Example 4 1A

a Find how many negative terms there are in the sequence $T_n = 12n - 100$.
b Find the first positive term of the sequence $T_n = 7n - 60$.

SOLUTION

a Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms.

b Put $T_n > 0$.
Then $7n - 60 > 0$
$7n > 60$
$n > 8\frac{4}{7}$.
Thus the first positive term is $T_9 = 3$.

Note: The question, 'Find the first positive term' requires two answers:
- Which number term is it?
- What is its value?

Thus the correct answer is, 'The first positive term is $T_9 = 3$'.

Exercise 1A

FOUNDATION

1 Alex collects stamps. He found a collection of 700 stamps in the attic a few years ago, and every month since then he has been buying 150 interesting stamps to add to his collection. Thus the numbers of stamps at the end of each month after his discovery form a sequence

$$850,\ 1000,\ \ldots$$

a Copy and continue the sequence to at least 12 terms followed by dots . . . .
b After how many months did his collection first exceed 2000 stamps?

2 Write down the next four terms of each sequence.
a $5,\ 10,\ 15,\ \ldots$ **b** $6,\ 16,\ 26,\ \ldots$ **c** $2,\ 4,\ 8,\ \ldots$ **d** $3,\ 6,\ 12,\ \ldots$
e $38,\ 34,\ 30,\ \ldots$ **f** $39,\ 30,\ 21,\ \ldots$ **g** $24,\ 12,\ 6,\ \ldots$ **h** $81,\ 27,\ 9,\ \ldots$
i $-1,\ 1,\ -1,\ \ldots$ **j** $1,\ 4,\ 9,\ \ldots$ **k** $\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots$ **l** $16,\ -8,\ 4,\ \ldots$

3 Find the first four terms of each sequence. You will need to substitute $n = 1$, $n = 2$, $n = 3$ and $n = 4$ into the formula for the nth term T_n.
a $T_n = 6n$ **b** $T_n = 5n - 2$ **c** $T_n = 2^n$ **d** $T_n = 5^n$
e $T_n = 20 - n$ **f** $T_n = 6 - 2n$ **g** $T_n = 3 \times 2^n$ **h** $T_n = 7 \times 10^n$
i $T_n = n^3$ **j** $T_n = n(n + 1)$ **k** $T_n = (-1)^n$ **l** $T_n = (-3)^n$

4 Write down the first four terms of each sequence described below.
a The first term is 6, and every term after that is 2 more than the previous term.
b The first term is 11, and every term after that is 50 more than the previous term.
c The first term is 15, and every term after that is 3 less than the previous term.
d The first term is 12, and every term after that is 8 less than the previous term.
e The first term is 5, and every term after that is twice the previous term.
f The first term is $\frac{1}{3}$, and every term after that is three times the previous term.
g The first term is 18, and every term after that is half the previous term.
h The first term is -100, and every term after that is one fifth of the previous term.

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5 Write out the first twelve terms of the sequence 7, 12, 17, 22, ...
a How many terms are less than 30?
b How many terms are less than 60?
c How many terms lie between 20 and 40?
d How many terms lie between 10 and 50?
e What is the 10th term?
f What number term is 37?
g Is 87 a term in the sequence?
h Is 201 a term in the sequence?
i Find the first term greater than 45.
j Find the last term less than 43.

6 Write out the first twelve terms of the sequence $\frac{3}{4}$, $1\frac{1}{2}$, 3, 6, ...
a How many terms are less than 30?
b How many terms are less than 400?
c How many terms lie between 20 and 100?
d How many terms lie between 1 and 1000?
e What is the 10th term?
f What number term is 192?
g Is 96 a term in the sequence?
h Is 100 a term in the sequence?
i Find the first term greater than 200.
j Find the last term less than 50.

DEVELOPMENT

7 For each sequence, write out the first five terms. Then explain how each term is obtained from the previous term.
a $T_n = 12 + n$
b $T_n = 4 + 5n$
c $T_n = 15 - 5n$
d $T_n = 3 \times 2^n$
e $T_n = 7 \times (-1)^n$
f $T_n = 80 \times \left(\frac{1}{2}\right)^n$

8 The nth term of a sequence is given by $T_n = 3n + 1$.
a Put $T_n = 40$, and hence show that 40 is the 13th term of the sequence.
b Put $T_n = 30$, and hence show that 30 is not a term of the sequence.
c Similarly, find whether 100, 200 and 1000 are terms of the sequence.

9 Answer each question by forming an equation and solving it.
a Find whether 16, 35 and 111 are terms of the sequence $T_n = 2n - 5$.
b Find whether 44, 200 and 306 are terms of the sequence $T_n = 10n - 6$.
c Find whether 40, 72 and 200 are terms of the sequence $T_n = 2n^2$.
d Find whether 8, 96 and 128 are terms of the sequence $T_n = 2^n$.

10 The nth term of a sequence is given by $T_n = 10n + 4$.
a Put $T_n < 100$, and hence show that the nine terms T_1 to T_9 are less than 100.
b Put $T_n > 56$, and hence show that the first term greater than 56 is $T_6 = 64$.
c Similarly, find how many terms are less than 500.
d Find the first term greater than 203, giving its number and its value.

11 Answer each question by forming an inequation and solving it.
a How many terms of the sequence $T_n = 2n - 5$ are less than 100?
b How many terms of the sequence $T_n = 4n + 6$ are less than 300?
c What is the first term of the sequence $T_n = 3n + 5$ greater than 127?
d What is the first term of the sequence $T_n = 7n - 44$ greater than 100?

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12 In each part, the two lines define a sequence T_n. The first line gives the first term T_1. The second line defines how each subsequent term T_n is obtained from the previous term T_{n-1}. Write down the first four terms of each sequence.

a $T_1 = 3,$
$T_n = T_{n-1} + 2, \quad \text{for } n \ge 2.$

b $T_1 = 5,$
$T_n = T_{n-1} + 12, \quad \text{for } n \ge 2.$

c $T_1 = 6,$
$T_n = T_{n-1} - 3, \quad \text{for } n \ge 2.$

d $T_1 = 12,$
$T_n = T_{n-1} - 10, \quad \text{for } n \ge 2.$

e $T_1 = 5,$
$T_n = 2T_{n-1}, \quad \text{for } n \ge 2.$

f $T_1 = 4,$
$T_n = 5T_{n-1}, \quad \text{for } n \ge 2.$

g $T_1 = 20,$
$T_n = \frac{1}{2}T_{n-1}, \quad \text{for } n \ge 2.$

h $T_1 = 1,$
$T_n = -T_{n-1}, \quad \text{for } n \ge 2.$

CHALLENGE

13 Write down the first four terms of each sequence. Then state which terms of the whole sequence are zero.

a $T_n = \sin 90n°$ **b** $T_n = \cos 90n°$ **c** $T_n = \cos 180n°$ **d** $T_n = \sin 180n°$

14 The *Fibonacci sequence* is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \ge 3.$$

Write out the first 12 terms of the sequence. Explain why every third term of the sequence is even and the rest are odd.

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1B Arithmetic sequences

A simple type of sequence is an *arithmetic sequence*. This is a sequence such as

$$3,\ 13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93,\ \dots,$$

in which the difference between successive terms is constant — in this example each term is 10 more than the previous term. Notice that all the terms can be generated from the *first term* 3 by repeated addition of this *common difference* 10.

In the context of successive terms of sequences, the word *difference* will always mean some term minus the previous term.

Definition of an arithmetic sequence

Arithmetic sequences are called APs for short. The initials stand for 'arithmetic progression' — an old name for the same thing.

2 ARITHMETIC SEQUENCES

- The *difference* between successive terms in a sequence T_n always means some term minus the previous term, that is,

  $$\text{difference} = T_n - T_{n-1}, \quad \text{where } n \geq 2.$$

- A sequence T_n is called an *arithmetic sequence* or AP if

  $$T_n - T_{n-1} = d, \quad \text{for all } n \geq 2,$$

  where d is a constant, called the *common difference*.
- The terms of an arithmetic sequence can be generated from the first term by repeated addition of this common difference,

  $$T_n = T_{n-1} + d, \quad \text{for all } n \geq 2.$$

Example 5 — 1B

Test whether each sequence is an AP. If the sequence is an AP, find its first term a and its common difference d.

a $46,\ 43,\ 40,\ 37,\ \dots$

b $1,\ 4,\ 9,\ 16,\ \dots$

SOLUTION

a $T_2 - T_1 = 43 - 46 = -3$ $\quad T_3 - T_2 = 40 - 43 = -3$ $\quad T_4 - T_3 = 37 - 40 = -3$

Hence the sequence is an AP with $a = 46$ and $d = -3$.

b $T_2 - T_1 = 4 - 1 = 3$ $\quad T_3 - T_2 = 9 - 4 = 5$ $\quad T_4 - T_3 = 16 - 9 = 7$

The differences are not all the same, so the sequence is not an AP.

