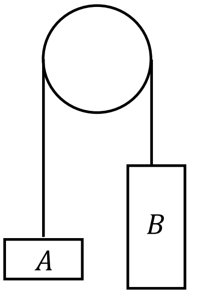
EXERCISE 6.2

BASIC

1 A particle of mass m is moving in a straight line under the action of a force, $F = \phi m(x-2)$, where ϕ is a positive constant. Given the particle starts from rest at the origin, prove $v = -\sqrt{\phi(x^2-4x)}$.

2 Particle A of mass m kg and Particle B of mass $5m$ kg are connected by a light inextensible string passing over a frictionless pulley. Initially the particles are at rest. After Particle A has travelled x metres in an upwards direction it is travelling at v metres per second.



Prove $v = \sqrt{\dfrac{4gx}{3}}$

3 A body is projected vertically downwards from a height of $2R$ (from the centre of the Earth) with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth. Let the radius of the Earth be R, and the acceleration due to gravity at the surface be $-g$.

Prove that the speed at any position x is given by $v^2 = u^2 + 2gR^2\left(\dfrac{1}{x} - \dfrac{1}{2R}\right)$

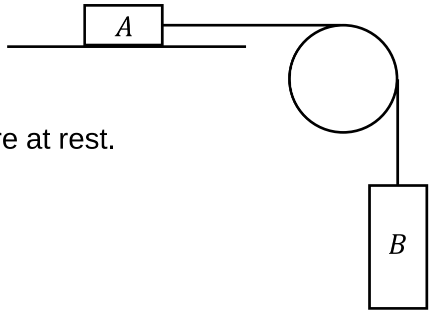
4 A particle of mass m is moving in a straight line under the action of a force, $F = \frac{m}{x^3}(6+10x)$. Find an expression for velocity as a function of its displacement x, if the particle starts from rest at $x = 1$.

5 A $20\sqrt{2}$ kilogram box sits on a slippery ramp which is inclined at an angle of $45°$ to the horizontal. If the box starts from rest, find:

a its velocity after t seconds.

b its displacement after t seconds

6 Particle A of mass m kg and Particle B of mass $2m$ kg are connected by a light inextensible string passing over a frictionless pulley. Particle A is on a horizontal surface, with constant friction F*. Initially the particles are at rest. After Particle A has travelled x metres to the right it is travelling at v metres per second.



Given $v = \sqrt{\frac{gx}{2}}$, find an expression for F as a function of m and g.

* Friction isn't constant, so this is not a real world example, but good integration practice!

7 A particle of mass m moves in a straight line under the action of a resultant force F where $F = F(x)$. Given that the velocity v is v_0 when the position x is x_0, and that v is v_1 when x is x_1, prove $|v_1| = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)\, dx + v_0^2}$

CHALLENGING

8 A particle is initially at rest at the point B which is b metres to the right of O. The particle then moves in a straight line towards O. For $x \neq 0$, the acceleration of the particle is given by $-\frac{\mu^2}{x^2}$, where x is the distance from O and μ is a positive constant.

i Prove that $\frac{dx}{dt} = -\mu\sqrt{2}\sqrt{\frac{b-x}{bx}}$

ii Using the substitution $x = b\cos^2\theta$, show that the time taken to reach a distance d metres to the right of O is given by $t = \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\, d\theta$

iii It can be shown that $t = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(\sqrt{bd - d^2} + b\cos^{-1}\sqrt{\frac{d}{b}}\right)$ (Do NOT prove this.)

What is the limiting time taken for the particle to reach O?

9 In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3}, where x is the distance between their centres. A particle is projected upward from the surface of the planet with velocity u at time $t = 0$. Its distance x from the centre of the planet satisfies the equation $\ddot{x} = -\frac{k}{x^3}$.

i Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet.

ii Show that v, the velocity of the particle, is given by $v^2 = \dfrac{gR^3}{x^2} - (gR - u^2)$

iii It can be shown that $x = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$. (Do NOT prove this.)

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet.

iv If $u < \sqrt{gR}$ the particle reaches a point whose distance from the centre of the planet is D, and then falls back.

(α) Use the formula in part (ii) to find D in terms of u, R and g.

(β) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u, R and g.

