EXERCISE 6.8

BASIC

1 A particle is projected upwards from ground level with initial velocity $\frac{1}{2}\sqrt{\frac{g}{k}}$ ms^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v ms^{-1} and experiences a resistive force. The acceleration of the particle is given by $\ddot{x} = -g - kv^2$. Do NOT prove this. The particle reaches a maximum height, H, before returning to the ground.

Using $\ddot{x} = v\frac{dv}{dx}$, or otherwise, show that $H = \dfrac{1}{2k}\log_e\left(\dfrac{5}{4}\right)$ metres.

2 A particle is projected with velocity 60 ms^{-1} at an angle of $30°$ to the horizontal in a resistive medium. It reached a maximum height of 17.9 m and lands 75.7 m away from its projection point. Which of the following statements cannot be true?

A The maximum height occurs when the particle has travelled 47 m horizontally
B The velocity at impact is 60.0 ms^{-1}.
C The angle to the horizontal at impact is $58°$

3 A particle of unit mass is projected in a medium where air resistance is proportional to velocity, at 50 ms^{-1} at an angle of θ to the horizontal where $\tan\theta = \frac{3}{4}$. The vertical equation of motion of is $\dot{y} = 230e^{-\frac{t}{20}} - 200$, where $\dot{y}$ is in metres per second.

Find the time taken to reach maximum height, to 1 decimal place.

CHALLENGING

4 A ball of mass m is projected vertically into the air from the ground with initial velocity u. After reaching the maximum height H it falls back to the ground. While in the air, the ball experiences a resistive force kv^2, where v is the velocity of the ball and k is a constant. The equation of motion when the ball falls can be written as $m\dot{v} = mg - kv^2$ (Do NOT prove this.)

i Show that the terminal velocity v_T of the ball when it falls is $\sqrt{\dfrac{mg}{k}}$

ii Show that when the ball goes up, the maximum height H is $H = \dfrac{v_T^2}{2g}\ln\left(1 + \dfrac{u^2}{v_T^2}\right)$

iii When the ball falls from height H it hits the ground with velocity w.

Show that $\dfrac{1}{w^2} = \dfrac{1}{u^2} + \dfrac{1}{v_T^2}$

5 A particle of mass 1 kg is projected vertically upwards from the ground with a speed of 20 m/s. The particle is under the effect of both gravity (g) and an air resistance of magnitude $\frac{1}{40}v^2$ where v is the current velocity of the particle at any time. The effects of gravity and air resistance remain the same no matter which direction the particle is going.

i Explain why the acceleration of the particle at any time whilst traveling upwards is given by $\ddot{x} = -g - \dfrac{1}{40}v^2$

(For the remainder of this question you may use $g = 10$ ms^{-2})

ii Calculate the greatest height reached by the particle

iii Write an expression for the acceleration of the particle as it returns to earth.

iv Find the speed of the particle just before it strikes the ground.

6 A rubber ball of mass 7 kg, falls from rest, from the top of a building. While falling the ball experiences a resistive force $\frac{7v^2}{10}$, where v is the velocity of the ball. Take g, acceleration due to gravity, as $g = 10$ ms^{-2}.

i Show that $\ddot{x} = 10 - \frac{v^2}{10}$, where x is the distance the ball has fallen.

ii Find the terminal velocity of the ball as it falls.

iii Show that $v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$

iv After hitting the ground the ball rises vertically such that $\ddot{X} = -10 - \frac{V^2}{10}$, where V is the velocity of the ball as it rises and X is the distance the ball rises. Find the time that it takes for the ball to rise to its maximum height if initially $V = \frac{10}{\sqrt{3}}$ ms^{-1}.

7 A particle is projected from the origin with an initial velocity $60\,\mathrm{ms}^{-1}$ at $30°$ to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -\frac{\dot{x}}{10} \quad \text{and} \quad \ddot{y} = -\frac{\dot{y}}{10} - 10,$$

(You are NOT required to show these.)

i Find an expression for horizontal displacement as a function of time.

ii Find an expression for vertical displacement as a function of time.

iii Find the Cartesian equation of the trajectory of the particle

iv Find the value of t when the particle reaches its maximum height. to 1 decimal place.

8 A body of unit mass is projected vertically upwards in a medium that has a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20-g)$.

i Show that the equation of motion of the projectile is $\dfrac{dv}{dt}=-g-\dfrac{v}{10}$

ii Show that the time T for the particle to reach its greatest height is given by $T=10\ln\left(\dfrac{20}{g}\right)$

iii Show that the maximum height H is given by $H=2000-10g[10+T]$

iv If the particle then falls from this height, find the terminal velocity in this medium.

