EXERCISE 2.10

BASIC

1 Find the fifth roots of 1 by sketching the roots first.

2 Find the fifth roots of 1 using de Moivre's theorem.

3 Prove that if ω is a non-real cube root of unity, then $(1-\omega+\omega^2)^4 = 16\omega$

4 Find the cube roots of -1 in polar form.

5 Find the cube roots of i

6 Find the eighth roots of 1.

MEDIUM

7 Find the fourth roots of $8\sqrt{2}-8\sqrt{2}i$, leaving answers in exponential form.

8 Find the cube roots of 8 in Cartesian form.

9 Find the remainder when $P(x)$ is divided by $x+i$ if $P(x) = (x^2+1)Q(x)+4x-2$

10 If w is a non-real cube root of unity prove

$$\frac{1}{1+w}-\frac{1}{1+w^2} = -(1+2w)$$

11 If ω is a non-real cube root of unity, prove that $(a-b)(a-\omega b)(a-\omega^2 b) = a^3-b^3$

12 Given $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$ let w be a solution to $z^5-1=0$ where $w \neq -1$.

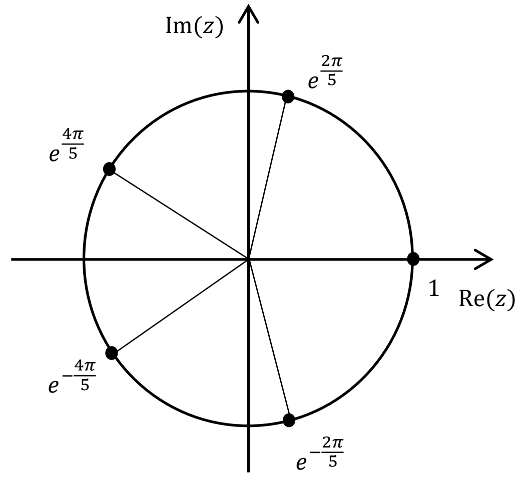
a Prove that $1+w^2+w^4 = -(w+w^3)$

b Hence show that $\cos\frac{2\pi}{5}-\cos\frac{\pi}{5} = -\frac{1}{2}$

SOLUTIONS - EXERCISE 2.10

1

1 is one of the roots, and the others are spread evenly around the unit circle, so there is $\frac{2\pi}{5}$ between each.



$\therefore$ the roots are $e^{-\frac{4\pi}{5}}, e^{-\frac{2\pi}{5}}, 1, e^{\frac{2\pi}{5}}, e^{\frac{4\pi}{5}}$

2

Let a root be $z = r(\cos\theta + i\sin\theta)$

$$\therefore z^5 = 1$$

$$r^5(\cos\theta + i\sin\theta)^5 = \cos 2k\pi + i\sin 2k\pi \quad \text{for } k = -2, -1, 0, 1, 2$$

$$r^5(\cos 5\theta + i\sin 5\theta) = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore r^5 = 1 \quad \text{and } 5\theta = 2k\pi$$

$$\therefore r = 1 \quad \text{and } \theta = \frac{2k\pi}{5}$$

$$\therefore z = \operatorname{cis}\left(-\frac{4\pi}{5}\right), \operatorname{cis}\left(-\frac{2\pi}{5}\right), 1, \operatorname{cis}\left(\frac{2\pi}{5}\right), \operatorname{cis}\left(\frac{4\pi}{5}\right)$$

3

$1 + \omega + \omega^2 = 0$

$\therefore (1 - \omega + w^2)^4$

$= (1 + \omega + \omega^2 - 2\omega)^4$

$= (-2\omega)^4$

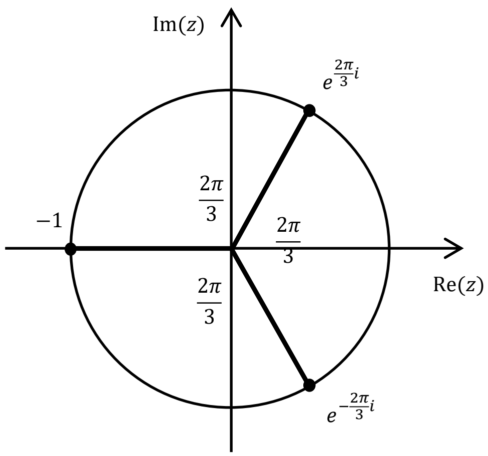
$= 16\omega^4$

$= 16\omega^3\omega$

$= 16\omega$

4

-1 is a cube root of -1, and the other roots must be spaced by $\frac{2\pi}{3}$ as shown. The cube roots of -1 are $-1, e^{\frac{2\pi}{3}i}$ and $e^{-\frac{2\pi}{3}i}$.



The third root could be written as $e^{\frac{5\pi}{3}i}$ instead.

