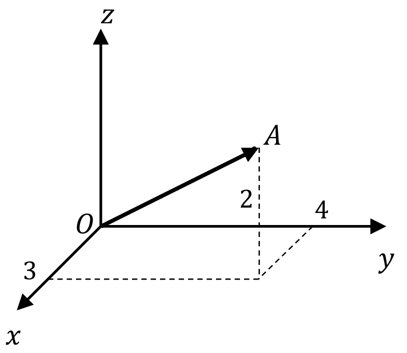
EXERCISE 5.1

BASIC

1 Write $\overrightarrow{OA}$ in component form, as an ordered triple and in column vector notation



2 Simplify the following in each of the three forms.

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right)$ **b** $(1,3,-2) + (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

3 Simplify the following in each of the three forms.

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) - \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right)$ **b** $(1,3,-2) - (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

4 Simplify.

a $3\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ **b** $3(1,3,-2)$ **c** $3\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

5 Find the magnitude of these vectors:

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ **b** $(4,3,-2)$ **c** $\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$

6 Find the unit vector for each vector in Question 5.

MEDIUM

7 Prove that if $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ then $\left|\underset{\sim}{u}\right| = \sqrt{x^2 + y^2 + z^2}$

8 If $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 7 \\ 6 \\ 2 \end{pmatrix}$ find $\left|\overrightarrow{PQ}\right|$

9 Show that $\underset{\sim}{a} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$ is a unit vector

10 Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ that has a magnitude of 3.

11 Find the following scalar products

a $\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + 6\underset{\sim}{k}\right)$ **b** $(3,2,-6) \cdot (2,-4,1)$ **c** $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

12 Use the two formula for the dot product to prove that the angle between

$\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ is given by

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}$$

13 Find the angle between

a $3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

b $\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $-2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

c $\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}$

d $0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$ and $\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

CHALLENGING

14 A triangular based pyramid has three of its vertices at $A(2,0,0), B(0,2,0)$ and $C(0,0,2)$. If its fourth vertex is at $D(a,a,a)$, where $a > 0$, find the value of a. You are given the three triangles forming the sides of the pyramid are equilateral.

15 A rectangular prism with sides of length 6, 8 and 10 units has both ends of one of its longest diagonals along the x-axis. Prove that all points on the surface of the prism satisfy $|z| \leq 5\sqrt{2}$.

SOLUTIONS - EXERCISE 5.1

1

$$\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} = (3,4,2) = \begin{pmatrix}3\\4\\2\end{pmatrix}$$

2

a $(1+2)\underset{\sim}{i} + (3-4)\underset{\sim}{j} + (-2+1)\underset{\sim}{k}$

$= 3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

b $(1+2, 3-4, -2+1)$

$= (3,-1,-1)$

c $\begin{pmatrix}1+2\\3-4\\-2+1\end{pmatrix} = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$

3

a $(1-2)\underset{\sim}{i} + (3+4)\underset{\sim}{j} + (-2-1)\underset{\sim}{k}$

$= -\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$

b $(1-2, 3+4, -2-1)$

$= (-1,7,-3)$

c $\begin{pmatrix}1-2\\3+4\\-2-1\end{pmatrix} = \begin{pmatrix}-1\\7\\-3\end{pmatrix}$

4

a $3\underset{\sim}{i} + 9\underset{\sim}{j} - 6\underset{\sim}{k}$

b $(3,9,-6)$

c $\begin{pmatrix}3\\9\\-6\end{pmatrix}$

5

a $\left|\underset{\sim}{a}\right| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$

b $\left|\underset{\sim}{b}\right| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

c $\left|\underset{\sim}{c}\right| = \sqrt{1^2 + 5^2 + (-2)^2} = \sqrt{30}$

6

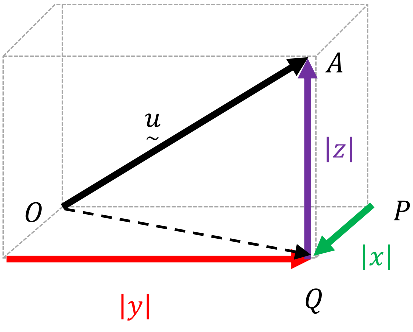
a $\left|\underset{\sim}{a}\right| = \sqrt{14}$, $\quad \hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{14}}\underset{\sim}{i} + \dfrac{3}{\sqrt{14}}\underset{\sim}{j} - \dfrac{2}{\sqrt{14}}\underset{\sim}{k}$

b $\left|\underset{\sim}{b}\right| = \sqrt{29}$, $\quad \hat{\underset{\sim}{b}} = \left(\dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}, -\dfrac{2}{\sqrt{29}}\right)$

c $\left|\underset{\sim}{c}\right| = \sqrt{30}$, $\quad \hat{\underset{\sim}{c}} = \begin{pmatrix}\dfrac{1}{\sqrt{30}}\\[2mm] \dfrac{5}{\sqrt{30}}\\[2mm] -\dfrac{2}{\sqrt{30}}\end{pmatrix}$

