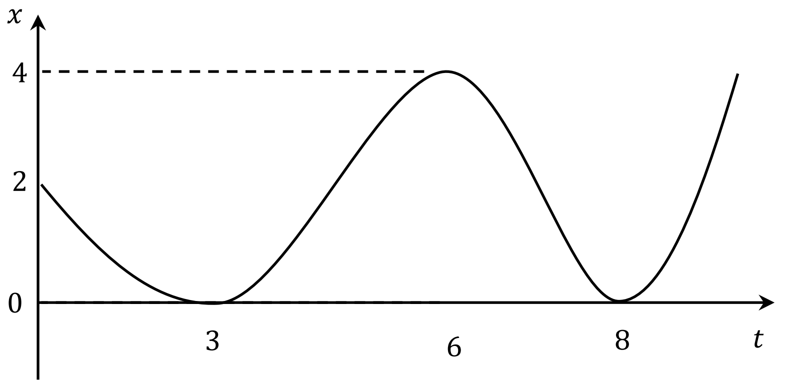
EXERCISE 6.1

BASIC

1 The graph below shows the displacement of a particle moving horizontally along the x-axis over time.

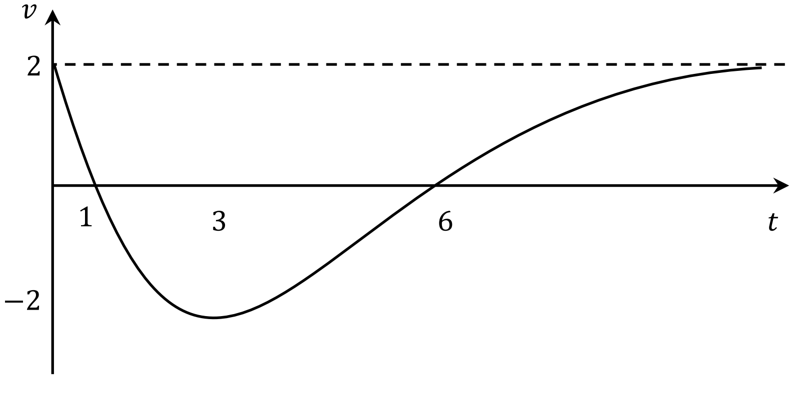


a What can we determine about the initial displacement, velocity and acceleration of the particle?
b When does the resultant force on the particle equal zero?
c When is the force directed to the right?

For the rest of the question assume that the displacement function is a polynomial of degree 4.

d What are the degrees of the functions of velocity and acceleration?
e How many times is the particle at the origin?
f How many times is the particle at rest?

2 The graph below shows the velocity of a particle moving horizontally along the x-axis over time.



a What is the initial velocity and acceleration of the particle?
b Can we tell the initial displacement without further information?
c When is the particle furthest to the left?
d When is the force at a minimum/maximum?
e When is the force directed to the left/right?
f What would the graph of the displacement of the particle look like as $t \to \infty$?

3 Prove $\ddot{x} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

4 The velocity of a particle is given by $\dot{x} = 2x^2 + 3$ metres per second. What is the acceleration when the particle is at $x = 1$?

5 The velocity of a particle, in metres per second, is given by $v = x^2 + 2$, where x is its displacement in metres from the origin. What is the acceleration of the particle at $x = 1$?

6 At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$.
Find the acceleration of the particle as function of x.

MEDIUM

7 A particle moves along a straight line with displacement x metre and velocity v metres per second. The acceleration of the particle is given by $\ddot{x} = 2 - e^{-\frac{x}{2}}$. Given that $v = 4$ when $x = 0$, express v^2 in terms of x.

8 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 1$. Its acceleration is given by $\ddot{x} = x + 4$. Find the speed of the particle at $x = 2$.

9 A particle has velocity given by $\dot{x} = -x^2$. If it is initially at $x = 2$, find the displacement of the particle after 1 second.

10 The acceleration of a particle is given by $\ddot{x} = v^2 + v$. If the particle has a velocity of $2\ \mathrm{ms}^{-1}$ at the origin, find an expression for the velocity in terms of displacement.

11 A particle moves in a straight line. At time t seconds the particle has a displacement of x metres, a velocity of v metres per second and acceleration of a metres per second squared. Initially the particle has displacement $0\ \mathrm{m}$ and velocity of $2\ \mathrm{ms}^{-1}$. The acceleration is given by $a = -2e^{-x}$. The velocity of the particle is always positive.

i Show that $v = 2e^{-\frac{x}{2}}$

ii Find an expression for x as a function of t.

