EXERCISE 4.8

BASIC

Use Integration by parts to find/evaluate the following integrals, unless told otherwise.

1 $\int x^2 \ln x \, dx$

2 $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$

3 $\int x^3 \sin x^2 \, dx$

4 $\int \ln x \, dx$

Hint: let $u = \ln x$ and $\frac{dv}{dx} = 1$.

You may wish to memorise this result.

5 $\int xe^{2x} \, dx$

6 $\int_0^{\pi} x \cos x \, dx$

7 $\int_0^2 te^{-t} \, dt$

8 $\int_0^1 \tan^{-1} x \, dx$

MEDIUM

9 $\int \ln|1 + x| \, dx$

10 $\int x2^x \, dx$

CHALLENGING

11 Find $\int x^2\sqrt{x - 1} \, dx$

i Using IBP **ii** Using a u^2 substitution

12 Find $\int \frac{\ln x - 2}{(\ln x - 1)^2} dx$

i Using IBP

ii By using the quotient rule to find a function whose derivative equals the integrand

13 Using the result from Q4, repeat Q1 but ignore DETAIL and let $u = x^2$ and $\frac{dv}{dx} = \ln x$

14 $\int x \sin x \cos x \, dx$

15 $\int \frac{xe^x}{(1 + x)^2} dx$

1

$$\int x^2 \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = x^2$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \times \frac{1}{x} dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

2

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$u = \cos x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = -\sin x \quad v = e^x$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left(\left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx\right)$$

$$u = \sin x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = \cos x \qquad v = e^x$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \left[e^x \sin x\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx = \frac{1}{2}\left(\left(0 - 1\right) + \left(e^{\frac{\pi}{2}} - 0\right)\right)$$

$$= \frac{e^{\frac{\pi}{2}} - 1}{2}$$

3

$$\int x^3 \sin x^2 \, dx$$

$$u = \frac{x^2}{2} \qquad \frac{dv}{dx} = 2x \sin x^2$$
$$\frac{du}{dx} = x \qquad v = -\cos x^2$$

$$= -\frac{x^2 \cos x^2}{2} + \int x \cos x^2 \, dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{1}{2}\int 2x \ \cos x^2 \, dx$$

$$= -\frac{x^2 \cos x^2}{2} + \frac{\sin x^2}{2} + c$$

4

$$\int \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$= x \ln x - \int x \times \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

5

$$\int x e^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$\boxed{\begin{array}{ll} u = x & dv = e^{2x} \\ \dfrac{du}{dx} = 1 & v = \dfrac{1}{2}e^{2x} \end{array}}$$

6

$$\int_0^{\pi} x\cos x\,dx$$

$$= \Big[x\sin x\Big]_0^{\pi} - \int_0^{\pi} \sin x\,dx$$

$$= 0 - 0 + \Big[\cos x\Big]_0^{\pi}$$

$$= -1 - 1$$

$$= -2$$

$$\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = \cos x \\ \dfrac{du}{dx} = 1 & v = \sin x \end{array}}$$

7

$$\int_0^{2} te^{-t}\,dt$$

$$= -\Big[te^{-t}\Big]_0^{2} + \int_0^{2} e^{-t}\,dt$$

$$= -\frac{2}{e^2} - \Big[e^{-t}\Big]_0^{2}$$

$$= -\frac{2}{e^2} - \frac{1}{e^2} + 1$$

$$= 1 - \frac{3}{e^2}$$

$$\boxed{\begin{array}{ll} u = t & \dfrac{dv}{dt} = e^{-t} \\ \dfrac{du}{dx} = 1 & v = -e^{-t} \end{array}}$$

8

$$\int_0^{1} \tan^{-1} x\,dx$$

$$= \Big[x\tan^{-1} x\Big]_0^{1} - \int_0^{1} \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - 0 - \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^{1}$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

$$\boxed{\begin{array}{ll} u = \tan^{-1} x & dv = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}}$$

9

$$\int \ln(1+x)\,dx$$

$$\boxed{\begin{array}{ll} u = \ln(1+x) & dv = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x} & v = x \end{array}}$$

$$= x\ln(1+x) - \int \frac{x}{1+x}\,dx$$

$$= x\ln(1+x) - \int \frac{1+x-1}{1+x}\,dx$$

$$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\,dx$$

$$= x\ln(1+x) - x + \ln(1+x) + c$$

$$= (x+1)\ln(1+x) - x + c$$

10

$$\int x2^x\,dx$$

$$\boxed{\begin{array}{ll} u = x & \dfrac{dv}{dx} = 2^x \\ \dfrac{du}{dx} = 1 & v = \dfrac{2^x}{\ln 2} \end{array}}$$

