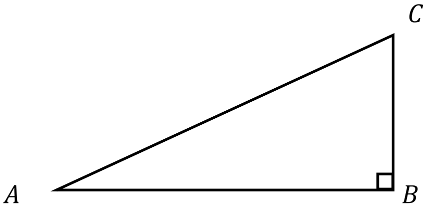
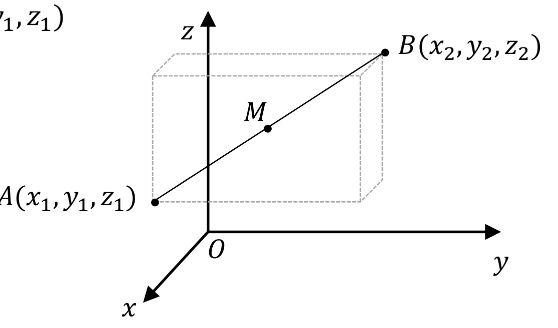
BASIC

1 Prove that the sum of the square of the hypotenuse equals the sum of the squares of the other two sides in a right angled triangle.



2 What type of triangle is formed by the points $\mathrm{A}(1,1,1), B(1,-1,1)$ and $O(0,0,0)$?

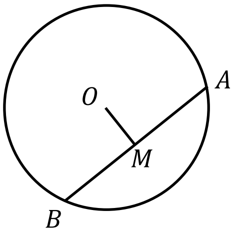
3 Prove that the midpoint of the interval from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$



4 A mass exerts a downward force of $50\ \mathrm{N}$. It is being held in a steady position by four drones, exerting forces in Newtons of $(10,20,10), (20,-10,20), (-15,-15,20)$ and (a, b, c). Find the value of a, b and c.

MEDIUM

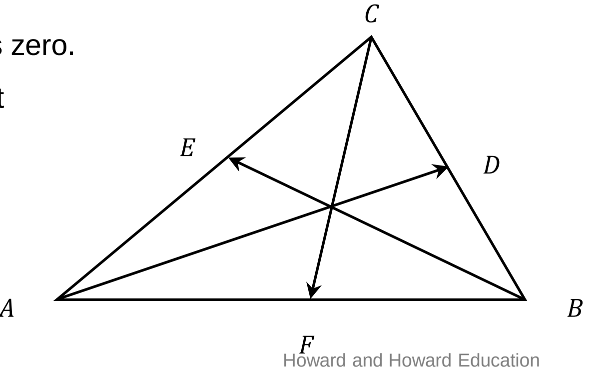
5 Prove that the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.



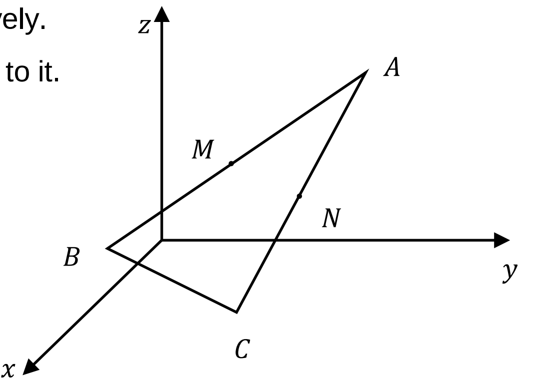
6 Find the point that divides $P(1,-2,4)$ and $Q(5,6,0)$ in the ratio $1:3$.

7 Three vertices of a parallleogram are $O(0,0,0), A(1,1,1)$ and $\mathrm{B}(1,-1,1)$. Find the possible positions of the fourth vertex.

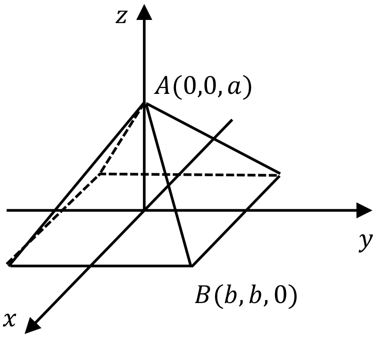
8 Prove that the sum of the medians of a triangle is zero. A median is a line joining a vertex to the midpoint of the opposite side as shown.



