HSC Mathematics Extension 2

Steve Howard

A textbook for the Mathematics Extension 2 (2017) NSW Syllabus

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Published by
Steve Howard
Willowvale Road
COWRA NSW 2794 Australia

About the Author

Steve is a Mathematics Teacher at Cowra High School, a medium sized rural high school in Central West NSW, where he has taught for over 25 years. He has also taught gifted and talented students online through xsel and Aurora College.

He has a particular love and passion for Mathematics Extension 2, writing this textbook and the 1000 Extension 2 Revision Questions - see howardmathematics.com. Both the textbook and revision questions are completely free, although students may like to sign up to his course that includes fully recorded lessons that cover the entire Extension 2 course, including fully recorded solutions to every question in the textbook. Much cheaper than tutoring!

In his spare time Steve writes commercial online professional development courses for teachers covering the new Mathematics Advanced, Extension 1 and 2 syllabuses through TTA. The online courses for teachers cover this material in much greater depth and include hundreds of recorded examples. For the courses currently offered see https://ttacreativemarketi.wixsite.com/stevehowardtta

Steve loves finding more efficient techniques for solving mathematical questions, by trawling through other teachers' solutions or making up his own approaches when there must be a better way. Many of the approaches you see here are unique, with emphases on both understanding the work, plus little tricks to help students succeed in exams.

Steve did 4 Unit Mathematics as a student way back in 1988, gaining a mark of 198/200 and training as an actuary. Working in an office in the city wasn’t for him, so he went back to uni to retrain as a teacher then headed to the country, where he lives in an owner built mud house, with chickens, goats and rescued native birds (which you will sometimes hear in his recordings)!!

Contents

Introduction

Welcome! I hope all students and teachers find this book useful and enjoyable in your journey in Extension 2 Maths. This textbook and the matching '1000 Revision Questions' were first released for the 2020 HSC, and will be continually changed and updated over time - the two digital books work best together. For free downloads of the latest versions please visit howardmathematics.com.

This textbook for the new syllabus evolved from the textbook I wrote for the old syllabus which itself evolved over a few years from recorded lessons I was making for my classes, and sharing with a few others around the state. I was originally using some of the commercially published textbooks for the old course, plus one with restricted availability, and found they weren't meeting the needs of my students or myself. This textbook is the end result of that. It is mainly a passion project, but indirectly earns me a bit of pocket money through school copyright licencing and increasing traffic to my commercial online professional development courses for teachers through TTA.

Some of the features of this textbook that students and teachers may find useful:

- The questions are in the style of past HSC questions where possible, so students are learning the right type of questions throughout the course.
- The questions are graded in difficulty so students can work at their own level.
- Each exercise starts with Basic questions that match the examples, so students can ease themselves into the new concepts if they need to. More confident students can skip these first questions and start with Medium.
- All questions have fully worked solutions.
- There are lots of diagrams that help understand concepts that would normally only be dealt with algebraically.
- There are many hints and tips on how to more easily answer questions, and to see the way past tricks in exam questions.
- Some of the chapters have appendices, where extra content that may sometimes be useful is available. This content is not necessary to succeed in the course, but can help high ability students eliminate mistakes in exams or is sometimes just plain interesting!

The textbook is paired with '1000 Revision Questions in Mathematics Extension 2'. The first 500 questions are arranged topic by topic matching the chapters from the textbook to help students study and revise, while the last 500 questions are from mixed topics to help students prepare for their Trials and the HSC.

The course is currently set out in 42 lessons. The aim is for you to be able to finish the course as soon as possible so that you can have months of revision before the HSC to master the content.

If you find any mistakes, or have any ideas that would make either the textbook or the revision questions better, please contact me via email below.

Cheers
Steve Howard
steve@howardmathematics.com

Thanks to the following teachers and students for letting me know errors or improvements:
Gavin S, Matthew D, Drew S, Brailey S, Charlene C, Ian B, Glenn B, Daniel P, Kingsley H, Abi R, Chris B.

And to my current or past students who have earned lots of Mars Bars by picking up errors:
Alan, Brendan, Kurt, Andrew, Sam, Xavier, Luke, El, Sean and Aeryc.

Chapter 1
The Nature of Proof

MEX-P1: The Nature of Proof
All topics in Extension 2 rely on your ability to construct logically sound and convincing proofs, so it is a fitting topic with which we begin this textbook. In the Nature of Proof we will focus on the logic of proofs, and use alternative ways to prove arguments where direct proof or mathematical induction are not appropriate.

The topic tests our ability to use mathematical language and plain language to reason and communicate, promoting clear, simple and logical thought processes.