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A formula for the *n*th term of an AP

Let the first term of an AP be a and the common difference be d. Then the first few terms of the sequence are

$$T_1 = a, \qquad T_2 = a + d, \qquad T_3 = a + 2d, \qquad T_4 = a + 3d, \qquad \ldots$$

From this pattern, the general formula for the nth term is clear:

3 THE *n*TH TERM OF AN AP

$$T_n = a + (n - 1)d$$

where a is the first term and d is the common difference.

Example 6 1B

Write out the first five terms, and calculate the 20th term, of the AP with:

a $a = 2$ and $d = 5$,

b $a = 20$ and $d = -3$.

SOLUTION

a $2,\ 7,\ 12,\ 17,\ 22,\ \ldots$

$$\begin{aligned} T_{20} &= a + 19d \\ &= 2 + 19 \times 5 \\ &= 97 \end{aligned}$$

b $20,\ 17,\ 14,\ 11,\ 8,\ \ldots$

$$\begin{aligned} T_{20} &= a + 19d \\ &= 20 + 19 \times (-3) \\ &= -37 \end{aligned}$$

Example 7 1B

a Find a formula for the nth term of the sequence 26, 35, 44, 53, ...

b How many terms are there in the sequence 26, 35, 44, 53, . . . , 917?

SOLUTION

a The sequence is an AP with $a = 26$ and $d = 9$.

$$\begin{aligned} \text{Hence } T_n &= a + (n - 1)d \\ &= 26 + 9(n - 1) \\ &= 26 + 9n - 9 \\ &= 17 + 9n. \end{aligned}$$

b
$$\begin{aligned} \text{Put} \qquad T_n &= 917. \\ \text{Then } 17 + 9n &= 917 \\ 9n &= 900 \\ n &= 100, \end{aligned}$$

so there are 100 terms in the sequence.

Solving problems involving APs

Now that we have the formula for the nth term T_n, many problems can be solved by forming an equation and solving it.

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Example 8 1B

a Show that the sequence 200, 193, 186, . . . is an AP.
b Find a formula for the nth term.
c Find the first negative term.

SOLUTION

a Because $T_2 - T_1 = -7$
and $T_3 - T_2 = -7$,
it is an AP with $a = 200$ and $d = -7$.

b Hence $T_n = 200 - 7(n - 1)$
$= 200 - 7n + 7$
$= 207 - 7n$.

c Put $T_n < 0$.
Then $207 - 7n < 0$
$207 < 7n$
$n > 29\frac{4}{7}$,
so the first negative term is $T_{30} = -3$.

Example 9 1B

The first term of an AP is 105 and the 10th term is 6. Find the common difference and write out the first five terms.

SOLUTION

First, we know that $T_1 = 105$,
that is, $a = 105$. (1)
Secondly, we know that $T_{10} = 6$,
so using the formula for the 10th term, $a + 9d = 6$. (2)
Substituting (1) into (2), $105 + 9d = 6$
$9d = -99$
$d = -11$,
so the common difference is $d = -11$ and the sequence is 105, 94, 83, 72, 61, . . .

Arithmetic sequences and linear functions

Take a linear function such as $f(x) = 30 - 8x$, and substitute the positive integers. The result is an arithmetic sequence

$$22,\ 14,\ 6,\ -2,\ -10,\ \ldots .$$

x	1	2	3	4	5
$f(x)$	22	14	6	-2	-10

The formula for the nth term of this AP is $T_n = 22 - 8(n - 1) = 30 - 8n$. This is a function whose domain is the set of positive integers, and its equation is the same as the linear function above, with only a change of pronumeral from x to n.

Every arithmetic sequence can be generated in this way.

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Exercise 1B

FOUNDATION

1 Write out the next three terms of these sequences. They are all APs.

a $2,\ 6,\ 10,\ \ldots$ b $3,\ 8,\ 13,\ \ldots$ c $35,\ 25,\ 15,\ \ldots$

d $11,\ 5,\ -1,\ \ldots$ e $4\frac{1}{2},\ 6,\ 7\frac{1}{2},\ \ldots$ f $8,\ 7\frac{1}{2},\ 7,\ \ldots$

2 Write out the first four terms of the APs whose first terms and common differences are:

a $a = 3$ and $d = 2$ b $a = 7$ and $d = 2$ c $a = 7$ and $d = -4$

d $a = 17$ and $d = 11$ e $a = 30$ and $d = -11$ f $a = -9$ and $d = 4$

g $a = 4\frac{1}{2}$ and $d = -\frac{1}{2}$ h $a = 3\frac{1}{2}$ and $d = -2$ i $a = 0.9$ and $d = 0.7$

3 Find the differences $T_2 - T_1$ and $T_3 - T_2$ for each sequence to test whether it is an AP. If the sequence is an AP, state the values of the first term a and the common difference d.

a $3,\ 7,\ 11,\ \ldots$ b $11,\ 7,\ 3,\ \ldots$ c $10,\ 17,\ 24,\ \ldots$

d $10,\ 20,\ 40,\ \ldots$ e $50,\ 35,\ 20,\ \ldots$ f $23,\ 34,\ 45,\ \ldots$

g $-12,\ -7,\ -2,\ \ldots$ h $-40,\ 20,\ -10,\ \ldots$ i $1,\ 11,\ 111,\ \ldots$

j $8,\ -2,\ -12,\ \ldots$ k $-17,\ 0,\ 17,\ \ldots$ l $10,\ 7\frac{1}{2},\ 5,\ \ldots$

4 Use the formula $T_n = a + (n - 1)d$ to find the 11th term T_{11} of the APs in which:

a $a = 7$ and $d = 6$ b $a = 15$ and $d = -7$ c $a = 10\frac{1}{2}$ and $d = 4$

5 Use the formula $T_n = a + (n - 1)d$ to find the eighth term T_8 of the APs in which:

a $a = 1$ and $d = 4$ b $a = 100$ and $d = -7$ c $a = -13$ and $d = 6$

6 a Find the first term a and the common difference d of the AP $6,\ 16,\ 26,\ \ldots$

b Find the ninth term T_9, the 21st term T_{21} and the 100th term T_{100}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

7 a Find the first term a and the common difference d of the AP $-20,\ -9,\ 2,\ \ldots$

b Find the eighth term T_8, the 31st term T_{31} and the 200th term T_{200}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

8 a Find the first term a and the common difference d of the AP $300,\ 260,\ 220,\ \ldots$

b Find the seventh term T_7, the 51st term T_{51} and the 1000th term T_{1000}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

DEVELOPMENT

9 Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If the sequence is an AP, use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

a $8,\ 11,\ 14,\ \ldots$ b $21,\ 15,\ 9,\ \ldots$ c $8,\ 4,\ 2,\ \ldots$

d $-3,\ 1,\ 5,\ \ldots$ e $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \ldots$ f $12,\ -5,\ -22,\ \ldots$

g $\sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2},\ \ldots$ h $1,\ 4,\ 9,\ 16,\ \ldots$ i $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \ldots$

10 a Use the formula $T_n = a + (n - 1)d$ to find the nth term T_n of $165,\ 160,\ 155,\ \ldots$

b Solve $T_n = 40$ to find the number of terms in the finite sequence $165,\ 160,\ 155,\ \ldots,\ 40$.

c Solve $T_n < 0$ to find the first negative term of the sequence $165,\ 160,\ 155,\ \ldots$

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11 Use the formula $T_n = a + (n - 1)d$ to find the number of terms in each finite sequence.

a $10, 12, 14, \ldots, 30$ **b** $1, 4, 7, \ldots, 100$ **c** $105, 100, 95, \ldots, 30$

d $100, 92, 84, \ldots, 4$ **e** $-12, -10\frac{1}{2}, -9, \ldots, 0$ **f** $2, 5, 8, \ldots, 2000$

12 Find T_n for each AP. Then solve $T_n < 0$ to find the first negative term.

a $20, 17, 14, \ldots$ **b** $50, 45, 40, \ldots$ **c** $67, 60, 53, \ldots$

d $82, 79, 76, \ldots$ **e** $345, 337, 329, \ldots$ **f** $24\frac{1}{2}, 24, 23\frac{1}{2}, \ldots$

13 The nth term of an arithmetic sequence is $T_n = 7 + 4n$.

a Write out the first four terms, and hence find the values of a and d.

b Find the sum and the difference of the 50th and the 25th terms.

c Prove that $5T_1 + 4T_2 = T_{27}$.

d Which term of the sequence is 815?

e Find the last term less than 1000 and the first term greater than 1000.

f Find which terms are between 200 and 300, and how many of them there are.

