EXERCISE 6.6

BASIC

1 A mass of 1 kg is released from rest at the surface in which the retardation on the mass is proportional to the distance fallen (x). The net force for this motion is $g - kx$ Newtons, with the downward direction as positive. How far will the mass fall before acceleration is zero?

2 A particle of unit mass falls from rest from the top of a cliff in a medium where the resistive force is kv^2 . How far has it fallen when it reaches a speed half its terminal velocity?

3 A ball of mass m is projected vertically upwards with speed u. The acceleration acting against the ball is gravity plus air resistance proportional to its speed, kv. Find the time (t) taken to reach the greatest height.

MEDIUM

4 A particle of mass m is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle or mkv^2. Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x} = -(g + kv^2)$ where g ms^{-2} is the acceleration due to gravity. Assume $k = 10$ and the acceleration due to gravity is 10 ms^{-2}. Find an expression for the time taken as a function of velocity.

5 **i** Find the constants a, b and c such that:

$$\frac{300x}{1000 + x^3} = \frac{a}{10 + x} + \frac{bx + c}{100 - 10x + x^2}$$

ii A particle of mass m kg is projected vertically upwards in a highly resistive medium at a velocity of 5 m/s. The particle is subjected to the force of gravity and to a resistance due to the medium of magnitude $\frac{mv^3}{100}$ Newtons. Given the acceleration due to gravity is 10 ms^{-2},

$(\boldsymbol{\alpha})$ State the equation of motion (if upwards is the positive direction)

$(\boldsymbol{\beta})$ Hence find the maximum height reached by the particle, (giving your answer correct to 1 decimal place).

6 A particle of unit mass falls from rest under gravity and the resistance to its motion is kv^2, where v is its speed and k is a positive constant. Prove $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$.

7 A rock of mass m is dropped under gravity g, from rest, at the top of a cliff. The vertical distance travelled is represented by x in time t. Air resistance is proportional to the velocity v of the rock, $R = -kv$.

i Explain why $\frac{dv}{dt} = g - \frac{k}{m}v$

ii Show that $v = \frac{g}{k}\left(1 - e^{-\frac{k}{m}t}\right)$ when $t \geq 0$.

iii Show that $x = -\frac{m}{k}v + \frac{m^2 g}{k^2}\log_e\left(\frac{mg}{mg - kv}\right)$

8 A body of mass m in falling from rest, experiences air resistance of magnitude kv^2 per unit mass, where k is a positive constant.

i Write the equation of motion of the body and find the value of the terminal velocity V of the body in terms of k and g (acceleration due to gravity).

ii If w is the velocity of the body when it reaches the ground, show that the distance S fallen is given by $S = -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right)$

iii With air resistance remaining the same, prove that if the body is projected vertically upwards from the ground with velocity U, then it will attain its greatest height H where

$H = \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$ and return to the ground with velocity w given by $w^{-2} = U^{-2} + V^{-2}$

9 A particle A of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point O with initial speed u. The resistance on particle A due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant. When $t = 0$, a second particle B of equal mass is projected vertically upwards from O with the same initial speed u through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle B is kw^2, where w is the velocity of the particle B at time t. The acceleration due to gravity is g.

i Show that the velocity v of particle A is given by $\frac{1}{v} = kt + \frac{1}{u}$

ii By considering the velocity w of particle B, show that

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right)$$

iii Show that the velocity V of particle A when particle B is at rest is given by

$$\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\,tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

iv Hence, if u is very large, explain why $V \approx \frac{2}{\pi}\sqrt{\frac{g}{k}}$

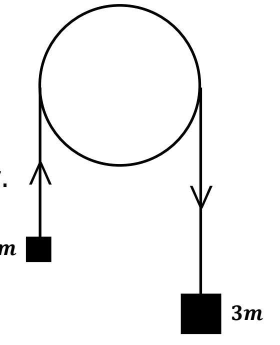
10 A particle of unit mass is projected vertically upwards. The resistance to the motion is proportional to the square of the velocity. The velocity of projection is V.

i Show that the acceleration is given by: $\ddot{x} = -(g + kv^2)$.

ii Show that the maximum height H reached is: $H = \dfrac{1}{2k} \ln \left\{ \dfrac{g + kV^2}{g} \right\}$

iii Show that T, the time taken to reach H is: $T = \dfrac{1}{\sqrt{kg}} \tan^{-1} \left(\dfrac{\sqrt{k}}{\sqrt{g}} V \right)$

11 Particles of mass m and $3m$ kilograms are connected by a light inextensible string which passes over a smooth fixed pulley. The string hangs vertically on each side, as shown in the diagram.