Lessons
The Nature of Proof is covered in 6 lessons.

1.1 The Language of Proof and Simple Proofs
1.2 Proof by Contrapositive
1.3 Proof by Contradiction
1.4 Equivalence and Disproofs
1.5 Inequality Proofs
1.6 Arithmetic Mean - Geometric Mean Inequality

Revision Questions
In '1000 Revision Questions', the revision book that goes with this textbook you will find the following questions matching this chapter:

- Revision Exercise 1
  60 graded questions on this topic only
- Revision Exercises 7 (Basic), 8 (Medium) and 9 (Challenging)
  Another 60 questions mixed through other topics for when you finish the course.

Don't forget to do any questions from the exercises in this textbook you haven't done.

In Lesson 1 we look at the language of proof and simple proofs, covering:

- How should we use the Language of Proof?
- Statements
- Implication
- Quantifiers
- Other Terminology
- Symbols and Set Notation
- Simple Proofs Involving Numbers

HOW SHOULD WE USE THE LANGUAGE OF PROOF?

You should make sure that you know what the symbols that we cover in this lesson mean, but use them sparingly in your own proofs. It may be tempting to cram as many of the symbols at any possible point in a proof, yet if we look at the solutions to the NESA sample questions and many university level references to proofs for first year students they use no symbols whatsoever, instead using whole words and sentences!

Using the symbols can save you time, but they make your proof harder to read and thus less convincing for your reader. Use them sparingly! Never forget that a good proof needs to convince the reader of the truth of the argument, and preferably explain to them why it is true.

STATEMENTS

A statement is an assertion that can be true or false but not both. Some examples are:

- 6 is an even number (which is a true statement)
- 6 is an odd number (which is a false statement)
- The square of a number is even (which is a false statement since it is not true for all numbers).
- 6 is an even number **and** 3 is an odd number (which is true since both parts are true).
- 6 is an even number **and** 4 is an odd number (which is false since one or more parts are false).
- 6 is an even number **or** 4 is an odd number (which is true since one or more parts is true).
- 6 is an odd number **or** 4 is an odd number (which is false since neither part is true).

We often use the letters P, Q, R or S as shorthand to represent a statement. If there is a variable used in the statement then we often add it in brackets or as a subscript, such as

$P(n)$: The sum of the first n positive integers is $\frac{n(n+1)}{2}$.

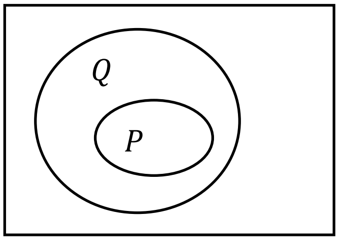
If there is a number used for the variable in the question we often use it, such as

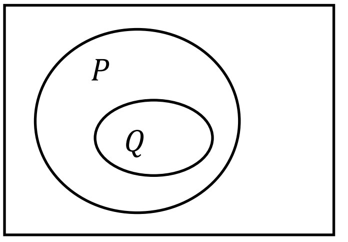
$P(10)$: The sum of the first 10 positive integers is $\frac{10\times 11}{2}$

IMPLICATION

To say that P implies Q means that **if** P is true **then** Q must be true. It is an if-then statement. We are not saying that Q is true, just promising that if P is true then Q must be true.

We can see how implication looks like using an Euler Diagram at left:





If you are in the P ellipse then you must be in the Q ellipse as well, so $P \Rightarrow Q$ (if P is true then Q must be true).

We can also write $P \Leftarrow Q$ if Q implies P (P is implied by Q) if the relationship is reversed, as seen at right.

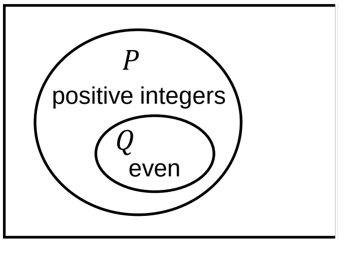
Example 1

Consider the statements:
P: n is a positive integer.
Q: n is an even number greater than 0
Which of the following is true? There is more than one correct answer.

a $P \Rightarrow Q$ **b** $P \Leftarrow Q$ **c** $Q \Rightarrow P$ **d** P implies Q **e** P is implied by Q

Solution

Consider the Euler diagram at right, where every even number is also an integer.



a False Since if a number is an integer it might not be even
b True Since all even numbers are integers
c True Since a number must be an integer if it is even
d False Since if a number is an integer it might not be even
e True Since a number must be an integer if it is even

QUANTIFIERS

For all ∀

Some statements are true or false for all values of a variable, and we can abbreviate this with the symbol ∀, an upside down A.

