EXERCISE 3.2

BASIC

1 Prove that for any positive integer n, $\frac{d}{dx}(x^n) = nx^{n-1}$

2 Prove that for any positive integer n, $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$

3 Prove that the sum of the interior angles of an n-sided plane convex polygon is $180(n-2)°$

4 Given $a_1 = 2$ and $a_n = 5a_{n-1}$ for $n \geq 2$, prove that $a_n = 2 \times 5^{n-1}$ for $n \geq 1$.

5 Given $a_0 = A$ and $a_n = (1+r)a_{n-1}$, show that $a_n = A(1+r)^n$ for $n \geq 0$

MEDIUM

6 Prove that for any positive integer n, $\frac{d}{dx}(\cos^n x) = -n\cos^{n-1} x \sin x$

7 Prove that for any positive integer $n \geq 1$ that $\binom{n}{1} = n$.

You may assume $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ and $\binom{n}{0} = 1$

8 A plane is divided into regions by one or more intersecting circles. Prove that it is possible to colour the regions with only two colours, such that no two regions sharing an edge are the same colour.

9 Given $a_1 = 2$ and $a_n = \frac{a_{n-1}}{n}$ for $n \geq 2$, prove that $a_n = \frac{2}{n!}$ for $n \geq 1$

CHALLENGING

10 Chessboard Problem Prove that it is possible to cover a $2^n \times 2^n$ grid with L tiles consisting of 3 squares if 1 square is removed.

11 Tower of Hanoi You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top of each other according to size on the first peg

- the largest disk being on the bottom and the smallest on top, as shown above. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissible moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg. Prove that it takes $2^n - 1$ moves to move n disks from the first peg to the third peg.

12 Postage Stamp Problem Prove any integer amount of postage in cents $n \geq 12$ can be paid for exactly using only 4 cent and 5 cent stamps.

Question 1-3

See the lesson for solutions

4 Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = 2 \times 5^{1-1} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = 2 \times 5^{k-1}$

RTP $P(k+1)$ $\qquad a_{k+1} = 2 \times 5^k$

$\text{LHS} = a_{k+1}$

$= 5a_k$ $\qquad$ from the recursive formula

$= 5\left(2 \times 5^{k-1}\right)$ $\qquad$ from $P(k)$

$= 2 \times 5^k$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

5 Let $P(n)$ represent the proposition.

$P(0)$ is true since $a_0 = A$ and $a_0 = A(1+r)^0 = A$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = A(1+r)^k$

RTP $P(k+1)$ $\qquad a_{k+1} = A(1+r)^{k+1}$

$\text{LHS} = a_{k+1}$

$= (1+r)a_k$ $\qquad$ from the recursive formula

$= (1+r) \times A(1+r)^k$ $\qquad$ from $P(k)$

$= A(1+r)^{k+1}$

$= \text{RHS}$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 0$ by induction $\square$

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\frac{d}{dx}(\cos^1 x) = -\sin x$ and $-1\cos^{1-1} x \sin x = -\sin x$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\frac{d}{dx}\left(\cos^k x\right) = -k\cos^{k-1} x \sin x$

RTP $P(k+1)$ $\qquad \frac{d}{dx}\left(\cos^{k+1} x\right) = -(k+1)\cos^k x \sin x$

$$\begin{aligned}
\text{LHS } &= \frac{d}{dx}\left(\cos x \cos^k x\right) \\
&= \cos x \frac{d}{dx}\left(\cos^k x\right) + \cos^k x \times \frac{d}{dx}(\cos x) \\
&= \cos x\left(-k\cos^{k-1} x \sin x\right) + \cos^k x\,(-\sin x) \qquad \text{from } P(k) \\
&= -k\cos^k x \sin x - \cos^k x \sin x \\
&= -(k+1)\cos^k x \sin x \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

7 Let $P(n)$ represent the proposition.

$P(1)$ is true since $\binom{1}{1} = \frac{1!}{0!1!} = 1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\binom{k}{1} = k$

RTP $P(k+1)$ $\binom{k+1}{1} = k+1$

$$\begin{aligned}
\text{LHS} &= \binom{k+1}{1} \\
&= \binom{k}{1} + \binom{k}{0} \qquad \text{since} \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \\
&= k+1 \qquad \text{from } P(k) \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

8 Let $P(n)$ represent the proposition.

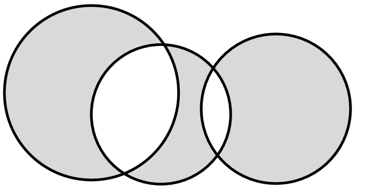
$P(1)$ is true since with one circle we can colour the inside of the circle with one colour and the outside the other.

If $P(k)$ is true for some arbitrary, then with k circles we can colour the resulting regions with only two colours so no two regions sharing an edge are the same colour.

