EXERCISE 1.3

BASIC

1 Prove by contradiction that if n is an even integer then n^2 is even.

2 Prove by contradiction that if n is an integer and $n^2 - 1$ is even then n is odd.

MEDIUM

3 Prove $\sqrt{5} + \sqrt{7} < 5$ by contradiction

4 Prove that $\sqrt{2}$ is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Prove by contradiction that if a, b are integral and $a + b \leq 5$ then $a \leq 2$ or $b \leq 2$.

7 Prove by contradiction that there are no integers m, n which satisfy $4n + 8m = 102$

CHALLENGING

8 Prove by contradiction that the square root of π is also irrational.

9 Prove $\sin x + \cos x \geq 1$ for all $0 \leq x \leq \frac{\pi}{2}$ by contradiction.

10 If a is rational and b is irrational, prove $a + b$ is irrational.

11 Prove that for positive integers a, b and $a > 1$ that either b is not divisible by a or $b + 1$ is not divisible by a.

12 Prove that there are no positive integers x, y such that $x^2 - y^2 = 1$.

1 Suppose n is an even integer and n^2 is odd. *

Let $n = 2k$ for integral k

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$= 2p \quad$ for integral p

$\therefore n^2$ is even #

Which contradicts (*) since n^2 cannot be both odd and even, hence if n is an even integer then n^2 is even.

2 Suppose $n^2 - 1$ is even and n is even. *

Let $n = 2k$ for integral k

$\therefore n^2 - 1 = (2k)^2 - 1$

$= 4k^2 - 1$

$= 2(2k^2 - 1) + 1$

$= 2p + 1 \quad$ for integral p

$\therefore n^2 - 1$ is odd #

Which contradicts (*) since $n^2 - 1$ cannot be both odd and even, hence if $n^2 - 1$ is even then n is odd.

3 Suppose $\sqrt{5} + \sqrt{7} \geq 5$

$\left(\sqrt{5} + \sqrt{7}\right)^2 \geq 25 \quad$ since $\sqrt{5}, \sqrt{7}, 5 > 0$

$5 + 2\sqrt{35} + 7 \geq 25$

$2\sqrt{35} \geq 13$

$\sqrt{140} \geq 13$

$140 \geq 169$ #

Which is a contradiction, so $\sqrt{5} + \sqrt{7} < 5$

4 Suppose that $\sqrt{2}$ is rational.

$\therefore \sqrt{2} = \frac{p}{q}$ where p, q are integers with no common factor except 1 (*) $2q^2 = p^2$.

Now 2 is even

$\therefore 2q^2$ is even

$\therefore p^2$ is even

$\therefore p$ is even

Let $p = 2k$ for some integer k.

$\therefore 2q^2 = 4k^2$

$q^2 = 2k^2$

Now $2k^2$ is even

$\therefore q^2$ is even

$\therefore q$ is even. #

This contradicts (*), since if p and q are both even they have a common factor of 2, hence $\sqrt{2}$ is irrational.

5 Suppose that $\log_2 5$ is rational.

$\therefore \log_2 5 = \frac{p}{q}$ where p, q are integers with no common factor except 1

$q \log_2 5 = p$

$\log_2 5^q = p$

$5^q = 2^p$ #

Now the LHS is odd and the RHS is even which is a contradiction, hence $\log_2 5$ is irrational.

6 Suppose $a + b \leq 5$ and $a > 2$ and $b > 2$ *

$\therefore a + b \geq 3 + 3 \quad \text{since } a, b \text{ integral}$

≥ 6 #

Which contradicts (*) since $a + b$ cannot be ≤ 5 and ≥ 6, hence $a \leq 2$ or $b \leq 2$.

7 Suppose m, n are integers which do satisfy $4n + 8m = 102$

$\therefore 4(n + 2m) = 102$

$4p = 4 \times 25 + 2 \quad \text{for integral } p$ #

Which is a contradiction since the LHS is a multiple of 4 but the RHS is not, hence there are no integers m, n which satisfy $4n + 8m = 102$

8 Suppose that $\sqrt{\pi}$ is rational.

$\therefore \sqrt{\pi} = \frac{p}{q}$ where p, q are integers with no common factor except 1

$\pi q^2 = p^2$ #

Now the RHS is an integer but the LHS is not since π is irrational which is a contradiction, hence the square root of the irrational number m is also irrational.

9 Suppose $\sin x + \cos x < 1$

$$\therefore (\sin x + \cos x)^2 < 1 \quad \text{since } \sin x, \cos x \geq 0 \text{ in the given domain}$$

$$\sin^2 x + 2\sin x\cos x + \cos^2 x < 1$$

$$1 + 2\sin x\cos x < 1$$

$$2\sin x\cos x < 0 \qquad \#$$

But $\sin x, \cos x > 0$ for $0 \leq x \leq \frac{\pi}{2}$ so this is a contradiction, hence $\sin x + \cos x \geq 1$.

10 Suppose by contradiction that a is rational, b irrational and $a + b$ rational (*).

Let $a = \frac{p}{q}, a + b = \frac{j}{k}$ for integral p, q, j, k

$$\therefore \frac{p}{q} + b = \frac{j}{k}$$

$$b = \frac{j}{k} - \frac{p}{q}$$

$$= \frac{jq - kp}{kq}$$

$$= \frac{m}{n} \text{ for integral } m, n \text{ since } p, q, j, k \text{ are integral}$$

$\therefore b$ is rational #

This contradicts (*) since b cannot be rational and irrational, $\therefore$ if a is rational and b is irrational, then $a + b$ is irrational

11 Suppose by contradiction that b and $b + 1$ are both divisible by a.

Let $b = ma$ $\quad(1)$ and $b + 1 = na$ $\quad(2)$ for integral m, n.

$\therefore ma + 1 = na$

$na - ma = 1$

$$n - m = \frac{1}{a} \qquad \#$$

This is a contradiction since the LHS is an integer but the RHS is not since $a > 1$, $\therefore$ for positive integers a, b and $a > 1$ b is not divisible by a or $b + 1$ is not divisible by a.

12 Suppose by contradiction that $x^2 - y^2 = 1$ has solutions x, y positive integers (*)

$$\therefore (x+y)(x-y) = 1$$

$$x + y = 1 \text{ and } x - y = 1 \text{ since } x \text{ and } y \text{ are integers} \quad \#$$

This contradicts (*) since $x + y = 1$ has no positive integral solutions, so there are no positive integers x, y such that $x^2 - y^2 = 1$