For example, '$\forall \text{ real numbers } x, x^2 \geq 0$' means that the square of any real number must be non-negative.

There exists ∃

The term 'there exists' indicates that there is at least one number for which the statement is true.

We represent the words with a back to front E.

For example '$\exists x$ for which x^2 is odd' is saying there is at least one value of x for which x^2 is odd.

Example 2

If m is an integer, add the most relevant quantifier to the start of the statement to make it true as often as possible.

a $m, m^2 \geq 0$ **b** $m, m = 0$ **c** $m, \frac{m}{2}$ is integral **d** $m, -1 \leq \sin m \leq 1$

Solution

a ∀ Since the square of all integers are non-negative
b ∃ Since only one integer is equal to zero
c ∃ Since only the even integers are divisible by 2
d ∀ Since the sine ratio is always from -1 to 1

SIMPLE PROOFS INVOLVING NUMBERS

We can prove some simple results using direct proofs, rather than mathematical induction. Many of these results could also be proved by induction, but the direct proofs we will use here are simpler.

Example 3

If m is odd and n is even, prove mn is even.

Solution

Let $m = 2k + 1$ and $n = 2j$ for integral j, k

$$\begin{aligned} mn &= (2k + 1) \times 2j \\ &= 2(2jk + j) \\ &= 2p \qquad \text{where } p \text{ is integral since } j \text{ and } k \text{ are integral} \end{aligned}$$

$\therefore mn$ is even □

Example 4

If m is a multiple of 3, prove m^2 is a multiple of 9.

Solution

Let $m = 3k$ where k is integral

$$\begin{aligned} \therefore m^2 &= (3k)^2 \\ &= 9k^2 \\ &= 9j \qquad \text{where } j \text{ is integral since } k \text{ is integral} \end{aligned}$$

$\therefore m^2$ is a multiple of 9 □

Example 5

Prove that the sum of any two consecutive numbers is always odd.

Solution

Let the consecutive numbers be k and $k + 1$ for integral k

$S = k + k + 1$

$\quad = 2k + 1$ which is odd □

EXERCISE 1.1

BASIC

1 Consider the statements:

P: n is a multiple of 9. Q: n is a multiple of 3.

Which of the following is true? There is more than one correct answer.

a) $P \Rightarrow Q$ b) $P \Leftarrow Q$ c) $Q \Rightarrow P$ d) P implies Q

2 If m is a positive integer, add the most relevant quantifier to the start of the statement to make it true as often as possible.

a) __ $m,\quad 2m$ is even b) __ $m,\quad m^2 = 4$

c) __ $m,\quad m^2 \leq 2$ d) __ $m,\quad 1 + \cos m \geq 0$

3 If m and n are odd, prove
a mn is odd
b $m + n$ is even

4 If m is a multiple of 4, prove m^2 is a multiple of 16

5 Prove that the sum of any two consecutive numbers equals the difference of their squares.

6 If m is even, prove m^2 is even.

7 Prove that the product of any three consecutive numbers is even.

8 Prove that the sum of any four consecutive numbers is even.

MEDIUM

9 Given $a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 \ldots + b^{k-1}\right)$ prove
a $\frac{3^k}{2}$ always has a remainder of 1.
b $3^{2n} - 1$ is divisible by 4

10 If $a + b = 2$ prove $a^2 + 2b = b^2 + 2a$

11 Prove the expression $a^3 - a + 1$ is odd for all positive integer values of a.

12 Prove $n^2 - 1$ is divisible by 3 if n is not a multiple of 3.

13 Prove $(2^m - 1)^2 - 1$ is divisible by 2^{m+1}

14 **a** Given a is integral and not divisible by 5, prove the remainder when a^2 is divided by 5 is either 1 or 4
b Hence given that a, b are integral and not divisible by 5, prove that $a^4 - b^4$ is divisible by 5.

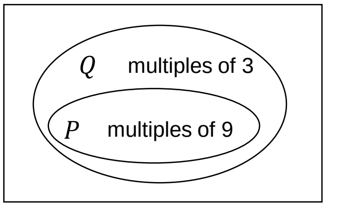
15 Prove $(k^3 - k)(2k^2 + 5k - 3)$ is divisible by 5 without using induction. You may assume that the product of 5 consecutive numbers is divisible by 5.

16 A triangular number is in the form $T = \frac{k(k+1)}{2}$. Prove the square of any odd positive integer greater than 1 is of the form $8T + 1$ where T is a triangular number.

17 Prove that an irrational number raised to the power of an irrational number can be rational, by considering $\sqrt{2}^{\sqrt{2}}$. You may assume $\sqrt{2}$ is irrational.