14 **a** Let T_n be the sequence $8, 16, 24, \ldots$ of positive multiples of 8.

i Show that the sequence is an AP, and find a formula for T_n.

ii Find the first term of the sequence greater than 500 and the last term less than 850.

iii Hence find the number of multiples of 8 between 500 and 850.

b Use the same steps to find the number of multiples of 11 between 1000 and 2000.

c Use the same steps to find the number of multiples of 7 between 800 and 2000.

15 **a** The first term of an AP is $a = 7$ and the fourth term is $T_4 = 16$. Use the formula $T_n = a + (n - 1)d$ to find the common difference d. Then write down the first four terms.

b The first term of an AP is $a = 100$ and the sixth term is $T_6 = 10$. Find the common difference d using the formula $T_n = a + (n - 1)d$. Then write down the first five terms.

c Find the 20th term of an AP with first term 28 and 11th term 108.

d Find the 100th term of an AP with first term 32 and 20th term -6.

16 Ionian Windows charges \$500 for the first window, then \$300 each additional window.

a Write down the cost of 1 window, 2 windows, 3 windows, 4 windows, . . .

b Show that this is an AP, and write down the first term a and common difference d.

c Use the formula $T_n = a + (n - 1)d$ to find the cost of 15 windows.

d Use the formula $T_n = a + (n - 1)d$ to find a formula for the cost of n windows.

e Form an inequation and solve it to find the maximum number of windows whose total cost is less than \$10 000.

17 Many years ago, 160 km of a railway line from Nevermore to Gindarinda was built. On 1st January 2001, work was resumed, with 20 km of new track completed each month.

a Write down the lengths of track 1 month later, 2 months later, 3 months later, ...

b Show that this is an AP, and write down the first term a and common difference d.

c Use the formula $T_n = a + (n - 1)d$ to find how much track there was after 12 months.

d Use the formula $T_n = a + (n - 1)d$ to find a formula for the length after n months.

e The distance from Nevermore to Gindarinda is 540 km. Form an equation and solve it to find how many months it took to complete the track.

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18 **a** Write down the first few terms of the AP generated by substituting the positive integers into the linear function $f(x) = 12 - 3x$. Then write down a formula for the nth term.

b **i** Find the formula of the nth term T_n of the AP $-3,\ -1,\ 1,\ 3,\ 5,\ \ldots$ Then write down the linear function $f(x)$ that generates this AP when the positive integers are substituted into it.

ii Graph the function and mark the points $(1, -3)$, $(2, -1)$, $(3, 1)$, $(4, 3)$, $(5, 5)$.

CHALLENGE

19 Find the common difference of each AP. Then find x if $T_{11} = 36$.

a $5x - 9,\ 5x - 5,\ 5x - 1,\ \ldots$

b $16,\ 16 + 6x,\ 16 + 12x,\ \ldots$

20 Find the common difference of each AP. Then find a formula for the nth term T_n.

a $\log_3 2,\ \log_3 4,\ \log_3 8,\ \ldots$

b $\log_a 54,\ \log_a 18,\ \log_a 6,\ \ldots$

c $x - 3y,\ 2x + y,\ 3x + 5y,\ \ldots$

d $5 - 6\sqrt{5},\ 1 + \sqrt{5},\ -3 + 8\sqrt{5},\ \ldots$

e $1.36,\ -0.52,\ -2.4,\ \ldots$

f $\log_a 3x^2,\ \log_a 3x,\ \log_a 3,\ \ldots$

21 **a** What are the first term and difference of the AP generated by substituting the positive integers into the linear function with gradient m and y-intercept b?

b What are the gradient and y-intercept of the linear function that generates an AP with first term a and difference d when the positive integers are substituted into it?

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1C Geometric sequences

A *geometric sequence* is a sequence like this:

$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ \ldots$$

in which the *ratio* of successive terms is constant — in this example, each term is 3 times the previous term. Because the ratio is constant, all the terms can be generated from the *first term* 2 by repeated multiplication by this *common ratio* 3.

In the context of successive terms of sequences, the word *ratio* will always mean some term divided by the previous term.

Definition of a geometric sequence

The old name was 'geometric progression' and geometric sequences are called GPs for short.

4 GEOMETRIC SEQUENCES

- The *ratio* of successive terms in a sequence T_n always means some term divided by the previous term, that is,
$$\text{ratio} = \frac{T_n}{T_{n-1}}, \text{ where } n \geq 2.$$
- A sequence T_n is called a *geometric sequence* if
$$\frac{T_n}{T_{n-1}} = r, \text{ for all } n \geq 2,$$
where r is a non-zero constant, called the *common ratio*.
- The terms of a geometric sequence can be generated from the first term by repeated multiplication by this common ratio,
$$T_n = T_{n-1} \times r, \text{ for all } n \geq 2.$$

Thus arithmetic sequences have a common difference and geometric sequences have a common ratio, so the methods of dealing with them are quite similar.

Example 10 1C

Test whether each sequence is a GP. If the sequence is a GP, find its first term a and its ratio r.

a $40,\ 20,\ 10,\ 5,\ \ldots$

b $5,\ 10,\ 100,\ 200,\ \ldots$

SOLUTION

a Here $\frac{T_2}{T_1} = \frac{20}{40} = \frac{1}{2}$ and $\frac{T_3}{T_2} = \frac{10}{20} = \frac{1}{2}$ and $\frac{T_4}{T_3} = \frac{5}{10} = \frac{1}{2}$,

so the sequence is a GP with $a = 40$ and $r = \frac{1}{2}$.

b Here $\frac{T_2}{T_1} = \frac{10}{5} = 2$ and $\frac{T_3}{T_2} = \frac{100}{10} = 10$ and $\frac{T_4}{T_3} = \frac{200}{100} = 2$.

The ratios are not all the same, so the sequence is not a GP.

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A formula for the *n*th term of a GP

Let the first term of a GP be a and the common ratio be r. Then the first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad T_5 = ar^4, \quad \dots$$

From this pattern, the general formula for the nth term is clear:

5 THE *n*TH TERM OF A GP

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 11 — 1C

Write out the first five terms, and calculate the 10th term, of the GP with:

a $a = 3$ and $r = 2$,

b $a = 7$ and $r = 10$.

SOLUTION

a 3, 6, 12, 24, 48, ...

$$T_{10} = ar^9 = 3 \times 2^9 = 1536$$

b 7, 70, 700, 7000, 70 000, ...

$$T_{10} = a \times r^9 = 7 \times 10^9 = 7\,000\,000\,000$$

Zeroes and GPs don't mix

No term of a GP can be zero. For example, if $T_2 = 0$, then $\dfrac{T_3}{T_2}$ would be undefined, contradicting the definition that $\dfrac{T_3}{T_2} = r$.

Similarly, the ratio of a GP cannot be zero. Otherwise $T_2 = ar$ would be zero, which is impossible, as we have explained above.

Negative ratios and alternating signs

The sequence

$$2, \ -6, \ 18, \ -54, \ \dots$$

is an important type of GP. Its ratio is $r = -3$, which is negative, so the terms are alternately positive and negative.

Example 12 — 1C

a Show that $2, \ -6, \ 18, \ -54, \ \dots$ is a GP and find its first term a and ratio r.

b Find a formula for the nth term, and hence find T_6 and T_{15}.

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SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{-6}{2} = -3$ and $\dfrac{T_3}{T_2} = \dfrac{18}{-6} = -3$ and $\dfrac{T_4}{T_3} = \dfrac{-54}{18} = -3,$

so the sequence is a GP with $a = 2$ and $r = -3$.

b Using the formula for the nth term,

$$T_n = ar^{n-1} = 2 \times (-3)^{n-1}.$$

Hence $T_6 = 2 \times (-3)^5 = -486$ and $T_{15} = 2 \times (-3)^{14} = 2 \times 3^{14}$, because 14 is even.

Using a switch to alternate the sign

Here are two classic GPs with ratio -1:

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \dots \quad \text{and} \quad 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

The first has formula $T_n = (-1)^n$, and the second has formula $T_n = (-1)^{n-1}$.

