EXERCISE 6.7

BASIC

1 Given the parametric equations $x = Vt\cos\theta$ and $y = -\frac{gt^2}{2} + Vt\sin\theta$, prove the Cartesian equation of motion of a projectile fired from the Origin is $y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$

2 A stone is thrown from the top of a cliff. Its parametric equations of motion are $x = 5\sqrt{2}t$ and $y = 10 + 2\sqrt{2}t - 5t^2$. What is its Cartesian equation?

3 A cricket player hits a ball at a velocity of $30\ \mathrm{ms}^{-1}$ and the ball just clears a 1 metre high fence which is $50\ \mathrm{m}$ away. Find the two possible angles at which the ball could have been hit, to the nearest degree. Assume there is no air resistance and that $g = 10\ \mathrm{ms}^{-2}$).

The equation of motion is $y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$

MEDIUM

4 A ball is kicked on level ground to clear a fence 2 metres high and 40 metres away. The initial velocity is 30 metres per second and the angle of projection is α. The displacement equations are $x = 30t\cos\alpha$ and $y = -5t^2 + 30t\sin\alpha$. (Do NOT prove these).

i Show that $y = -\frac{x^2}{180}\sec^2\alpha + x\tan\alpha$

ii Hence, or otherwise, find the angles of projection that allow the ball to clear the fence. Answer to the nearest degree.

5 A stone is thrown from the top of a $50\ \mathrm{m}$ high cliff and lands in the sea $30\ \mathrm{m}$ from the base. If the stone was thrown at a velocity of $20\ \mathrm{ms}^{-1}$ what are the possible angles of projection? Assume $g = -9.8\ \mathrm{ms}^{-2}$.

6 A paint ball is fired at a velocity of $20\ \mathrm{ms}^{-1}$, at an angle of θ to the horizontal, at a target $2.5\ \mathrm{m}$ above the ground which is $25\ \mathrm{m}$ horizontally from the point of projection. The paint ball is fired from a height of $1.5\ \mathrm{m}$. Assume $g = -9.8\ \mathrm{ms}^{-2}$.

i The equation of horizontal motion is given by $x = 20t\cos\theta$.
Derive the equations of vertical motion.

ii To avoid overhead power lines the paintball must be fired at an angle less than $45°$.
At what does it need to fired to hit the target?

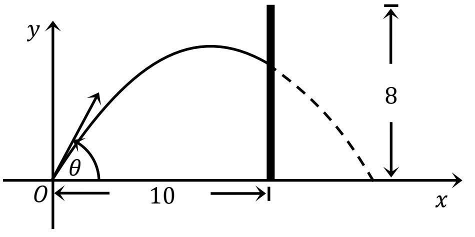
7 A ball is hit at a velocity of $50\ \mathrm{ms}^{-1}$ at an angle of projection θ where $\tan\theta = \frac{3}{4}$.

i Taking the origin as the point of projection and $g = 10\ \mathrm{ms}^{-2}$ show that $\dot{x} = 40$ and $\dot{y} = -10t + 30$, and then find x and y in terms of t.

ii A tall building is $100\ \mathrm{m}$ from where the ball is hit on horizontal ground. If the ball passes through a small open window in the building find the height of the window.

iii Find the velocity and angle that the ball makes with the horizontal as it passes through the window.

8 A small paintball is fired from the origin with initial velocity 14 metres per second towards an eight metre high barrier. The origin is at ground level 10 metres from the base of the barrier. The equations of motion are $x = 14t\cos\theta$ and $y = 14t\sin\theta - 4.9t^2$ where θ is the angle to the horizontal at which the paintball is fired and t is the time in seconds. (Do NOT prove these equations of motion.)



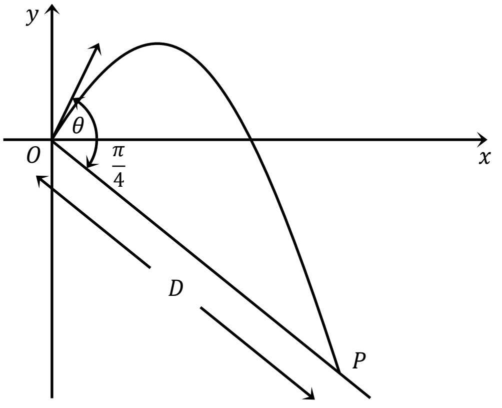
i Show that the equation of trajectory of the paintball is $y = mx - \left(\dfrac{1+m^2}{40}\right)x^2$ where $m = \tan\theta$.

ii Show that the paintball hits the barrier at height h metres when $m = 2 \pm \sqrt{3 - 0.4h}$

iii There is a large hole in the barrier. The bottom of the hole is 3.9 metres above the ground and the top of the hole is 5.9 metres above the ground. The paintball passes through the hole if m is in one of two intervals. One interval is $2.8 \le m \le 3.2$. Find the other interval.

iv Show that, if the paintball passes through the hole, the range is $\dfrac{40m}{1+m^2}$ metres. Hence find the width of the two intervals in which the paintball can land at ground level on the other side of the barrier.