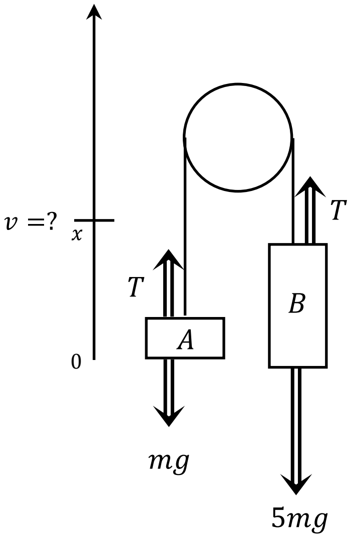
1

$$m\ddot{x} = \phi m(x-2)$$
$$\ddot{x} = \phi(x-2)$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \phi(x-2)$$
$$\frac{1}{2}v^2 = \phi\int_0^x (x-2)\,dx$$
$$v^2 = 2\phi\left[\frac{x^2}{2} - 2x\right]_0^x$$
$$\therefore v^2 = \phi x^2 - 4\phi x$$
$$v = -\sqrt{\phi(x^2-4x)}$$

negative root since initially $\ddot{x} < 0$ and $\dot{x} = 0$ at the origin so the particle moves left.

2

$$(m+5m)\ddot{x} = (5mg - T) - (mg - T)$$
$$6m\ddot{x} = 4mg$$
$$\ddot{x} = \frac{2g}{3}$$
$$v\frac{dv}{dx} = \frac{2g}{3}$$
$$\frac{dv}{dx} = \frac{2g}{3v}$$
$$\frac{dx}{dv} = \frac{3v}{2g}$$
$$x = \frac{3}{2g}\int_0^v v\,dv$$
$$= \frac{3}{4g}\left[v^2\right]_0^v$$
$$= \frac{3v^2}{4g}$$
$$v^2 = \frac{4gx}{3}$$
$$v = \sqrt{\frac{4gx}{3}} \quad \text{since Particle A only moves up}$$



3

$$\ddot{x} = -\frac{k}{x^2}$$
when $x = R$ $\ddot{x} = -g$
$$-g = -\frac{k}{R^2}$$
$$k = gR^2$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$$
$$= -gR^2x^{-2}$$
$$\therefore \frac{1}{2}v^2 - \frac{1}{2}u^2 = -gR^2\int_{2R}^x x^{-2}\,dx$$
$$v^2 - u^2 = -2gR^2\left[\frac{x^{-1}}{-1}\right]_{2R}^x$$
$$v^2 = 2gR^2\left(\frac{1}{2R} - \frac{1}{x}\right) + u^2$$
$$= u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{2R}\right)$$

4

$$a = \frac{F}{m}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{-3} + 10x^{-2}$$

$$\frac{1}{2}v^2 = \int_1^x (6x^{-3} + 10x^{-2})\,dx$$

$$v^2 = 2\left[-3x^{-2} - 10x^{-1}\right]_1^x$$

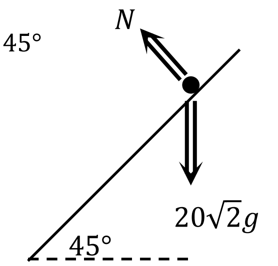
$$= 2\big((-3x^{-2} - 10x^{-1}) - (-3 - 10)\big)$$

$$= \frac{2(-3 - 10x + 13x^2)}{x^2}$$

$$v = \frac{1}{x}\sqrt{26x^2 - 20x - 6}$$

positive root since initially $F > 0$ and $\dot{x} = 0$ so the particle moves to the right.

5 **a**



$$20\sqrt{2}\ddot{x} = 20\sqrt{2}g \times \sin 45°$$

$$\frac{dv}{dt} = \frac{g}{\sqrt{2}}$$

$$v = \frac{g}{\sqrt{2}}\int_0^t dt$$

$$= \frac{gt}{\sqrt{2}}$$

b

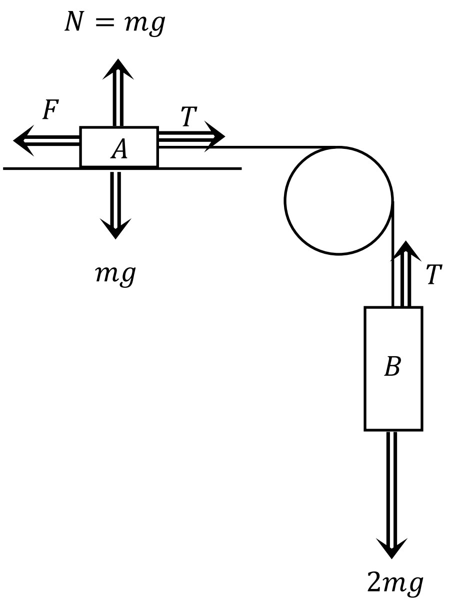
$$\therefore \frac{dx}{dt} = \frac{gt}{\sqrt{2}}$$

$$x = \frac{g}{\sqrt{2}}\int_0^t t\,dt$$

$$= \frac{g}{2\sqrt{2}}\left[t^2\right]_0^t$$

$$= \frac{gt^2}{2\sqrt{2}}$$

6



$$(m + 2m)\ddot{x} = (2mg - T) - (F - T)$$

$$3m\ddot{x} = 2mg - F$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{2mg - F}{3m}$$