The particles are released from rest and move under the influence of gravity. The air resistance on each particle is kv Newtons, when the speed of the particles is v ms^{-1} and the acceleration due to gravity is g ms^{-2} and is taken as positive throughout the question and is assumed to be constant. k is a positive constant.

i Draw diagrams to show the forces acting on each particle.

ii Show that the equation of motion is: $\ddot{x} = \dfrac{mg - kv}{2m}$

iii Find the terminal velocity V or maximum speed of the system stating your answer in terms of m, g and k.

iv Prove that the time since the beginning of the motion is given by: $t = \dfrac{2m}{k} \ln \left(\dfrac{mg}{mg - kv} \right)$

v If the bodies attain a velocity equal to half of the terminal speed, show by using the results in iii. and iv. that the time elapsed is equal to $\frac{V}{g} \ln 4$, where V is the terminal velocity.

12 A weight is oscillating on the end of a spring under water. Because of the resistance by the water (proportional to speed), the equation of the particle is : $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$, where x is the distance in metres above equilibrium position at time t seconds. Initially the particle is at the equilibrium position, moving upwards with a speed of 3 ms^{-1}

i Find the first and second derivatives of $x = Ae^{-\sqrt{3}t} \sin t$, where A is the constant, and hence show that $x = Ae^{-\sqrt{3}t} \sin t$, is a solution of the differential equation, $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$, then substitute the initial conditions to find A.

ii At what times during the first 2π seconds is the particle moving downwards?

1 constant velocity when force is zero

$$0 = g - kx$$
$$kx = g$$
$$x = \frac{g}{k}$$

2

$$\ddot{x} = g - kv^2$$
$$0 = g - kV_T^2 \Rightarrow V_T^2 = \frac{g}{k} \Rightarrow V_T = \sqrt{\frac{g}{k}}$$
$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$x = \int_0^{\frac{1}{2}\sqrt{\frac{g}{k}}} \frac{v}{g - kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g - kv^2)\Big]_{\frac{1}{2}\sqrt{\frac{g}{k}}}^{0}$$
$$= \frac{1}{2k}\left(\ln g - \ln\left(g - k\left(\frac{g}{4k}\right)\right)\right)$$
$$= \frac{1}{2k}\left(\ln g - \ln\frac{3g}{4}\right)$$
$$= \frac{1}{2k}\ln\left(\frac{4}{3}\right)$$

3

$$m\ddot{x} = -mg - kv$$
$$\frac{dv}{dt} = -\frac{mg + kv}{m}$$
$$\frac{dt}{dv} = -\frac{m}{mg + kv}$$
$$t = -\int_u^0 \frac{m}{mg + kv}\,dv$$
$$= \frac{m}{k}\Big[\ln(mg + kv)\Big]_0^u$$
$$= \frac{m}{k}\Big(\ln(mg + ku) - \ln mg\Big)$$
$$= \frac{m}{k}\ln\left(\frac{mg + ku}{mg}\right)$$

4

$$\frac{dv}{dt} = -(10 + 10v^2)$$
$$\frac{dt}{dv} = -\frac{1}{10}\times\frac{1}{1 + v^2}$$
$$t = -\frac{1}{10}\int_u^v \frac{1}{1 + v^2}\,dv$$
$$= \frac{1}{10}\Big[\tan^{-1}(v)\Big]_v^u$$
$$= \frac{1}{10}\Big(\tan^{-1}u - \tan^{-1}v\Big)$$

5

$$a = \frac{300(-10)}{100 - 10(-10) + (-10)^2} = -10$$

equating coefficients of x^2: $a + b = 0 \Rightarrow$
$b = 10$

equating constants: $100a + 10c = 0 \Rightarrow$
$c = 100$

$$\therefore \frac{300x}{1000 + x^3} = -\frac{10}{10 + x} + \frac{10x + 100}{100 - 10x + x^2}$$