5

For i $r = 1, \theta = \frac{\pi}{2}$

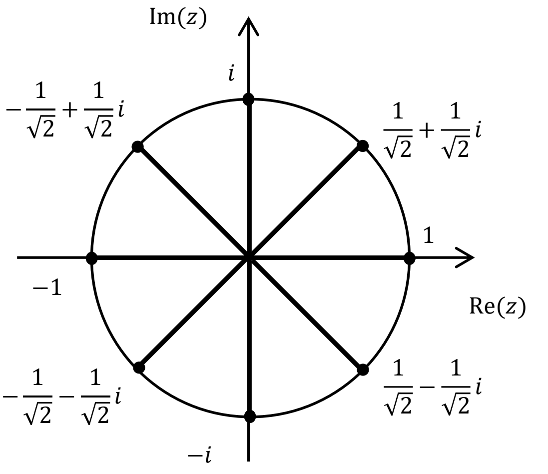
The first root has modulus $\sqrt[3]{1} = 1$ and argument $\frac{\pi}{2} \div 3 = \frac{\pi}{6}$.

The other roots have arguments which differ by $\frac{2\pi}{3} = \frac{4\pi}{6}$

$\therefore$ the fifth roots of i are $e^{\frac{\pi}{6}i}, e^{\frac{5\pi}{6}i}, e^{\frac{9\pi}{6}i}$ which simplify to $e^{\frac{\pi}{6}i}, e^{\frac{5\pi}{6}i}, -i$.

6

1 is an eighth root of 1, and the other roots must be spaced by $\frac{2\pi}{8} = \frac{\pi}{4}$ as shown. The eighth roots of 1 are $\pm 1, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, \pm i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$.



Alternatively in exponential form the roots are

$1, e^{\pm\frac{\pi}{4}i}, \pm i, e^{\pm\frac{3\pi}{4}i}$

7

For $4 - i$, $r = \sqrt{(8\sqrt{2})^2 + \left(-8\sqrt{2}\right)^2} = 16$ and $\theta = -\frac{\pi}{4}$

The first root has modulus $\sqrt[4]{16} = 2$ and argument $-\frac{\pi}{4} \div 4 = -\frac{\pi}{16}$.

The other roots have arguments which differ by $\frac{2\pi}{4} = \frac{8\pi}{16}$

$\therefore$ the cube roots of i are $2e^{-\frac{\pi}{16}i}, 2e^{\frac{7\pi}{16}i}, 2e^{\frac{15\pi}{16}i}, 2e^{\frac{23\pi}{16}i}$

8

Let z be a cube root of 8

$$\therefore z^3 = 8$$

$$\therefore z^3 - 8 = 0$$

$$(z-2)(z^2+2z+4) = 0$$

$z - 2 = 0$ or $z^2 + 2z + 4 = 0$

$$z = 2 \qquad z = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

The three cube roots of 8 are 2 and $-1 \pm \sqrt{3}i$

9

$$P(-i) = \left((-i)^2+1\right)Q(-i) + 4(-i) - 2$$

$$= (-1+1)Q(i) - 4i - 2$$

$$= -4i - 2$$

10

$$\frac{1}{1+w}-\frac{1}{1+w^2}$$

$$=\frac{1+w^2-1-w}{1+w^2+1+w}$$

$$=\frac{w^2-w}{1+w+w^2+1}$$

$$=\frac{1+w+w^2-1-2w}{0+1}$$

$$=-(1+2w) \quad \square$$

11

$(a-b)(a-\omega b)(a-\omega^2 b)$

$=(a-b)(a^2-ab\omega^2-ab\omega+b^2\omega^3)$

$=(a-b)(a^2-ab(1+\omega+\omega^2-1)+b^2)$

$=(a-b)(a^2-ab(0-1)+b^2)$

$=(a-b)(a^2+ab+b^2)$

$=a^3-b^3 \quad \square$

12a

$w^5-1=0$

$\therefore (w-1)(w^4+w^3+w^2+w+1)=0$

$\therefore w^4+w^3+w^2+w+1=0 \qquad \text{since } w\neq -1$

$\therefore 1+w^2+w^4=-(w+w^3)$

b

Let $z=r\operatorname{cis}\theta$

$$z^5-1=0$$

$$z^5=1$$

$$\therefore r^5\operatorname{cis}^5\theta=\operatorname{cis}2k\pi$$

$$r^5(\cos 5\theta+i\sin 5\theta)=\cos(2k\pi)+i\sin(2k\pi)$$

$$\therefore r=1 \text{ and } 5\theta=(2k\pi)$$

$$\theta=\frac{2k\pi}{5}$$

$$=0,\pm\frac{2\pi}{5},\pm\frac{4\pi}{5}$$

Let $w=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

$$\therefore 1+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^2+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^4=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^3\right)$$

$$1+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right)$$

$$1-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}\right)$$

$$1-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=-\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}+\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$$

$$\therefore 1-2\cos\frac{\pi}{5}+2\cos\frac{2\pi}{5}=0$$

$$\therefore \cos\frac{2\pi}{5}-\cos\frac{\pi}{5}=-\frac{1}{2}$$