9 M and N are the midpoints of AB and AC respectively. Prove that MN is half the length of BC and parallel to it.



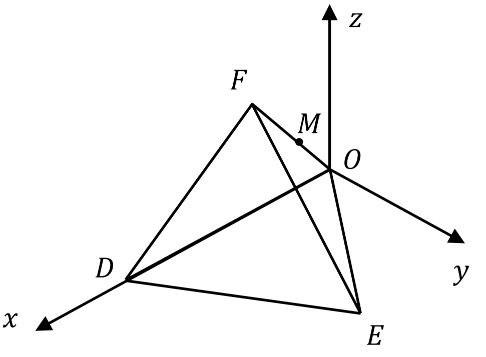
10 A square based pyramid has its base on the $x-y$ plane, with its apex at $A(0,0,a)$. The four triangles forming its sides are isosceles with sides in the ratio $2:2:1$, the short side being the bottom side. One of the four vertices of the square base is $\mathrm{B}(b,b,0)$, where $b>0$. Find b in terms of a.



CHALLENGING

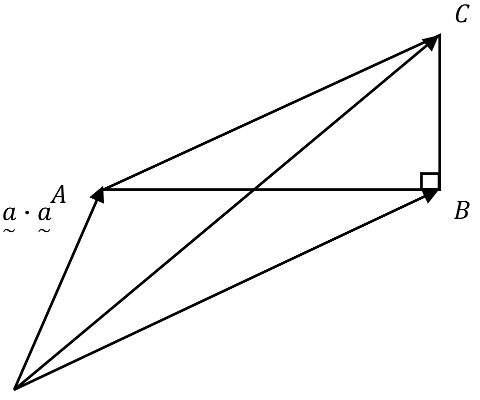
11 Prove that the medians of a triangle are concurrent (intersect at one point).

12 The faces of tetrahedron $ODEF$ are comprised of equilateral triangles of side length 1 unit. Its base lies flat on the $x-y$ plane with vertices at O, $\mathrm{D}(1,0,0)$ and $E\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)$ as shown. Prove the coordinates of M, the midpoint of FO, is $\left(\frac{1}{4},\frac{\sqrt{3}}{12},\frac{\sqrt{6}}{6}\right)$.



1

$$\overrightarrow{BC} \cdot \overrightarrow{AB} = 0 \quad \text{(perpendicular)}$$
$$\left(\underset{\sim}{c} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$$
$$\underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$$
$$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{b} \quad (1)$$



$$\begin{aligned}
\left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AB}\right|^2 &= \left(\underset{\sim}{c} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right) + \left(\underset{\sim}{b} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) \\
&= \underset{\sim}{c} \cdot \underset{\sim}{c} - 2\underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b} - 2\,\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a} \\
&= \underset{\sim}{c} \cdot \underset{\sim}{c} + 2\left(\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c}\right) + \underset{\sim}{a} \cdot \underset{\sim}{a} \\
&= \underset{\sim}{c} \cdot \underset{\sim}{c} - 2\underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{a} \\
&= \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \\
&= \left|\overrightarrow{AC}\right|^2
\end{aligned}$$

∴ The square on the hypotenuse of a right angled triangle equals the sum of the squares on the other two sides. □

2

$$\left|\overrightarrow{OA}\right| = \sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2} = \sqrt{3}$$

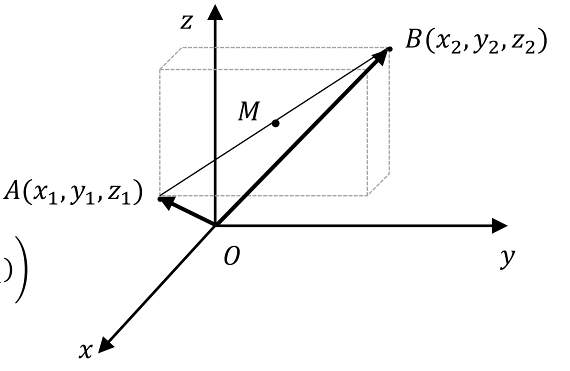
$$\left|\overrightarrow{OB}\right| = \sqrt{(1-0)^2 + (-1-0)^2 + (1-0)^2} = \sqrt{3}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(1-1)^2 + (1+1)^2 + (1-1)^2} = 2$$

ΔABC is isosceles.