$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx$$

$$= \frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right) + c$$

$$= \frac{x2^x}{\ln 2} - \frac{2^x}{\ln^2 2} + c$$

11

i $\int x^2\sqrt{x-1}\,dx$

$$\boxed{\begin{array}{ll} u = x^2 & dv = (x-1)^{\frac{1}{2}} \\ \dfrac{du}{dx} = 2x & v = \dfrac{2}{3}(x-1)^{\frac{3}{2}} \end{array}}$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\int x(x-1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\left[\frac{2x(x-1)^{\frac{5}{2}}}{5} - \frac{2}{5}\int (x-1)^{\frac{5}{2}}\,dx\right]$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8x(x-1)^{\frac{5}{2}}}{15} + \frac{16}{105}(x-1)^{\frac{7}{2}} + c$$

$$= \frac{2x^2\sqrt{(x-1)^3}}{3} - \frac{8x\sqrt{(x-1)^5}}{15} + \frac{16\sqrt{(x-1)^7}}{105} + c$$

ii $\int x^2\sqrt{x-1}\,dx$

$$\boxed{\begin{array}{c} u^2 = x-1 \\ 2u\,du = dx \end{array}}$$

$$= \int x^2u \times 2u\,du$$

$$= 2\int (u^2+1)^2u^2\,du$$

$$= 2\int (u^6 + 2u^4 + u^2)\,du$$

$$= \frac{2u^7}{7} + \frac{4u^5}{5} + \frac{2u^3}{3} + c$$

$$= \frac{2\sqrt{(x-1)^7}}{7} + \frac{4\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + c$$

12

i $\int \frac{\ln x - 2}{(\ln x - 1)^2}\,dx$

$u = x(\ln x - 2)$	$\frac{dv}{dx} = \frac{1}{x}(\ln x - 1)^{-2}$
$\frac{du}{dx} = x\left(\frac{1}{x}\right) + (\ln x - 2)(1)$	$v = -(\ln x - 1)^{-1}$
$= \ln x - 1$	$= -\frac{1}{\ln x - 1}$

$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int \frac{\ln x - 1}{\ln x - 1}\,dx$$

$$= -\frac{x(\ln x - 2)}{\ln x - 1} + \int dx$$

$$= \frac{2x - x\ln x}{\ln x - 1} + x + c$$

$$= \frac{2x - x\ln x}{\ln x - 1} + \frac{x\ln x - x}{\ln x - 1} + c$$

$$= \frac{x}{\ln x - 1} + c$$

ii

$$\therefore \int \frac{\ln x - 2}{(\ln x - 1)^2}\,dx = \int \frac{\ln x - 1 - 1}{(\ln x - 1)^2}\,dx$$

$$= \int \frac{(\ln x - 1)(1) - (x)\left(\frac{1}{x}\right)}{(\ln x - 1)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{x}{\ln x - 1}\right)dx$$

$$= \frac{x}{\ln x - 1} + c$$

13

$u = x^2$	$\frac{dv}{dx} = \ln x$
$\frac{du}{dx} = 2x$	$v = x\ln x - x$

$$\int x^2 \ln x\,dx = x^2(x\ln x - x) - 2\int x(x\ln x - x)\,dx$$

$$= x^3\ln x - x^3 - 2\int (x^2\ln x - x^2)\,dx$$

$$= x^3\ln x - x^3 - 2\int x^2\ln x\,dx + 2\int x^2 dx$$

$$\therefore 3\int x^2\ln x\,dx = x^3\ln x - x^3 + \frac{2x^3}{3} + c$$

$$\int x^2\ln x\,dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$$

14

$$I = \int x\sin x\cos x\,dx$$

$$= x\sin^2 x - \int (\sin^2 x + x\sin x\cos x)\,dx$$

$$= x\sin^2 x - \int \sin^2 x\,dx - \int x\sin x\cos x\,dx$$

$$\therefore 2I = x\sin^2 x - \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$I = \frac{x\sin^2 x}{2} - \frac{1}{4}\left(x - \frac{1}{2}\sin 2x\right) + c$$

$$= \frac{x\sin^2 x}{2} - \frac{x}{4} + \frac{\sin 2x}{8} + c$$

$u = x\sin x$	$\frac{dv}{dx} = \cos x$
$\frac{du}{dx} = \sin x + x\cos x$	$v = \sin x$

15

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

$$= -\frac{xe^x}{1+x} + \int e^x\,dx$$

$$= -\frac{xe^x}{1+x} + e^x + c$$

$$= \frac{-xe^x + e^x + xe^x}{1+x} + c$$

$$= \frac{e^x}{1+x} + c$$

$u = xe^x$	$\frac{dv}{dx} = \frac{1}{(1+x)^2}$
$\frac{du}{dx} = xe^x + e^x$	$v = -\frac{1}{1+x}$
$\quad = e^x(x+1)$	

Alternatively

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

$$= \int \frac{(1+x)e^x - e^x}{(1+x)^2}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{e^x}{1+x}\right)dx$$

$$= \frac{e^x}{1+x} + c$$