9 The take-off point O on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\frac{\pi}{4}$. A skier takes off from O with velocity $V\ \mathrm{ms}^{-1}$ at angle θ to the horizontal, where $0 \leq \theta < \frac{\pi}{4}$. The skier lands on the downslope at some point P, a distance D metres from O.



The flight path of the skier is given by $\underset{\sim}{r} = \begin{pmatrix} Vt\cos\theta \\ -\frac{1}{2}gt^2 + Vt\sin\theta \end{pmatrix}$, where t is the time in seconds after take-off. (Do NOT prove this.)

i Show that the Cartesian equation of the flight path of the skier is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

ii Show that $D = 2\sqrt{2}\dfrac{V^2}{g}\cos\theta\left(\cos\theta + \sin\theta\right)$

iii Show that $\dfrac{dD}{d\theta} = 2\sqrt{2}\dfrac{V^2}{g}\left(\cos 2\theta - \sin 2\theta\right)$

iv Show that D has a maximum value and find the value of θ for which this occurs.

1

$x = Vt\cos\theta \qquad (1)$

$y = -\dfrac{gt^2}{2} + Vt\sin\theta \qquad (2)$

From (1):

$t = \dfrac{x}{V\cos\theta}$

Substituting into (2):

$$y = -\frac{g\left(\frac{x}{V\cos\theta}\right)^2}{2} + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$y = -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$$

2

$x = 5\sqrt{2}t \qquad (1)$

$y = 10 + 2\sqrt{2}t - 5t^2 \qquad (2)$

$t = \dfrac{x}{5\sqrt{2}} \qquad$ from (1)

sub in (2):

$$y = 10 + 2\sqrt{2}\left(\frac{x}{5\sqrt{2}}\right) - 5\left(\frac{x}{5\sqrt{2}}\right)^2$$

$$= 10 + \frac{2x}{5} - \frac{x^2}{10}$$

3

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$$

$$1 = -\frac{10\times 50^2}{2\times 30^2}(1+\tan^2\theta) + 50\tan\theta$$

$$1 = -13.\dot{8}(1+\tan^2\theta) + 50\tan\theta$$

$$13.\dot{8}\tan^2\theta - 50\tan\theta + 14.\dot{8} = 0$$

$$\tan\theta = \frac{50 \pm \sqrt{50^2 - 4(13.\dot{8})(14.\dot{8})}}{2(13.\dot{8})}$$

$$= 3.2724, \quad 0.3276$$

$$\theta = 89°, \quad 18°$$

4

i

$x = 30t\cos\alpha \ \rightarrow\ t = \dfrac{x}{30\cos\alpha}$

$$\therefore y = -5\left(\frac{x}{30\cos\alpha}\right)^2 + 30\left(\frac{x}{30\cos\alpha}\right)\sin\alpha$$

$$= -\frac{5x^2}{900}\sec^2\alpha + x\tan\alpha$$

$$= -\frac{x^2}{180}\sec^2\alpha + x\tan\alpha$$

ii

Let $x = 40, y = 2$

$$2 = -\frac{40^2}{180}(\tan^2\alpha + 1) + 40\tan\alpha$$

$$2 = -\frac{80}{9}\tan^2\alpha - \frac{80}{9} + 40\tan\alpha$$

$$80\tan^2\alpha - 360\tan\alpha + 98 = 0$$

$$\tan\alpha = \frac{360 \pm \sqrt{(-360)^2 - 4(80)(98)}}{2(80)}$$

$$= -0.2910, 4.2090$$

$$= 16°, 77°$$

The ball will clear the fence for any angle from 16° to 77°.

