EXERCISE 4.4

BASIC

1 **i** Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ by equating coefficients.

ii Hence find $\displaystyle\int \frac{2x+1}{(x+1)(x+2)}\,dx$

2 **i** Express $\dfrac{2x+3}{x^2(x+1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ by elimination by substitution.

ii Hence evaluate $\displaystyle\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx$

3 Express $\dfrac{2x+1}{(x+1)(x+2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x+2}$ using the cover up method

4 Express $\dfrac{2x+3}{x^2(x+1)}$ in the form $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ using the cover up method.

MEDIUM

5 Express $\dfrac{5x-5}{(x+2)(x^2+1)}$ in the form $\dfrac{a}{x+2}+\dfrac{bx+c}{x^2+1}$ and hence evaluate $\displaystyle\int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx$

6 It can be shown that $\dfrac{2}{x^3+x^2-x+1}=\dfrac{1}{x+1}-\dfrac{x}{x^2+1}+\dfrac{1}{x^2+1}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$

7 It can be shown that $\dfrac{8(1-x)}{(2-x^2)(2-2x+x^2)}=\dfrac{4-2x}{2-2x+x^2}-\dfrac{2x}{2-x^2}$ (Do NOT prove this.)

Use this result to evaluate $\displaystyle\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$

8 Evaluate $\displaystyle\int_2^5 \frac{x-6}{x^2+3x-4}\,dx$

9 **i** Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{16x-43}{(x-3)^2(x+2)}=\dfrac{a}{(x-3)^2}+\dfrac{b}{x-3}+\dfrac{c}{x+2}$ where a, b and c are real numbers, find a, b and c.

ii Hence find $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$

10 Find $\int \frac{3x^2+8}{x(x^2+4)}dx$

11 Find $\int \frac{2x^3-x^2-8x-2}{x(x-2)}dx$

12 Use partial fractions to show that $\frac{3!}{x(x+1)(x+2)(x+3)} = \frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2} - \frac{1}{x+3}$

13 It is given that $x^4+4=(x^2+2x+2)(x^2-2x+2)$.

i Find A and B so that $\frac{16}{x^4+4} = \frac{A+2x}{x^2+2x+2} + \frac{B-2x}{x^2-2x+2}$

ii Hence, or otherwise, show that for any real number m,

$$\int_0^m \frac{16}{x^4+4}dx = \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1).$$

iii Find the limiting value as $m \to \infty$ of $\int_0^m \frac{16}{x^4+4}dx$

1

i $\dfrac{2x+1}{(x+1)(x+2)} = \dfrac{a}{x+1} + \dfrac{b}{x+2}$

$$\frac{2x+1}{(x+1)(x+2)} = \frac{a(x+2)+b(x+1)}{(x+1)(x+2)}$$

$$2x+1 = (a+b)x + 2a + b$$

$$\therefore 2 = a + b \qquad (1)$$

$$1 = 2a + b \qquad (2)$$

$$(2)-(1) \quad -1 = a \quad \rightarrow \quad a = -1$$

$$\text{sub in } (1) \quad 2 = -1 + b \quad \rightarrow \quad b = 3$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

ii $\therefore \displaystyle\int \frac{2x+1}{(x+1)(x+2)}\,dx = \int \left(-\frac{1}{x+1} + \frac{3}{x+2}\right) dx$

$$= -\ln|x+1| + 3\ln|x+2| + c$$

2

i $\dfrac{2x+3}{x^2(x+1)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x+1}$

$$(2x-3) = ax(x+1) + b(x+1) + cx^2$$

Let $x = 0 \;\therefore 2(0)+3 = a(0) + b(0+1) + c(0) \;\rightarrow\; 3 = b \;\rightarrow\; b = 3$

Let $x = -1 \;\therefore 2(-1)+3 = a(0) + b(0) + c(1) \;\rightarrow\; 1 = c \;\rightarrow c = 1$

equating coefficients of x^2: $0 = a + c \;\rightarrow\; 0 = a + 1 \;\rightarrow\; a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

ii $\therefore \displaystyle\int_2^3 \frac{2x+3}{x^2(x+1)}\,dx = \int_2^3 \left(-\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}\right) dx$

$$= \left[-\ln|x| - \frac{3}{x} + \ln|x+1|\right]_2^3$$

$$= \left(-\ln 3 - 1 + \ln 4\right) - \left(-\ln 2 - \frac{3}{2} - \ln 3\right)$$

$$= \ln 4 + \ln 2 - 2\ln 3 + \frac{1}{2}$$

$$= \ln\left(\frac{8}{9}\right) + \frac{1}{2}$$

3

$$a = \frac{2(-1)+1}{-1+2} = -1$$

$$b = \frac{2(-2)+1}{(-2)+1} = 3$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{3}{x+2}$$

