EXERCISE 5.5

BASIC

1 Find the equations of the spheres with:
a radius 2 and centre at the origin **b** radius 3 and centre at $(-1,1,3)$

2 Sketch $x = 3\cos t\,, y = 2\sin t$

3 Sketch $x = t^2, y = t$

4 Sketch $x = 5\cos t - 2, y = 4\sin t + 1$

5 Sketch $x = t^2 - 1, y = t + 1$

6 Sketch $x = \frac{1}{t}, y = t + 2$

7 Sketch $x = \sec\theta\,, y = \tan\theta$

8 Sketch $x = t, y = t^2$ for $0 \leq t \leq 2$

MEDIUM

9 Complete the square to determine the radius and centre of the sphere

$$x^2 - 2x + y^2 + z^2 + 4z + 4 = 0$$

10 The parameterised equation of a sphere is $x = r\sin\alpha\sin\beta\,, y = r\cos\alpha\,, z = r\sin\alpha\cos\beta$.
Prove that it satisfies $x^2 + y^2 + z^2 = r^2$.

11 Sketch $x = -|2\cos t|, y = |2\sin t|$

12 Sketch $x = t^2, y = \frac{1}{t^2}$

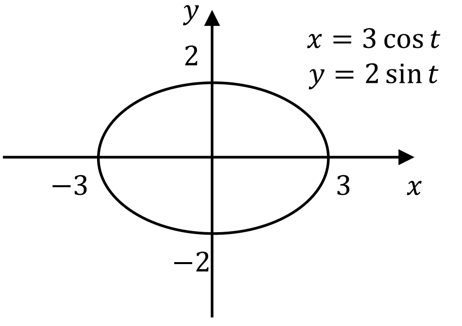
13 Sketch $x = t\sin t\,, y = t\cos t$

CHALLENGING

14 The parametric equations $x = \cos t\,, y = \sin t$ gives a unit circle, and as t increases from zero the point moves anticlockwise from $(1,0)$. Find the parametric equations of a circle where as t increases from zero the point moves clockwise from $\left(\sqrt{3}, 1\right)$, centred about the origin.

1 **a** $x^2 + y^2 + z^2 = 4$ **b** $(x+1)^2 + (y-1)^2 + (z-3)^2 = 9$

2 From the common curves this is an ellipse centred at the origin with a horizontal semi-major axis of 3 and a semi-minor axis of 3.



Alternatively

$x = 3\cos t \ \rightarrow \cos t = \dfrac{x}{3}$

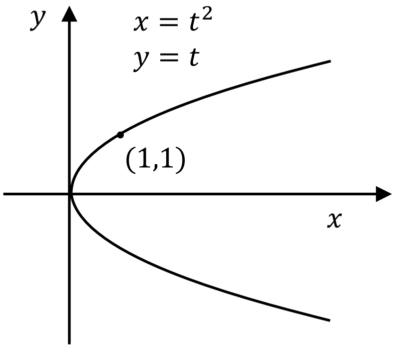
$y = 2\sin t \ \rightarrow \sin t = \dfrac{y}{2}$

$\sin^2\theta + \cos^2\theta = 1$ Pythagorean Identity

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Which is the ellipse centred at the origin with horizontal semi major axis 3 and semi minor axis 2

3 From the common curves this is a parabola with x and y values swapped, so concave right with vertex at the origin



Alternatively

$x = t^2 \ \rightarrow \ t = \pm\sqrt{x} \quad (1)$

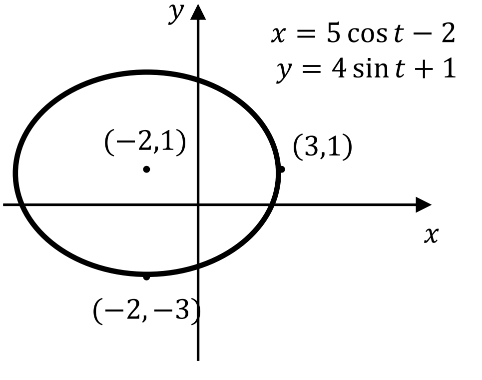
$y = t \quad (2)$

sub (1) in (2)

$y = \pm\sqrt{x}$

This is the parabola which is concave right with vertex at the origin.