12 A particle is moving horizontally. Initially the particle is at the origin O moving with velocity $1\ \mathrm{ms}^{-1}$. The acceleration of the particle is given by $\ddot{x} = x - 1$, where x is its displacement at time t.

i Show that the velocity of the particle is given by $\dot{x} = 1 - x$.

ii Find an expression for x as a function of t.

iii Find the limiting position of the particle.

13 A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$. Initially $x = -2$ and the velocity, v, is -6.

i Show that $v^2 = 9x^2(1+x)^2$

ii Hence, or otherwise, show that $\int \frac{1}{x(1+x)}\,dx = -3t$

iii It can be shown that for some constant c, $\log_e\left(1+\frac{1}{x}\right) = 3t + c$ (Do NOT prove this)

Using this equation and the initial conditions, find x as a function of t.

14 The acceleration of a particle moving along a straight path is given by $\ddot{x} = -\frac{e^x + 1}{e^{2x}}$ where x is in metres. Initially the particle is at the origin with a velocity of $2\ \mathrm{ms}^{-1}$, and its velocity remains positive.

i Show that $v = e^{-x} + 1$

ii Find the equation of the displacement, x, in terms of t.

15 **i** Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

ii Prove $\frac{d}{dx}(x\ln x) = 1 + \ln x$

iii The acceleration of a particle moving in a straight line and starting from rest at $1\ \mathrm{cm}$ on the positive side of the origin is given by $\frac{d^2x}{dt^2} = 1 + \ln x$

(α) Derive the equation relating v and x

(β) Hence, evaluate v when $x = e^2$.

1 a The particle is initially 2 metres to the right of the origin, moving to the left (since the gradient is negative), with positive acceleration (since it is concave up).

b After approximately 4.5 and 7 seconds at the points of inflexion, since the curve is neither concave up or down so the net force is zero.

c From $t=0$ to approx. $t=4.5$ seconds, and from $t=7$ onwards since the curve is concave up.

d The velocity is of degree 3 and acceleration of degree 2, since they are the first and second derivative of displacement.

e Only the two times shown, as the particle will continue to move to the right.

f Only the three turning points shown, as the particle will continue moving to the right.

2 a The particle is initially at a velocity of 2 metres per second to the right (since the height is 2) with negative acceleration (since the curve then has a negative gradient)

b No, as there could be an infinite number of displacement-time graphs where the gradient matches the height of this velocity-time graph.

c After 6 seconds when the particle is at rest, after having negative velocity.

d The force (and acceleration) is at a minimum at the turning point (approx. $t=3$), and at a maximum at about $t=0$ when it is steepest.

e The force is always to the right from $t=0$ to $t=3$ where the gradient of v is negative and to the right for $t>3$ as the velocity increases.

f Since velocity is approaching a constant value of 2, the gradient of the displacement would be approaching a straight line with gradient 2.

3

$$
\begin{aligned}
\ddot{x} &= \frac{d^2x}{dt^2} \\
&= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\
&= \frac{d}{dt}(v) \\
&= \frac{dv}{dt} && (1) \\
&= \frac{dv}{dx}\times\frac{dx}{dt} \\
&= \frac{dv}{dx}\times v \rightarrow = v\frac{dv}{dx} && (2) \\
&= \frac{dv}{dx}\times\frac{d}{dv}\left(\frac{1}{2}v^2\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}v^2\right) && (3)
\end{aligned}
$$

$\ddot{x} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ from $(1), (2), (3)$

4

$$
\begin{aligned}
\ddot{x} &= v\frac{dv}{dx} \\
&= (2x^2+3)\times(4x) \\
&= 8x^3+12x
\end{aligned}
$$

Let $x=1$

$$
\begin{aligned}
\ddot{x} &= 8(1)^3+12(1) \\
&= 20 \text{ ms}^{-2}
\end{aligned}
$$

5

$$
\begin{aligned}
\ddot{x} &= v\frac{dv}{dx} \\
&= (x^2+2)(2x) \\
&= (1^2+2)(2(1)) \\
&= 6 \text{ ms}^{-2}
\end{aligned}
$$

6

$$\begin{aligned} t &= 4 - e^{-2x} \\ \frac{dt}{dx} &= 2e^{-2x} \\ v &= \frac{dx}{dt} \\ &= \frac{e^{2x}}{2} \\ a &= v\frac{dv}{dx} \\ &= \frac{e^{2x}}{2} \times e^{2x} \\ &= \frac{e^{4x}}{2} \end{aligned}$$

