EXERCISE 4.7

BASIC

1 Find $\displaystyle\int \frac{dx}{1+\sin x}$ using $t=\tan\frac{x}{2}$

2 Using the substitution $t=\tan\frac{x}{2}$, or otherwise, find $\displaystyle\int \frac{5}{12+13\cos x}\,dx$

3 Using the substitution $x=\sin\theta$, or otherwise, find $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

4 Using the substitution $x=2\sec\theta$, or otherwise, find $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

MEDIUM

5 Using the substitution $t=\tan\frac{\theta}{2}$ find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$

6 Using the substitution $x=4\tan^2 u$, or otherwise, find $\displaystyle\int \sqrt{x+4}\,dx$

7 Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x-4\cos x+5}\,dx$

8 Using the substitution $x=\sin^2\theta$, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$

9 Use the substitution $t=\tan\frac{\theta}{2}$ to show that $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta}=\frac{1}{2}\log_e 3$

CHALLENGING

Find the following indefinite integrals using a suitable trig substitution:

10 $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

11 $\displaystyle\int \frac{3+\cos x}{2-\cos x}\,dx$

12 $\displaystyle\int \sqrt{2x-x^2}\,dx$

1

$$\int \frac{dx}{1+\sin x}$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1+t^2+2t}\,dt$$

$$= 2\int \frac{1}{(t+1)^2}\,dt$$

$$= 2\int (t+1)^{-2}\,dt$$

$$= -\frac{2}{t+1} + c$$

$$= -\frac{2}{\tan\frac{x}{2}+1} + c$$

2

$$\int \frac{5}{12+13\cos x}\,dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} \\ dx &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int \frac{5}{12+13\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= 10\int \frac{1}{12+12t^2+13-13t^2}\,dt$$

$$= 10\int \frac{1}{25-t^2}\,dt$$

$$= \int \left(\frac{1}{5-t}+\frac{1}{5+t}\right)dt$$

$$= -\ln|5-t| + \ln|5+t| + c$$

$$= \ln\left|\frac{5+\tan\frac{x}{2}}{5-\tan\frac{x}{2}}\right| + c$$

3

$$\int x^3\sqrt{1-x^2}\,dx$$

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$

$$= \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \int \sin^3\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$$

$$= \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int \sin\theta\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$$

$$= -\int(-\sin\theta)(\cos^2\theta-\cos^4\theta)\,d\theta$$

$$= \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + c$$

$$\boxed{\begin{aligned} \cos\theta &= \sqrt{1-\sin^2\theta} \\ &= \sqrt{1-x^2} \end{aligned}}$$

$$= \frac{\sqrt{(1-x^2)^5}}{5} - \frac{\sqrt{(1-x^2)^3}}{3} + c$$

4

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} x &= 2\sec u \\ dx &= 2\sec u\tan u\,du \end{aligned}}$$

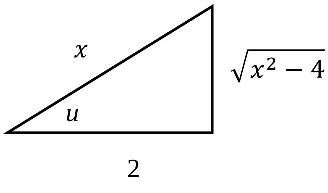
$$= \int \frac{1}{2\sec u\sqrt{4\sec^2 u - 4}} \times 2\sec u\tan u\,du$$

$$= \int \frac{\tan u}{\sqrt{4\tan^2 u}}\,du$$

$$= \int \frac{\tan u}{2\tan u}\,du$$

$$= \frac{1}{2}\int du$$

$$= \frac{u}{2} + c$$



$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$

Alternatively:

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2 - 4 \\ 2u\,du &= 2x\,dx \\ dx &= \frac{u\,du}{x} \end{aligned}}$$

$$= \int \frac{1}{xu} \times \left(u\frac{du}{x}\right)$$

$$= \int \frac{1}{x^2}\,du$$

$$= \int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + c$$

$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + c$$

Hint: Integrands in the form $\frac{1}{x\sqrt{x^2-c}}$ can be solved using a u^2 substitution.

5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$$

$$\boxed{\begin{aligned} t &= \tan\frac{\theta}{2} \\ d\theta &= \frac{2dt}{1+t^2} \end{aligned}}$$

$$= \int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{1}{2+2t^2-1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}t)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}\pi}{9}$$

6

$$\int \sqrt{x+4}\,dx$$

$$\boxed{\begin{aligned} x &= 4\tan^2 u \\ dx &= 8\tan u\sec^2 u\,du \end{aligned}}$$