9 A particle of mass m is projected from the origin with an initial velocity $V\ \mathrm{ms}^{-1}$ at an angle of θ to the horizontal. The particle experiences the effect of gravity and a resistance proportional to its velocity in both the horizontal and vertical directions.

Prove the following results, where k is the coefficient of drag and g is gravitational acceleration.

i $\dot{x}=V\cos\theta\, e^{-\frac{k}{m}t}$

ii $x=\dfrac{mV\cos\theta}{k}\left(1-e^{-\frac{k}{m}t}\right)$

iii $\dot{y}=\left(\dfrac{mg}{k}+V\sin\theta\right)e^{-\frac{k}{m}t}-\dfrac{mg}{k}$

iv $y=\dfrac{m}{k}\left(\dfrac{mg}{k}+V\sin\theta\right)\left(1-e^{-\frac{k}{m}t}\right)-\dfrac{mgt}{k}$

v $y=\left(\dfrac{mg}{kV\cos\theta}+\tan\theta\right)x+\dfrac{m^2g}{k^2}\ln\left(1-\dfrac{kx}{mV\cos\theta}\right)$

10 A particle is moving in a medium where resistance to motion is proportional to the square of velocity, so $R=-kv^2$. At some point in its flight $\dot{x}=7\ \mathrm{ms}^{-1}$ and $\dot{y}=24\ \mathrm{ms}^{-1}$.

a Use similar triangles to find the horizontal and vertical components of resistance at the point, and prove that the total resistance and its components satisfy Pythagoras' Theorem.

b Show that the horizontal and vertical components at the point can be found using $R_x=-kv\dot{x}$ and $R_y=-kv\dot{y}$

c Show that $R_x=-k\dot{x}^2$ and $R_y=-k\dot{y}^2$ do not match the horizontal and vertical components at the point.

d Prove that $R=-kv^2$ cannot be split into horizontal and vertical components of $R_x=-k\dot{x}^2$ and $R_y=-k\dot{y}^2$ if the particle is moving at angle to the horizontal (ie unless $\dot{x}=0$ and/or $\dot{y}=0$ so the particle is moving vertically or horizontally, or is stationary).

1

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\int_{\frac{1}{2}\sqrt{\frac{g}{k}}}^{0} \frac{2kv}{g + kv^2}\,dv$$

$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^{\frac{1}{2}\sqrt{\frac{g}{k}}}$$

$$= \frac{1}{2k}\left(\ln\left(g + k\left(\frac{g}{4k}\right)\right) - \ln g\right)$$

$$= \frac{1}{2k}\ln\left(1 + \frac{1}{4}\right)$$

$$= \frac{1}{2k}\ln\frac{5}{4} \text{ metres}$$

2

A The maximum height occurs more than half way along the range, so is possible

B The impact velocity is equal to the projection velocity, which is impossible.

C The impact angle is greater than the projection angle, so is possible.

ANSWER (B)

3

Let $\dot{y} = 0$

$$\therefore 230e^{-\frac{t}{20}} - 200 = 0$$

$$230e^{-\frac{t}{20}} = 200$$

$$e^{-\frac{t}{20}} = \frac{200}{230}$$

$$e^{\frac{t}{20}} = \frac{230}{200}$$

$$\frac{t}{20} = \ln\frac{230}{200}$$

$$t = 20\ln\frac{230}{200}$$

$= 2.8$ seconds (1 dp)

4

i At terminal velocity $\dot{v} = 0$

$$\therefore mg - kv_T^2 = 0$$

$$v_T = \sqrt{\frac{mg}{k}}$$

ii

$$m\dot{v} = -(mg + kv^2)$$

$$\therefore v\frac{dv}{dx} = -\frac{(mg + kv^2)}{m}$$

$$\frac{dv}{dx} = -\frac{mg + kv^2}{mv}$$

$$\frac{dx}{dv} = -\frac{mv}{mg + kv^2}$$

$$\therefore H = -\int_u^0 \frac{mv}{mg + kv^2}\,dv$$

$$= -\frac{m}{2k}\Big[\ln(mg + kv^2)\Big]_u^0$$

$$= -\frac{m}{2k}\left(\ln mg - \ln(mg + ku^2)\right)$$

$$= \frac{m}{2k}\ln\frac{mg + ku^2}{mg}$$

$$= \frac{m}{2\left(\frac{mg}{V_T^2}\right)}\ln\frac{mg + \left(\frac{mg}{V_T^2}\right)u^2}{mg}$$

