EXERCISE 4.5

BASIC

Find the following indefinite integrals:

1 $\int x^5\sqrt{x^2+1}\,dx$

2 $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx$

3 $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

4 $\int \frac{1}{2+\sqrt{x}}\,dx$

MEDIUM

5 $\int \frac{x^3}{(x^2+1)^2}\,dx$

6 $\int x^5\sqrt{2-x^3}\,dx$

7 $\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

CHALLENGING

8 $\int \frac{\sec x\tan x}{\sec x+\sec^2 x}\,dx$

9 Let $I=\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$

i Use the substitution $u=4-x$ to show that $I=\int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$

ii Hence, find the value of I.

1

$$\int x^5\sqrt{x^2+1}\,dx$$

$$\boxed{\begin{aligned} u^2 &= x^2+1 \\ 2udu &= 2xdx \\ dx &= \frac{udu}{x} \end{aligned}}$$

$$= \int x^4 \times x \times u \times \frac{udu}{x}$$

$$= \int (u^2-1)^2 u^2 du$$

$$= \int (u^4-2u^2+1)u^2 du$$

$$= \int (u^6-2u^4+u^2)\,du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$$

$$= \frac{\sqrt{(x^2+1)^7}}{7} - \frac{2\sqrt{(x^2+1)^5}}{5} + \frac{\sqrt{(x^2+1)^3}}{3} + c$$

2

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}dx$$

$$\boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$= \int \frac{u}{1-u} \times 2u\ \ du$$

$$= 2\int \frac{u^2}{1-u}du$$

$$= 2\int \frac{-u(1-u)-(1-u)+1}{1-u}du$$

$$= 2\int \left(-u-1+\frac{1}{1-u}\right)du$$

$$= -u^2 - 2u - 2\ln|1-u| + c$$

$$= -x - 2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + c$$

3

$$\int \frac{x^3}{\sqrt{1+x^2}}dx$$

$$\boxed{\begin{aligned} u^2 &= 1+x^2 \\ 2u\,du &= 2xdx \\ dx &= \frac{u\,du}{x} \end{aligned}}$$

$$= \int \frac{x^2 \times x}{u} \times \left(\frac{u\,du}{x}\right)$$

$$= \int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u + c$$

$$= \frac{\sqrt{(1+x^2)^3}}{3} - \sqrt{1+x^2} + c$$

4

$$\int \frac{1}{2+\sqrt{x}}dx$$

$$\boxed{\begin{aligned} u^2 &= x \\ 2u\,du &= \ dx \end{aligned}}$$

$$= \int \frac{1}{2+u} \times 2u\,du$$

$$= 2\int \frac{u}{u+2}du$$

$$= 2\int \frac{u+2-2}{u+2}du$$

$$= 2\int \left(1-\frac{2}{u+2}\right)du$$

$$= 2u - 4\ln|u+2| + c$$

$$= 2\sqrt{x} - 4\ln\left|2+\sqrt{x}\right| + c$$

5

$$\int \frac{x^3}{(x^2+1)^2}\,dx$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$= \int \frac{x^2 \times x}{u^2} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \frac{u-1}{u^2}\,du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right)du$$

$$= \frac{1}{2}\ln|u| + \frac{1}{2u} + c$$

$$= \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + c$$

6

$$\int x^5\sqrt{2-x^3}\,dx$$

$$\boxed{\begin{aligned} u^2 &= 2-x^3 \\ 2u\,du &= -3x^2dx \\ dx &= -\frac{2u}{3x^2}du \end{aligned}}$$

$$= \int x^5 \times u \times \left(-\frac{2u}{3x^2}\right)du$$

$$= -\frac{2}{3}\int x^3u^2\,du$$

$$= -\frac{2}{3}\int (2-u^2)u^2\,du$$

$$= -\frac{2}{3}\int (2u^2-u^4)\,du$$

$$= -\frac{2}{3}\left(\frac{2u^3}{3} - \frac{u^5}{5}\right) + c$$

$$= -\frac{4\sqrt{(2-x^3)^3}}{9} + \frac{2\sqrt{(2-x^3)^5}}{15} + c$$

7

$$\int x\sqrt{\frac{1+x^2}{1-x^2}}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$= \int x\sqrt{\frac{1+u}{1-u}} \times \frac{du}{2x}$$

$$= \frac{1}{2}\int \sqrt{\frac{1+u}{1-u}} \times \sqrt{\frac{1+u}{1+u}}\,du$$

$$= \frac{1}{2}\int \frac{1+u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du + \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{4}\int (-2u)(1-u^2)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\sin^{-1}u - \frac{1}{2}\sqrt{1-u^2} + c$$

$$= \frac{\sin^{-1}x^2}{2} - \frac{\sqrt{1-x^4}}{2} + c$$

8

$$\int \frac{\sec x\tan x}{\sec x + \sec^2 x}\,dx$$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x\tan x\,dx \\ dx &= \frac{du}{\sec x\tan x} \end{aligned}}$$

$$= \int \frac{\sec x\tan x}{u+u^2} \times \frac{du}{\sec x\tan x}$$

$$= \int \frac{1}{u(1+u)}\,du$$

$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right)du$$

$$= \ln|u| - \ln|1+u| + c$$

$$= \ln\left|\frac{u}{1+u}\right| + c$$

$$= \ln\left|\frac{\sec x}{1+\sec x}\right| + c$$

$$= \ln\left|\frac{1}{\cos x + 1}\right| + c$$

$$= -\ln|\cos x + 1| + c$$

9

$$I=\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$$

$$=-\int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)u}\,du$$

$$=\int_1^3 \frac{\cos^2\left(\frac{\pi}{2}-\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$=\int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

ii

$$2I=\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx+\int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

$$2I=\int_1^3 \frac{1}{x(4-x)}\,dx$$

$$2I=\frac{1}{4}\int_1^3 \left(\frac{1}{x}+\frac{1}{4-x}\right)dx$$

$$I=\frac{1}{8}\Big[\ln x-\ln|4-x|\Big]_1^3$$

$$=\frac{1}{8}(\ln 3-\ln 1-\ln 1+\ln 3)$$

$$=\frac{\ln 3}{4}$$