SOLUTIONS - EXERCISE 1.1

1 Consider the Euler diagram at right, where every multiple of 9 is also a multiple of 3.



a True Since if a number is a multiple of 9 it must also be a multiple of 3
b False Since a number is not necessarily a multiple of 9 if it is a multiple of 3
c False Since if a number is a multiple of 3 it is not necessarily a multiple of 9
d True Since if a number is a multiple of 9 it must also be a multiple of 3

2 **a** ∀ Since twice any integer is even
b ∃ Since only two integers (± 2) when squared give 4
c ∃ Since only $0, \pm 1$ when squared give an answer less than or equal to 2
d ∀ Since the cosine ratio is always from -1 to 1

3 **a** Let $m = 2j + 1$ and $n = 2k + 1$ for integral j, k

$mn = (2j + 1) \times (2k + 1)$

$= 4jk + 2j + 2k + 1$

$= 2(2jk + j + k) + 1$

$= 2p + 1 \qquad$ for integral p

$\therefore mn$ is odd □

b Let $m = 2j + 1$ and $n = 2k + 1$ for integral j, k

$mn = (2j + 1) + (2k + 1)$

$= 2j + 2k + 12$

$= 2(j + k + 1)$

$= 2p \qquad$ for integral p

$\therefore m + n$ is even □

4 Let $m = 4k$ where k is integral

$\therefore m^2 = (4k)^2$

$= 16k^2$

$= 16p \qquad$ for integral p

$\therefore m^2$ is a multiple of 16 □

5 Let the consecutive numbers be k and $k + 1$ for integral k

$(k + 1)^2 - k^2$

$= k^2 + 2k + 1 - k^2$

$= 2k + 1$

$= k + (k + 1)$ □

6 Let $m = 2k$ for integral k

$\therefore m^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p \qquad$ for integral p

$\therefore$ If m is even then m^2 is even □

7 Let the consecutive numbers be $k-1, k$ and $k+1$ for integral k
$P = k(k-1)(k+1) = k^3 - k$

Case 1 - k is even
Let $k = 2m$ for integral m
$\therefore P = (2m)^3 - 2m$
$= 2(4m^3 - m)$
$= 2p \quad \text{for integral } p$
$\therefore$ true if k is even

Case 2 - k is odd
Let $k = 2m + 1$ for integral m
$\therefore P = (2m+1)^3 - (2m+1)$
$= 8m^3 + 12m^2 + 6m + 1 - 2m - 1$
$= 2(4m^3 + 6m^2 + 2m)$
$= 2p \quad \text{for integral } p$
$\therefore$ true if k is odd
$\therefore$ The product of any three consecutive numbers is even □

8 Let the consecutive numbers be $k, k+1, k+2$ and $k+3$ for integral k

$S = k + k + 1 + k + 2 + k + 3$

$= 4k + 6$

$= 2(2k + 3)$

$= 2p \quad \text{for integral } p$

$\therefore$ The sum of any four consecutive numbers is always even. □

9 **a** $3^k = 3^k - 1^k + 1$
$= (3-1)\left(3^{k-1} + 3^{k-2} \times 1 + 3^{k-3} \times 1^2 \ldots + 1^{k-1}\right) + 1$
$= 2\left(3^{k-1} + 3^{k-2} \times 1 + 3^{k-3} \times 1^2 \ldots + 1^{k-1}\right) + 1$
$= 2p + 1 \quad \text{for integral } p$
$\therefore \frac{3^k}{2}$ always has remainder 1 □

b $3^{2n} - 1 = (3^n - 1)(3^n + 1)$

$= (3-1)(3^{n-1} + 3^{n-2} + \ldots + 3^0)(3^n + 1)$

$= 2(3^{n-1} + 3^{n-2} + \ldots + 3^0)(2k) \quad \text{for integral } k \text{ since } 3^n \text{ is odd } 3^n + 1 \text{ is even}$

$= 4k(3^{n-1} + 3^{n-2} + \ldots + 3^0)$

$= 4p \quad \text{for integral } p$

$\therefore 3^{2n} - 1$ is divisible by 4 □

10 $\text{LHS} - \text{RHS} = a^2 - b^2 + 2b - 2a$
$= (a+b)(a-b) - 2(a-b)$
$= (a-b)(a+b-2)$
$= (a-b)(0) \quad \text{since } a+b=2$
$= 0$
$\therefore a^2 - b^2 + 2b - 2a = 0$
$\therefore a^2 + 2b = b^2 + 2a$ □

