EXERCISE 6.4

BASIC

1 A particle is moving in simple harmonic motion with displacement x. Its velocity v is given by $v^2 = 16(9 - x^2)$. What is the amplitude and period of the motion?

2 A particle is moving in simple harmonic motion. The displacement of the particle is x and its velocity, v, is given by the equation $v^2 = n^2(2kx - x^2)$, where n and k are constants. The particle is initially at $x = k$. Find a possible equation for the displacement of the particle as a function of time.

3 A particle moves with equation of motion $x = \sqrt{3}\cos 3t - \sin 3t - 2$ metres. Prove that the particle is in SHM, and find the centre and amplitude of its motion.

4 A particle is moving in SHM about the point $x = 1$ with period $\frac{\pi}{4}$, and initially the particle is at rest at the origin.

i Derive an equation for v^2 as a function of displacement, x.

ii Find all values of x for which the particle is at rest.

iii Find the maximum velocity of the particle

5 A particle is moving in SHM with $v^2 = 8 - 4x - 4x^2$.

i Find an expression for the acceleration of the particle in terms of x.

ii Find the centre of motion and period

6 A particle is moving with equation of motion $v^2 + x^2 = 4$.
Show that the particle is in SHM with period 2π.

7 A particle is moving in SHM with $v^2 = 25(3 - 2x - x^2)$. Find a possible equation for displacement as a function of time.

8 A particle is moving in SHM with $v^2 = -4(x - 5)(x + 1)$. For what value of x is acceleration a maximum.

9 The displacement x of a particle at time t is given by $x = 5\sin 4t + 12\cos 4t$. What is the maximum velocity of the particle?

MEDIUM

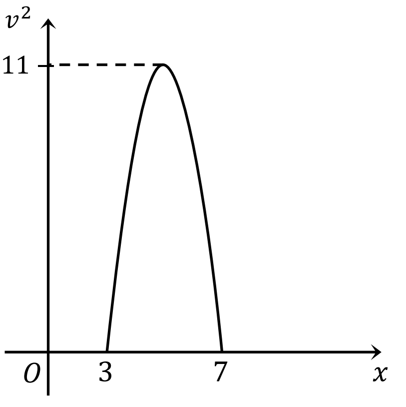
10 Prove for a particle in SHM about a point c with amplitude a that $v^2 = n^2(a^2 - (x - c)^2)$.

11 A particle is moving in a straight line according to the equation $x = 5 + 6\cos 2t + 8\sin 2t$, where x is the displacement in metres and t is the time in seconds.

i Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$.

ii When is the displacement of the particle zero for the first time?

12 A particle is moving along the x-axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is $v \text{ ms}^{-1}$. The parabola at right shows v^2 as a function of x.



i For what value(s) of x is the particle at rest?

ii What is the maximum speed of the particle?

iii The velocity of the particle is given by the equation $v^2 = n^2(a^2 - (x - c)^2)$, where a, c and n are positive constants.

What are the values of a, c and n?

13 **i** Verify that a particle with displacement given by $x = A\cos nt + B\sin nt$, where A and B are constants, is in simple harmonic motion.

ii The particle is initially at the origin and moving with velocity $2n$.

Find the values of A and B.

iii When is the particle first at its greatest distance from the origin?

iv What is the total distance the particle travels between $t = 0$ and $t = \frac{2\pi}{n}$?

14 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$, where n is a positive constant, x is the displacement of the particle and t is time.

i Show that the square of the velocity of the particle is given by $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dv}{dt}$ and a is the amplitude of the motion.

ii Find the maximum speed of the particle.

iii Find the maximum acceleration of the particle.

iv The particle is initially at the origin. Write a formula for x as a function of t, and hence find the first time that the particle's speed is half its maximum speed.

15 The velocity, $v \text{ ms}^{-1}$, of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 8 - 2x - x^2$, where x is in metres.

i Find the centre of the motion, and the two extreme points of motion

ii Find the maximum speed

iii Find an expression for the acceleration of the particle in terms of x.

16 A particle P is moving in simple harmonic motion. At time t seconds, its acceleration is given by $\ddot{x} = -9(x - 2)$, where x metres is the displacement from the origin O. Initially the particle is at O and its velocity is 8 ms^{-1}.

i Find the centre and period of motion

ii Show that $v^2 = 64 + 36x - 9x^2$.

iii Find the maximum speed of the particle.

17 A particle is moving in a straight line and performing simple harmonic motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by

$x = 2\cos\left(2t - \frac{\pi}{4}\right)$, velocity v ms^{-1} and acceleration $\ddot{x}$ ms^{-2}.

i Show that $v^2 - x\ddot{x} = 16$

ii Sketch the graph of x as a function of t for $0 \leq t \leq \pi$ clearly showing the coordinates of the endpoints.

iii Show that the particle first returns to its starting point after one quarter of its period.

iv Find the time taken by the particle to travel the first 100 metres of its motion.