$$= \frac{V_T^2}{2g}\ln\left(1 + \frac{u^2}{V_T^2}\right)$$

iii

$$mv\frac{dv}{dx} = mg - \frac{mg}{V_T^2}v^2$$

$$\frac{dv}{dx} = \frac{g(V_T^2 - v^2)}{v.V_T^2}$$

$$\frac{dx}{dv} = \frac{V_T^2}{g} \times \frac{v}{V_T^2 - v^2}$$

$$\therefore H = \frac{V_T^2}{g}\int_0^w \frac{v}{V_T^2 - v^2}\,dv$$

$$= -\frac{V_T^2}{2g}\Big[\ln(V_T^2 - v^2)\Big]_0^w$$

$$= -\frac{V_T^2}{2g}\left(\ln(V_T^2 - w^2) - \ln(V_T^2)\right)$$

$$= \frac{V_T^2}{2g}\ln\left(\frac{V_T^2}{V_T^2 - w^2}\right)$$

$$\therefore \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right) = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - w^2}\right)$$

$$\therefore 1 + \frac{u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$\frac{v_T^2 + u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$v_T^4 - w^2v_T^2 + u^2v_T^2 - u^2w^2 = v_T^4$$

$$u^2v_T^2 = w^2v_T^2 + w^2u^2$$

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^2}$$

$(\div \text{ both sides by } u^2V_T^2w^2)$

5

i

$$m\ddot{x} = -mg - R$$
$$1 \times \ddot{x} = -1 \times g - \frac{1}{40}v^2$$
$$\ddot{x} = -g - \frac{1}{40}v^2$$

ii

$$v\frac{dv}{dx} = -\left(g + \frac{v^2}{40}\right)$$
$$\frac{dv}{dx} = -\frac{40g + v^2}{40v}$$
$$\frac{dx}{dv} = -\frac{40v}{40g + v^2}$$
$$x = -\int_{20}^{0} \frac{40v}{40g + v^2}\,dv$$
$$= 20\left[\ln(40g + v^2)\right]_0^{20}$$
$$= 20(\ln 800 - \ln 400)$$
$$= 20 \ln 2$$

iii

$$\ddot{x} = g - \frac{1}{40}v^2$$

iv

Let w be the velocity just before impact.

$$v\frac{dv}{dx} = g - \frac{1}{40}v^2$$
$$\frac{dv}{dx} = \frac{40g - v^2}{40v}$$
$$\frac{dx}{dv} = \frac{40v}{40g - v^2}$$
$$x = \int_0^w \frac{40v}{40g - v^2}\,dv$$
$$= 20\left[\ln(40g - v^2)\right]_w^0$$
$$= 20\left(\ln(40g) - \ln(40g - w^2)\right)$$
$$= 20\ln\left(\frac{400}{400 - w^2}\right)$$
$$\therefore 20 \ln 2 = 20\ln\left(\frac{400}{400 - w^2}\right)$$
$$\frac{400}{400 - w^2} = 2$$
$$400 = 800 - 2w^2$$
$$2w^2 = 400$$
$$w = \sqrt{200}$$
$$= 10\sqrt{2}\ \text{ms}^{-1}$$

6

i

$$m\ddot{x} = mg - R$$
$$7\ddot{x} = 7 \times 10 - \frac{7v^2}{10}$$
$$\ddot{x} = 10 - \frac{v^2}{10}$$

ii

$$\ddot{x} = 0$$
$$0 = 10 - \frac{v_T^2}{10}$$
$$V_T^2 = 100$$
$$V_T = 10\ \text{ms}^{-1}$$

iii

$$v\frac{dv}{dx} = 10 - \frac{v^2}{10}$$
$$\frac{dv}{dx} = \frac{100 - v^2}{10v}$$
$$\frac{dx}{dv} = \frac{10v}{100 - v^2}$$
$$x = \int_0^v \frac{10v}{100 - v^2}\,dv$$
$$= -5\left[\ln(100 - v^2)\right]_0^v$$
$$= 5\ln\left(\frac{100}{100 - v^2}\right)$$
$$e^{\frac{x}{5}} = \frac{100}{100 - v^2}$$
$$100e^{\frac{x}{5}} - e^{\frac{x}{5}}v^2 = 100$$
$$e^{\frac{x}{5}}v^2 = 100\left(e^{\frac{x}{5}} - 1\right)$$
$$v^2 = 100\left(1 - e^{-\frac{x}{5}}\right)$$