Alternatively
$\text{LHS} = a^2 + 2b$
$= (2-b)^2 + 2b \qquad \text{since } a+b=2$
$= 4 - 4b + b^2 + 2b$
$= b^2 + 4 - 2b$
$= b^2 + 2(2-b)$
$= b^2 + 2a$
$= \text{RHS}$ □

11 $a^3 - a + 1$
$= a(a^2-1) + 1$
$= a(a+1)(a-1) + 1$
$= 2k + 1$ where k is integral,
since we have proved that the product of three consecutive numbers is even.
$\therefore a^3 - a + 1$ is odd for all positive integer values of a. □

12 Let $n = 3k \pm 1$ for integral k
$n^2 - 1 = (3k \pm 1)^2 - 1$
$= 9k^2 \pm 6k + 1 - 1$
$= 9k^2 \pm 6k$
$= 3k(3k \pm 2)$
$= 3p \quad \text{for integral } p$
$\therefore n^2 - 1$ is divisible by 3 if n is not a multiple of 3 □

13 $(2^m - 1)^2 - 1 = (2^m - 1 + 1)(2^m - 1 - 1)$
$= 2^m(2^m - 2)$
$= 2^m \cdot 2(2^{m-1} - 1)$
$= 2^{m+1}(2^{m-1} - 1)$
$\therefore (2^m - 1)^2 - 1$ is divisible by 2^{m+1} □

14 **a** Let $a = 5k + m$, where $m = 1, 2, 3$ or 4 and k is integral
$a^2 = (5k+m)^2$
$= 25k^2 + 20km + m^2$
$= 5(5k^2 + 4km) + m^2$
Now $m^2 = 1, 4, 9$ or 16
$\therefore a^2 = 5(5k^2 + 4km) + 1, 5(5k^2 + 4km) + 4, 5(5k^2 + 4km + 1) + 4$ or $5(5k^2 + 4km + 3) + 1$
$\therefore a^2 = 5j + 1, 5j + 4 \text{ where } j \text{ is integral}$ □

b $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$
Case 1 - $a^2 = 5j + r, b^2 = 5i + r \quad r = 1$ or 4 and j, i integral, have the same remainder when divided by 5
$a^4 - b^4$
$= \big(5j + r - (5i + r)\big)(5j + r + 5i + r)$
$= 5(j - i)(5j + 5i + 2r)$
So $a^4 - b^4$ is divisible by 5.
Case 2 - $a^2 = 5j + 1, b^2 = 5i + 4$
$a^4 - b^4$
$= \big((5j+1) - (5i+4)\big)\big((5j+1) + (5i+4)\big)$
$= (5j - 5i - 3)(5j + 5i + 5)$
$= 5(5j - 5i - 3)(j + i + 1)$
So $a^4 - b^4$ is divisible by 5.

$\therefore a^4 - b^4$ is divisible by 5. □

15 $(k^3-k)(2k^2+5k-3)$
$= k(k^2-1)(2k^2+6k-k-3)$
$= k(k+1)(k-1)\big(2k(k+3)-(k+3)\big)$
$= k(k+1)(k-1)(2k-1)(k+3)$
$= k(k+1)(k-1)\left(2(k+2)-5\right)(k+3)$
$= 2(k-1)k(k+1)(k+2)(k+3)-5(k-1)k(k+1)(k+3)$
$= 2(5m)-5n \qquad m, n$ integral, since the product of 5 consecutive numbers is divisible by 5
$= 5(2m-n)$ □

16 $(2k+1)^2$

$= 4k^2+4k+1$

$= 8\left(\dfrac{k^2+k}{2}\right)+1$

$= 8\left(\dfrac{k(k+1)}{2}\right)+1$

$= 8T+1$ □

17 Case 1 - $\sqrt{2}^{\sqrt{2}}$ is rational

$\sqrt{2}^{\sqrt{2}}$ is an irrational number to the power is an irrational number, so if $\sqrt{2}^{\sqrt{2}}$ is rational then we have proved the problem.

Case 2 - $\sqrt{2}^{\sqrt{2}}$ is irrational

Consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, which in this case is an irrational number to the power is an irrational number.

$$\begin{aligned}\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} &= \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} \\ &= \sqrt{2}^2 \\ &= 2\end{aligned}$$

which is rational, so we have proved the problem.

∴ whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational we have proved that an irrational number raised to the power of an irrational number can be rational. □

1.2: PROOF BY CONTRAPOSITIVE

In Lesson 2 we look at the first of two methods of indirect proof - Proof by Contrapositive. We will cover:

- Converse
- Negation
- Contrapositive
- Proof by Contrapositive

CONVERSE

The converse of a statement 'If P then Q' is 'If Q then P'. The statements can be represented as 'the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$', or 'the converse of $P \Rightarrow Q$ is $P \Leftarrow Q$'.