4 This is the ellipse with horizontal semi major axis 5 and semi minor axis 4 moved 2 units to the left and 1 unit up.

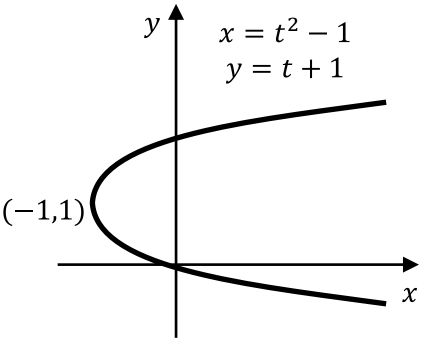


5 Safest to find the Cartesian equation:

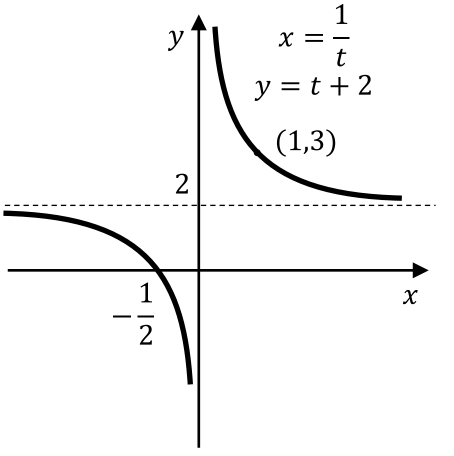
$x = t^2 - 1 \ \rightarrow t = \pm\sqrt{x+1}$

$y = t + 1 = \pm\sqrt{x+1} + 1$

This is the concave right parabola moved 1 unit to the left and up 1.



6 This is the basic rectangular hyperbola moved 2 units up.

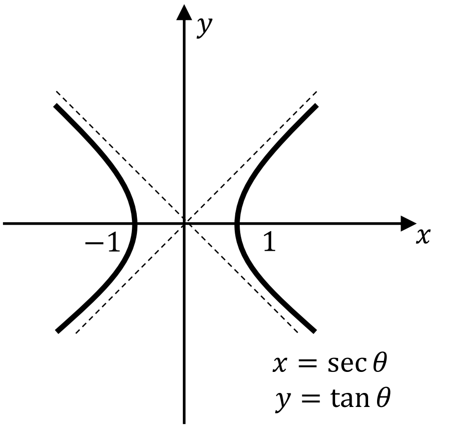


Alternatively we could find the Cartesian equation:

$x = \frac{1}{t} \to t = \frac{1}{x}$

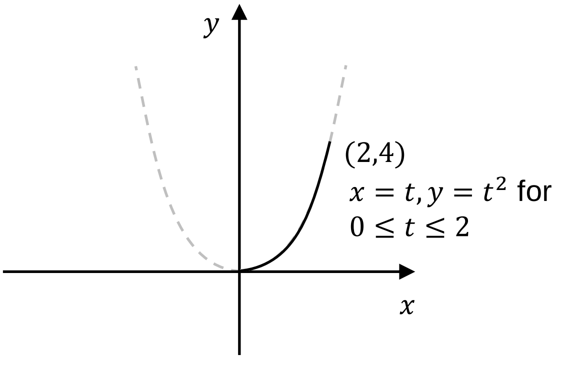
$y = t + 2 = \frac{1}{x} + 2$

7 This is the hyperbola $x^2 - y^2 = 1$.



The asymptotes are $y = \pm x$

8 This is the basic parabola.



We only take the section from $t = 0$ to $t = 2$, so from $(0,0)$ to $(2,4)$

9

$$\begin{aligned} x^2 - 2x + y^2 + z^2 + 4z + 4 &= 0 \\ x^2 - 2x + 1 + y^2 + z^2 + 4z + 4 &= 1 \\ (x-1)^2 + y^2 + (z+2)^2 &= 1 \end{aligned}$$

A sphere with radius 1 and centre $(1,0,-2)$.

