EXERCISE 1.6

BASIC

1 For $a, b \geq 0$ prove $\frac{a+b}{2} \geq \sqrt{ab}$ by starting with $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$

2 For $a, b \geq 0$ prove $\frac{a+b}{2} \geq \sqrt{ab}$ by starting with $(a-b)^2 \geq 0$

3 For $x \neq 0$ prove $x^2 + \frac{1}{x^2} \geq 2$

4 Prove $x^2 \geq 2\sqrt{(x-1)(x+1)}$ for $x \geq 1$

5 For $a, b > 0$ prove $(a+2b)^2 \geq 8ab$

MEDIUM

6 For $a, b, c > 0$ prove $a^2c^2 + \frac{b^2}{c^2} \geq 2ab$

7 Given $\frac{a+b}{2} \geq \sqrt{ab}$, prove $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ for $x, y, z > 0$

8 Prove $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2 + z^4} + \frac{1}{xz}\sqrt{y^2z^2 + x^4}$ for $x, y, z > 0$

9 For $a, b, c, d \geq 0$ prove $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

10 If $a, b, c, d > 0$ then prove $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{d} + \frac{d^3}{a} \geq 4\sqrt{abcd}$

11 For $a, b, c \geq 0$ prove $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

12 If $a, b, c > 0$ then prove $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$

13 Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2a_3\ldots a_n = 1$.
Prove that $(a_1^2 + a_1)(a_2^2 + a_2)\ldots(a_n^2 + a_n) \geq 2^n$

14 i) For $a, b, c \geq 0$ prove $a^2 + b^2 + c^2 \geq ab + bc + ca$

ii) Hence prove $(a+b+c)^2 \geq 3(ab + bc + ca)$

15 Let $a+b=1$ prove that $a^4+b^4 \geq \frac{1}{8}$

16 Let $a,b,c>0$ such that $abc=1$, prove that $a^2+b^2+c^2 \geq a+b+c$

17 If $a,b,c>0$ then prove $a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab$

18 If $a,b,c>0$ satisfy $abc=1$, prove

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq 3$$

19 Prove the Harmonic Mean $\leq$ the Geometric Mean $\leq$ the Arithmetic Mean $\leq$ the Quadratic Mean for 2 numbers, ie:

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

20 **a** For $a,b,c>0$ prove $3(a^2+b^2+c^2) \geq (a+b+c)^2 \geq 3(ab+bc+ac)$
b If $a+b+c=3$, hence prove that $a^2+b^2+c^2+ab+bc+ca \geq 6$

1

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$
$$a - 2\sqrt{ab} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab} \quad \square$$

2

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a + b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

3 Method 1

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$
$$x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 2

Let $a = x^2, b = \frac{1}{x^2}$ in $\frac{a+b}{2} \geq \sqrt{ab}$

$$\therefore \frac{x^2 + \frac{1}{x^2}}{2} \geq \sqrt{x^2 \times \frac{1}{x^2}}$$
$$x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 3

$$\text{LHS} - \text{RHS} = x^2 - 2 + \frac{1}{x^2}$$
$$= \left(x - \frac{1}{x}\right)^2$$
$$\geq 0$$
$$\therefore x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$\therefore x^2 + \frac{1}{x^2} \geq 2 \quad \square$$

Method 4

On working out paper:

$$x^2 + \frac{1}{x^2} \geq 2$$
$$x^2 - 2 + \frac{1}{x^2} \geq 0$$
$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

Now rewrite like Method 1

4

$$\left(\sqrt{x^2-1} - 1\right)^2 \geq 0$$
$$x^2 - 1 - 2\sqrt{x^2-1} + 1 \geq 0$$

$x^2 \geq 2\sqrt{(x-1)(x+1)}$ for $x \geq 1$ $\square$

5 LHS − RHS

$$= (a+2b)^2 - 8ab$$
$$= a^2 + 4ab + 4b^2 - 8ab$$
$$= a^2 - 4ab + 4b^2$$
$$= (a-2b)^2$$

≥ 0 since $\mathbb{R}^2 \geq 0$

$$\therefore (a+2b)^2 - 8ab \geq 0$$
$$\therefore (a+2b)^2 \geq 8ab \quad \square$$

6 Method 1

$$\left(ac - \frac{b}{c}\right)^2 \geq 0$$
$$a^2c^2 - 2ab + \frac{b^2}{c^2} \geq 0$$
$$\therefore a^2c^2 + \frac{b^2}{c^2} \geq 2ab \quad \square$$