These sequences provide a way of writing any GP that alternates in sign using a *switch*. For example, the sequence $2,\ -6,\ 18,\ -54,\ \dots$ in the previous worked example has formula $T_n = 2 \times (-3)^{n-1}$, which can also be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^{n-1}$$

to emphasise the alternating sign, and $-2,\ 6,\ -18,\ 54,\ \dots$ can be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^n.$$

Solving problems involving GPs

As with APs, the formula for the nth term allows many problems to be solved by forming an equation and solving it.

Example 13 1C

a Find a formula for the nth term of the geometric sequence $5,\ 10,\ 20,\ \dots$.

b Hence find whether 320 and 720 are terms of this sequence.

SOLUTION

a The sequence is a GP with $a = 5$ and $r = 2$.

Hence $T_n = ar^{n-1} = 5 \times 2^{n-1}$.

b Put $T_n = 320$.

Then $5 \times 2^{n-1} = 320$

$2^{n-1} = 64$

$n - 1 = 6$

$n = 7$,

so 320 is the seventh term T_7.

c Put $T_n = 720$.

Then $5 \times 2^{n-1} = 720$

$2^{n-1} = 144$.

But 144 is not a power of 2, so 720 is not a term of the sequence.

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Example 14 1C

The first term of a GP is 448 and the seventh term is 7. Find the common ratio and write out the first seven terms.

SOLUTION

First, we know that $T_1 = 448$

that is, $a = 448.$ (1)

Secondly, we know that $T_7 = 7$

so using the formula for the 7th term, $ar^6 = 7.$ (2)

substituting (1) into (2),

$$448r^6 = 7$$
$$r^6 = \tfrac{1}{64}$$
$$r = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Thus either the ratio is $r = \frac{1}{2}$, and the sequence is

$$448,\ 224,\ 112,\ 56,\ 28,\ 14,\ 7,\ \ldots$$

or the ratio is $r = -\frac{1}{2}$, and the sequence is

$$448,\ -224,\ 112,\ -56,\ 28,\ -14, 7,\ \ldots$$

Geometric sequences and exponential functions

Take the exponential function $f(x) = 54 \times 3^{-x}$, and substitute the positive integers. The result is a geometric sequence

$$18,\ 6,\ 2,\ \tfrac{2}{3},\ \tfrac{2}{9},\ \ldots$$

x	1	2	3	4	5
$f(x)$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$

The formula for the nth term of this GP is $T_n = 18 \times \left(\frac{1}{3}\right)^{n-1} = 54 \times 3^{-n}$. This is a function whose domain is the set of positive integers, and its equation is the same as the exponential function, with only a change of pronumeral from x to n.

Thus the graph of an arithmetic sequence is the positive integer points on the graph of a linear function, and the graph of a geometric sequence is the positive integer points on the graph of an exponential function.

Exercise 1C

FOUNDATION

1 Write out the next three terms of each sequence. They are all GPs.

a $1,\ 2,\ 4,\ \ldots$

b $81,\ 27,\ 9,\ \ldots$

c $-7,\ -14,\ -28$

d $-2500,\ -500,\ -100,\ \ldots$

e $3,\ -6,\ 12,\ \ldots$

f $-25,\ 50,\ -100,\ \ldots$

g $5,\ -5,\ 5,\ \ldots$

h $-1000,\ 100,\ -10,\ \ldots$

i $0.04,\ 0.4,\ 4,\ \ldots$

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2 Write out the first four terms of the GPs whose first terms and common ratios are:

a $a = 1$ and $r = 3$ **b** $a = 12$ and $r = 2$ **c** $a = 5$ and $r = -2$
d $a = 18$ and $r = \frac{1}{3}$ **e** $a = 18$ and $r = -\frac{1}{3}$ **f** $a = 50$ and $r = \frac{1}{5}$
g $a = 6$ and $r = -\frac{1}{2}$ **h** $a = -13$ and $r = 2$ **i** $a = -7$ and $r = -1$

3 Find the ratios $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ for each sequence to test whether it is a GP. If the sequence is a GP, write down the first term a and the common ratio r.

a $4,\ 8,\ 16,\ \dots$ **b** $16,\ 8,\ 4,\ \dots$ **c** $7,\ 21,\ 63,\ \dots$
d $-4,\ -20,\ -100,\ \dots$ **e** $2,\ 4,\ 6,\ \dots$ **f** $-1000,\ -100,\ -10,\ \dots$
g $-80,\ 40,\ -20,\ \dots$ **h** $29,\ 29,\ 29,\ \dots$ **i** $1,\ 4,\ 9,\ \dots$
j $-14,\ 14,\ -14,\ \dots$ **k** $6,\ 1,\ \frac{1}{6},\ \dots$ **l** $-\frac{1}{3},\ 1,\ -3,\ \dots$

4 Use the formula $T_n = ar^{n-1}$ to find the fourth term of the GP with:

a $a = 5$ and $r = 2$ **b** $a = 300$ and $r = \frac{1}{10}$ **c** $a = -7$ and $r = 2$
d $a = -64$ and $r = \frac{1}{2}$ **e** $a = 11$ and $r = -2$ **f** $a = -15$ and $r = -2$

5 Use the formula $T_n = ar^{n-1}$ to find an expression for the 70th term of the GP with:

a $a = 1$ and $r = 3$ **b** $a = 5$ and $r = 7$ **c** $a = 8$ and $r = -3$

6 **a** Find the first term a and the common ratio r of the GP $7,\ 14,\ 28,\ \dots$
b Find the sixth term T_6 and an expression for the 50th term T_{50}.
c Find a formula for the nth term T_n.

7 **a** Find the first term a and the common ratio r of the GP $10,\ -30,\ 90,\ \dots$
b Find the sixth term T_6 and an expression for the 25th term T_{25}.
c Find a formula for the nth term T_n.

8 **a** Find the first term a and the common ratio r of the GP $-80,\ -40,\ -20,\ \dots$
b Find the 10th term T_{10} and an expression for the 100th term T_{100}.
c Find a formula for the nth term T_n.

DEVELOPMENT

9 Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If the sequence is a GP, use the formula $T_n = ar^{n-1}$ to find a formula for the nth term, then find T_6.

a $10,\ 20,\ 40,\ \dots$ **b** $180,\ 60,\ 20,\ \dots$ **c** $64,\ 81,\ 100,\ \dots$
d $35,\ 50,\ 65,\ \dots$ **e** $\frac{3}{4},\ 3,\ 12,\ \dots$ **f** $-48,\ -24,\ -12,\ \dots$

10 Find the common ratio of each GP, find a formula for T_n, and find T_6.

a $1,\ -1,\ 1,\ \dots$ **b** $-2,\ 4,\ -8,\ \dots$ **c** $-8,\ 24,\ -72,\ \dots$
d $60,\ -30,\ 15,\ \dots$ **e** $-1024,\ 512,\ -256,\ \dots$ **f** $\frac{1}{16},\ -\frac{3}{8},\ \frac{9}{4},\ \dots$

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11 Use the formula $T_n = ar^{n-1}$ to find how many terms there are in each finite sequence.

a $1, 2, 4, \ldots, 64$ **b** $-1, -3, -9, \ldots, -81$ **c** $8, 40, 200, \ldots, 125\,000$

d $7, 14, 28, \ldots, 224$ **e** $2, 14, 98, \ldots, 4802$ **f** $\frac{1}{25}, \frac{1}{5}, 1, \ldots, 625$

12 **a** The first term of a GP is $a = 25$ and the fourth term is $T_4 = 200$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.

b The first term of a GP is $a = 3$ and the sixth term is $T_6 = 96$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first six terms.

c The first term of a GP is $a = 1$ and the fifth term is $T_5 = 81$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.

13 Use the formula $T_n = ar^{n-1}$ to find the common ratio r of a GP for which:

a $a = 486$ and $T_5 = \frac{2}{27}$ **b** $a = 1000$ and $T_7 = 0.001$

c $a = 32$ and $T_6 = -243$ **d** $a = 5$ and $T_7 = 40$

14 The nth term of a geometric sequence is $T_n = 25 \times 2^n$.

a Write out the first six terms and hence find the values of a and r.

b Which term of the sequence is 6400?

c Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.

d Prove that $T_9 \times T_{11} = 25 \times T_{20}$.

e Write out the terms between 1000 and 100 000. How many of them are there?

f Verify by calculations that $T_{11} = 51\,200$ is the last term less than 100 000 and that $T_{12} = 102\,400$ is the first term greater than 100 000.