$$\frac{d}{dx}\left(\frac{gx}{4}\right) = \frac{2mg - F}{3m}$$

$$\frac{g}{4} = \frac{2mg - F}{3m}$$

$$3mg = 8mg - 4F$$

$$4F = 5mg$$

$$F = \frac{5mg}{4}$$

7

$$m\ddot{x} = F(x)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{m} \times F(x)$$

integrating from $x = x_0, v^2 = v_0^2$

to $x = x_1, v^2 = v_1^2$

$$\frac{1}{2}\int_{v_0^2}^{v_1^2} v^2\ dv^2 = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$

$$\frac{1}{2}\left[v^2\right]_{v_0^2}^{v_1^2} = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$

$$\frac{1}{2}v_1^2 - \frac{1}{2}v_0^2 = \frac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$$

$$v_1^2 - v_0^2 = \frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx$$

$$v_1^2 = \frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + v_0^2$$

$$|v_1| = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + v_0^2}$$

8 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{\mu^2}{x^2}$$
$$\frac{1}{2}v^2 = -\mu^2 \int_b^x \frac{1}{x^2}\,dx$$
$$\frac{1}{2}v^2 = \mu^2\left[\frac{1}{x}\right]_b^x$$
$$= \mu^2\left(\frac{1}{x} - \frac{1}{b}\right)$$
$$v^2 = 2\mu^2\left(\frac{b-x}{bx}\right)$$
$$\frac{dx}{dt} = -\mu\sqrt{2}\sqrt{\frac{b-x}{bx}}$$

Negative root since particle initially moves to the left.

ii

$$\frac{dt}{dx} = -\frac{1}{\mu\sqrt{2}}\sqrt{\frac{bx}{b-x}}$$
$$t = -\frac{1}{\mu\sqrt{2}}\int_b^d \sqrt{\frac{bx}{b-x}}\,dx \qquad \boxed{\begin{aligned} x &= b\cos^2\theta \\ dx &= -2b\cos\theta\sin\theta\,d\theta \end{aligned}}$$
$$= -\frac{1}{\mu\sqrt{2}}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b^2\cos^2\theta}{b - b\cos^2\theta}}\,(-2b\cos\theta\sin\theta\,d\theta)$$
$$= \frac{2b}{\mu\sqrt{2}}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \sqrt{\frac{b\cos^2\theta}{\sin^2\theta}}\cos\theta\sin\theta\,d\theta$$
$$= \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\,d\theta$$

iii

Let $d = 0$ in the given equation

$$t = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(0 + b\left(\frac{\pi}{2}\right)\right)$$
$$= \frac{b\sqrt{b}\pi}{2\sqrt{2}\mu}\ sec$$

9 i When $x = R,\ \ddot{x} = -g$

$$\therefore -g = -\frac{k}{R^3}$$
$$k = gR^3$$

ii $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^3}{x^3}$

$$\frac{1}{2}v^2 - \frac{1}{2}u^2 = -gR^3\int_R^x \frac{1}{x^3}\,dx$$
$$v^2 - u^2 = 2gR^3\left[\frac{1}{x^2}\right]_R^x$$
$$v^2 = 2gR^3\left(\frac{1}{x^2} - \frac{1}{R^2}\right) + u^2$$
$$= \frac{gR^3}{x^2} - (gR - u^2)$$

iii Let $x = R$

$R = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$

$R^2 = R^2 + 2uRt - (gR - u^2)t^2$

$\therefore t(2uR - (gR - u^2)t) = 0$

$\therefore t_1 = 0$ or $2uR - (gR - u^2)t = 0$

$$t_2 = \frac{2uR}{gR - u^2}$$

now $0 < \sqrt{gR} \leq u$ so the numerator is positive and the denominator negative so $t_2 < 0$ and is not a valid solution. The particle does not return to the planet.

$\boldsymbol{iv}\ \boldsymbol{\alpha}\ let\ v = 0$

$$0 = \frac{gR^3}{D^2} - (gR - u^2)$$

$$\frac{gR^3}{D^2} = gR - u^2$$
$$D^2 = \frac{gR^3}{gR - u^2}$$
$$D = \sqrt{\frac{gR^3}{gR - u^2}}$$

$\boldsymbol{\beta}\ \therefore \sqrt{\frac{gR^3}{gR - u^2}} = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$

$$gR^3 = gR^3 - u^2R^2 + 2uR(gR - u^2)t - (gR - u^2)^2t^2$$
$$\therefore (gR - u^2)^2t^2 - 2uR(gR - u^2)t + u^2R^2 = 0$$
$$\left((gR - u^2)t - uR\right)^2 = 0$$

$$t = \frac{uR}{gr - u^2}$$