EXERCISE 2.3

BASIC

1 Convert the following complex numbers from Cartesian form into polar form:

a $1+i$ **b** $-1+\sqrt{3}i$ **c** $-\sqrt{3}-i$

2 Convert the following complex numbers from polar into Cartesian form:

a $4\text{ cis}\frac{\pi}{6}$ **b** $\text{cis}\left(-\frac{\pi}{3}\right)$

3 If $u = 4\text{ cis}\frac{\pi}{6}$ and $v = \text{cis}\left(-\frac{\pi}{3}\right)$, find w where $w = uv$.

4 If $u = 4\text{ cis}\frac{\pi}{6}$ and $v = \text{cis}\left(-\frac{\pi}{3}\right)$, find w where $w = \frac{u}{v}$.

5 State the conjugate of each complex number, leaving your answer in polar form

a $\mathrm{z} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **b** $\mathrm{z} = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

6 Convert $\sqrt{3}+i$ into polar form in radians using a calculator (see Appendix 1)

7 Convert $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ into rectangular form using a calculator (see Appendix 1)

MEDIUM

8 Prove $z_1 \times z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$.

Hint: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

9 Prove $z_1 \div z_2 = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

Hint: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

10

a Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in the form $a+ib$ where a and b are real.

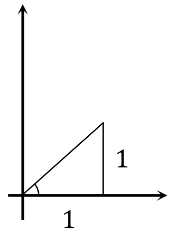
b By expressing $\sqrt{3}+i$ and $\sqrt{3}-i$ in polar form, write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in polar form.

c Hence find $\sin\frac{\pi}{3}$ in surd form

11 Multiplying a non-zero complex number by $\frac{1-i}{-1+i}$ results in a rotation about the origin. What is the angle of rotation, and in what direction?

12 Given $\mathrm{z} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ find $(\bar{z})^{-1}$in polar form.

1 a

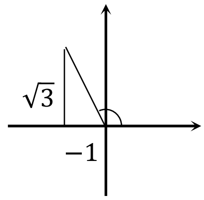


$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$$

b



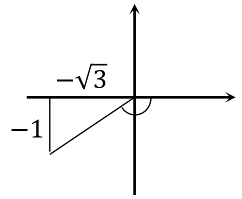
$$|z| = \sqrt{(-1)^2+\left(\sqrt{3}\right)^2} = 2$$

$$\arg(z) = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\therefore \arg(z) = \frac{2\pi}{3}$$

$$\therefore -1+\sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3}$$

c



$$|z| = \sqrt{(-\sqrt{3})^2+(-1)^2} = 2$$

$$\arg(z) = -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore \arg(z) = -\frac{5\pi}{6}$$

$$\therefore -1-\sqrt{3}i = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$$

2 a

$$\begin{aligned}
&4\operatorname{cis}\frac{\pi}{6}\\
&= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\
&= 4\times\left(\frac{\sqrt{3}}{2} + i\times\frac{1}{2}\right)\\
&= 2\sqrt{3}+2i
\end{aligned}$$

b

$$\begin{aligned}
&\operatorname{cis}\left(-\frac{\pi}{3}\right)\\
&= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\\
&= \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\\
&= \frac{1}{2} - i\times\frac{\sqrt{3}}{2}\\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}$$

3

$$\begin{aligned}
w &= uv\\
&= 4\operatorname{cis}\frac{\pi}{6}\times\operatorname{cis}\left(-\frac{\pi}{3}\right)\\
&= (4\times 1)\operatorname{cis}\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\\
&= 4\operatorname{cis}\left(-\frac{\pi}{6}\right)
\end{aligned}$$

4

$$\begin{aligned}
w &= \frac{u}{v}\\
&= 4\operatorname{cis}\frac{\pi}{6}\div\operatorname{cis}\left(-\frac{\pi}{3}\right)\\
&= (4\div 1)\operatorname{cis}\left(\frac{\pi}{6}- -\frac{\pi}{3}\right)\\
&= 4\operatorname{cis}\frac{\pi}{2}\\
&= 4i
\end{aligned}$$

5 a

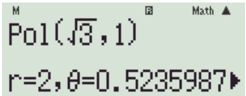
$$\overline{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)} = 2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

b

$$\overline{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)} = \cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$$

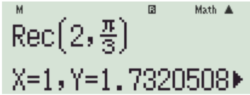
6 Keystrokes: $\boxed{\text{SHIFT}}$ $\boxed{\text{Pol}}$ $\boxed{\sqrt{\ }}$ $\boxed{3}$ $\boxed{\rightarrow}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{1}$ $\boxed{)}$ $\boxed{=}$

To find the argument in exact form, $\boxed{\text{SHIFT}}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{Y}}$ $\boxed{=}$ gives 6, so $\frac{\pi}{6}$



7 Keystrokes: $\boxed{\text{SHIFT}}$ $\boxed{\text{Rec}}$ $\boxed{2}$ $\boxed{\text{SHIFT}}$ $\boxed{,}$ $\boxed{\text{SHIFT}}$ $\boxed{\pi}$ $\boxed{\frac{\blacksquare}{\square}}$ $\boxed{\triangledown}$ $\boxed{3}$ $\boxed{)}$ $\boxed{=}$

To find the imaginary part in exact form, $\boxed{\text{ALPHA}}$ $\boxed{\text{Y}}$ $\boxed{x^2}$ $\boxed{=}$ gives 3, so $\sqrt{3}$



8

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} z_1 \times z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= r_1 r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \qquad \square \end{aligned}$$

$\therefore |z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

9

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\ &= \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \end{aligned}$$

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \square$

10a

$$\begin{aligned} &\frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} \\ &= \frac{3 + 2\sqrt{3}i + 1}{3+1} \\ &= \frac{4 + 2\sqrt{3}i}{4} \\ &= 1 + \frac{\sqrt{3}}{2}i \end{aligned}$$

b

$$\begin{aligned} \sqrt{3}+i &= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ \sqrt{3}-i &= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \\ \frac{\sqrt{3}+i}{\sqrt{3}-i} &= \frac{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} \\ &= \frac{2}{2}\left(\cos\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6} - -\frac{\pi}{6}\right)\right) \\ &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) \end{aligned}$$

c

Equating imaginary parts of **a** and **b**

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

11

$$\frac{1-i}{-1+i} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)} = \cos(-\pi) + i\sin(-\pi) = -1$$

The rotation is $180°$.

12

$$\begin{aligned} (\bar{z})^{-1} &= \frac{1}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} \times \frac{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)}{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)} \\ &= \frac{\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)}{2} \\ &= \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \qquad \text{since cosine is even and sine is odd} \end{aligned}$$