \\ -3 + 2p &= 0 \\ \therefore p &= \frac{3}{2} \end{aligned}$$

16 All diagonals pass through $(1,1,1)$, plus one of the four base vertices, $A(0,0,0), B(2,0,0), C(2,2,0)$ and $D(0,2,0)$

$$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1-0 \\ 1-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1-2 \\ 1-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1-2 \\ 1-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r_4} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1-0 \\ 1-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

None of the dot products of the direction vectors give zero, so the diagonals are not perpendicular.

For example $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -1 + 1 + 1 = 1 \neq 0$