iv

$$\frac{dV}{dt} = -\left(10 + \frac{V^2}{10}\right)$$
$$\frac{dt}{dV} = -\frac{10}{100 + V^2}$$
$$t = -10\int_{\frac{10}{\sqrt{3}}}^{0} \frac{1}{100 + V^2}\,dV$$
$$= 10 \times \frac{1}{10}\left[\tan^{-1}\frac{V}{10}\right]_0^{\frac{10}{\sqrt{3}}}$$
$$= \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0$$
$$= \frac{\pi}{6}$$

7 **i**

$$\frac{d\dot{x}}{dt} = -\frac{\dot{x}}{10}$$
$$\frac{dt}{d\dot{x}} = -\frac{10}{d\dot{x}}$$
$$t = -10\int_{60\cos 30^\circ}^{\dot{x}} \frac{1}{\dot{x}}\,d\dot{x}$$
$$-\frac{t}{10} = \Big[\ln \dot{x}\Big]_{30\sqrt{3}}^{\dot{x}}$$
$$\ln|\dot{x}| - \ln\left(30\sqrt{3}\right) = -\frac{t}{10}$$
$$\ln|\dot{x}| = \ln(30\sqrt{3}) - \frac{t}{10}$$
$$\dot{x} = 30\sqrt{3}e^{-\frac{t}{10}}$$
$$x = 30\sqrt{3}\int_0^t e^{-\frac{t}{10}}\,dt$$
$$= -300\sqrt{3}\left[e^{-\frac{t}{10}}\right]_0^t$$
$$= -300\sqrt{3}\left(e^{-\frac{t}{10}} - 1\right)$$
$$= 300\sqrt{3}\left(1 - e^{-\frac{t}{10}}\right)$$

ii

$$\frac{d\dot{y}}{dt} = -\frac{\dot{y}}{10} - 10$$
$$= -\frac{\dot{y} + 100}{10}$$
$$\frac{dt}{d\dot{y}} = -\frac{10}{\dot{y} + 100}$$
$$t = -10\int_{60\sin 30^\circ}^{\dot{y}} \frac{1}{\dot{y} + 100}\,d\dot{y}$$
$$-\frac{t}{10} = \Big[\ln(\dot{y} + 100)\Big]_{30}^{\dot{y}}$$
$$\ln|\dot{y} + 100| - \ln(130) = -\frac{t}{10}$$
$$\ln|\dot{y} + 100| = \ln 130 - \frac{t}{10}$$
$$\dot{y} + 100 = 130e^{-\frac{t}{10}}$$
$$\dot{y} = 130e^{-\frac{t}{10}} - 100$$
$$y = \int_0^t \left(130e^{-\frac{t}{10}} - 100\right)dt$$
$$= \left[-1300e^{-\frac{t}{10}} - 100t\right]_0^t$$
$$= -1300e^{-\frac{t}{10}} - 100t + 1300$$
$$= 1300 - 1300e^{-\frac{t}{10}} - 100t$$

7.. **iii**

$$\frac{x}{300\sqrt{3}} = 1 - e^{-\frac{t}{10}} \qquad (1)$$

$$e^{-\frac{t}{10}} = 1 - \frac{x}{300\sqrt{3}}$$

$$= \frac{300\sqrt{3} - x}{300\sqrt{3}}$$

$$e^{\frac{t}{10}} = \frac{300\sqrt{3}}{300\sqrt{3} - x}$$

$$t = 10\ln\left(\frac{300\sqrt{3}}{300\sqrt{3} - x}\right) \qquad (2)$$

from (1) and (2):

$$y = 1300\left(1 - e^{-\frac{t}{10}}\right) - 100t$$

$$= 1300\left(\frac{x}{300\sqrt{3}}\right)$$

$$= \frac{13x}{3\sqrt{3}} + 1000\ln\left(\frac{300\sqrt{3} - x}{300\sqrt{3}}\right)$$