3

$$\begin{aligned}
\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\
&= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} \\
&= (x_1, y_1, z_1) + \frac{1}{2}(x_2 - x_1, y_2 - y_1, z_2 - z_1) \\
&= \left(x_1 + \frac{1}{2}(x_2 - x_1), x_1 + \frac{1}{2}(x_2 - x_1), x_1 + \frac{1}{2}(x_2 - x_1)\right) \\
&= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \quad □
\end{aligned}$$



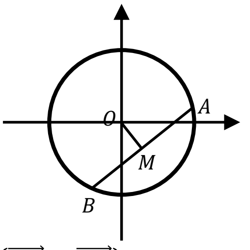
4

$$(10,20,10) + (20,-10,20) + (-15,-15,20) + (a,b,c) + (0,0,-50) = (0,0,0)$$
$$(10 + 20 - 15 + a + 0, 20 - 10 - 15 + b + 0, 10 + 20 + 20 + c - 50) = (0,0,0)$$
$$(15 + a, b - 5, c) = (0,0,0)$$

∴ $a = -15, b = 5$ and $c = 0$.

5

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$



$$\overrightarrow{OM}\cdot\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)\cdot\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\left(\left(\overrightarrow{OB}\right)\cdot\left(\overrightarrow{OB}\right) - \left(\overrightarrow{OA}\right)\cdot\left(\overrightarrow{OA}\right)\right)$$
$$= \frac{1}{2}\left(\left|\overrightarrow{OB}\right|^2 - \left|\overrightarrow{OA}\right|^2\right)$$
$$= \frac{1}{2}\left(r^2 - r^2\right)$$
$$= 0$$

$\therefore OM \perp AB$

6

Let $X(a,b,c)$ be the point that divides PQ in the ratio $1:3$.

$$\therefore \overrightarrow{PX} = \frac{1}{4}\overrightarrow{PQ}$$

$$\begin{pmatrix} a-1 \\ b+2 \\ c-4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5-1 \\ 6+2 \\ 0-4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore a-1=1 \ \rightarrow \ a=2$

$b+2=2 \ \rightarrow \ b=0$

$c-4=-1 \ \rightarrow \ c=3$

$\therefore X(2,0,3)$

7

There are three possible vectors for $\overrightarrow{OD}$ that would create a parallelogram: $\overrightarrow{OA} + \overrightarrow{OB}, \overrightarrow{OA} - \overrightarrow{OB}$ and $\overrightarrow{OB} - \overrightarrow{OA}$.

$\therefore \overrightarrow{OD} = (1,1,1) + (1,-1,1) = (2,0,2)$ or

$= (1,1,1) - (1,-1,1) = (0,2,0)$ or

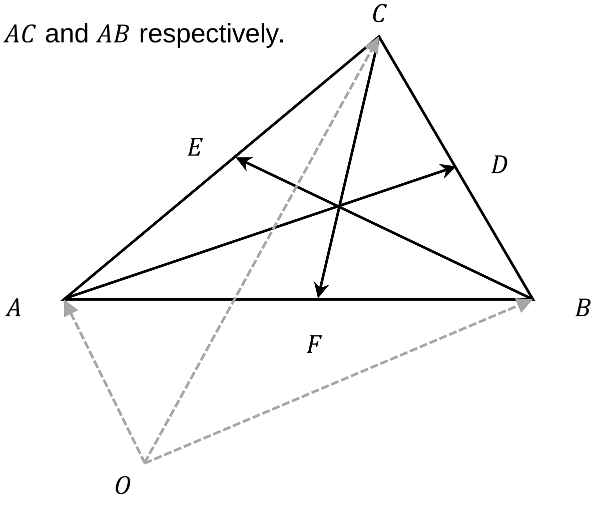
$= (1,-1,1) - (1,1,1) = (0,-2,0)$

The fourth vertex is at $(2,0,2), (0,2,0)$ or $(0,-2,0)$.

