EXERCISE 2.4

BASIC

1 Simplify $3(4e^{3i})$

2 Simplify $-(4e^{3i})$

3 Simplify **a** $i(2e^{\frac{\pi}{3}i})$ **b** $-i\left(3e^{-\frac{\pi}{2}i}\right)$

4 Simplify $2e^{-2i} \times 3e^{i}$

5 Simplify **a** $\left(3e^{i}\right)^3$ **b** $\left(3e^{-i}\right)^2$

6 Simplify $4e^{\frac{\pi}{2}i} \div 2$

7 Simplify **a** $2e^{\frac{\pi}{3}i} \div i$ **b** $5e^{-\frac{\pi}{2}i} \div (-i)$

8 Simplify $2e^{-2i} \div 4e^{i}$

9 Find the conjugate of $z = 3e^{-\frac{\pi}{3}i}$

10 Convert the following complex numbers from polar form to exponential form:

a $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **b** $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

11 Convert the following complex numbers from Cartesian form into exponential form:

a $1 + i$ **b** $-\sqrt{3} + i$

12 Convert the following complex numbers from exponential into Cartesian form:

a $e^{\frac{2\pi}{3}i}$ **b** $e^{-\frac{\pi}{4}i}$

MEDIUM

13 Prove $e^{ix} = \cos x + i\sin x$ using the Power Series

14 Prove $e^{i\pi} + 1 = 0$

15 Simplify each expression and mark on a complex plane

a i^{-i} **b** $\sqrt{e^{-i\pi}}$

16 Prove $(-1)^{\frac{1}{n}} = e^{\frac{\pi}{n}i}$

17 Find the square roots of $9e^{\frac{\pi}{3}i}$

CHALLENGING

18 Prove $e^{ix} = \cos x + i\sin x$ using the Maclaurin Series

1 $3(4e^{3i}) = 12e^{3i}$

2 $-(4e^{3i}) = e^{-i\pi} \times 4e^{3i} = 4e^{(3-\pi)i}$

3
a $i\left(2e^{\frac{\pi}{3}i}\right) = e^{\frac{\pi}{2}i} \times 2e^{\frac{\pi}{3}i} = 2e^{\left(\frac{\pi}{2}+\frac{\pi}{3}\right)i} = 2e^{\frac{5\pi}{6}i}$
b $-i\left(3e^{-\frac{\pi}{2}i}\right) = e^{-\frac{\pi}{2}i} \times 3e^{-\frac{\pi}{2}i} = 3e^{\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)i} = 3e^{-i\pi} = 3 \times -1 = -3$

4 $2e^{-2i} \times 3e^{i} = 6e^{-i}$

5
a $\left(3e^{i}\right)^3 = 3^3 \times e^{i\times 3} = 27e^{3i}$ **b** $\left(3e^{-i}\right)^2 = 3^2 \times e^{-i\times 2} = 9e^{-2i}$

6 $4e^{\frac{\pi}{2}i} \div 2 = 2e^{\frac{\pi}{2}i}$

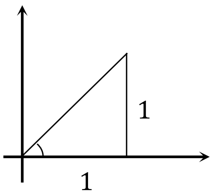
7
a $2e^{\frac{\pi}{3}i} \div i = 2e^{\frac{\pi}{3}i} \div e^{\frac{\pi}{2}i} = 2e^{\left(\frac{\pi}{3}-\frac{\pi}{2}\right)i} = 2e^{-\frac{\pi}{6}i}$
b $5e^{-\frac{\pi}{2}i} \div (-i) = -5i \div (-i) = 5$

8 $2e^{-2i} \div 4e^{i} = \dfrac{1}{2}e^{-3i}$

9 $\bar{z} = 3e^{\frac{\pi}{3}i}$

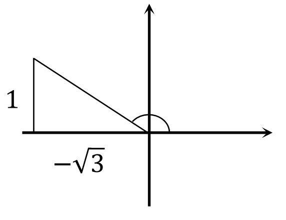
10
a $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{\frac{\pi}{3}i}$ **b** $\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = e^{-\frac{\pi}{4}i}$

11
a



$$|z| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right)$$
$$= \frac{\pi}{4}$$
$$\therefore 1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$$

b



$$|z| = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= 2$$
$$\arg(z) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$\therefore \arg(z) = \frac{5\pi}{6}$$
$$\therefore -\sqrt{3} + i = 2\,e^{\frac{5\pi}{6}i}$$

12
a $e^{\frac{2\pi}{3}i} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + i \times \frac{\sqrt{3}}{2}$

b $e^{-\frac{\pi}{4}i} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - i \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

13

$$e^{ix} = e^{0i} + ie^{0i}x + \frac{i^2e^{0i}x^2}{2!} + \frac{i^3e^{0i}x^3}{3!} + \frac{i^4e^{0i}x^4}{4!} + \ldots$$
$$= 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \ldots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$
$$= \cos x + i\sin x \quad \square$$

14 Let $x=\pi$ in Euler's formula $\rightarrow \therefore e^{i\pi}=\cos\pi+i\sin\pi \;\rightarrow\; e^{i\pi}=-1+i(0) \;\rightarrow\; e^{i\pi}+1=0$

15

a $i^{-i}=\left(e^{-\frac{\pi}{2}i}\right)^{i}=e^{-\frac{\pi}{2}i^2}=e^{\frac{\pi}{2}}\approx 4.8$

b $\sqrt{e^{-i\pi}}=\sqrt{(e^{i\pi})^{-1}}=\sqrt{(-1)^{-1}}=\sqrt{-1}=i$

16

$(-1)^{\frac{1}{n}}=\left(e^{i\pi}\right)^{\frac{1}{n}}=e^{\frac{\pi}{n}i}$ Since $\arg\left(e^{\frac{\pi}{n}i}\right)=\frac{\pi}{n}$, this is equivalent to a rotation of $\frac{\pi}{n}$.

17 The square roots are $3e^{\frac{\pi}{6}i}$ and $2e^{-\frac{5\pi}{6}i}$

18

For $f(x)=e^{ix}$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$e^{ix}=e^{0i}+ie^{0i}x+\frac{i^2e^{0i}x^2}{2!}+\frac{i^3e^{0i}x^3}{3!}+\frac{i^4e^{0i}x^4}{4!}+\ldots$$

$$=1+xi-\frac{x^2}{2!}-\frac{x^3}{3!}i+\frac{x^4}{4!}+\frac{x^5}{5!}i-\frac{x^6}{6!}-\frac{x^7}{7!}i+\ldots \qquad *$$

$$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right) \qquad (1)$$

For $f(x)=\cos x$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$\cos x=\cos(0)-\sin(0)\,x-\frac{\cos(0)\,x^2}{2!}+\frac{\sin(0)\,x^3}{3!}+\frac{\cos(0)\,x^4}{4!}+\ldots$$

$$=1-0x-\frac{x^2}{2!}+0x^3+\frac{x^4}{4!}+0x^5-\frac{x^6}{6!}+\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (2)$$

For $f(x)=\sin x$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(a)}{3!}x^3+\ldots$$

$$\sin x=\sin(0)+\cos(0)\,x-\frac{\sin(0)\,x^2}{2!}-\frac{\cos(0)\,x^3}{3!}+\frac{\sin(0)\,x^4}{4!}+\ldots$$

$$=0+x+0x^2-\frac{x^3}{3!}+0x^4+\frac{x^5}{5!}+0x^6+\ldots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \qquad (3)$$

So we have the following equations:

$$e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right) \qquad (1)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (2)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \qquad (3)$$

From (1), (2) and (3) we see that $e^{ix}=\cos x+i\sin x$ $\quad\square$