7

Splitting $\underset{\sim}{u}$ into its component vectors $x\underset{\sim}{i}$, $y\underset{\sim}{j}$ and $z\underset{\sim}{k}$, we can see that ΔOPQ is right angled with hypotenuse $|\overrightarrow{OQ}|$ and short sides of $\left|x\underset{\sim}{i}\right| = |x|$ and $\left|y\underset{\sim}{j}\right| = |y|$



$\therefore |\overrightarrow{OQ}|^2 = |x|^2 + |y|^2 \quad$ (Pythagoras)

Similarly we have ΔOAQ also right angled

$$\left|\underset{\sim}{u}\right|^2 = |\overrightarrow{OQ}|^2 + |\overrightarrow{AQ}|^2$$

$$= |x|^2 + |y|^2 + |z|^2$$

$$= x^2 + y^2 + z^2$$

$\therefore \left|\underset{\sim}{u}\right| = \sqrt{x^2 + y^2 + z^2} \quad \square$

8

$$\overrightarrow{PQ} = \begin{pmatrix}7\\6\\2\end{pmatrix} - \begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}6\\1\\4\end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{6^2 + 1^2 + 4^2} = \sqrt{53}$$

9

$$\left|\underset{\sim}{a}\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}}$$

$$= 1$$

$\therefore \underset{\sim}{a}$ is a unit vector

10 Find the unit vector parallel to $2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ and multiply it by ± 3

$$\left|\underset{\sim}{u}\right| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$$

$$\hat{\underset{\sim}{u}} = \frac{2}{\sqrt{21}}\underset{\sim}{i} - \frac{1}{\sqrt{21}}\underset{\sim}{j} + \frac{4}{\sqrt{21}}\underset{\sim}{k}$$

$$\underset{\sim}{v} = \hat{\underset{\sim}{u}} \times (\pm 3)$$

$$= \pm\left(\frac{6}{\sqrt{21}}\underset{\sim}{i} - \frac{3}{\sqrt{21}}\underset{\sim}{j} + \frac{12}{\sqrt{21}}\underset{\sim}{k}\right)$$

11 **a** $1(2) + 3(-4) + 2(6) = 2$
b $3(2) + 2(-4) - 6(1) = -8$
c $1(2) + 3(0) - 2(1) = 0$

12 From the scalar product we have

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2 + z_1z_2 \quad (1)$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right| \cos\theta \quad (2)$$

$$\therefore \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right| \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|} \qquad \square$$

13

a $\cos\theta = \dfrac{3(2) + 1(-3) - 2(-1)}{\sqrt{3^2 + 1^2 + (-2)^2} \times \sqrt{2^2 + (-3)^2 + (-1)^2}}$

$\theta = 69°05'$

b $\cos\theta = \dfrac{1(-2) + 2(-4) - (2)}{\sqrt{1^2 + 2^2 + (-1)^2} \times \sqrt{(-2)^2 + (-4)^2 + 2^2}}$

$\theta = 180 \quad \text{(parallel vectors)}$

c $\cos\theta = \dfrac{1(1) - (1) + (0)}{\sqrt{4^2 + (-1)^2 + 3^2} \times \sqrt{2^2 + 8^2} + 0^2}$

$\theta = 90° \quad \text{(perpendicular vectors)}$

d The first vector is the zero vector. There is no angle, as an angle needs two arms and the zero vector does not form a line. The vectors are orthogonal.

14 D is the far corner of a cube whose other end is at the origin, and using the other vertices we can easily see $a = 2$.

15 The longest diagonal is $\sqrt{6^2 + 8^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$. The centre of the prism is on the long diagonal, so lies on the x-axis, and every point on the prism must be at most half of the length of the long diagonal from the centre, and thus $|z| \leq 5\sqrt{2}$. We could also say $|y| \leq 5\sqrt{2}$.