For example:

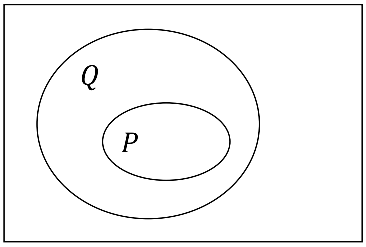
Statement: 'If a number is even it is an integer'

Converse: 'If a number is an integer it is even'

The converse of a true statement is not necessarily true, which is a major source of improperly constructed proofs.

A common fault involves square roots. For example the converse of the statement 'if $x = 4$ then $x^2 = 16$' is the statement 'if $x^2 = 16$ then $x = 4$'. Now the original statement is true, but the converse is false since it does not include $x = -4$ as a possible solution.

We can see that the converse is not necessarily true using an Euler Diagram:



$P \Rightarrow Q$ is true since all of P lies within Q, but $Q \Rightarrow P$ is not necessarily correct as not all of Q lies within P.

The only time that it is appropriate to prove a converse is as one half of a proof where we are proving equivalence.

NEGATION

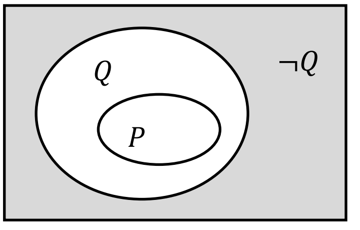
If P is a statement then the statement 'not P' is called the negation of P. The negation of P is denoted by $\neg P$ or $\sim P$.

For example the negation of the statement 'x is even' would be 'x is not even' which we could also write as 'x is odd' if x is a positive integer. The negation can be true or false, as the original statement could be false or true.

Negation is an important technique that we will use in indirect proofs - proof by contrapositive and proof by contradiction.

We can see negations using an Euler Diagram:

We can see that $\neg Q$ is the area outside the Q ellipse.



Note that $\neg P$ is not shown on this diagram to avoid confusion, but includes everything outside the P ellipse, which includes parts of Q and all of $\neg Q$.

When we negate equality statements or inequalities, the negation involves everything else - similar to the concept of the complement in probability.

Original Statement	Negation
$=$	$\neq$
$>$	$\leq$
$\geq$	$<$
$<$	$\geq$
$\leq$	$>$

For example, the negation of $x > 2$ is $x \leq 2$.

Example 1

Find the negation of the following:

a $x = 2$ **b** $x < 2$ **c** $x \geq 2$

Solution

a $x \neq 2$ **b** $x \geq 2$ **c** $x < 2$

NEGATING STATEMENTS INVOLVING 'FOR ALL' OR 'THERE EXISTS'

When we negate statements involving 'For All' or 'There exists', then the original statement and its negation swap the two terms, as well as negating any other part of the statement as shown above.

For example:
The negation of '$\forall$ real numbers $x, x^2 \geq 0$' is '$\exists$ a real number $x, x^2 < 0$'. The original statement is saying that $x^2 \geq 0$ for every value of x, while the negation is just saying, hang on, there is at least one value where that isn't true.

Notice the negation is not saying that it the original statement is false for every value of x, just for one or more. In this case the original statement is true and its negation is false.

The negation of '$\exists$ a real number $x, x^2 - 4 = 0$' is '$\forall$ real numbers $x, x^2 - 4 \neq 0$'. The original statement is saying that $x^2 - 4 = 0$ for at least one value of x, while the negation is just saying, hang on, there are no values of x for which it is true, in other words $x^2 - 4 \neq 0$ for every value of x.

Notice the negation is not saying that there is one or more values of x for which $x^2 - 4 \neq 0$. Again the original statement is true and its negation is false.

Example 2

Find the negation of the following:

a $\forall$ integers $x, 2x$ is even

b $\exists$ a real number $x, x = 4m$ for integral m

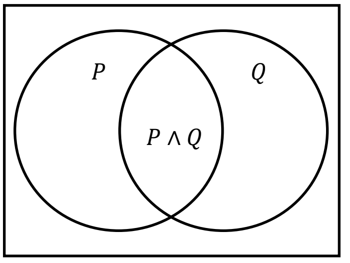
Solution

a $\exists$ integers $x, 2x$ is odd

b $\forall$ real numbers $x, x \neq 4m$ for integral m

NEGATING COMPOUND STATEMENTS (AND/OR)

Say we have to negate a compound statement like 'x is even and less than 10', let's look at the Euler Diagram and find the complement. So we break the compound statement into two statements, '$P: x$ is even', and '$Q: x$ is less than 10'. The original statement is '$R: P \wedge Q$' using the 'and' proof symbol, or represented by the intersection below.