Method 2

Let $m = a^2c^2, y = \frac{b^2}{c^2}$ in $\frac{m+n}{2} \geq \sqrt{mn}$

$$\therefore \frac{a^2c^2 + \frac{b^2}{c^2}}{2} \geq 2\sqrt{a^2c^2 \cdot \frac{b^2}{c^2}}$$
$$= 2\sqrt{a^2b^2}$$
$$= 2ab$$
$$\therefore a^2c^2 + \frac{b^2}{c^2} \geq 2ab \quad \square$$

7 $$\frac{\frac{x}{yz}+\frac{y}{xz}}{2} \geq \sqrt{\frac{xy}{xyz^2}} \qquad \text{(AM−GM)}$$

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right) \geq \frac{1}{z} \qquad (1)$$

Similarly

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right) \geq \frac{1}{y} \qquad (2)$$

$$\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right) \geq \frac{1}{x} \qquad (3)$$

$(1)+(2)+(3)$:

$$\frac{1}{2}\left(\frac{x}{yz}+\frac{y}{xz}\right)+\frac{1}{2}\left(\frac{x}{yz}+\frac{z}{xy}\right)+\frac{1}{2}\left(\frac{y}{xz}+\frac{z}{xy}\right) \geq \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \qquad \square$$

8 $$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \sqrt{\frac{x^2}{y^2}\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)} \qquad \text{(AM − GM)}$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \sqrt{\frac{x^2}{y^2}\left(\frac{x^2y^2+z^4}{x^2z^2}\right)}$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4} \qquad (1)$$

Similarly

$$\frac{\frac{y^2}{z^2}+\left(\frac{x^2}{y^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{xz}\sqrt{y^2z^2+x^4} \qquad (2)$$

$(1)+(2)$:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4}+\frac{1}{xz}\sqrt{y^2z^2+x^4} \qquad \square$$

9 Let $x=\frac{a+b}{2}, y=\frac{c+d}{2}$ in $\frac{x+y}{2} \geq \sqrt{xy}$

$$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \geq \sqrt{\frac{a+b}{2}\times\frac{c+d}{2}}$$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt{\sqrt{ab}\times\sqrt{cd}} \qquad \text{since } \frac{(a+b)}{2} \geq \sqrt{ab}, \frac{c+d}{2} \geq \sqrt{cd}$$

$$\therefore \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd} \qquad \square$$

10 $$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4} \geq \sqrt[4]{\frac{a^3}{b}\times\frac{b^3}{c}\times\frac{c^3}{d}\times\frac{d^3}{a}} \quad (\mathrm{AM}-\mathrm{GM})$$

$$\frac{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a}}{4} \geq \sqrt[4]{a^2b^2c^2d^2}$$

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{d}+\frac{d^3}{a} \geq 4\sqrt{abcd} \quad \square$$

11 Let $w=a,\ x=b, y=c, z=\frac{a+b+c}{3}$ in $\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}$

$$\therefore \frac{a+b+c+\left(\frac{a+b+c}{3}\right)}{4} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{\frac{4}{3}(a+b+c)}{4} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{1} \geq (abc)^{\frac{1}{4}}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}\times\frac{4}{3}} \geq (abc)^{\frac{1}{4}\times\frac{4}{3}}$$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \square$$

12 $$\frac{a^3+a^3+b^3}{3} \geq \sqrt[3]{a^6b^3}$$

$$\therefore \frac{2a^3+b^3}{3} \geq a^2b \quad (1)$$

Similarly

$$\frac{2b^3+a^3}{3} \geq b^2c \quad (2) \qquad \frac{2c^3+a^3}{3} \geq c^2a \quad (3)$$

$(1)+(2)+(3)$: $\quad a^3+b^3+c^3 \geq a^2b+b^2c+c^2a \quad \square$

13 $(a_1-\sqrt{a_1})^2 \geq 0$

$a_1^2 - 2a_1\sqrt{a_1} + a_1 \geq 0$

$a_1^2 + a_1 \geq 2a_1\sqrt{a_1} \quad (1)$

Similarly for a_2 to a_n $\quad (2)\ldots(n)$

$(1)\times(2)\times(3)\times\ldots\times(n)$:

$\therefore (a_1^2+a_1)(a_2^2+a_2)(a_3^2+a_3)\ldots(a_n^2+a_n) \geq 2a_1\sqrt{a_1}\times 2a_2\sqrt{a_2}\times 2a_3\sqrt{a_3}\times\ldots\times 2a_n\sqrt{a_n}$

$\geq 2^n(a_1a_2a_3\ldots a_n)\sqrt{a_1a_2a_3\ldots a_n}$

$\geq 2^n \quad \square$

14 **i** $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab \quad (1)$

Similarly:

$a^2 + c^2 \geq 2ac \quad (2)$

$b^2 + c^2 \geq 2bc \quad (3)$

$(1)+(2)+(3)$:

$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$a^2+b^2+c^2 \geq ab+bc+ca \quad \square$

Alternatively

$a^2+b^2+c^2-ab-bc-ca$

$=\frac{1}{2}(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2)$

$=\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$

≥ 0

$\therefore a^2+b^2+c^2 \geq ab+bc+ca \quad \square$

ii $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$\geq ab+bc+ca+2(ab+bc+ca)$ from (i).

$\geq 3(ab+bc+ca) \quad \square$

15 $(a^2-b^2)^2 \geq 0$

$a^4 - 2a^2b^2 + b^4 \geq 0$

$2(a^4+b^4)-(a^2+b^2)^2 \geq 0$

$\therefore a^4+b^4 \geq \frac{(a^2+b^2)^2}{2} \quad (1)$

Similarly

$a^2+b^2 \geq \frac{(a+b)^2}{2}$

$\therefore a^2+b^2 \geq \frac{1^2}{2}$

$a^2+b^2 \geq \frac{1}{2} \quad (2)$

From (1) and (2):

$a^4+b^4 \geq \frac{\left(\frac{1}{2}\right)^2}{2}$

$a^4+b^4 \geq \frac{1}{8} \quad \square$

16 $a+b+c \geq 3\sqrt[3]{abc}$

$a+b+c \geq 3 \qquad (1)$

$$\frac{a^2+1}{2} \geq \sqrt{a^2 \cdot 1}$$
$$\geq a$$

Similarly

$$\frac{b^2+1}{2} \geq b \qquad \frac{c^2+1}{2} \geq c$$

Summing the inequalities:

$$\begin{aligned}
\frac{a^2+1}{2}+\frac{b^2+1}{2}+\frac{c^2+1}{2} &\geq a+b+c \\
a^2+b^2+c^2+3 &\geq 2(a+b+c) \\
a^2+b^2+c^2 &\geq 2(a+b+c)-3 \\
&\geq 2(a+b+c)-(a+b+c) \quad \text{from } (1) \\
&\geq a+b+c
\end{aligned}$$

17 $$\frac{a^4+a^4+b^4+c^4}{4} \geq \sqrt[4]{a^8b^4c^4}$$

$$\therefore \frac{2a^4+b^4+c^4}{4} \geq a^2bc$$

Similarly

$$\frac{2b^4+a^4+c^4}{4} \geq b^2ac$$

$$\frac{2c^4+a^4+b^4}{4} \geq c^2ab$$

Summing the above gives

$a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab \quad \square$

18 $$\frac{1+ab}{1+a}=\frac{abc+ab}{abc+a}=\frac{ab(c+1)}{a(bc+1)}=\frac{b(c+1)}{bc+1} \qquad (3)$$

Similarly

$$\frac{1+bc}{1+b}=\frac{c(a+1)}{ac+1} \qquad (2)$$

$$\frac{1+ac}{1+c}=\frac{a(b+1)}{ab+1} \qquad (3)$$

$$\begin{aligned}
\text{LHS} &= \frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \\
&= \frac{1}{2}\left(\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}+\frac{b(c+1)}{bc+1}+\frac{c(a+1)}{ac+1}+\frac{a(b+1)}{ab+1}\right) \\
&\geq \frac{1}{2}\left(6 \times \sqrt[6]{\frac{1+ab}{1+a}\cdot\frac{1+bc}{1+b}\cdot\frac{1+ac}{1+c}\cdot\frac{b(c+1)}{bc+1}\cdot\frac{c(a+1)}{ac+1}\cdot\frac{a(b+1)}{ab+1}}\right) \\
&\geq \frac{1}{2} \times 6 \times \sqrt{1} \\
&\geq 3 \quad \square
\end{aligned}$$