15 A piece of paper 0.1 mm thick is folded successively 100 times. How thick is it now?

16 **a** Write down the first few terms of the GP generated by substituting the positive integers into the exponential function $f(x) = \frac{4}{25} \times 5^x$. Then write down a formula for the nth term.

b **i** Find the formula of the nth term T_n of the GP $5, 10, 20, 40, 80 \ldots$ Then write down the exponential function $f(x)$ that generates this GP when the positive integers are substituted into it.

ii Graph the function (without the same scale on both axes) and mark the points $(1, 5)$, $(2, 10)$, $(3, 20)$, $(4, 40)$, $(5, 80)$.

CHALLENGE

17 Find the nth term of each GP.

a $\sqrt{6}, 2\sqrt{3}, 2\sqrt{6}, \ldots$ **b** $ax, a^2x^3, a^3x^5, \ldots$ **c** $-\frac{x}{y}, -1, -\frac{y}{x}, \ldots$

18 **a** Find a formula for T_n in $2x, 2x^2, 2x^3, \ldots$ Then find x if $T_6 = 2$.

b Find a formula for T_n in $x^4, x^2, 1, \ldots$ Then find x if $T_6 = 3^6$.

c Find a formula for T_n in $2^{-16}x, 2^{-12}x, 2^{-8}x, \ldots$ Then find x if $T_6 = 96$.

19 **a** What are the first term and common ratio of the GP generated by substituting the positive integers into the exponential function $f(x) = cb^x$?

b What is the equation of the exponential function that generates a GP with first term a and ratio r when the positive integers are substituted into it?

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1D Solving problems involving APs and GPs

This section deals with APs and GPs together and presents some further approaches to problems about the terms of APs and GPs.

A condition for three numbers to be in AP or GP

The three numbers $10,\ 25,\ 40$ form an AP because the differences $25 - 10 = 15$ and $40 - 25 = 15$ are equal.

Similarly, $10,\ 20,\ 40$ form a GP because the ratios $\frac{20}{10} = 2$ and $\frac{40}{20} = 2$ are equal.

These situations occur quite often and a formal statement is worthwhile:

6 THREE NUMBERS IN AP OR GP

- Three numbers a, b and c form an AP if
  $$b - a = c - b$$
- Three numbers a, b and c form a GP if
  $$\frac{b}{a} = \frac{c}{b}$$

Example 15 1D

a Find the value of x if $3,\ x,\ 12$ form an AP.
b Find the value of x if $3,\ x,\ 12$ form a GP.

SOLUTION

a Because $3,\ x,\ 12$ form an AP,

$$\begin{aligned} x - 3 &= 12 - x \\ 2x &= 15 \\ x &= 7\tfrac{1}{2}. \end{aligned}$$

b Because $3,\ x,\ 12$ form a GP,

$$\begin{aligned} \frac{x}{3} &= \frac{12}{x} \\ x^2 &= 36 \\ x &= 6 \text{ or } -6. \end{aligned}$$

Solving problems leading to simultaneous equations

Many problems about APs and GPs lead to simultaneous equations. These are best solved by elimination.

7 PROBLEMS ON APS AND GPS LEADING TO SIMULTANEOUS EQUATIONS

- With APs, eliminate a by subtracting one equation from the other.
- With GPs, eliminate a by dividing one equation by the other.

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Example 16 1D

The third term of an AP is 16 and the 12th term is 79. Find the 41st term.

SOLUTION

Let the first term be a and the common difference be d.

Because $T_3 = 16$, $\quad a + 2d = 16$, (1)

and because $T_{12} = 79$, $\quad a + 11d = 79$. (2)

Subtracting (1) from (2), $\quad 9d = 63$ (the key step that eliminates a)

$d = 7$.

Subtracting into (1), $\quad a + 14 = 16$

$a = 2$.

Hence $\quad T_{41} = a + 40d$

$= 282$.

Example 17 1D

Find the first term a and the common ratio r of a GP in which the fourth term is 6 and the seventh term is 162.

SOLUTION

Because $T_4 = 6$, $\quad ar^3 = 6$, (1)

and because $T_7 = 162$, $\quad ar^6 = 162$. (2)

Dividing (2) by (1), $\quad r^3 = 27$ (the key step that eliminates a)

$r = 3$.

Substituting into (1), $\quad a \times 27 = 6$

$a = \frac{2}{9}$.

Solving GP problems involving trial-and-error or logarithms

Equations and inequations involving the terms of a GP are index equations, so logarithms are needed for a systematic approach.

Trial-and-error, however, is quite satisfactory for simpler problems, and the reader may prefer to leave the application of logarithms until Chapter 8.

Example 18 1D

a Find a formula for the nth term of the geometric sequence $2, 6, 18, \ldots$.

b Use trial-and-error to find the first term greater than 1 000 000.

c Use logarithms to find the first term greater than 1 000 000.

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SOLUTION

a This is a GP with $a = 2$ and $r = 3$,

so $T_n = ar^{n-1}$
$= 2 \times 3^{n-1}$.

b Put $T_n > 1000000$.

Using the calculator, $T_{12} = 354294$

and $T_{13} = 1062882$.

Hence the first term over 1000000 is $T_{13} = 1062882$.

c Put $T_n > 1000000$.

$$
\begin{aligned}
\text{Then } 2 \times 3^{n-1} &> 1000000 \\
3^{n-1} &= 500000 \\
n - 1 &> \log_3 500000 \quad \text{(remembering that } 2^3 = \text{ means } 3 = \log_2 8\text{)} \\
n - 1 &> \frac{\log_{10} 500000}{\log_{10} 3} \quad \text{(the change-of-base formula)} \\
n - 1 &> 11.94\ldots \\
n &> 12.94\ldots
\end{aligned}
$$

Hence the first term over 1000000 is $T_{13} = 1062882$.

Exercise 1D

FOUNDATION

1 Find the value of x if each set of numbers below forms an arithmetic sequence.
(Hint: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find x.)

a $5,\ x,\ 17$ **b** $32,\ x,\ 14$ **c** $-12,\ x,\ -50$

d $-23,\ x,\ 7$ **e** $x,\ 22,\ 32$ **f** $-20,\ -5,\ x$

2 Each triple of number forms a geometric sequence. Find the value of x. (Hint: Form an equation using the identity $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$, then solve it to find x.)

a $2,\ x,\ 18$ **b** $48,\ x,\ 3$ **c** $-10,\ x,\ -90$

d $-98,\ x,\ -2$ **e** $x,\ 20,\ 80$ **f** $-1,\ 4,\ x$

3 Find x if each triple of three numbers forms: **i** an AP, **ii** a GP.

a $4,\ x,\ 16$ **b** $1,\ x,\ 49$ **c** $16,\ x,\ 25$ **d** $-5,\ x,\ -20$

e $x,\ 10,\ 50$ **f** $x,\ 12,\ 24$ **g** $x,\ -1,\ 1$ **h** $x,\ 6,\ -12$

i $20,\ 30,\ x$ **j** $-36,\ 24,\ x$ **k** $-\frac{1}{4},\ -3,\ x$ **l** $7,\ -7,\ x$

DEVELOPMENT

4 In these questions, substitute the last term into $T_n = a + (n - 1)d$ or $T_n = ar^{n-1}$.

a Find the first six terms of the AP with first term $a = 7$ and sixth term $T_6 = 42$.

b Find the first four terms of the GP with first term $a = 27$ and fourth term $T_4 = 8$.

c Find the first eleven terms of the AP with $a = 40$ and $T_{11} = 5$.

d Find the first seven terms of the GP with $a = 1$ and $T_7 = 1000000$.

e Find the first five terms of the AP with $a = 3$ and $T_5 = 48$.

f Find the first five terms of the GP with $a = 3$ and $T_5 = 48$.

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5 Use simultaneous equations and the formula $T_n = a + (n - 1)d$ to solve these problems.
 a Find the first term and common difference of the AP with $T_{10} = 18$ and $T_{20} = 48$.
 b Find the first term and common difference of the AP with $T_2 = 3$ and $T_{10} = 35$.
 c Find the first term and common difference of the AP with $T_5 = 24$ and $T_9 = -12$.
 d Find the first term and common difference of the AP with $T_4 = 6$ and $T_{12} = 34$.

6 Use simultaneous equations and the formula $T_n = ar^{n-1}$ to solve these problems.
 a Find the first term and common ratio of the GP with $T_3 = 16$ and $T_6 = 128$.
 b Find the first term and common ratio of the GP with $T_3 = 1$ and $T_6 = 64$.
 c Find the first term and common ratio of the GP with $T_2 = \frac{1}{3}$ and $T_6 = 27$.
 d Find the first term and common ratio of the GP with $T_5 = 6$ and $T_9 = 24$.