8

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.



$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$

$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$

$$= \left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2} - \underset{\sim}{a}\right) + \left(\frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} - \underset{\sim}{b}\right) + \left(\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} - \underset{\sim}{c}\right)$$

$$= \left(-1 + \frac{1}{2} + \frac{1}{2}\right)\underset{\sim}{a} + \left(\frac{1}{2} - 1 + \frac{1}{2}\right)\underset{\sim}{b} + \left(\frac{1}{2} + \frac{1}{2} - 1\right)\underset{\sim}{c}$$

$$= 0$$

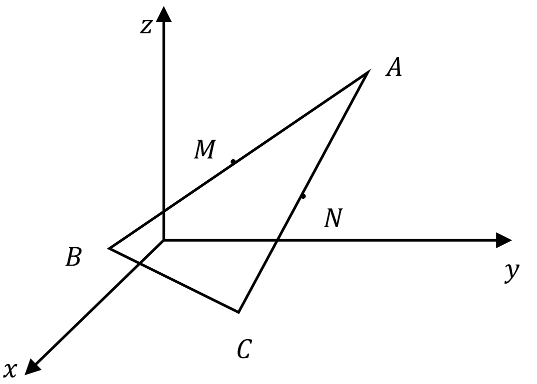
9 $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN}$

$= \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AC}\right)$

$= \frac{1}{2}\overrightarrow{BC}$

$\therefore MN$ is half the length of BC and parallel to it.



10 $\left|\overrightarrow{AB}\right| = \sqrt{(-b)^2 + (-b)^2 + a^2} = \sqrt{a^2 + 2b^2}$

The side length of the base is $2b$.

$\therefore 2(2b) = \sqrt{a^2 + 2b^2}$

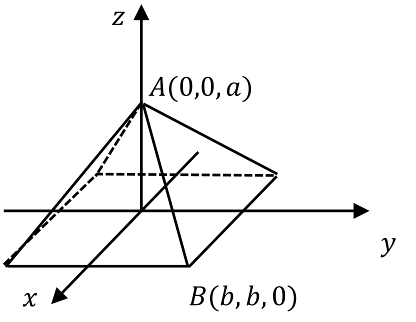
$4b = \sqrt{a^2 + 2b^2}$

$16b^2 = a^2 + 2b^2$

$a^2 = 14b^2$

$b^2 = \frac{a^2}{14}$

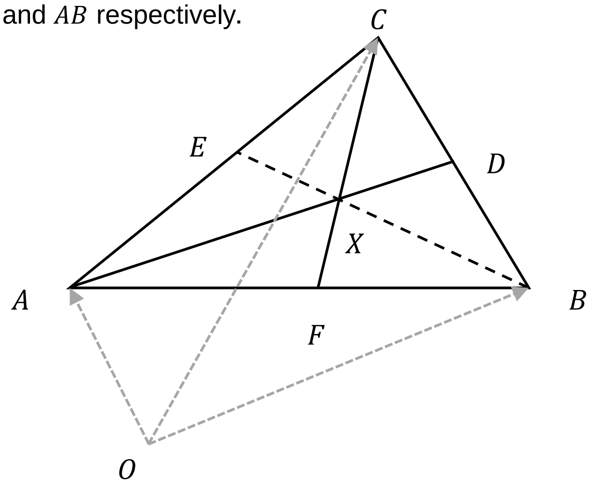
$b = \frac{a}{\sqrt{14}}$



11 We will let X be the intersection of AD and CF and prove that it lies on BE.

Let $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

In ΔABC let D, E, F be the midpoints of the sides BC, AC and AB respectively.