$\boldsymbol{\alpha}$) $m\ddot{x} = -mg - \dfrac{mv^3}{100}$

$$\ddot{x} = -\left(g + \frac{v^3}{100}\right)$$
$$= -\left(10 + \frac{v^3}{100}\right)$$

5

$$\boldsymbol{\beta})\ v\frac{dv}{dx} = -\left(10 + \frac{v^3}{100}\right)$$

$$\frac{dv}{dx} = -\frac{1000 + v^3}{100v}$$

$$\frac{dx}{dv} = -\frac{100v}{1000 + v^3}$$

$$x = -\int_5^0 \frac{100v}{1000 + v^3}\,dv$$

$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10+v} + \frac{10v + 100}{100 - 10v + v^2}\right) dv$$

$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10+v} + \frac{5(2v - 10) + 150}{100 - 10v + v^2}\right) dv$$

$$= \frac{1}{3}\int_0^5 \left(-\frac{10}{10+v} + \frac{5(2v - 10)}{100 - 10v + v^2} + \frac{150}{(v-5)^2 + \left(\sqrt{75}\right)^2}\right) dv$$

$$= \frac{1}{3}\left[-10\ln(10 + v) + 5\ln(100 - 10v + v^2) + \frac{150}{\sqrt{75}}\tan^{-1}\left(\frac{v-5}{\sqrt{75}}\right)\right]_0^5$$

$$= \frac{1}{3}\left((-10\ln 15 + 5\ln 75 + 0) - \left(-10\ln 10 + 5\ln 100 + \frac{150}{\sqrt{75}}\tan^{-1}\left(-\frac{5}{\sqrt{75}}\right)\right)\right)$$

$$= 1.19197\ldots$$

$$= 1.2 \text{ m } \ (1 \text{ dp})$$

6

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = \int_0^v \frac{v}{g - kv^2}\,dv$$

$$= -\frac{1}{2k}\left[\ln(g - kv^2)\right]_0^v$$

$$= \frac{1}{2k}\left(\ln g - \ln(g - kv^2)\right)$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$g - kv^2 = ge^{-2kx}$$

$$kv^2 = g\left(1 - e^{-2kx}\right)$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

7 **i**

$$m\ddot{x} = mg - kv$$
$$\therefore \frac{dv}{dt} = g - \frac{k}{m}v$$

ii

$$\therefore \frac{dt}{dv} = \frac{m}{mg - kv}$$
$$t = \int_0^v \frac{m}{mg - kv}$$
$$= -\frac{m}{k}\Big[\ln(mg - kv)\Big]_0^v$$
$$= \frac{m}{k}\Big(\ln mg - \ln(mg - kv)\Big)$$
$$\frac{k}{m}t = \ln\frac{mg}{mg - kv}$$
$$e^{\frac{k}{m}t} = \frac{mg}{mg - kv}$$
$$mge^{kt} - kve^{\frac{k}{m}t} = mg$$
$$kve^{\frac{k}{m}t} = mg\left(e^{\frac{k}{m}t} - 1\right)$$
$$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

iii

$$v\frac{dv}{dx} = \frac{mg - kv}{m}$$
$$\frac{dv}{dx} = \frac{mg - kv}{mv}$$
$$\frac{dx}{dv} = \frac{mv}{mg - kv}$$
$$x = m\int_0^v \frac{v}{mg - kv}\,dv$$
$$= -\frac{m}{k}\int_0^v \frac{mg - kv - mg}{mg - kv}\,dv$$
$$= -\frac{m}{k}\int_0^v \left(1 + \frac{mg}{k} \times \frac{-k}{mg - kv}\right)dv$$
$$= \frac{m}{k}\left[v + \frac{mg}{k}\ln(mg - kv)\right]_v^0$$
$$= \frac{m}{k}\left(\left(0 + \frac{mg}{k}\ln mg\right)\right.$$
$$\left. - \left(v + \frac{mg}{k}\ln(mg - kv)\right)\right)$$
$$= -\frac{m}{k}v + \frac{m^2 g}{k^2}\ln\left(\frac{mg}{mg - kv}\right)$$