18 A particle moves in such a way that its displacement, x cm, from the origin at any time is given by the function $x = 2 + \cos^2 t$, where t is in seconds.

i Show that acceleration is given by $\ddot{x} = 10 - 4x$

ii Prove $v^2 = -4x^2 + 20x - 24$

1
$v^2 = 4^2(3^2 - x^2)$
$\therefore n = 4, a = 3, T = \dfrac{2\pi}{n} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

2
$$\begin{aligned} v^2 &= n^2\left(-(-2kx + x^2)\right) \\ &= n^2\left(k^2 - (k^2 - 2kx + x^2)\right) \\ &= n^2(k^2 - (k - x)^2) \end{aligned}$$
$\therefore n^2(k^2 - (x - k)^2) \equiv n^2(a^2 - (x - b)^2)$
The amplitude is k and the centre of motion is k. Possible equations of motion include $x = k \sin nt + k$ and $x = k \cos\left(nt + \frac{\pi}{2}\right) + k$

3
Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$
$\therefore R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$
$\alpha = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$
$\therefore x = 2\cos\left(3t + \dfrac{\pi}{6}\right) - 2$
This is in the form $x = a\cos(nt + \alpha) + c$, so the particle is in SHM. The centre of motion is -2 and the amplitude is 2.

Alternatively
$x = \sqrt{3}\cos 3t - \sin 3t - 2$
$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$
$$\begin{aligned} \ddot{x} &= -9\sqrt{3}\cos 3t + 9\sin 3t \\ &= -9\left(\sqrt{3}\cos 3t - \sin 3t - 2 + 2\right) \\ &= -3^2(x + 2) \end{aligned}$$
$\therefore$ the particle is in SHM with centre -2.

Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$
$\therefore R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$
$\therefore$ the amplitude of the motion is 2.

4
i $\frac{2\pi}{n} = \frac{\pi}{4} \rightarrow \quad n = 8$
$$\begin{aligned} \therefore \ddot{x} &= -64(x - 1) \\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= -64(x - 1) \\ \frac{1}{2}v^2 &= -64\int_0^x (x - 1)\,dx \\ v^2 &= -128\left[\frac{x^2}{2} - x\right]_0^x \\ &= -64x^2 + 128x \\ &= 64x(2 - x) \end{aligned}$$

ii Let $v = 0$
$\therefore 0^2 = 64x(2 - x)$
$x = 0,2$

iii The maximum velocity occurs at the centre of motion, $x = 1$, when the particle is moving to the right.
$v_{\text{max}}^2 = 64(1)\left(2 - (1)\right)$
$v_{\text{max}} = \sqrt{64}$
$= 8$

5
i
$$\begin{aligned} \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{d}{dx}\left(4 - 2x - 2x^2\right) \\ &= -2 - 4x \\ &= -4\left(x + \frac{1}{2}\right) \\ &= -2^2\left(x - \left(-\frac{1}{2}\right)\right) \end{aligned}$$

ii
$c = -\dfrac{1}{2}, n = 2$
$T = \dfrac{2\pi}{2} = \pi$

6
$v^2 = 4 - x^2$
$$\begin{aligned} \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{1}{2} \times \frac{d}{dx}\left(4 - x^2\right) \\ &= \frac{1}{2}(-2x) \\ &= -x \end{aligned}$$
$\therefore$ the particle is in SHM with $n = 1$, so the period is 2π.

7 $v^2 = 25(3-2x-x^2)$
$= 5^2\left(-(x^2+2x-3)\right)$
$= 5^2\left(-(x^2+2x+1-4)\right)$
$= 5^2\left(2^2-(x+1)^2\right)$
$\therefore n=5, a=2$ and $c=-1$
A possible equation of motion is
$x = 2\sin(5t)-1$.

8 The extremes of the motion are at $x=5, x=-1$, with maximum (positive) acceleration to the left, so $x=-1$

9
$x = 5\sin 4t + 12\cos 4t = r\sin(4t+\alpha)$
$r = \sqrt{5^2+12^2} = 13$
$x = 13\sin(4t+\alpha)$
$\dot{x} = 52\cos(4t+\alpha)$
since the maximum value of $\cos\theta$ is 1, the maximum value of $\dot{x}$ is 52