7

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= 2 - e^{-\frac{x}{2}} \\ \frac{1}{2}v^2 - \frac{1}{2}(4)^2 &= \int_0^x \left(2 - e^{-\frac{x}{2}}\right) dx \\ \frac{1}{2}v^2 - 8 &= \left[2x + 2e^{-\frac{x}{2}}\right]_0^x \\ \frac{1}{2}v^2 &= \left(2x + 2e^{-\frac{x}{2}}\right) - (0 + 2) + 8 \\ &= 2x^{-\frac{x}{2}} + 2x + 6 \\ v^2 &= 4e^{-\frac{x}{2}} + 4x + 12 \end{aligned}$$

8

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= x + 4 \\ \frac{1}{2}v^2 &= \int_1^x (x + 4)\, dx \\ &= \left[\frac{x^2}{2} + 4x\right]_1^x \\ v^2 &= 2\left(\left(\frac{x^2}{2} + 4x\right) - \left(\frac{1}{2} + 4\right)\right) \\ &= x^2 + 8x - 9 \end{aligned}$$

when $x = 2$

$$\begin{aligned} v^2 &= 2^2 + 8(2) - 9 \\ &= 11 \end{aligned}$$

$\therefore$ speed $= \sqrt{11}$

9

$$\begin{aligned} \frac{dx}{dt} &= -x^2 \\ \frac{dt}{dx} &= -x^{-2} \\ t &= -\int_2^x x^{-2}\, dx \\ &= \left[\frac{1}{x}\right]_2^x \\ &= \frac{1}{x} - \frac{1}{2} \\ \frac{2t + 1}{2} &= \frac{1}{x} \\ x &= \frac{2}{2t + 1} \end{aligned}$$

Let $t = 1$

$$x = \frac{2}{2(1) + 1} = \frac{2}{3}$$

10

$$\begin{aligned} v\frac{dv}{dx} &= v^2 + v \\ \frac{dv}{dx} &= v + 1 \\ \frac{dx}{dv} &= \frac{1}{v + 1} \\ x &= \int_2^v \frac{1}{v + 1}\, dv \\ &= \Big[\ln(v + 1)\Big]_2^v \\ &= \ln(v + 1) - \ln 3 \\ \ln(v + 1) &= x - \ln 3 \\ v + 1 &= e^{x - \ln 3} \\ &= 3e^x \\ v &= 3e^x - 1 \end{aligned}$$

11 **i**

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= -2e^{-x} \\ \frac{1}{2}v^2 - \frac{1}{2}(2)^2 &= -2\int_0^x e^{-x}\, dx \\ \frac{1}{2}v^2 - 2 &= 2\left[e^{-x}\right]_0^x \\ &= 2(e^{-x} - 1) \\ \therefore \frac{1}{2}\, v^2 &= 2e^{-x} \\ v^2 &= 4e^{-x} \\ \therefore v &= 2e^{-\frac{x}{2}} \end{aligned}$$

positive root since $v = 2$ when $x = 0$

ii

$$\begin{aligned} \therefore \frac{dx}{dt} &= 2e^{-\frac{x}{2}} \\ \frac{dt}{dx} &= \frac{1}{2}e^{\frac{x}{2}} \\ t &= \frac{1}{2}\int_0^x e^{\frac{x}{2}}\, dx \\ &= \left[e^{\frac{x}{2}}\right]_0^x \\ &= e^{\frac{x}{2}} - 1 \\ e^{\frac{x}{2}} &= t + 1 \\ \frac{x}{2} &= \ln(t + 1) \\ x &= 2\ln(t + 1) \end{aligned}$$