4

$$b = \frac{2(0)+3}{0+1} = 3$$

$$c = \frac{2(-1)+3}{(-1)^2} = 1$$

equating coefficients of x^2: $0 = a + c$

$\rightarrow\ 0 = a + 1\ \rightarrow\ a = -1$

$$\therefore \frac{2x+3}{x^2(x+1)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x+1}$$

5

$$a = \frac{5(-2)-5}{(-2)^2+1} = -3$$

Let $x = i$ $\quad bi + c = \frac{5i-5}{i+2} \times \frac{i-2}{i-2} = \frac{-5-10i-5i+10}{i^2-2^2}$

$$bi + c = -1 + 3i$$

$\therefore c = -1, b = 3$

$$\therefore \frac{5x-5}{(x+2)(x^2+1)} = -\frac{3}{x+2} + \frac{3x-1}{x^2+1}$$

$$\therefore \int_3^4 \frac{5x-5}{(x+2)(x^2+1)}\,dx = \int_3^4 \left(-\frac{3}{x+2} + \frac{3x-1}{x^2+1}\right)dx$$

$$= \int_3^4 \left(-\frac{3}{x+2} + \frac{3}{2} \times \frac{2x}{x^2+1} - \frac{1}{1+x^2}\right)dx$$

$$= \left[-3\ln|x+2| + \frac{3}{2}\ln|x^2+1| - \tan^{-1} x\right]_3^4$$

$$= \left(-3\ln 6 + \frac{3}{2}\ln 17 - \tan^{-1} 4\right) - \left(-3\ln 5 + \frac{3}{2}\ln 10 - \tan^{-1} 3\right)$$

$$= \frac{3}{2}\ln\frac{17}{10} + 3\ln\frac{5}{6} + \tan^{-1} 3 - \tan^{-1} 4$$

6

$$\int_{\frac{1}{2}}^{2} \frac{2}{x^3 + x^2 + x + 1}\, dx$$

$$= \int_{\frac{1}{2}}^{2} \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

$$= \left[\ln\left|x+1\right| - \frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1} x \right]_{\frac{1}{2}}^{2}$$

$$= \left(\ln 3 - \frac{1}{2}\ln 5 + \tan^{-1} 2 \right) - \left(\ln\frac{3}{2} - \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}\left(\frac{1}{2}\right) \right)$$

$$= \ln 3 - \ln\frac{3}{2} - \frac{1}{2}\ln 5 + \frac{1}{2}\ln\frac{5}{4} + \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

$$= \ln 2 - \frac{1}{2}\ln 4 + \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$$

7

$$\int_{0}^{1} \frac{8(1-x)}{(2-x^2)(2-2x+x^2)} dx$$

$$= \int_{0}^{1} \left(\frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2} \right) dx$$

$$= \int_{0}^{1} \left(-\frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1^2} - \frac{2x}{2-x^2} \right) dx$$

$$= \left[-\ln|x^2-2x+2| + 2\tan^{-1}(x-1) + \ln|2-x^2| \right]_{0}^{1}$$

$$= (0+0+0) - \left(-\ln 2 + 2\left(-\frac{\pi}{4}\right) + \ln 2 \right)$$

$$= \frac{\pi}{2}$$

8

$$\frac{x-6}{x^2+3x-4} = \frac{x-6}{(x+4)(x-1)} = \frac{a}{x+4} + \frac{b}{x-1}$$

$$a = \frac{(-4)-6}{(-4)-1} = 2$$

$$b = \frac{(1)-6}{(1)+4} = -1$$

$$\therefore \int_2^5 \frac{x-6}{x^2+3x-4}\,dx$$

$$= \int_2^5 \left(\frac{2}{x+4} - \frac{1}{x-1}\right) dx$$

$$= \Big[2\ln|x+4| - \ln|x-1|\Big]_2^5$$

$$= (2\ln 9 - \ln 4) - (2\ln 6 - 0)$$

$$= \ln 81 - \ln 4 - \ln 36$$

$$= \ln\frac{81}{144}$$

$$= \ln\frac{9}{16}$$

9

i $$\frac{16x-43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2}$$

$$a = \frac{16(3)-43}{(3)+2} = 1$$

$$c = \frac{16(-2)-43}{\big((-2)-3\big)^2} = -3$$

equating coefficients of x^2: $b + c = 0 \;\rightarrow\; b = 3$

ii $$\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$$

$$= \int \left(\frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\right) dx$$