8 **i**

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$
$$\therefore 0 = g - kV^2$$
$$V^2 = \frac{g}{k}$$
$$V = \sqrt{\frac{g}{k}}$$

ii

$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$S = \int_0^w \frac{v}{g - kv^2}\,dv$$
$$= -\frac{1}{2k}\Big[\ln(g - kv^2)\Big]_0^w$$
$$= -\frac{1}{2k}\Big(\ln(g - kw^2) - \ln g\Big)$$
$$= -\frac{1}{2k}\ln\left(\frac{g - kw^2}{g}\right)$$
$$= -\frac{1}{2k}\left(1 - \frac{k}{g}w^2\right)$$
$$= -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right) \quad \text{from (i)}$$

iii

$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -(g + kv^2)$$
$$v\frac{dv}{dx} = -(g + kv^2)$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$H = -\int_U^0 \frac{v}{g + kv^2}\,dv$$
$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^U$$
$$= \frac{1}{2k}(\ln(g + kU^2) - \ln g)$$
$$= \frac{1}{2k}\ln\left(\frac{g + kU^2}{g}\right)$$
$$= \frac{1}{2k}\ln\left(1 + \frac{k}{g}U^2\right)$$
$$= \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$$

but $H = S$

$$\therefore \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right) = -\frac{1}{2k}\ln\left(1 - \frac{w^2}{V^2}\right)$$
$$\ln\left(1 + \frac{U^2}{V^2}\right) = -\ln\left(1 - \frac{w^2}{V^2}\right)$$
$$\therefore \frac{V^2 + U^2}{V^2} = \frac{V^2}{V^2 - w^2}$$
$$V^4 + U^2V^2 - V^2w^2 - U^2w^2 = V^4$$
$$U^2V^2 = V^2w^2 + U^2w^2$$
$$\frac{1}{w^2} = \frac{1}{U^2} + \frac{1}{V^2}$$
$$\therefore w^{-2} = U^{-2} + V^{-2}$$

9 **i**

$$\frac{dv}{dt} = -kv^2$$
$$\frac{dt}{dv} = -\frac{1}{k} \times \frac{1}{v^2}$$
$$t = -\frac{1}{k}\int_u^v v^{-2}\,dv$$
$$= \frac{1}{k}\left[\frac{1}{v}\right]_u^v$$
$$= \frac{1}{k}\left(\frac{1}{v} - \frac{1}{u}\right)$$
$$kt = \frac{1}{v} - \frac{1}{u}$$
$$\frac{1}{v} = kt + \frac{1}{u}$$

ii

$$\frac{dw}{dt} = -g - kw^2$$
$$\frac{dt}{dw} = -\frac{1}{g + kw^2}$$
$$t = -\int_w^u \frac{1}{g + kw^2}\,dw$$
$$= \frac{1}{k}\int_w^u \frac{1}{\left(\sqrt{\frac{g}{k}}\right)^2 + w^2}\,dw$$
$$= \frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\sqrt{\frac{k}{g}}w\right)\right]_w^u$$
$$= \frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right)$$

iii

Let $w = 0$

$$t = \frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right)$$
$$\therefore \frac{1}{v} = k\left(\frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right)\right) + \frac{1}{u}$$
$$= \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

iv

as $u \to \infty$ $\frac{1}{u} \to 0$ and $\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \to \frac{\pi}{2}$

$$\frac{1}{v} \to \frac{\sqrt{k}}{\sqrt{g}}\tan^{-1}\left(u\frac{\sqrt{k}}{\sqrt{g}}\right) = \frac{\pi\sqrt{k}}{2\sqrt{g}}$$
$$\therefore v \to \frac{2}{\pi}\sqrt{\frac{g}{k}}$$

10 **i**

$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -(g + kv^2)$$

ii

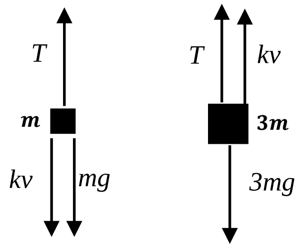
$$v\frac{dv}{dx} = -(g + kv^2)$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$H = -\int_v^0 \frac{v}{g + kv^2}$$
$$= \frac{1}{2k}\left[\ln(g + kv^2)\right]_0^V$$
$$= \frac{1}{2k}\left(\ln(g + kV^2) - \ln g\right)$$
$$= \frac{1}{2k}\ln\left\{\frac{g + kV^2}{g}\right\}$$

iii

$$\frac{dv}{dt} = -(g + kv^2)$$
$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$
$$t = -\int_V^0 \frac{1}{g + kv^2}\,dv$$
$$= \frac{1}{k}\int_0^V \frac{1}{\left(\sqrt{\frac{g}{k}}\right)^2 + v^2}\,dv$$
$$= \frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\sqrt{\frac{k}{g}}v\right)\right]_0^V$$
$$= \frac{1}{\sqrt{kg}}\tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}V\right)$$

