EXERCISE 4.10

BASIC

1 Prove $\int_0^a f(a-x)dx = \int_0^a f(x)dx$ and hence find $\int_0^{2\pi} x\cos x\,dx$

2 Evaluate $\int_0^2 x\sqrt{2-x}\,dx$

3 Evaluate $\int_{-2}^{2}(x+x^3+x^5)(1+x^2+x^4)dx$

4 Evaluate $\int_{-\pi}^{0}\sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)dx$

MEDIUM

5 A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$.

Given $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ prove that $\int_0^a f(x)\,dx = \frac{a}{2}f(a)$

6 Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x^2}\cos x\,dx$ (B) $\int_{-\pi}^{\pi} x^3\cos x\,dx$ (C) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sin^2 x-\cos^2 x)\,dx$ (D) $\int_{-1}^{1}\sin^{-1}(x^3)\,dx$

7 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\,dx$

(A) $\int_0^a f(x-a)-f(-x)\,dx$

(B) $\int_0^a f(a+x)-f(x)\,dx$

(C) $\int_0^a f(x)+f(-x)\,dx$

(D) $\int_0^a f(x-a)-f(a-x)\,dx$

8 It is given that $f(x)$ is a non-zero even function and $g(x)$ is a non-zero odd function.

Which expression is equal to $\int_{-a}^{a} f(x)+g(x)\,dx$?

(A) $\int_0^a g(x)+g(-x)\,dx$

(B) $2\int_0^a g(x)+g(-x)\,dx$

(C) $\int_0^a f(x)+f(-x)\,dx$

(D) $2\int_0^a f(x)+f(-x)\,dx$

9 Which of these integrals has the smallest value?

A $\int_0^{\frac{\pi}{6}} \sin x \, dx$ **B** $\int_0^{\frac{\pi}{6}} \sin^2 x \, dx$ **C** $\int_0^{\frac{\pi}{6}} (1 - \sin x) \, dx$ **D** $\int_0^{\frac{\pi}{6}} (1 - \sin^2 x) \, dx$

10 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

11 Evaluate $\int_0^{2} (1 + \sin(\pi(1 - x)^3)) \, dx$

CHALLENGING

12 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx$

1

$$\int_0^a f(a-x)dx$$

$$\boxed{\begin{aligned} u &= a-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$= \int_a^0 f(u) \times (-du)$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

$$\int_0^{2\pi} x\cos x\,dx$$

$$= \int_0^{2\pi} (2\pi - x)\cos(2\pi - x)\,dx$$

$$= 2\pi\int_0^{2\pi} \cos(2\pi - x)\,dx - \int_0^{2\pi} x\cos(2\pi - x)\,dx$$

$$= 2\pi\int_0^{2\pi} \cos x\,dx - \int_0^{2\pi} x\cos x\,dx$$

$$\therefore 2\int_0^{2\pi} x\cos x\,dx = 2\pi\int_0^{2\pi} \cos x\,dx$$

$$\int_0^{2\pi} x\cos x\,dx = \pi\Big[\sin x\Big]_0^{2\pi}$$

$$= \pi(0-0)$$

$$= 0$$

2

$$\int_0^2 x\sqrt{2-x}\,dx$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3}\times 2\sqrt{2} - \frac{2}{5}\times 4\sqrt{2}$$

$$= \frac{16\sqrt{2}}{15}$$

3

$$\int_{-2}^2 (x + x^3 + x^5)(1 + x^2 + x^4)dx = 0$$

[since an odd function × an even function is an odd function].

4

$$\int_{-\pi}^0 \sin\left(x + \frac{\pi}{2}\right)\cos\left(x + \frac{\pi}{2}\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\cos x\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx \quad \text{where } f(x) \text{ is odd}$$

$$= 0$$

5

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore 2\int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx$$

$$\begin{aligned}\int_0^a f(x)\,dx &= \frac{1}{2}\int_0^a \big(f(x)+f(a-x)\big)\,dx \\ &= \frac{1}{2}\int_0^a f(a)\,dx \\ &= \frac{f(a)}{2}\Big[x\Big]_0^a \\ &= \frac{f(a)}{2}(a-0) \\ &= \frac{a}{2}f(a)\end{aligned}$$

6

A: $e^{x^2} > 0$ and $\cos x \geq 0$ in the domain. True.

B: $x^3 \cos x$ is odd since x^3 is odd and $\cos x$ is even. False

C: $\sin^2 x - \cos^2 x = -\cos 2x$ which is negative in the domain. False

D; $\sin^{-1}(x^3)$ is the odd function of an odd function so is odd. False.

ANSWER (A)

7

$$\int_{-a}^a f(x)\,dx$$

$$= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

ANSWER (C)

8

$$\int_{-a}^a f(x) + g(x)\,dx$$

$$= \int_{-a}^a f(x)\,dx \quad \text{since } g(x) \text{ is odd}$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad \text{from Q7}$$

ANSWER (C)

9

for $0 \leq x \leq \frac{\pi}{6}$ $\sin^2 x < \sin x < 1 - \sin x < 1 - \sin^2 x$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\,dx \text{ is the smallest integral}$$

ANSWER (B)

10

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx \qquad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \qquad (2)$$

From (1) and (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{1}{2}\Big[x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

11

$$\int_0^2 (1 + \sin(\pi(1-x)^3))\,dx$$

$$= \int_0^2 (1 - \sin(\pi(x-1)^3))\,dx$$

$$= \int_{-1}^1 (1 - \sin \pi x^3)\,dx$$

$$= \int_{-1}^1 1\,dx - \int_{-1}^1 \sin \pi x^3\,dx$$

$$= \Big[x\Big]_{-1}^1 - 0$$

$$= \big(1 - (-1)\big)$$

$$= 2$$

12

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x + \cos^2 x - 2\sin x \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + (\sin x - \cos x)^2}\,dx \qquad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \left(\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\cos x - \sin x)^2}\,dx \qquad (2)$$

(1) + (2):

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + (\sin x - \cos x)^2}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x}\,dx = \frac{1}{2}\Big[\tan^{-1}(\sin x - \cos x)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4}$$