10

$$\text{Let } \ddot{x} = -n^2(x-c)$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$$
$$\frac{1}{2}v^2 = -n^2\int_{c-a}^{x}(x-c)\,dx$$
$$v^2 = -2n^2\left[\frac{x^2}{2}-cx\right]_{c-a}^{x}$$
$$= -2n^2\left(\left(\frac{x^2}{2}-cx\right)-\left(\frac{1}{2}(c-a)^2-c(c-a)\right)\right)$$
$$= -2n^2\left(\frac{x^2}{2}-cx-\frac{1}{2}c^2+ac-\frac{1}{2}a^2+c^2-ac\right)$$
$$= n^2(-x^2+2cx-c^2+a^2)$$
$$= n^2\left(a^2-(x^2-2cx+c^2)\right)$$
$$= n^2(a^2-(x-c)^2)$$

Alternatively
Let $x = a\sin(nt+\alpha)+c$
$\therefore \dot{x} = an\cos(nt+\alpha)$
$\therefore v^2 = a^2n^2\cos^2(nt+\alpha)$
$= a^2n^2\left(1-\sin^2(nt+\alpha)\right)$
$= n^2(a^2-a^2\sin^2(nt+\alpha))$
$= n^2(a^2-(a\sin(nt+\alpha)+c$
$= n^2(a^2-(x-c)^2)$

11 **i**
$x = 5+6\cos 2t+8\sin 2t$
$\dot{x} = -12\sin 2t+16\cos 2t$
$\ddot{x} = -24\cos 2t-32\sin 2t$
$= -4(6\cos 2t+8\sin 2t)$
$= -2^2(x-5)$
ii
$5+6\cos 2t+8\sin 2t = 0$
$6\cos 2t+8\sin 2t = -5$
$r = \sqrt{6^2+8^2} = 10$
$\alpha = \tan^{-1}\left(\frac{8}{6}\right) = 0.9272\ldots$
$\therefore 10\cos(2t-0.9272) = -5$
$\cos(2t-0.9272) = -\frac{1}{2}$
$2t-0.9272 = \frac{2\pi}{3}$
$t = \frac{1}{2}\left(\frac{2\pi}{3}+0.9272\right)$
$= 1.510\ldots$
$t = 1.5\ s$ (1 dp)

12 **i**
$x = 3$ or 7

ii
$v = \sqrt{11}$

iii
$c = 5$ (centre of oscillation)
$a = 2$ (the amplitude of the motion)

$v^2 = n^2(a^2-(x-c)^2)$
maximum velocity when $x=5$

$11 = n^2(2^2-(5-5)^2)$
$11 = 4n^2$
$n = \frac{\sqrt{11}}{2}\quad (n>0)$

13 **i**
$x = A\cos nt + B\sin nt$
$\dot{x} = -An\sin nt + Bn\cos nt$
$\ddot{x} = -An^2\cos nt - Bn^2\sin nt$
$= -n^2(A\cos nt + B\sin nt)$
$= -n^2x$
ii
$0 = A\cos 0 + B\sin 0$
$0 = A+0$
$A = 0$
$2n = -An\sin 0 + Bn\cos 0$
$2n = 0+Bn$
$B = 2$

iii
Let $\dot{x} = 0$
$0\cos nt + 2n\cos nt = 0$
$\cos nt = 0$
$nt = \frac{\pi}{2}$
$t = \frac{\pi}{2n}$
iv
$t = 0\ to\ t = \frac{2\pi}{n}$ is one full cycle
$= 4\times\text{amplitude} = 4\times 2 = 8$

14 **i**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$$

$$\frac{1}{2}v^2 = -n^2\int_{-a}^{x} x\,dx$$

$$v^2 = -2n^2\left[\frac{x^2}{2}\right]_{-a}^{x}$$

$$= -2n^2\left(\frac{x^2}{2} - \frac{a^2}{2}\right)$$

$$= n^2(a^2 - x^2)$$

ii

Let $x = 0$

$v = \sqrt{n^2(a^2 - 0)} = an\ \ \text{ms}^{-1}$

iii

Let $x = a$

$$\frac{d^2x}{dt^2} = -n^2a$$

∴ maximum acceleration is n^2a

iv

$x = a\sin(nt)$

$\dot{x} = an\cos(nt)$

$let\ \dot{x} = \dfrac{an}{2}$

$\dfrac{an}{2} = an\cos(nt) \ \rightarrow\ \cos(nt) = \dfrac{1}{2}$

$nt = \dfrac{\pi}{3} \ \rightarrow\ t = \dfrac{\pi}{3n}$

15 **i**

$$v^2 = 8 - 2x - x^2$$
$$= -(x^2 + 2x - 8)$$
$$= -(x + 4)(x - 2)$$
$$= (x + 4)(2 - x)$$

The extremes of motion are $x = -4$ and $x = 2$, with the centre of motion halfway between at $c = \frac{-4+2}{2} = -1$.

