EXERCISE 5.4

BASIC

1 Prove the point $\binom{1}{-4}$ lies on the line $\underset{\sim}{r} = \binom{3}{-1} + \lambda\binom{2}{3}$

2 Prove the point $\begin{pmatrix}4\\3\\-3\end{pmatrix}$ lies on the line $\underset{\sim}{r} = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

3 Prove the point $\binom{1}{2}$ does not lie on $\underset{\sim}{r} = \binom{3}{-1} + \lambda\binom{2}{3}$

4 Prove the point $\begin{pmatrix}3\\4\\2\end{pmatrix}$ does not lie on $\underset{\sim}{r} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$

MEDIUM

5 Find the point of intersection of the lines $\underset{\sim}{r} = \binom{1}{1} + \lambda\binom{2}{3}$ and $\underset{\sim}{q} = \binom{4}{2} + \lambda\binom{-1}{2}$

6 Find the point of intersection of the lines
$\underset{\sim}{r} = \begin{pmatrix}0\\3\\-5\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\-1\end{pmatrix}$

7 Rewrite the following vector equations in Cartesian form by first finding expressions for x and y in terms of λ.
a $\underset{\sim}{r} = \binom{0}{1} + \lambda\binom{2}{-1}$
b $\underset{\sim}{r} = \binom{2}{-3} + \lambda\binom{-1}{-3}$

8 Prove that the vector equations $\underset{\sim}{r} = \binom{2}{4} + \lambda\binom{1}{3}$ and $\underset{\sim}{q} = \binom{0}{-2} + \lambda\binom{-2}{-6}$ have the same Cartesian equation.

9 Rewrite $y = 4x + 5$ as a vector equation.

CHALLENGING

10 Prove the lines $y = 2x + 1$ and $y = -\frac{1}{2}x$ are perpendicular
a Using the product of their gradients
b By first converting them to vector form

11 A triangle has vertices $A(0,0,0), B(0,2,4)$ and $C(4,2,0)$. Find the equations of the three medians and show that they are concurrent.

1

$$\begin{pmatrix}1\\-4\end{pmatrix}=\begin{pmatrix}3+2\lambda\\-1+3\lambda\end{pmatrix}$$
$$1=3+2\lambda\ \rightarrow \lambda=-1$$
$$-4=-1+3\lambda\ \rightarrow \lambda=-1$$
$\therefore \binom{1}{-4}$ lies on the line $\underset{\sim}{r}=\binom{3}{-1}+\lambda\binom{2}{3}$

2

$$\begin{pmatrix}4\\3\\-3\end{pmatrix}=\begin{pmatrix}2\lambda\\1+\lambda\\-1-\lambda\end{pmatrix}$$
$$4=2\lambda \rightarrow \lambda=2$$
$$3=1+\lambda \rightarrow \lambda=2$$
$$-3=-1-\lambda\ \rightarrow\ \lambda=2$$
$\therefore \begin{pmatrix}4\\3\\-3\end{pmatrix}$ lies on the line $\underset{\sim}{r}=\begin{pmatrix}0\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

3

$$\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3+2\lambda\\-1+3\lambda\end{pmatrix}$$
$$1=3+2\lambda\ \rightarrow \lambda=-1$$
$$2=-1+3\lambda\ \rightarrow \lambda=1$$
$\therefore\binom{1}{2}$ does not lie on $\underset{\sim}{r}=\binom{3}{-1}+\lambda\binom{2}{3}$

4

$$\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}3\lambda\\1+2\lambda\\2+\lambda\end{pmatrix}$$
$$3=3\lambda\ \rightarrow \lambda=1$$
$$4=1+2\lambda\ \rightarrow \lambda=\frac{3}{2}$$
$\therefore \begin{pmatrix}3\\4\\2\end{pmatrix}$ does not lie on $\underset{\sim}{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$

5

Change the second parameter to μ

$$\begin{pmatrix}1+2\lambda\\1+3\lambda\end{pmatrix}=\begin{pmatrix}4-\mu\\2+2\mu\end{pmatrix}$$

$1+2\lambda=4-\mu\ \ (1)\qquad 1+3\lambda=2+2\mu\ \ (2)$

$(2)+2\times(1)$:

$3+7\lambda=10\ \rightarrow\ 7\lambda=7\ \rightarrow\ \lambda=1$

The point of intersection is $\binom{1}{1}+\binom{2}{3}=\binom{3}{4}$

6

$$\begin{pmatrix}-\lambda\\3+\lambda\\-5+2\lambda\end{pmatrix}=\begin{pmatrix}-1-\mu\\2+2\mu\\3-\mu\end{pmatrix}$$

$-\lambda=-1-\mu\qquad (1)$

$3+\lambda=2+2\mu\qquad (2)$

$-5+2\lambda=3-\mu\qquad (3)$

$(1)+(2)$:

$3=1+\mu$

$\mu=2$

The point of intersection is

$$\begin{pmatrix}-1\\2\\3\end{pmatrix}+2\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}-3\\6\\1\end{pmatrix}$$

7 **a**

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\lambda\\1-\lambda\end{pmatrix}$$
$$x=2\lambda\ \rightarrow\ \lambda=\frac{x}{2}$$
$$y=1-\lambda=1-\frac{x}{2}$$
$$\therefore y=-\frac{1}{2}x+1$$

b

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2-\lambda\\-3-3\lambda\end{pmatrix}$$
$$x=2-\lambda\ \rightarrow\ \lambda=2-x$$
$$y=-3-3\lambda=-3-3(2-x)$$
$$\therefore y=3x-9$$

8

For $\underset{\sim}{r}=\binom{2}{4}+\lambda\binom{1}{3}$, we have $m=\frac{3}{1}=3$ and $b=4-2(3)=-2$, so its Cartesian equation is $y=3x-2$.
For $\underset{\sim}{q}=\binom{0}{-2}+\lambda\binom{-2}{-6}$, we have $m=\frac{-6}{-2}=3$ and $b=-2-0(3)=-2$, so its Cartesian equation is $y=3x-2$..
Both vector equations have the same Cartesian equation.

9

The y-intercept is $(0,5)$ so let $\underset{\sim}{a}=\binom{0}{5}$.
The gradient is $\frac{4}{1}$ so let $\underset{\sim}{b}=\binom{1}{4}$.

$y=4x+5$ is equivalent to $\underset{\sim}{r}=\binom{0}{5}+\lambda\binom{1}{4}$

10 **a**

$m_1 = 2, m_2 = -\frac{1}{2}$

$m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$

∴ The two lines are perpendicular

b

In vector form $y = 2x + 1 \ \rightarrow \ \underset{\sim}{r_1} = \binom{0}{1} + \lambda \binom{1}{2}$ and $y = -\frac{1}{2}x \ \rightarrow \ \underset{\sim}{r_2} = \lambda \binom{-2}{1}$

$\binom{1}{2} \cdot \binom{-2}{1} = -2 + 2 = 0$

∴ The two lines are perpendicular

11 The midpoints are $M_{AB} = (0,1,2), M_{AC} = (2,1,0), M_{BC} = (2,2,2)$. The equations of the medians through each vertex are:

$$\underset{\sim}{r_A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2-0 \\ 2-0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix} \qquad (1) \qquad \underset{\sim}{r_B} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2-0 \\ 1-2 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 2-\lambda \\ 4-4\lambda \end{pmatrix} \qquad (2)$$

$$\underset{\sim}{r_C} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0-4 \\ 1-2 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 4-4\lambda \\ 2-\lambda \\ 2\lambda \end{pmatrix} \qquad (3)$$

For (1) and (2):

$2\lambda = 2\mu \quad 2\lambda = 2 - \mu \qquad 2\lambda = 4 - 4\mu$

$\lambda = \mu \qquad 3\lambda = 2 \ \rightarrow \lambda = \frac{2}{3}$

∴ (1) and (2) intersect at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

For (1) and (3):

$2\lambda = 4 - 4\mu \quad 2\lambda = 2 - \mu \qquad 2\lambda = 2\mu$

$3\lambda = 2 \ \rightarrow \lambda = \frac{2}{3} \qquad \lambda = \mu$

∴ (1) and (3) intersect at$\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

Since (2) and (3) both intersect (1) at the same point, they must intersect there as well.

∴ all three medians intersect at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$