7 a The third term of an AP is 7 and the seventh term is 31. Find the eighth term.
 b The common difference of an AP is -7 and the 10th term is 3. Find the second term.
 c The common ratio of a GP is 2 and the sixth term is 6. Find the second term.

8 Use either trial-and-error or logarithms to solve these problems.
 a Find the smallest value of n such that $3^n > 1\,000\,000$.
 b Find the largest value of n such that $5^n < 1\,000\,000$.
 c Find the smallest value of n such that $7^n > 1\,000\,000\,000$.
 d Find the largest value of n such that $12^n < 1\,000\,000\,000$.

9 Let T_n be the sequence $2,\ 4,\ 8,\ \dots$ of powers of 2.
 a Show that the sequence is a GP, and show that the nth term is $T_n = 2^n$.
 b Find how many terms are less than 1 000 000. (You will need to solve the inequation $T_n < 1\,000\,000$ using trial-and-error or logarithms.)
 c Use the same method to find how many terms are less than 1 000 000 000.
 d Use the same method to find how many terms are less than 10^{20}.
 e How many terms are between 1 000 000 and 1 000 000 000?
 f How many terms are between 1 000 000 000 and 10^{20}?

10 Find a formula for T_n for these GPs. Then find how many terms exceed 10^{-6}. (You will need to solve the inequation $T_n > 10^{-6}$ using trial-and-error or logarithms.)
 a $98,\ 14,\ 2,\ \dots$ b $25,\ 5,\ 1,\ \dots$ c $1,\ 0.9,\ 0.81,\ \dots$

11 When light passes through one sheet of very thin glass, its intensity is reduced by 3%. (Hint: 97% of the light gets though each sheet.)
 a If the light passes through 50 sheets of this glass, find by what percentage (correct to the nearest 1%) the intensity will be reduced.
 b What is the minimum number of sheets that will reduce the intensity below 1%?

12 a Find a and d for the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.
 b Find a and r for the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.
 c The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

13 Each set of three numbers forms an AP. Find x and write out the numbers.
 a $x - 1,\ 17,\ x + 15$ b $2x + 2,\ x - 4,\ 5x$
 c $x - 3,\ 5,\ 2x + 7$ d $3x - 2,\ x,\ x + 10$

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14 Each set of three numbers forms a GP. Find x and write out the numbers.

a $x,\ x+1,\ x$ **b** $2-x,\ 2,\ 5-x$

15 Find x and write out the three numbers if they form: **i** an AP, **ii** a GP.

a $x,\ 24,\ 96$ **b** $0.2,\ x,\ 0.00002.$ **c** $x,\ 0.2,\ 0.002.$

d $x-4,\ x+1,\ x+11$ **e** $x-2,\ x+2,\ 5x-2$ **f** $\sqrt{5}+1,\ x,\ \sqrt{5}-1$

g $\sqrt{2},\ x,\ \sqrt{8}$ **h** $2^4,\ x,\ 2^6$ **i** $7,\ x,\ -7$

16 [The relationship between APs and GPs]

a The AP $1,\ 3,\ 5,\ 7,\ 9,\ \ldots$ has first term 1 and difference 2. Show that the sequence

$$2^1,\ 2^3,\ 2^5,\ 2^7,\ 2^9,\ \ldots$$

is a GP, and find its first term and common ratio.

b The GP $3,\ 6,\ 12,\ 24,\ 48\ \ldots$ has first term 3 and ratio 2. Show that the sequence

$$\log_2 3,\ \log_2 6,\ \log_2 12,\ \log_2 24,\ \log_2 48$$

is an AP, and find its first term and common difference.

CHALLENGE

17 **a** Find a and b if $a,\ b,\ 1$ forms a GP, and $b,\ a,\ 10$ forms an AP.

b Find a and b if $a,\ 1,\ a+b$ forms a GP, and $b,\ \frac{1}{2},\ a-b$ forms an AP.

c The first, second and fourth terms of an AP $T_n = a + (n-1)d$ form a geometric sequence. Show that either $d = 0$ or $d = a$.

d The first, second and fifth terms of an AP $T_n = a + (n-1)d$ form a geometric sequence. Show that either $d = 0$ or $d = 2a$.

18 [The relationship between APs and GPs]

a Show that $2^5,\ 2^2,\ 2^{-1},\ 2^{-4},\ \ldots$ is a GP. Then find its nth term.

b Show that $\log_2 96,\ \log_2 24,\ \log_2 6, \ldots$ is an AP. Then show that $T_n = 7 - 2n + \log_2 3$.

c Show that for any positive base $b \neq 1$, if T_n is an AP with first term a and difference d, then b^{T_n} is a GP with first term b^a and ratio b^d.

d Show that for any positive base $b \neq 1$, if T_n is a GP with first term $a > 0$ and ratio $r > 0$, then $\log_b T_n$ is an AP with first term $\log_b a$ and difference $\log_b r$.

19 [Extension — geometric sequences in musical instruments] The pipe lengths in a rank of organ pipes decrease from left to right. The lengths form a GP, and the 13th pipe along is exactly half the length of the first pipe (making an interval called an *octave*).

a Show that the ratio of the GP is $r = \left(\frac{1}{2}\right)^{\frac{1}{12}}$.

b Show that the eighth pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).

c Show that the fifth pipe along is just under four-fifths the length of the first pipe (a *major third*).

d Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.

e What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

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1E Adding up the terms of a sequence

Adding the terms of a sequence is often important. For example, a boulder falling from the top of a high cliff falls 5 metres in the first second, 15 metres in the second second, 25 metres in the third second, and so on. The distance that it falls in the first 10 seconds is the sum of the 10 numbers

$$5 + 15 + 25 + 35 + \cdots + 95 = 500.$$

A notation for the sums of terms of a sequence

For any sequence $T_1, T_2, T_3, \ldots$, define S_n to be the sum of the first n terms of the sequence.

8 THE SUM OF THE FIRST n TERMS OF A SEQUENCE

Given a sequence $T_1, T_2, T_3, \ldots$, define

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

The sum S_n is variously called:

- the *sum of the first n terms* of the sequence,
- the *sum to n terms* of the sequence,
- the *nth partial sum* of the sequence ('partial' meaning 'part of the sequence').

For example, the sum of the first 10 terms of the sequence $5, 15, 25, 35, \ldots$ is

$$\begin{aligned} S_{10} &= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 \\ &= 500, \end{aligned}$$

which is also called the 10th partial sum of the sequence.

The sequence $S_1, S_2, S_3, S_4, \ldots$ of sums

The partial sums $S_1, S_2, S_3, S_4, \ldots$ form another sequence. For example, with the sequence $5, 15, 25, 35, \ldots$,

$$S_1 = 5 \qquad \begin{aligned} S_2 &= 5 + 15 \\ &= 20 \end{aligned} \qquad \begin{aligned} S_3 &= 5 + 15 + 25 \\ &= 45 \end{aligned} \qquad \begin{aligned} S_4 &= 5 + 15 + 25 + 35 \\ &= 80 \end{aligned}$$

Example 19 1E

Copy and complete this table for the successive sums of a sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n										

SOLUTION

Each entry for S_n is the sum of all the terms T_n up to that point.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n	5	20	45	80	125	180	245	320	405	500

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Recovering the sequence from the partial sums

Suppose we know that the partial sums S_n of a sequence are the successive squares,

$$S_n\text{: } 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \dots$$

and we want to recover the terms T_n. The first term is $T_1 = S_1 = 1$, and then we can take successive differences, giving the sequence

$$T_n\text{: } 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ \dots$$

9 RECOVERING THE TERMS FROM THE PARTIAL SUMS

The original sequence T_n can be recovered from the sequence S_n of partial sums by taking successive differences,

$$T_1 = S_1$$
$$T_n = S_n - S_{n-1}, \text{ for } n \geq 2.$$

Example 20 — 1E

By taking successive differences, list the terms of the original sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n										
S_n	1	5	12	22	35	51	70	92	117	145

SOLUTION

Each entry for T_n is the difference between two successive sums S_n.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	4	7	10	13	16	19	22	25	28
S_n	1	5	12	22	35	51	70	92	117	145

Example 21 — 1E

Confirm the example given above by proving algebraically that if the partial sums S_n of a sequence are the successive squares, then the sequence T_n is the sequence of odd numbers.