11

i



ii

$m\ddot{x} + 3m\ddot{x}$

$= 3mg - kv - T - mg - kv + T$

$$4m\ddot{x} = 2mg - 2kv$$

$$\ddot{x} = \frac{mg - kv}{2m}$$

iii

Let $\ddot{x} = 0$

$\therefore mg - kV = 0$

$$V = \frac{mg}{k}$$

iv

$$\frac{dv}{dt} = \frac{mg - kv}{2m}$$

$$\frac{dt}{dv} = \frac{2m}{mg - kv}$$

$$\begin{aligned} t &= 2m\int_0^v \frac{1}{mg - kv}\,dv \\ &= -\frac{2m}{k}\Big[\ln(mg - kv)\Big]_0^v \\ &= \frac{2m}{k}\Big(\ln(mg) - \ln|(mg - kv)|\Big) \\ &= \frac{2m}{k}\ln\left(\frac{mg}{mg - kv}\right) \end{aligned}$$

v

Let $v = \frac{V}{2} = \frac{mg}{2k}$

$$\begin{aligned} t &= \frac{2m}{k}\ln\left(\frac{mg}{mg - k\left(\frac{mg}{2k}\right)}\right) \\ &= \frac{2m}{k}\ln\left(\frac{mg}{mg - \frac{mg}{2}}\right) \\ &= \frac{2m}{k}\ln 2 \\ &= \frac{m}{k}\ln 4 \\ &= \frac{V}{g}\ln 4 \quad \left(\text{since } V = \frac{mg}{k}\right) \end{aligned}$$

12

i

$$\begin{aligned} x &= Ae^{-\sqrt{3}t}\sin t \\ \dot{x} &= A\left(e^{-\sqrt{3}t}\cos t + \sin t\left(-\sqrt{3}e^{-\sqrt{3}t}\right)\right) \\ &= Ae^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right) \\ \ddot{x} &= A\left(e^{-\sqrt{3}t}\left(-\sin t - \sqrt{3}\cos t\right) + \left(\cos t - \sqrt{3}\sin t\right)\left(-\sqrt{3}e^{-\sqrt{3}t}\right)\right) \\ &= 2Ae^{-\sqrt{3}t}\left(\sin t - \sqrt{3}\cos t\right) \end{aligned}$$

$$\begin{aligned} &-4x - 2\sqrt{3}\dot{x} \\ &= -4\left(Ae^{-\sqrt{3}t}\sin t\right) - 2\sqrt{3}\left(Ae^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right)\right) \\ &= -4Ae^{-\sqrt{3}t}\sin t - 2\sqrt{3}Ae^{-\sqrt{3}t}\cos t + 6Ae^{-\sqrt{3}t}\sin t \\ &= 2Ae^{-\sqrt{3}t}\left(\sin t - \sqrt{3}\cos t\right) \\ &= \ddot{x} \end{aligned}$$

$\therefore x = Ae^{-\sqrt{3}t}\sin t$ is a solution of $\ddot{x} = -4x - 2\sqrt{3}\dot{x}$

ii

Let $\dot{x} < 0$

$3e^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right) < 0$

since $3e^{-\sqrt{3}t} > 0$ for all t, velocity is negative when

$\cos t - \sqrt{3}\sin t < 0$

Let $\cos t - \sqrt{3}\sin t = r\cos(t + \alpha)$ where

$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ and $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$$\begin{aligned} \therefore 2\cos\left(t + \frac{\pi}{3}\right) &< 0 \\ \cos\left(t + \frac{\pi}{3}\right) &< 0 \\ \frac{\pi}{2} < t + \frac{\pi}{3} &< \frac{3\pi}{2} \\ \frac{\pi}{6} < t &< \frac{7\pi}{6} \end{aligned}$$

The particle is moving downwards between $t = \frac{\pi}{6}$ and $t = \frac{7\pi}{6}$ seconds.