EXERCISE 2.6

BASIC

1 If $z = 2 - 3i$ find $z\bar{z}$.

2 If $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ find $\frac{1}{z}$

3 If $z = 2 - 3i$ find $z + \bar{z}$

4 If $z = 3 - 4i$ find $z - \bar{z}$.

5 If p and q are the roots of $3z^2 + (2 - i)z + 6i = 0$, find:

a $\overline{p} + \overline{q}$

b $\overline{p} \times \overline{q}$

6 Prove that $1 + i$ is a root of $z^3 - 2z + 4 = 0$, and hence find the other roots.

MEDIUM

7 Prove $z\bar{z} = |z|^2$. Hint: let $z = re^{i\theta}$

8 Prove that if $|z| = 1$ then $\frac{1}{z} = \bar{z}$. Hint: let $z = e^{i\theta}$

9 Prove $z + \bar{z} = 2\mathrm{Re}(z)$. Hint: let $z = a + ib$

10 Prove $z - \bar{z} = 2\mathrm{Im}(z)$. Hint: let $z = a + ib$

11 Prove $\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$. Hint: let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ etc

12 Prove $\overline{z_1 \times z_2 \times \ldots \times z_n} = \overline{z_1} \times \overline{z_2} \times \ldots \times \overline{z_n}$. Hint: let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ etc

13 The polynomial $P(z)$ has real coefficients, and $z = 2 + 3i$ is a root of $P(z)$. What quadratic must be a factor of $P(z)$?

14 The polynomial $P(z)$ has real coefficients, and the non-real numbers α and $-i\alpha$ are zeros of $P(z)$, where $\bar{\alpha} \neq i\alpha$. Explain why $\bar{\alpha}$ and $i\bar{\alpha}$ are also zeros of $P(z)$.

1

$$z\bar{z} = |z|^2$$
$$= a^2 + b^2$$
$$= 2^2 + (-3)^2$$
$$= 13$$

2

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\therefore \frac{1}{z} = \bar{z} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

3

$$z + \bar{z} = 2\mathrm{Re}(z)$$
$$= 2 \times 2$$
$$= 4$$

4

$$z - \bar{z} = 2\mathrm{Im}(z)$$
$$= 2 \times (-4i)$$
$$= -8i$$

5a

$$\bar{p} + \bar{q} = \overline{p+q}$$
$$= \overline{\left(-\frac{b}{a}\right)}$$
$$= -\overline{\left(\frac{2-i}{3}\right)}$$
$$= -\frac{2+i}{3}$$

b

$$\bar{p} \times \bar{q} = \overline{p \times q}$$
$$= \overline{\left(\frac{c}{a}\right)}$$
$$= \overline{\left(\frac{6i}{3}\right)}$$
$$= -2i$$

6

$$(1+i)^3 - 2(1+i) + 4$$
$$= 1^3 + 3(1)^2 i + 3(1)(i)^2 + i^3 - 2(1+i) + 4$$
$$= 1 + 3i - 3 - i - 2 - 2i + 4$$
$$= 0$$

$\therefore 1 + i$ is a root.

$\therefore 1 - i$ is a root.

$$\therefore 1 + i + 1 - i + \gamma = -\frac{b}{a}$$
$$= -\frac{0}{1}$$
$$= 0$$

$$\therefore 2 + \gamma = 0$$
$$\therefore \gamma = -2$$

$\therefore$ The roots of $z^3 - 2z + 4 = 0$ are $-2, 1 \pm i$.

7

Let $z = re^{i\theta}$

$$\therefore \bar{z} = re^{-i\theta}$$
$$z\bar{z} = re^{i\theta} \times re^{-i\theta}$$
$$= r^2 e^{i\theta - i\theta}$$
$$= r^2$$
$$= |z|^2$$

8

Let $z = e^{i\theta}$

$$\frac{1}{z} = \frac{1}{e^{i\theta}}$$
$$= e^{-i\theta}$$
$$= \bar{z} \qquad \square$$

9

$$z+\bar{z}=a+ib+a-ib$$
$$=2a$$
$$=2\operatorname{Re}(z) \quad \square$$

10

$$z-\bar{z}=a+ib-(a-ib)$$
$$=2bi$$
$$=2\operatorname{Im}(z) \quad \square$$

11

Let $z_1=a_1+ib_1, z_2=a_2+ib_2$ etc

$$\text{LHS}=\overline{a_1+ib_1+a_2+ib_2+\ldots+a_n+ib_n}$$
$$=\overline{(a_1+a_2+\ldots a_n)+i(b_1+b_2+\ldots+b_n)}$$
$$=(a_1+a_2+\ldots a_n)-i(b_1+b_2+\ldots+b_n)$$
$$=a_1-ib_1+a_2-ib_2+\ldots+a_n-ib_n$$
$$=\overline{z_1}+\overline{z_2}+\ldots+\overline{z_n}$$
$$=RHS \quad \square$$

12

Let $z_1=r_1e^{i\theta_1}, z_2=r_2e^{i\theta_2}$ etc

$$\text{LHS}=\overline{r_1e^{i\theta_1}\times r_2e^{i\theta_2}\times\ldots\times r_ne^{i\theta_n}}$$
$$=\overline{(r_1r_2\ldots r_n)e^{i\theta_1+i\theta_2+\ldots+i\theta_n}}$$
$$=(r_1r_2\ldots r_n)e^{-(i\theta_1+i\theta_2+\ldots+i\theta_n)}$$
$$=r_1e^{-i\theta_1}\times r_2e^{-i\theta_2}\times\ldots\times r_ne^{-i\theta_n}$$
$$=\overline{r_1e^{i\theta_1}}\times\overline{r_2e^{i\theta_2}}\times\ldots\times\overline{r_ne^{i\theta_n}}$$
$$=\overline{z_1}\times\overline{z_2}\times\ldots\times\overline{z_n}$$
$$=RHS \quad \square$$

13

$2-3i$ must also be a root

∴ the following quadratic must be a factor.

$$\big(z-(2+3i)\big)\big(z-(2-3i)\big)=(z-2-3i)(z-2+3i)$$
$$=(z-2)^2-(3i)^2$$
$$=z^2-4z+4+9$$
$$=z^2-4z+13$$

14

Since the coefficients are real then the conjugate of non-real zeros are also zeros.

The conjugate of α is $\bar{\alpha}$.

The conjugate of $-i\alpha$ is $\overline{-i\alpha}=\overline{-i}\times\bar{\alpha}=i\bar{\alpha}$

∴ $\bar{\alpha}$ and $i\bar{\alpha}$ are roots of $P(z)$ $\quad \square$