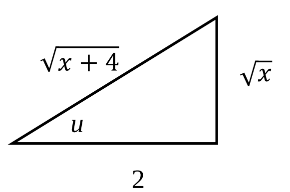
$$= \int \sqrt{4\tan^2 u + 4} \times 8\tan u\sec^2 u\,du$$

$$= 8\int \tan u\sec^2 u\sqrt{4\sec^2 u}\,du$$

$$= 16\int \tan u\sec^3 u\,du$$

$$= 16\int \tan u\sec u\,(\sec u)^2\,du$$

$$= \frac{16\sec^3 u}{3} + c$$



$$= \frac{2\sqrt{(x+4)^3}}{3} + c$$

7

$$t=\tan\frac{x}{2} \qquad dx=\frac{2dt}{1+t^2}$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{3\sin x-4\cos x+5}\,dx$$
$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{3\left(\frac{2t}{1+t^2}\right)-4\left(\frac{1-t^2}{1+t^2}\right)+5}\times\frac{2dt}{1+t^2}$$
$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2}{6t-4+4t^2+5+5t^2}\,dt$$
$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2}{9t^2+6t+1}\,dt$$
$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2}{(3t+1)^2}\,dt$$
$$=2\left[\frac{(3t+1)^{-1}}{-1\times 3}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$=-\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$=-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{\sqrt{3}+1}\right)$$
$$=-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$
$$=-\frac{2}{3}\left(\frac{1}{4}-\frac{\sqrt{3}-1}{2}\right)$$
$$=-\frac{2}{3}\frac{\left(3-2\sqrt{3}\right)}{4}$$
$$=\frac{2\sqrt{3}-3}{6}$$

8

$$x=\sin^2\theta \qquad dx=2\sin\theta\cos\theta\,d\theta$$

$$\int_0^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,dx$$
$$=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\times 2\sin\theta\cos\theta\,d\theta$$
$$=\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos\theta}\times 2\sin\theta\cos\theta\,d\theta$$
$$=2\int_0^{\frac{\pi}{4}}\sin^2\theta\,d\theta$$
$$=\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\,d\theta$$
$$=\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$=\left(\left(\frac{\pi}{4}-\frac{1}{2}\right)-(0-0)\right)$$
$$=\frac{\pi}{4}-\frac{1}{2}$$

9

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\frac{d\theta}{\sin\theta}$$
$$=\int_1^{\sqrt{3}}\frac{1+t^2}{2t}\times\frac{2dt}{1+t^2}$$
$$=\int_1^{\sqrt{3}}\frac{dt}{t}$$
$$=\left[\ln t\right]_1^{\sqrt{3}}$$
$$=\ln\sqrt{3}-\ln 1$$
$$=\ln 3^{\frac{1}{2}}$$
$$=\frac{1}{2}\ln 3$$

10

$$x=2\sin\theta \qquad dx=2\cos\theta\,d\theta$$

$$\int\frac{x^2}{\sqrt{4-x^2}}\,dx$$
$$=\int\frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\times 2\cos\theta\,d\theta$$
$$=\int\frac{4\sin^2\theta}{2\cos\theta}\times 2\cos\theta\,d\theta$$
$$=4\int\sin^2\theta\,d\theta$$
$$=4\int\frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$=2\left(\theta-\frac{1}{2}\sin 2\theta\right)+c$$
$$=2\theta-2\sin\theta\cos\theta+c$$
$$=2\sin^{-1}\left(\frac{x}{2}\right)-2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)+c$$
$$=2\sin^{-1}\left(\frac{x}{2}\right)-\frac{x\sqrt{4-x^2}}{2}+c$$

11

$$t=\tan\frac{x}{2} \qquad dx=\frac{2dt}{1+t^2}$$

$$\int\frac{3+\cos x}{2-\cos x}\,dx$$
$$=\int\frac{3+\frac{1-t^2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=2\int\frac{3+3t^2+1-t^2}{2+2t^2-1+t^2}\times\frac{dt}{1+t^2}$$
$$=2\int\frac{2t^2+4}{(3t^2+1)(1+t^2)}\,dt$$
$$=2\int\left(\frac{5}{3t^2+1}-\frac{1}{1+t^2}\right)dt$$
$$=\frac{10}{\sqrt{3}}\int\frac{\sqrt{3}}{3t^2+1}\,dt-2\int\frac{1}{1+t^2}\,dt$$
$$=\frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right)-2\tan^{-1}t+c$$
$$=\frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right)-x+c$$

12 $\int \sqrt{2x - x^2}\, dx$

$$= \int \sqrt{-(x^2 - 2x + 1 - 1)}\, dx$$

$$= \int \sqrt{1 - (x-1)^2}\, dx$$

$$\boxed{\begin{aligned} x - 1 &= \sin\theta \\ dx &= \cos\theta\, d\theta \end{aligned}}$$

$$= \int \sqrt{1 - \sin^2\theta} \times \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1 + \cos 2\theta) d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$$

$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + c$$

$$= \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + c$$

$$= \frac{\sin^{-1}(1-x)}{2} + \frac{(x-1)\sqrt{2x - x^2}}{2} + c$$