$\wedge$ = 'and'
$\vee$='or'

Now the complement of $P \wedge Q$ is everything outside the intersection, which we can most easily write as '$\neg P \vee \neg Q$' using the 'or' proof symbol. So the negation of 'x is even and less than 10' is 'x is not even or not less than 10', so 'x is odd or greater than or equal to 10'.

> To negate a compound statement, negate each of the original statements and swap 'and' for 'or'.

You might come across this as DeMorgan's Laws in some texts.

Example 3

Find the negation of the following:

a $\forall$ integers $x, 2x$ is even and $2x+1$ is odd

b $\exists$ a real number $x, x^2 = 9$ or $x > 2$

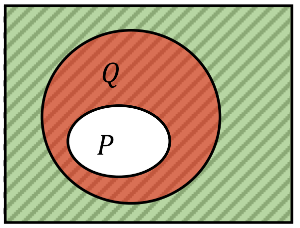
Solution

a $\exists$ integers $x, 2x$ is odd or $2x+1$ is even

b $\forall$ a real number $x, x^2 \neq 9$ and $x \leq 2$

CONTRAPOSITIVE

The contrapositive of the conditional statement 'If P then Q' is 'If not Q then not P', so the negation of Q implies the negation of P. Using symbols the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. The contrapositive is true if and only if the statement itself is also true.



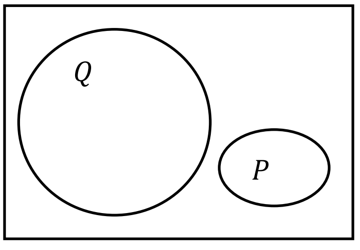
$\neg Q$ green shading

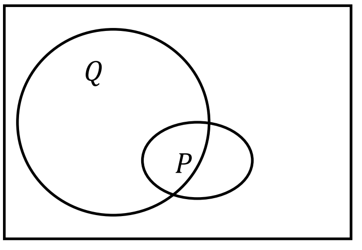
$\neg P$ diagonal stripes

It is hard to draw this clearly on an Euler diagram, but we can see above that if $P \Rightarrow Q$ then P is inside Q, so $\neg Q$ (shaded in green) must all be within $\neg P$ (diagonal stripes).

So '$\neg Q \Rightarrow \neg P$' is true when '$P \Rightarrow Q$' is true.

Now if the original statement is false, then P is either completely outside Q (below left) or overlaps Q (below right).





In each case $\neg Q$ is not completely inside $\neg P$. So when '$P \Rightarrow Q$' is false then '$\neg Q \Rightarrow \neg P$' is also false.

So we have seen is that if:

- $P \Rightarrow Q$ is true then $\neg Q \Rightarrow \neg P$ is true
- $P \Rightarrow Q$ is false then $\neg Q \Rightarrow \neg P$ is false.

We can say that:

> The original statement and its contrapositive are logically equivalent, so to prove a conditional statement we can prove its contrapositive instead.

Example 4

Find the contrapositive of the following:

a If x is even then $x + 1$ is odd.

b If $x^2 = 25$ then $x = \pm 5$

Solution

a If $x + 1$ is even then x is odd.

b If $x \neq \pm 5$ then $x^2 \neq 25$

PROOF BY CONTRAPOSITIVE

Now the syllabus states that students need to ‘use proof by contradiction’, but there is no similar statement about using proof by contrapositive. The only mention of contrapositive is that students ‘understand that a statement is equivalent to its contrapositive’ as we have just seen.

Proof by Contrapositive is generally considered easier than proof by contradiction, so I would certainly get students to use it and expect to see it in exams. It is just a pity the wording of the syllabus isn’t clearer!

In general we should always try direct proof first, then proof by contrapositive (if it is a conditional statement), then proof by contradiction as a last resort.

Example 5

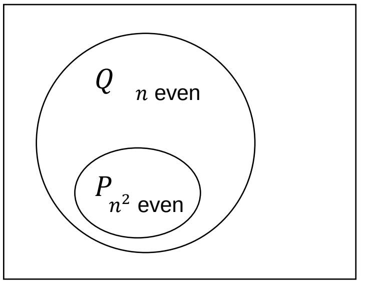
Prove by contrapositive for integral n that if n^2 is even then n is even.

Solution

Suppose n is odd.
Let $n = 2k + 1$ for integral k.

$$\begin{aligned}\therefore n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2j + 1 \quad \text{for integral } j \text{ since } 2, k \text{ are integral}\end{aligned}$$

∴ if n is odd then n^2 is odd.
∴ if n^2 is even then n is even by contrapositive.