19

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0$$
$$a-2\sqrt{ab}+b \ge 0$$
$$a+b \ge 2\sqrt{ab}$$
$$\sqrt{ab} \le \frac{a+b}{2} \qquad (1)$$

$(1)\times\sqrt{ab}$: $\quad ab \le \dfrac{(a+b)\sqrt{ab}}{2}$
$$\frac{2ab}{a+b} \le \sqrt{ab} \qquad (2)$$

$$\left(\frac{a}{2}-\frac{b}{2}\right)^2 \ge 0$$
$$\frac{a^2-2ab+b^2}{4} \ge 0$$
$$\frac{a^2-2ab+b^2}{4}+\frac{(a+b)^2}{4} \ge \frac{(a+b)^2}{4}$$
$$\frac{2a^2+2b^2}{4} \ge \frac{(a+b)^2}{4}$$
$$\frac{(a+b)^2}{4} \le \frac{a^2+b^2}{2}$$
$$\sqrt{\frac{(a+b)^2}{4}} \le \sqrt{\frac{a^2+b^2}{2}}$$
$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \qquad (3)$$

Alternative for GM and QM
$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

$$\left(\frac{a+b}{2}\right)^2-\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$=\frac{a^2+2ab+b^2}{4}-\frac{a^2+b^2}{2}$$
$$=-\frac{a^2-2ab+b^2}{4}$$
$$=-\left(\frac{a-b}{2}\right)^2$$
$$\le 0$$
$$\therefore \left(\frac{a+b}{2}\right)^2 \le \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2$$
$$\therefore \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

From $(2), (1)$and (3):
$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \qquad \square$$

20 **a**

$3(a^2+b^2+c^2)-(a+b+c)^2$
$= 3a^2+3b^2+3c^2-\left((a^2+b^2+c^2)+2(ab+bc+ca)\right)$
$= 2(a^2+b^2+c^2)-2(ab+bc+ca)$
$= (a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)$
$= (a-b)^2+(b-c)^2+(c-a)^2$
≥ 0
$\therefore 3(a^2+b^2+c^2) \ge (a+b+c)^2 \qquad (1)$

$(a+b+c)^2-3(ab+bc+ac)$
$= a^2+b^2+c^2+2(ab+ca+ca)-3(ab+bc+ca)$
$= a^2+b^2+c^2-(ab+bc+ca)$
$= \dfrac{a^2-2ab+b^2}{2}+\dfrac{b^2-2bc+c^2}{2}+\dfrac{c^2-2ac+a^2}{2}$
$= \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$
≥ 0
$\therefore (a+b+c)^2 \ge 3(ab+bc+ac) \qquad (2)$
From (1) and (2):
$3(a^2+b^2+c^2) \ge (a+b+c)^2 \ge 3(ab+bc+ac)$ $\square$

20 **b**

$$\begin{aligned}
\text{LHS} - \text{RHS} &= a^2 + b^2 + c^2 + ab + bc + ca - 6 \\
&= \frac{a^2 + 2ab + b^2}{2} + \frac{b^2 + 2bc + c^2}{2} + \frac{c^2 + 2ac + a^2}{2} - 2(a + b + c) \\
&= \frac{(a + b)^2 + (b + c)^2 + (c + a)^2}{2} - \frac{12}{2} \quad \text{since } a + b + c = 3 \\
&= \frac{(3 - c)^2 + (3 - a)^2 + (3 - b)^2}{2} - \frac{12}{2} \\
&= \frac{\left((3 - c)^2 - 4\right) + \left((3 - a)^2 - 4\right) + \left((3 - b)^2 - 4\right)}{2} \\
&= \frac{(5 - c)(1 - c) + (5 - a)(1 - a) + (5 - b)(1 - b)}{2} \\
&= \frac{(5 - 6c + c^2) + (5 - 6a + a^2) + (5 - 6b + b^2)}{2} \\
&= \frac{(a^2 + b^2 + c^2) - 6(a + b + c) + 15}{2} \\
&= \frac{a^2 + b^2 + c^2 - 18 + 15}{2} \\
&= \frac{a^2 + b^2 + c^2 - 3}{2}
\end{aligned}$$

Now $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$ from (a)

$$\begin{aligned}
\therefore a^2 + b^2 + c^2 &\geq \frac{(a + b + c)^2}{3} \\
&\geq \frac{3^3}{3} \\
&\geq 3
\end{aligned}$$

≥ 0

$\therefore a^2 + b^2 + c^2 + ab + bc + ca \geq 6$ □