SOLUTION

We are given that $S_n = n^2$.

Hence $T_1 = S_1$
$= 1$, which is the first odd number,

and for $n \geq 2$, $T_n = S_n - S_{n-1}$
$= n^2 - (n-1)^2$
$= 2n - 1$, which is the nth odd number.

Note: Taking successive differences in a sequence is analogous to differentiation in calculus, and the results have many similarities to differentiation. For example, in the worked example above, taking finite differences of a quadratic function yields a linear function. The last question in the Challenge has further analogies, which are not pursued in this course.

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Sigma notation

This is a concise notation for the sums of a sequence. For example:

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2 \qquad \sum_{n=6}^{10} n^2 = 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

$$= 4 + 9 + 16 + 25 \qquad = 36 + 49 + 64 + 81 + 100$$

$$= 54 \qquad = 330$$

The first sum says 'evaluate the function n^2 for all the integers from $n = 2$ to $n = 5$, then add up the resulting values'. There are 4 terms, and their sum is 54.

10 SIGMA NOTATION

Suppose that $T_1,\ T_2,\ T_3,\ \ldots$ is a sequence. Then

$$\sum_{n=5}^{20} T_n = T_5 + T_6 + T_7 + T_8 + \cdots + T_{20}$$

(Any two integers can obviously be substituted for the numbers 5 and 20.)

We used the symbol $\sum$ before in Chapter 11 of the Year 11 book. It stands for the word 'sum', and is a large version of the Greek capital letter Σ called 'sigma' and pronounced 's'. The superscripts and subscripts on the sigma sign, however, are used for the first time in this chapter.

Example 22 1E

Evaluate these sums.

a $\displaystyle\sum_{n=4}^{7} (5n + 1)$

b $\displaystyle\sum_{n=1}^{5} (-2)^n$

SOLUTION

a $$\sum_{n=4}^{7} (5n + 1) = 21 + 26 + 31 + 36$$
$$= 114$$

b $$\sum_{n=1}^{5} (-2)^n = -2 + 4 - 8 + 16 - 32$$
$$= -22$$

Series

The word *series* is often used imprecisely, but it always refers to the activity of adding up terms of a sequence. For example, the phrase

'the series $1 + 4 + 9 + \cdots$'

means that one is considering the successive partial sums $S_1 = 1,\ S_2 = 1 + 4,\ S_3 = 1 + 4 + 9,\ \ldots$ of the sequence of positive squares.

The precise definition is that a *series* is the sequence of partial sums of a sequence. That is, given a sequence $T_1,\ T_2,\ T_3,\ \ldots,$ the corresponding series is the sequence

$$S_1 = T_1,\ S_2 = T_1 + T_2,\ S_3 = T_1 + T_2 + T_3,\ S_4 = T_1 + T_2 + T_3 + T_4,\ \ldots$$

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Exercise 1E

FOUNDATION

1 Find the sum S_4 of the first four terms of each sequence.

a $3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots$

b $2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \ldots$

c $6,\ 2,\ -2,\ -6,\ -10,\ -14,\ \ldots$

d $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$

2 Find the sum S_3 of the first three terms of each series.

a $200 + 150 + 100 + 50 + 0 + \cdots$

b $32 - 16 + 8 - 4 + 2 - 1 + \cdots$

c $-24 - 18 - 12 - 6 + 0 + 6 + \cdots$

d $5.1 + 5.2 + 5.3 + 5.4 + 5.5 + 5.6 + \cdots$

3 Find the sums S_1, S_2, S_3, S_4 and S_5 for each sequence.

a $10,\ 20,\ 30,\ 40,\ 50,\ 60,\ \ldots$

b $1,\ -3,\ 9,\ -27,\ 81,\ -243,\ \ldots$

c $1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \ldots$

d $3,\ 4\frac{1}{2},\ 6,\ 7\frac{1}{2},\ 9,\ 10\frac{1}{2},\ 12,\ \ldots$

4 Find the sums S_4, S_5 and S_6 for each series. (You will need to continue each series first.)

a $1 - 2 + 3 - 4 + \cdots$

b $81 + 27 + 9 + 3 + \cdots$

c $30 + 20 + 10 + \cdots$

d $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

5 Copy and complete these tables of a sequence and its partial sums.

T_n	2	5	8	11	14	17	20
S_n							

T_n	2	−4	6	−8	10	−12	14
S_n							

T_n	40	38	36	34	32	30	28
S_n							

T_n	7	−7	7	−7	7	−7	7
S_n							

DEVELOPMENT

6 Each table below gives the successive sums $S_1,\ S_2,\ S_3,\ \ldots$ of a sequence. By taking successive differences, write out the terms of the original sequence.

T_n							
S_n	1	4	9	16	25	36	49

T_n							
S_n	−3	−8	−15	−24	−35	−48	−63

T_n							
S_n	2	6	14	30	62	126	254

T_n							
S_n	8	0	8	0	8	0	8

7 [The Fibonacci and Lucas sequences] Each table below gives the successive sums S_n of a sequence. By taking successive differences, write out the terms of the original sequence.

T_n								
S_n	1	2	3	5	8	13	21	34

T_n								
S_n	3	4	7	11	18	29	47	76

8 Rewrite each partial sum without sigma notation, then evaluate it.

a $\sum_{n=1}^{6} 2n$

b $\sum_{n=1}^{6} (3n + 2)$

c $\sum_{k=3}^{7} (18 - 3n)$

d $\sum_{n=5}^{8} n^2$

e $\sum_{n=1}^{4} n^3$

f $\sum_{n=0}^{5} 2^n$

g $\sum_{n=2}^{4} 3^n$

h $\sum_{\ell=1}^{31} (-1)^\ell$

i $\sum_{\ell=1}^{40} (-1)^{\ell-1}$

j $\sum_{n=5}^{105} 4$

k $\sum_{n=0}^{4} (-1)^n (n + 5)$

l $\sum_{n=0}^{4} (-1)^{n+1} (n + 5)$

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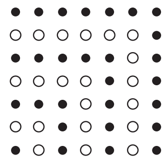
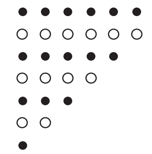
9 **a** Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



b Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.



CHALLENGE

10 Rewrite each sum in sigma notation, starting each sum at $n = 1$. Do not evaluate it.

a $1^3 + 2^3 + 3^3 + \cdots + 40^3$

b $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$

c $3 + 4 + 5 + \cdots + 22$

d $2 + 2^2 + 2^3 + \cdots + 2^{12}$

e $-1 + 2 - 3 + \cdots + 10$

f $1 - 2 + 3 - \cdots - 10$

11 **a** The partial sums of a sequence T_n are given by $S_n = 2^n$. Use the formula in Box 9 to find a formula for T_n.

b Confirm your answer by writing out the calculation in table form, as in Question 6.

c In Chapter 9 of the Year 11 book, you differentiated $y = e^x$. What is the analogy to these results?

12 **a** Prove that $n^3 - (n-1)^3 = 3n^2 - 3n + 1$.

b The partial sums of a sequence T_n are given by $S_n = n^3$. Use the formula in Box 9 to find a formula for T_n.

c The terms of the sequence T_n are the partial sums of a third sequence U_n. Use the formula in Box 9 to find a formula for T_n.

d Confirm your answer by writing out in table form the successive taking of differences in **b** and **c**.

e In Chapter 8 of the Year 11 volume, you differentiated powers of x. What is the analogy to these result?

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1F Summing an arithmetic series

There are two formulae for adding up the first n terms of an AP.

Adding the terms of an AP

Consider adding the first six terms of the AP

$5 + 15 + 25 + 35 + 45 + 55 + \cdots$

$$\begin{aligned}
&\text{Writing out the sum,} & S_6 &= 5 + 15 + 25 + 35 + 45 + 55.\\
&\text{Reversing the sum,} & S_6 &= 55 + 45 + 35 + 25 + 15 + 5,\\
&\text{and adding the two,} & 2S_6 &= 60 + 60 + 60 + 60 + 60 + 60\\
& & &= 6 \times 60,\ \text{because there are 6 terms in the series.}\\
&\text{Dividing by 2,} & S_6 &= \tfrac{1}{2} \times 6 \times 60\\
& & &= 180.
\end{aligned}$$

Notice that 60 is the sum of the first term $T_1 = 5$ and the last term $T_6 = 55$.