RTP $P(k+1)$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

Consider the k circles on the plane, with no two regions sharing a border being the same colour as shown (top right).

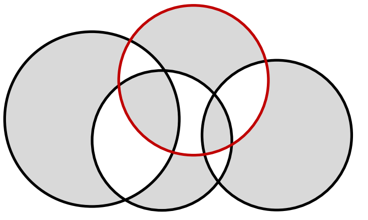
k circles



Add a circle, here shown in red, cutting some regions in two, and possibly leaving others unaffected.

For all regions on the inside of the circle swap their colour, as shown (bottom right).

$k+1$ circles



No two regions sharing an edge are the same colour.

$\therefore$ We can colour the regions that result from $k+1$ circles with only two colours so no two regions sharing an edge are the same colour.

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

9 Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = \frac{2}{1!} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = \frac{2}{k!}$

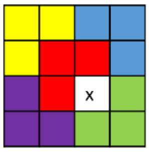
RTP $P(k+1)$ $\quad a_{k+1} = \frac{2}{(k+1)!}$

$$\begin{aligned}
\text{LHS} &= a_{k+1} \\
&= \frac{a_k}{k+1} && \text{from the recursive formula} \\
&= \frac{2}{k!} \div (k+1) && \text{from } P(k) \\
&= \frac{2}{(k+1)!} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

10



Here we have a $2^2 \times 2^2$ grid with 1 square removed, with the remainder covered in 5 L tiles.

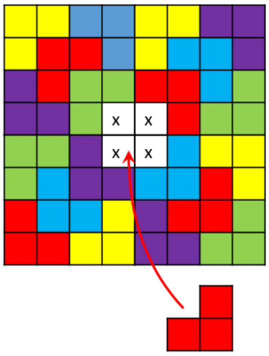
Let $P(n)$ represent the proposition.

$P(1)$ is true since a $2^1 \times 2^1$ grid with one square removed can be covered by one L tile.

If $P(k)$ is true for some arbitrary k then it is possible to completely cover a $2^k \times 2^k$ grid with L tiles consisting of 3 squares if 1 square is removed.

RTP $P(k+1)$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.

Arrange 4 of the $2^k \times 2^k$ grids in a square, removing the square from each grid that is in the centre of the new square, which is a $2^{k+1} \times 2^{k+1}$ grid, as shown at right.



Place one extra tile in the centre, so only 1 square has been removed from the $2^{k+1} \times 2^{k+1}$ grid and the rest is covered.

$\therefore$ It is possible to completely cover a $2^{k+1} \times 2^{k+1}$ grid with L tiles consisting of 3 squares if 1 square is removed.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

11

Let $P(n)$ represent the proposition.

$P(1)$ is true since the one disk can be moved to the third peg in 1 move, and $2^1 - 1 = 1$

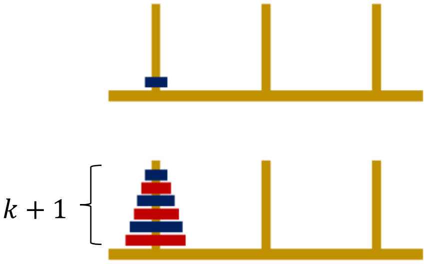
If $P(k)$ is true for some arbitrary k then it takes $2^k - 1$ moves to move k disks from the first peg to the third peg.

RTP $P(k+1)$ It takes $2^{k+1} - 1$ moves to move k disks from the first peg to the third peg.

Move the first k disks to the second peg instead of the third peg, in $2^k - 1$ moves, from $P(k)$.
Move the bottom disk to the third peg in one move.
Move the first k disks again until they are on the third peg on top of the largest disk, in $2^k - 1$ moves, again from $P(k)$.

$$\begin{aligned}\text{Total moves} &= (2^k - 1) + 1 + (2^k - 1)\\ &= 2 \cdot 2^k - 1\\ &= 2^{k+1} - 1 \quad \text{as required}\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$



$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

12 Let $P(n)$ represent the proposition.

$P(12)$ is true since we can use three 4 cent stamps.

If $P(k)$ is true for some arbitrary $k \geq 12$ then we can pay for it exactly using only 4 cent and 5 cent stamps.

RTP $P(k+1)$
We can pay for $k+1$ cents of postage exactly using only 4 cent and 5 cent stamps.

Case 1: We have used at least one 4 cent stamp to make k cents postage
Remove one 4 cents stamp and replace it with a 5 cent stamp, increasing the postage by 1 cent from k to $k+1$ cents.

Case 2: There are no 4 cent stamps used to make k cents postage.
There must be at least three 5 cent stamps here, and if we remove three 5 cent stamps and replace them with four 4 cent stamps the postage increases by 1 cent from k to $k+1$ cents.

$\therefore$ We can pay for $k+1$ cents postage.
$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 12$ by induction $\square$