iv

$$\text{Let } \dot{y} = 0$$

$$130e^{-\frac{t}{10}} - 100 = 0$$

$$e^{-\frac{t}{10}} = \frac{100}{130}$$

$$e^{\frac{t}{10}} = \frac{130}{100}$$

$$t = 10\ln 1.3$$

$$= 2.6 \text{ sec}$$

8 **i**

$$m\ddot{x} = -mg - \frac{mv}{10}$$

$$\therefore \frac{dv}{dt} = -g - \frac{v}{10}$$

ii

$$\frac{dt}{dv} = -\frac{10}{10g + v}$$

$$t = -\int_{10(20-g)}^{v} \frac{10}{10g + v}\,dv$$

$$= 10\Big[\ln(10g + v)\Big]_{v}^{10(20-g)}$$

$$= 10\Big(\ln(10g + 200 - 10g) - \ln(10g + v)\Big)$$

$$= 10\ln\left(\frac{200}{10g + v}\right)$$

when $t = T, v = 0$

$$\therefore T = 10\ln\left(\frac{200}{10g}\right)$$

$$= 10\ln\left(\frac{20}{g}\right)$$

iii

$$v\frac{dv}{dx} = -\frac{10g + v}{10}$$

$$\frac{dv}{dx} = -\frac{10g + v}{10v}$$

$$\frac{dx}{dv} = -\frac{10v}{10g + v}$$

$$H = -10\int_{10(20-g)}^{0} \frac{v}{10g + v}\,dv$$

$$= 10\int_{0}^{10(20-g)} \frac{10g + v - 10g}{10g + v}\,dv$$

$$= 10\int_{0}^{10(20-g)} \left(1 - \frac{10g}{10g + v}\right)dv$$

$$= 10\Big[v - 10g\ln(10g + v)\Big]_{0}^{10(20-g)}$$

$$= 10\big((200 - 10g - 10g\ln(200))$$
$$- (0 - 10g\ln(10g))\big)$$

$$= 10\left(200 - 10g\left(1 + \ln\left(\frac{20}{g}\right)\right)\right)$$

$$= 2000 - 10g\left[10 + 10\ln\left(\frac{20}{g}\right)\right]$$

$$= 2000 - 10g[10 + T]$$

iv

$$\ddot{x} = g - \frac{v}{10}$$

$$0 = g - \frac{V_T}{10}$$

$$V_T = 10g$$

9 **i**

$$m\ddot{x} = -k\dot{x}$$
$$\frac{d\dot{x}}{dt} = -\frac{k}{m}\dot{x}$$
$$\frac{dt}{d\dot{x}} = -\frac{m}{k} \times \frac{1}{\dot{x}}$$
$$t = -\frac{m}{k}\int_{V\cos\theta}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$
$$-\frac{k}{m}t = \Big[\ln\dot{x}\Big]_{V\cos\theta}^{\dot{x}}$$
$$= \ln\dot{x} - \ln(V\cos\theta)$$
$$\ln\dot{x} = \ln(V\cos\theta) - \frac{k}{m}t$$
$$\dot{x} = V\cos\theta\, e^{-\frac{k}{m}t}$$

ii

$$\frac{dx}{dt} = V\cos\theta\, e^{-\frac{k}{m}t}$$
$$x = V\cos\theta\int_0^t e^{-\frac{k}{m}t}\,dt$$
$$= V\cos\theta \times \left(-\frac{m}{k}\right)\left[e^{-\frac{k}{m}t}\right]_0^t$$
$$= \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

iii

$$m\ddot{y} = -k\dot{y} - mg$$
$$\frac{d\dot{y}}{dt} = -\frac{k}{m}\dot{y} - g$$
$$= -\frac{k\dot{y} + mg}{m}$$
$$\frac{dt}{d\dot{y}} = -\frac{m}{k\dot{y} + mg}$$
$$t = -\frac{m}{k}\int_{V\sin\theta}^{\dot{y}} \frac{k}{k\dot{y} + mg}\,d\dot{y}$$
$$= -\frac{m}{k}\Big[\ln(k\dot{y} + mg)\Big]_{V\sin\theta}^{\dot{y}}$$
$$= -\frac{m}{k}(\ln(k\dot{y} + mg) - \ln(kV\sin\theta + mg))$$
$$-\frac{k}{m}t = \ln(k\dot{y} + mg) - \ln(kV\sin\theta + mg)$$
$$\ln(k\dot{y} + mg) = \ln(kV\sin\theta + mg) - \frac{k}{m}t$$
$$k\dot{y} + mg = (kV\sin\theta + mg)e^{-\frac{k}{m}t}$$
$$\dot{y} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$$

iv

$$\frac{dy}{dt} = \left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$$
$$y = \int_0^t \left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t} - \frac{mg}{k}\right)dt$$
$$= \left[-\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}\right]_0^t$$
$$= -\frac{m}{k}\left(\left(\frac{mg}{k} + V\sin\theta\right)e^{-\frac{k}{m}t}\right) - \frac{mgt}{k} + \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)$$
$$= \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mgt}{k}$$