In general, let $\ell = T_n$ be the last term of an AP with first term a and difference d.

$$\begin{aligned}
&\text{Then} & S_n &= a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.\\
&\text{Reversing the sum,} & S_n &= \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a,\\
&\text{and adding,} & 2S_n &= (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)\\
& & &= n \times (a + \ell),\ \text{because there are } n \text{ terms in the series.}\\
&\text{Dividing by 2,} & S_n &= \tfrac{1}{2}n(a + \ell).
\end{aligned}$$

Example 23 1F

Add up all the integers from 100 to 200 inclusive.

SOLUTION

The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms.

The first term is $a = 100$ and the last term is $\ell = 200$.

$$\begin{aligned}
\text{Using}\quad S_n &= \tfrac{1}{2}n(a + \ell),\\
S_{101} &= \tfrac{1}{2} \times 101 \times (100 + 200)\\
&= \tfrac{1}{2} \times 101 \times 300\\
&= 15\,150.
\end{aligned}$$

An alternative formula for summing an AP

This alternative form is equally important.

$$\begin{aligned}
&\text{The previous formula is} & S_n &= \tfrac{1}{2}n(a + \ell),\ \text{where } \ell = T_n = a + (n - 1)d.\\
&\text{Substituting } \ell = a + (n - 1)d, & S_n &= \tfrac{1}{2}n\big(a + a(n - 1)d\big)\\
&\text{so} & S_n &= \tfrac{1}{2}n\big(2a + (n - 1)d\big).
\end{aligned}$$

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11 TWO FORMULAE FOR SUMMING AN AP

Suppose that the first term a of an AP, and the number n of terms, are known.

- When the last term $\ell = T_n$ is known, use $S_n = \frac{1}{2}n(a + \ell)$.
- When the difference d is known, use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

If you have a choice, use the first because it is simpler.

Example 24 1F

Consider the arithmetic series $100 + 94 + 88 + 82 + \cdots$.

a Find S_{10}.

b Find S_{41}.

SOLUTION

The series is an AP with $a = 100$ and $d = -6$.

a Using $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$,

$$\begin{aligned} S_{10} &= \tfrac{1}{2} \times 10 \times (2a + 9d) \\ &= 5 \times (200 - 54) \\ &= 730. \end{aligned}$$

b Similarly,

$$\begin{aligned} S_{41} &= \tfrac{1}{2} \times 41 \times (2a + 40d) \\ &= \tfrac{1}{2} \times 41 \times (200 - 240) \\ &= \tfrac{1}{2} \times 41 \times (-40) \\ &= -820. \end{aligned}$$

Example 25 1F

a Find how many terms are in the sum $41 + 45 + 49 + \cdots + 401$.

b Hence evaluate the sum $41 + 45 + 49 + \cdots + 401$.

SOLUTION

a The series is an AP with first term $a = 41$ and difference $d = 4$.

To find the numbers of terms, put $T_n = 401$

$$\begin{aligned} a + (n - 1)d &= 401 \\ 41 + 4(n - 1) &= 401 \\ 4(n - 1) &= 360 \\ n - 1 &= 90 \\ n &= 91. \end{aligned}$$

Thus there are 91 terms in the series.

b Because we now know both the difference d and the last term $\ell = T_{91}$, either formula can be used. It's always easier to use $S_n = \frac{1}{2}n(a + \ell)$ if you can.

Using $S_n = \frac{1}{2}n(a + \ell)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (41 + 401) \\ &= \tfrac{1}{2} \times 91 \times 442 \\ &= 20111. \end{aligned}$$

OR

Using $S_n = \frac{1}{2}n(2a + (n - 1)d)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (2a + 90d) \\ &= \tfrac{1}{2} \times 91 \times (82 + 360) \\ &= 20111. \end{aligned}$$

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Solving problems involving the sums of APs

Problems involving sums of APs are solved using the formulae developed for the nth term T_n and the sum S_n of the first n terms.

Example 26 1F

a Find an expression for the sum S_n of n terms of the series $40 + 37 + 34 + \cdots$.

b Hence find the least value of n for which the partial sum S_n is negative.

SOLUTION

The sequence is an AP with $a = 40$ and $d = -3$.

a
$$\begin{aligned} S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\ &= \tfrac{1}{2} \times n \times \big(80 - 3(n-1)\big) \\ &= \tfrac{1}{2} \times n \times (80 - 3n + 3) \\ &= \frac{n(83 - 3n)}{2} \end{aligned}$$

b
$$\begin{aligned} &\text{Put} && S_n < 0. \\ &\text{Then} && \frac{n(83-3n)}{2} < 0 \\ &\boxed{\times 2} && n(83 - 3n) < 0 \\ &\boxed{\div n} && 83 - 3n < 0, \quad \text{because } n \text{ is positive,} \\ &&& 83 < 3n \\ &&& n > 27\tfrac{2}{3}. \end{aligned}$$

Hence S_{28} is the first sum that is negative.

Example 27 1F

The sum of the first 10 terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION

$$\begin{aligned} &\text{The first piece of information given is} & S_{10} &= 0 \\ && 5(2a + 9d) &= 0 \\ &\boxed{\div 5} & 2a + 9d &= 0. \qquad (1) \\ &\text{The second piece of information given is} & T_1 + T_2 &= 24 \\ && a + (a + d) &= 24 \\ && 2a + d &= 24. \qquad (2) \\ &\text{Subtracting (2) from (1),} & 8d &= -24 \\ && d &= -3, \\ &\text{and substituting this into (2),} & 2a - 3 &= 24 \\ && a &= 13\tfrac{1}{2}. \end{aligned}$$

Hence the AP is $13\tfrac{1}{2} + 10\tfrac{1}{2} + 7\tfrac{1}{2} + \cdots$.

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Exercise 1F

FOUNDATION

1 Let $S_7 = 2 + 5 + 8 + 11 + 14 + 17 + 20$. By reversing the sum and adding in columns, evaluate S_7.

2 State how many terms each sum has, then find the sum using $S_n = \frac{1}{2}n(a + \ell)$.

a $1 + 2 + 3 + 4 + \cdots + 100$

b $1 + 3 + 5 + 7 + \cdots + 99$

c $2 + 4 + 6 + \cdots + 100$

d $3 + 6 + 9 + 12 + \cdots + 300$

e $101 + 103 + 105 + \cdots + 199$

f $1001 + 1002 + 1003 + \cdots + 10000$

3 Use $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum S_6 of the first 6 terms of the series with:

a $a = 5$ and $d = 10$

b $a = 8$ and $d = 2$

c $a = -3$ and $d = -9$

d $a = -7$ and $d = -12$

4 State the first term a and the difference d for each series below. Then use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum S_{21} of the first 21 terms of each series.

a $2 + 6 + 10 + \cdots$

b $3 + 10 + 17 + \cdots$

c $-6 - 1 + 4 + \cdots$

d $10 + 5 + 0 - \cdots$

e $-7 - 10 - 13 - \cdots$

f $1\frac{1}{2} + 3\frac{1}{2} + 5\frac{1}{2} + \cdots$

5 Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the stated number of terms.

a $2 + 5 + 8 + \cdots$ (12 terms)

b $40 + 33 + 26 + \cdots$ (21 terms)

c $-6 - 2 + 2 + \cdots$ (200 terms)

d $33 + 30 + 27 + \cdots$ (23 terms)

e $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)

f $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)

6 First use the formula $T_n = a + (n-1)d$ to find how many terms there are in each sum. Then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum, where ℓ is the last term T_n.

a $50 + 51 + 52 + \cdots + 150$

b $8 + 15 + 22 + \cdots + 92$

c $-10 - 3 + 4 + \cdots + 60$

d $4 + 7 + 10 + \cdots + 301$

e $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$

f $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$

7 Find these sums by any appropriate method.

a $2 + 4 + 6 + \cdots + 1000$

b $1000 + 1001 + \cdots + 3000$

c $1 + 5 + 9 + \cdots$ (40 terms)

d $10 + 30 + 50 + \cdots$ (12 terms)

DEVELOPMENT

8 Use $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find and simplify the sum of the first n terms of each series.

a $5 + 10 + 15 + \cdots$

b $10 + 13 + 16 + \cdots$

c $3 + 7 + 11 + \cdots$

d $-9 - 4 + 1 + \cdots$

e $5 + 4\frac{1}{2} + 4 + \cdots$

f $\big(1 - \sqrt{2}\big) + 1 + \big(1 + \sqrt{2}\big) + \cdots$

9 Use either standard formula for S_n to find a formula for the sum of the first n:

a positive integers,

b odd positive integers,

c positive integers divisible by 3,

d odd positive multiples of 100.

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