10 $x^2 + y^2 + z^2$

$= (r\sin\alpha\sin\beta)^2 + (r\cos\alpha)^2 + (r\sin\alpha\cos\beta)^2$

$= r^2\sin^2\alpha\sin^2\beta + r^2\cos^2\alpha + r^2\sin^2\alpha\cos^2\beta$

$= r^2(\sin^2\alpha\,(\sin^2\beta + \cos^2\beta) + \cos^2\alpha)$

$= r^2(\sin^2\alpha \times (1) + \cos^2\alpha)$

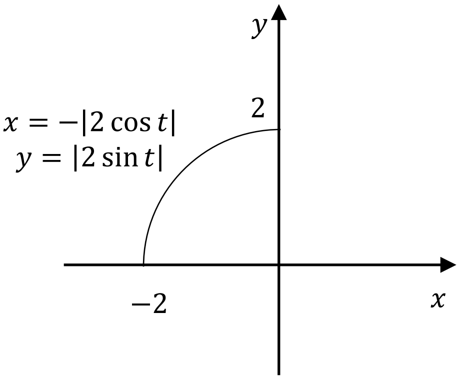
$= r^2(\sin^2\alpha + \cos^2\alpha)$

$= r^2$

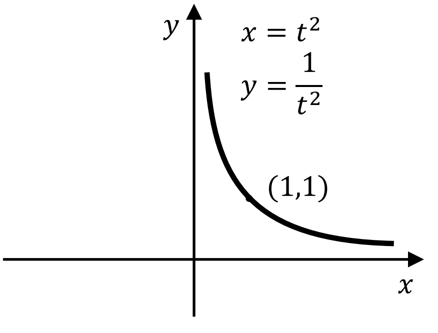
$\therefore x = r\sin\alpha\sin\beta\,, y = r\cos\alpha\,, z = r\sin\alpha\cos\beta$ satisfies

$x^2 + y^2 + z^2 = r^2$

11 $-2 \le x \le 0$ and $0 \le y \le 2$, so the top left quadrant of a circle of radius 2 centred at the origin.

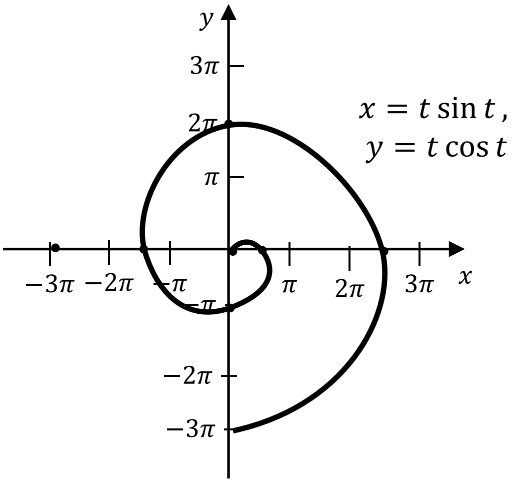


12 Since $t^2 \ge 0$, so $x > 0$ and $y > 0$. This is the top right branch of the hyperbola.



13 A clockwise Archimedean spiral.

t	x	y
0	0	0
$\frac{\pi}{2}$	$\frac{\pi}{2}$	0
π	0	$-\pi$
$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	0
2π	0	2π
3π	0	-3π



14

The radius of the new circle is $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$.

For the point to move clockwise we swap sine and cosine, so $x = 2\sin\big(f(t)\big), y = 2\cos\big(f(t)\big)$.

To move the starting position to $\left(\sqrt{3}, 1\right)$, or $\left(2\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}\right)$ we use $t + \frac{\pi}{3}$.

$\therefore x = 2\sin\left(t + \frac{\pi}{3}\right), y = 2\cos\left(t + \frac{\pi}{3}\right)$