$$= -\frac{1}{x-3} + 3\ln|x-3| - 3\ln|x+2| + c$$

$$= 3\ln\left|\frac{x-3}{x+2}\right| - \frac{1}{x-3} + c$$

10 $\dfrac{3x^2+8}{x(x^2+4)} = \dfrac{a}{x} + \dfrac{bx+c}{x^2+4}$

$a = \dfrac{3(0)+8}{(0)^2+4} = 2$

$b(2i) + c = \dfrac{3(2i)^2+8}{2i}$

$= -\dfrac{4}{2i} \times \dfrac{i}{i}$

$= 2i$

$\therefore b = 1, c = 0$

$$\int \frac{3x^2+8}{x(x^2+4)}\,dx = \int \left(\frac{2}{x} + \frac{x}{x^2+4}\right)dx$$

$$= 2\ln|x| + \frac{1}{2}\ln|x^2+4| + c$$

11 $\displaystyle\int \frac{2x^3 - x^2 - 8x - 2}{x(x-2)}\,dx$

$\displaystyle= \int \frac{2x(x^2-2x) + 3(x^2-2x) - 2x - 2}{x(x-2)}\,dx$

$\displaystyle= \int \left(2x + 3 - \frac{2x+2}{x(x-2)}\right)dx \qquad (*)$

Let $\dfrac{2x+2}{x(x-2)} = \dfrac{a}{x} + \dfrac{b}{x-2}$

$a = \dfrac{2(0)+2}{(0)-2} = -1$

$b = \dfrac{2(2)+2}{(2)} = 3$

sub in $(*)$

$\displaystyle\int \frac{2x^3 - x^2 - 8x - 2}{x(x-2)}\,dx$

$\displaystyle= \int \left(2x + 3 + \frac{1}{x} - \frac{3}{x-2}\right)dx$

$= x^2 + 3x + \ln|x| - 3\ln|x-2| + c$

12

Let $\dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{a}{x} + \dfrac{b}{x+1} + \dfrac{c}{x+2} + \dfrac{d}{x+3}$

$a = \dfrac{3!}{(0+1)(0+2)(0+3)} = 1$

$b = \dfrac{3!}{(-1)(-1+2)(-1+3)} = -3$

$c = \dfrac{3!}{(-2)(-2+1)(-2+3)} = 3$

$d = \dfrac{3!}{(-3)(-3+1)(-3+2)} = -1$

$\therefore \dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{1}{x} - \dfrac{3}{x+1} + \dfrac{3}{x+2} - \dfrac{1}{x+3}$

13

i $\text{RHS} = \dfrac{\big((A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)\big)}{(x^4+4)}$

equating coefficients of x $\quad -2A + 4x + 2B - 4x = 0 \quad \therefore A = B$

equating the constant terms $\; 2A + 2B = 16 \quad \therefore A = B = 4$

ii $\displaystyle\int_0^m \frac{16}{x^4+4}\,dx = \int_0^m \left(\frac{4+2x}{x^2+2x+2} + \frac{4-2x}{x^2-2x+2}\right) dx$

$\displaystyle = \int_0^m \left(\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right)\, dx$

$\displaystyle = \Big[\ln(x^2+2x+2) + 2\tan^{-1}(x+1) - \ln(x^2-2x+2) + 2\tan^{-1}(x-1)\Big]_0^m$

$= (\ln(m^2+2m+2) + 2\tan^{-1}(m+1) - \ln(m^2-2m+2) + 2\tan^{-1}(m-1))$

$- (\ln 2 + 2\tan^{-1} 1 - \ln 2 + 2\tan^{-1} 1)$

$\displaystyle = \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1)$

iii as $m \to \infty$

$\displaystyle \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1) \to$

$\ln(1) + 2\left(\dfrac{\pi}{2}\right) + 2\left(\dfrac{\pi}{2}\right) = 2\pi$

$\displaystyle \therefore \int_0^m \frac{16}{x^4+4}\,dx \to 2\pi$