Example 6

Prove by contrapositive that if mn is even then m or n must be even.

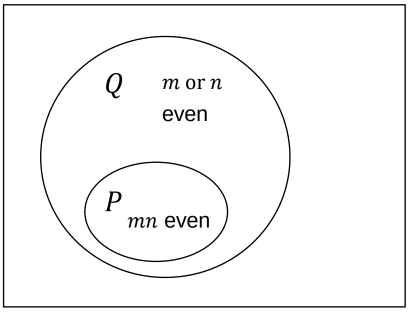
Solution

Note that we are negating a compound statement, so ‘or’ is replaced with ‘and’ and ‘even’ is replaced with ‘odd’.

Suppose m is odd and n is odd.
Let $m = 2k + 1, n = 2j + 1$ for integral k, j

$$\begin{aligned}mn &= (2k+1)(2j+1) \\ &= 4kj + 2k + 2j + 1 \\ &= 2(2kj + k + j) + 1 \\ &= 2p + 1 \quad \text{for integral } p \text{ since } 2, j, k \text{ are integral}\end{aligned}$$

∴ if m and n are both odd then mn is odd.
∴ if mn is even then m or n must be even.



EXERCISE 1.2

BASIC

1 Find the negation of the following:

a $x \leq 1$ **b** $x = 1$ **c** $x > 1$

2 Find the negation of the following:

a $\exists$ an integer $x, 2x = x^2$ **b** $\forall$ real numbers $x, x < 2m + 1$ for integral m

3 Find the negation of the following:

a $\forall$ integers $x, 2x$ is prime or $2x+1$ is odd **b** $\exists$ a real number $x, x^2 - 7 = 9$ and $x > 2$

4 Find the contrapositive of the following:

a If x is prime then $2x + 1$ is composite. **b** If $x^2 = 1$ then $\frac{1}{x} = 1$

5 Prove by contrapositive for integral n that if n^2 is odd then n is odd.

6 Prove by contrapositive for m, n positive integers that if mn is divisible by 5 then m or n must be divisible by 5.

7 Prove by contrapositive that if m is an integer and m^2 is not divisible by 4 then m is odd.

MEDIUM

8 Prove by contrapositive that if $\frac{mn}{2}$ is integral then m or n must be even.

1 **a** $x > 1$ **b** $x \neq 1$ c) $x \leq 1$

2 **a** $\forall$ integers $x, 2x \neq x^2$ **b** $\exists$ real numbers $x, x \geq 2m + 1$for integral m

3 **a** $\exists$ an integer x, $2x$ is not prime and $2x$+1 is even **b** $\forall$ real numbers $x, x^2 - 7 \neq 9$ or $x \leq 2$

4 **a** If $2x + 1$ is not composite then x is not prime **b** If $\frac{1}{x} \neq 1$ then $x^2 \neq 1$.

5 Suppose n is even.

Let $n = 2k$ for integral k.

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p$ for integral p

$\therefore$ if n is even then n^2 is even.

$\therefore$ if n^2 is odd then n is odd by contrapositive.

6 Suppose neither m nor n are divisible by 5.

Let $m = 5j + p$ and $n = 5k + q$ for integral j, k and $p, q = 1,2,3$ or 4.

$mn = (5j + p)(5k + q)$

$= 25jk + 5jq + 5kp + pq$

$= 5(5jk + jq + kp) + pq$

$= 5c + pq$ for integral c since j, k, p, q are integral

Now pq=1, 2, 3, 4, 6, 8, or 12, so none a multiple of 5, so mn is not a multiple of 5.

$\therefore$ if m and n are not divisible by 5 then mn is not divisible by 5.

$\therefore$ if mn is divisible by 5 then m or n must be divisible by 5 by contrapositive.

7 Suppose m is even.
Let $m = 2k$ for integral k
$m^2 = (2k)^2$
$= 4k^2$
$= 4p$ for integral p
$\therefore$ if m is even then m^2 is divisible by 4.
$\therefore$ if m^2 is not divisible by 4 then m must be odd by contrapositive.

8 Suppose m is odd and n is odd.

Let $m = 2k + 1, n = 2j + 1$ for integral k, j

$$\frac{mn}{2} = \frac{(2k+1)(2j+1)}{2}$$

$$= \frac{4kj + 2k + 2j + 1}{2}$$

$$= 2kj + k + j + \frac{1}{2}$$

Which is not integral since j and k are integral

$\therefore$ if m and n are both odd then $\frac{mn}{2}$ is not integral.

$\therefore$ if $\frac{mn}{2}$ is integral then m or n must be even.