$\therefore \overrightarrow{OE} = \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2}, \overrightarrow{OD} = \frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ and $\overrightarrow{OF} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$

$$\begin{aligned}
\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX} \\
&= \underset{\sim}{a} + \lambda\overrightarrow{AD} \\
&= \underset{\sim}{a} + \lambda\left(\frac{\underset{\sim}{b} + \underset{\sim}{c}}{2} - \underset{\sim}{a}\right) \\
&= (1-\lambda)\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b} + \frac{\lambda}{2}\underset{\sim}{c} \quad (1)
\end{aligned}$$

Also

$$\begin{aligned}
\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CX} \\
&= \underset{\sim}{c} + \mu\overrightarrow{CF} \\
&= \underset{\sim}{c} + \mu\left(\frac{\underset{\sim}{a} + \underset{\sim}{b}}{2} - \underset{\sim}{c}\right) \\
&= \frac{\mu}{2}\underset{\sim}{a} + \frac{\mu}{2}\underset{\sim}{b} + (1-\mu)\underset{\sim}{c} \quad (2)
\end{aligned}$$

From (1) and (2):

$$1 - \lambda = \frac{\mu}{2} \qquad \frac{\lambda}{2} = \frac{\mu}{2} \qquad \frac{\lambda}{2} = 1 - \mu$$

$$\lambda = \mu$$

$$\begin{aligned}
1 - \lambda &= \frac{\lambda}{2} \\
1 &= \frac{3\lambda}{2} \\
\lambda &= \mu = \frac{2}{3}
\end{aligned}$$

$$\therefore \overrightarrow{OX} = \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3}$$

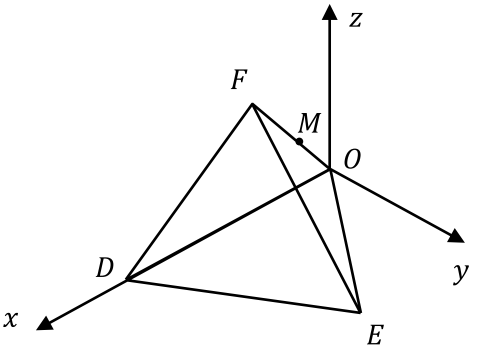
$$\begin{aligned}
\overrightarrow{BX} &= \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3} - \underset{\sim}{b} \\
&= \frac{\underset{\sim}{a} + \underset{\sim}{c} - 2\underset{\sim}{b}}{3}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{XE} &= \frac{\underset{\sim}{a} + \underset{\sim}{c}}{2} - \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3} \\
&= \frac{\underset{\sim}{a} + \underset{\sim}{c} - 2\underset{\sim}{b}}{6} \\
&= \frac{1}{2}\overrightarrow{BX}
\end{aligned}$$

$\therefore B, E, X$ are collinear

$\therefore$ the medians of a triangle are concurrent

12 Let $M(a,b,c)$



In ΔMOD

$$\cos\frac{\pi}{3} = \frac{(a,b,c)\cdot(1,0,0)}{\frac{1}{2}\times 1}$$

$$\frac{1}{4} = a + 0 + 0$$

$$a = \frac{1}{4}$$

In ΔMOE

$$\cos\frac{\pi}{3} = \frac{(a,b,c)\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)}{\frac{1}{2}\times 1}$$

$$\frac{1}{4} = \frac{a}{2} + \frac{\sqrt{3}}{2}b + 0$$

$$\frac{\sqrt{3}}{2}b = \frac{1}{4} - \frac{a}{2}$$

$$b = \frac{2}{\sqrt{3}}\left(\frac{1}{4} - \frac{1}{2}\times\frac{1}{4}\right)$$

$$= \frac{1}{4\sqrt{3}}$$

$$= \frac{\sqrt{3}}{12}$$

$$a^2 + b^2 + c^2 = \left(\frac{1}{2}\right)^2$$

$$\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{12}\right)^2 + c^2 = \frac{1}{4}$$

$$\frac{1}{16} + \frac{3}{144} + c^2 = \frac{1}{4}$$

$$c^2 = \frac{1}{6}$$

$$c = \frac{\sqrt{6}}{6}$$

$$\therefore M\left(\frac{1}{4},\frac{\sqrt{3}}{12},\frac{\sqrt{6}}{6}\right)$$