9.. **v**

$$x = \frac{mV\cos\theta}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\frac{kx}{mV\cos\theta} = 1 - e^{-\frac{k}{m}t} \qquad (1)$$

$$e^{-\frac{k}{m}t} = 1 - \frac{kx}{mV\cos\theta}$$

$$-\frac{k}{m}t = \ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$t = -\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right) \qquad (2)$$

sub (1), (2) in (iv):

$$y = \frac{m}{k}\left(\frac{mg}{k} + V\sin\theta\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{mg}{k}\left(\frac{m}{k}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)\right)$$

$$= \frac{m}{k}\left(\frac{mg + kV\sin\theta}{k}\right)\left(\frac{kx}{mV\cos\theta}\right) + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$= \left(\frac{mg + kV\sin\theta}{kV\cos\theta}\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

$$= \left(\frac{mg}{kV\cos\theta} + \tan\theta\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mV\cos\theta}\right)$$

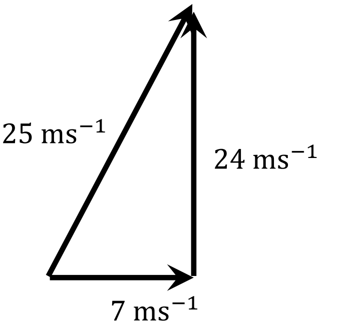
10

a

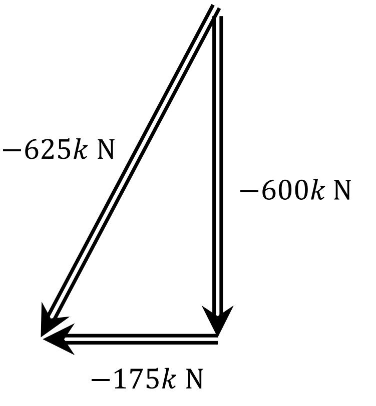
$v = \sqrt{7^2 + 24^2} = 25\ \text{ms}^{-1}$

$\therefore R = -k \times 25^2 = -625k\ \text{N}$

Now the triangle representing velocity and its components (above) and resistance and its components (below) must be similar.



The scale is $1: -25k$ so the horizontal component of resistance is $-175k$ N and the vertical component is $-600k$ N.



Using the magnitudes of the resistance and its components

$$175^2 + 600^2 = 390625$$

$$25^2 = 390625$$

$\therefore$ the resistance and its components satisfy Pythagoras' Theorem.

b

$R_x = -kv\dot{x} = -k \times 25 \times 7 = -175k$

$R_y = -kv\dot{y} = -k \times 25 \times 24 = -600k$

$\therefore$ the horizontal and vertical components at the point can be found using $R_x = -kv\dot{x}$ and $R_y = -kv\dot{y}$

c

$R_x = -k\dot{x}^2 = -k \times 7^2 = -49k \neq -175k$

$R_y = -k\dot{y}^2 = -k \times 24^2 = -576k \neq -600k$

$\therefore$ the horizontal and vertical components at the point cannot be found using $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$

d

If $-kv^2$ can be split into horizontal and vertical components $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$ then their magnitudes must satisfy Pythagoras' Theorem

$$\therefore (kv^2)^2 = (k\dot{x}^2)^2 + (k\dot{y}^2)^2$$

$$k^2(\dot{x}^2 + \dot{y}^2)^2 = k^2(\dot{x}^4 + \dot{y}^4)$$

$$\dot{x}^4 + 2\dot{x}^2\dot{y}^2 + \dot{y}^4 = \dot{x}^4 + \dot{y}^4$$

$$\therefore 2\dot{x}^2\dot{y}^2 = 0$$

$$\therefore \dot{x} = 0 \text{ and/or } \dot{y} = 0$$

$\therefore$ the resistance of $R = -kv^2$ cannot only be split into horizontal and vertical components $R_x = -k\dot{x}^2$ and $R_y = -k\dot{y}^2$ if the particle is moving vertically or horizontally, or is stationary