ii

Maximum speed at the centre

$v_{\text{max}}^2 = (-1 + 4)\big(2 - (-1)\big) = 9$

∴ $\text{speed}_{\text{max}} = 3\ \text{ms}^{-1}$

iii

$\ddot{x} = -n^2(x - c) = -(x + 1)$

16 **i**

$$\ddot{x} = -9(x - 2)$$
$$= -3^2(x - 2)$$

∴ $n = 3, c = 2$

$$T = \frac{2\pi}{3}$$

ii

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x - 2)$$

$$\frac{1}{2}v^2 - \frac{1}{2}(8)^2 = -9\int_{0}^{x}(x - 2)\,dx$$

$$v^2 - 64 = 18\left[\frac{x^2}{2} - 2x\right]_{x}^{0}$$

$$v^2 = \big(0 - (9x^2 - 36x)\big) + 64$$

$$= 64 + 36x - 9x^2$$

iii

Let $x = 2$

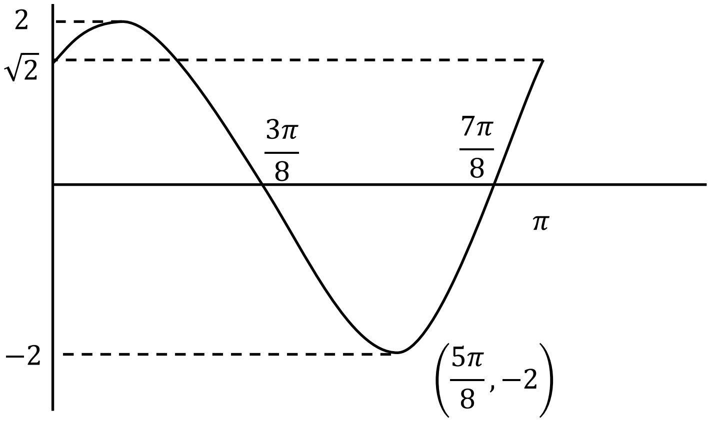
$$v_{\text{max}}^2 = 64 + 36(2) - 9(2)^2$$
$$= 100$$

$\text{speed}_{\text{max}} = 10\ \text{ms}^{-1}$

17 i

$$x = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$v = -4\sin\left(2t - \frac{\pi}{4}\right)$$
$$\ddot{x} = -8\cos\left(2t - \frac{\pi}{4}\right)$$
$$v^2 - x\ddot{x} = \left(-4\sin\left(2t - \frac{\pi}{4}\right)\right)^2 - 2\cos\left(2t - \frac{\pi}{4}\right)\left(-8\cos\left(2t - \frac{\pi}{4}\right)\right)$$
$$= 16\sin^2\left(2t - \frac{\pi}{4}\right) + 16\cos^2\left(2t - \frac{\pi}{4}\right)$$
$$= 16\left(\sin^2\left(2t - \frac{\pi}{4}\right) + \cos^2\left(2t - \frac{\pi}{4}\right)\right)$$
$$= 16$$

ii



iii

$$\sqrt{2} = 2\cos\left(2t - \frac{\pi}{4}\right)$$
$$\cos\left(2t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$2t - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$$
$$2t = 0, \frac{\pi}{2}, 2\pi, \ldots$$
$$t = 0, \frac{\pi}{4}, \pi, \ldots$$

The particle first returns to its starting point after $\frac{\pi}{4}$ seconds, which is one quarter of its period (from the graph).

iv

The amplitude is 2, so each cycle the particle travels $2 \times 4 = 8$ metres.

$\frac{100}{8} = 12.5$, so it will take 12.5 cycles to travel 100 metres

$$t = 12.5 \times T = \frac{25\pi}{2}$$

18 **i**

$$x = 2 + \cos^2 t$$
$$\dot{x} = -2\cos t \sin t$$
$$\ddot{x} = -2(\cos t \times \cos t + \sin t \times (-\sin t))$$
$$= -2(\cos^2 t - \sin^2 t)$$
$$= -2(2\cos^2 t - 1)$$
$$= -2(2(2 + \cos^2 t - 2) - 1)$$
$$= -4(x - 2) + 2$$
$$= -4x + 10$$
$$= 10 - 4x$$

ii

$$x = 2 + \cos^2 t$$
$$= 2 + \frac{1}{2}(1 + \cos 2t)$$
$$= \frac{1}{2}\cos 2t + \frac{5}{2}$$
$$a = \frac{1}{2}, n = 2, c = \frac{5}{2}$$
$$v^2 = (2)^2\left(\left(\frac{1}{2}\right)^2 - \left(x - \frac{5}{2}\right)^2\right)$$
$$= 1 - (2x - 5)^2$$
$$= 1 - 4x^2 + 20x - 25$$
$$= -4x^2 + 20x - 24$$