12 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 1$$
$$\frac{1}{2}v^2 - \frac{1}{2}(1)^2 = \int_0^x (x-1)\,dx$$
$$\frac{1}{2}v^2 - \frac{1}{2} = \left[\frac{x^2}{2} - x\right]_0^x$$
$$\frac{1}{2}v^2 - \frac{1}{2} = \frac{x^2}{2} - x$$
$$\frac{1}{2}v^2 = \frac{x^2}{2} - x - \frac{1}{2}$$
$$v^2 = x^2 - 2x + 1$$
$$= (x-1)^2$$
$$v = -(x-1) = 1 - x$$

negative root since $\dot{x} = 1$ when $x = 0$

ii

$$\frac{dx}{dt} = 1 - x$$
$$\frac{dt}{dx} = \frac{1}{1-x}$$
$$t = \int_0^x \frac{dx}{1-x}$$
$$= -\Big[\ln(1-x)\Big]_0^x$$
$$= -\ln(1-x)$$
$$e^{-t} = 1 - x$$
$$x = 1 - e^{-t}$$

iii

as $t \to \infty$ $e^{-t} \to 0$ $\therefore x \to 1$

13 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 27x^2 + 9x$$
$$\frac{1}{2}v^2 - \frac{1}{2}(-6)^2 = \int_{-2}^x (18x^3 + 27x^2 + 9x)\,dx$$
$$\frac{1}{2}v^2 - 18 = \left[\frac{9}{2}x^4 + 9x^3 + \frac{9}{2}x^2\right]_{-2}^{x}$$
$$v^2 - 36 = (9x^4 + 18x^3 + 9x^2)$$
$$-(144 - 144 + 36)$$
$$v^2 = 9x^4 + 18x^3 + 9x^2$$
$$= 9x^2(x^2 + 2x + 1)$$
$$= 9x^2(x+1)^2$$

ii

$\therefore v = -3x(1+x)$

(negative root since when $x = -2$ $v = -6$)

$$\frac{dx}{dt} = -3x(1+x)$$
$$\frac{dt}{dx} = -\frac{1}{3} \times \frac{1}{x(1+x)}$$
$$t = -\frac{1}{3}\int \frac{1}{x(1+x)}\,dx$$
$$\therefore \int \frac{1}{x(1+x)}\,dx = -3t$$

iii

$$\log_e\left(1 + \frac{1}{x}\right) = 3t + c$$

at $t = 0$ $x = -2$:

$$\log_e\left(1 - \frac{1}{2}\right) = 0 + c$$
$$c = -\log_e 2$$
$$\log_e\left(1 + \frac{1}{x}\right) = 3t - \log_e 2$$
$$1 + \frac{1}{x} = e^{3t - \log_e 2}$$
$$1 + \frac{1}{x} = \frac{e^{3t}}{2}$$
$$\frac{1}{x} = \frac{e^{3t} - 2}{2}$$
$$x = \frac{2}{e^{3t} - 2}$$

14 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{e^x+1}{e^{2x}}$$

$$\frac{1}{2}v^2 - \frac{1}{2}(2)^2 = -\int_0^x \frac{e^x+1}{e^{2x}}\,dx$$

$$v^2 - 4 = -2\int_0^x \left(e^{-x}+e^{-2x}\right)dx$$

$$= \left[2e^{-x}+e^{-2x}\right]_0^x$$

$$= \left(2e^{-x}+e^{-2x}\right) - \left(2+1\right)$$

$$v^2 = e^{-2x}+2e^x+1$$

$$\therefore v = e^{-x}+1$$

positive root since $v = 2$ when $x = 0$**.**

ii

$$\frac{dx}{dt} = e^{-x}+1$$

$$= \frac{1+e^x}{e^x}$$

$$\frac{dt}{dx} = \frac{e^x}{1+e^x}$$

$$t = \int_0^x \frac{e^x}{1+e^x}\,dx$$

$$= \left[\ln(e^x+1)\right]_0^x$$

$$= \ln(e^x+1) - \ln 2$$

$$\ln(e^x+1) = t + \ln 2$$

$$e^x + 1 = e^{t+\ln 2}$$

$$e^x + 1 = 2e^t$$

$$e^x = 2e^t - 1$$

$$x = \ln(2e^t - 1)$$

15 **i**

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d}{dt}(v)$$

$$= \frac{dv}{dx}\times\frac{dx}{dt}$$

$$= \frac{dv}{dx}\times v$$

$$= \frac{dv}{dx}\times\frac{d}{dv}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

ii

$$\frac{d}{dx}(x\ln x) = \ln x \times 1 + x \times \frac{1}{x}$$

$$= \ln x + 1$$

iii

α

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 + \ln x$$

$$\frac{1}{2}v^2 = \int_1^x (1+\ln x)\,dx$$

$$v^2 = 2\left[x\ln x\right]_1^x$$

$$= 2x\ln x$$

β

Let $x = e^2$

$$v^2 = 2(e^2)\ln(e^2)$$

$$= 4e^2$$

$$v = 2e$$

Positive root given initial conditions