5

$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$

At $t = 0$ $\underset{\sim}{v}_0 = \begin{pmatrix} 20\cos\theta \\ 20\sin\theta \end{pmatrix} \rightarrow \mathrm{c}_1 = 20\cos\theta, c_2 = 20\sin\theta$

$\therefore \underset{\sim}{v} = \begin{pmatrix} 20\cos\theta \\ -9.8t + 20\sin\theta \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 20t\cos\theta + c_3 \\ -4.9t^2 + 20t\sin\theta + c_4 \end{pmatrix}$

At $t = 0$ $\underset{\sim}{r}_0 = \begin{pmatrix} 0 \\ 50 \end{pmatrix} \rightarrow \mathrm{c}_3 = 0, c_4 = 50$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 20t\cos\theta \\ -4.9t^2 + 20t\sin\theta + 50 \end{pmatrix}$

At impact $x = 30, y = 0$

$\therefore 20t\cos\theta = 30 \rightarrow t = \dfrac{3}{2\cos\theta}$

$0 = -4.9\left(\dfrac{3}{2\cos\theta}\right)^2 + 20\left(\dfrac{3}{2\cos\theta}\right)\sin\theta + 50$

$0 = -\dfrac{4.9 \times 9}{4}\sec^2\theta + 30\tan\theta + 50$

$0 = -11.025(\tan^2\theta + 1) + 30\tan\theta + 50$

$0 = 11.025\tan^2\theta - 30\tan\theta - 38.975$

$\tan\theta = \dfrac{30 \pm \sqrt{(-30)^2 - 4(11.025)(-38.975)}}{2 \times 11.025}$

$\tan\theta = -0.9603, 3.6814,$

$\theta = -43°50', 74°48$

The stone can be thrown up at an angle of $74°48'$ or down at an angle of $43°50'$.

6

i

$\ddot{y} = -9.8$

$\dot{y} - 20\sin\theta = -9.8\int_0^t dt$

$= -9.8\Big[t\Big]_0^t$

$= -9.8t$

$\dot{y} = -9.8t + 20\sin\theta$

$y - 1.5 = \int_0^t (-9.8t + 20\sin\theta)\,dt$

$= \Big[-4.9t^2 + 20t\sin\theta\Big]_0^t$

$= -4.9t^2 + 20t\sin\theta - 0$

$y = -4.9t^2 + 20t\sin\theta + 1.5$

ii

At impact $x = 25, y = 2.5$

$t = \dfrac{25}{20\cos\theta} = \dfrac{5}{4\cos\theta}$

$2.5 = -4.9\left(\dfrac{5}{4\cos\theta}\right)^2 + 20\left(\dfrac{5}{4\cos\theta}\right)\sin\theta + 1.5$

$= -\dfrac{245}{32}\sec^2\theta + 25\tan\theta + 1.5$

$80 = -245(\tan^2\theta + 1) + 800\tan\theta + 48$

$245\tan^2\theta - 800\tan\theta + 277 = 0$

$\tan\theta = \dfrac{800 \pm \sqrt{(-800)^2 - 4(24)(277)}}{2(245)}$

$= 0.3937, 2.8716$

$\theta = 21°, 71°$

The paintball needs to be fired at an angle of $21°$ to miss the power lines and hit the target.

7 **i**

$$\underset{\sim}{v}_0 = \begin{pmatrix} 50\cos\left(\tan^{-1}\frac{3}{4}\right) \\ 50\sin\left(tan^{-1}\frac{3}{4}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 50 \times \frac{4}{\sqrt{3^2+4^2}} \\ 50 \times \frac{3}{\sqrt{3^2+4^2}} \end{pmatrix}$$
$$= \begin{pmatrix} 40 \\ 30 \end{pmatrix}$$
$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$
$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -10t + c_2 \end{pmatrix}$$
At $t = 0$ $\underset{\sim}{v}_0 = \begin{pmatrix} 40 \\ 30 \end{pmatrix}$

$\rightarrow \quad c_1 = 40, c_2 = 30$

$$\therefore \underset{\sim}{v} = \begin{pmatrix} 40 \\ -10t + 30 \end{pmatrix}$$
$$\underset{\sim}{r} = \begin{pmatrix} 40t + c_3 \\ -5t^2 + 30t + c_4 \end{pmatrix}$$
At $t = 0$ $\underset{\sim}{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\rightarrow \quad c_3 = 0, c_4 = 0$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 40t \\ -5t^2 + 30t \end{pmatrix}$$

ii

$$t = \frac{x}{40}$$
$$\therefore y = -5\left(\frac{x}{40}\right)^2 + 30\left(\frac{x}{4}\right)$$
$$= -5\left(\frac{100}{40}\right)^2 + 30\left(\frac{100}{40}\right)$$
$$= 43.75 \text{ m}$$

iii

$$t = \frac{100}{40} = 2.5$$
When $t = 2.5$

$$\dot{x} = 40$$
$$\dot{y} = -10(2.5) + 30 = 5$$
$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$
$$= \sqrt{40^2 + 5^2}$$
$$= 5\sqrt{65}$$
$$\tan\alpha = \frac{\dot{y}}{\dot{x}}$$
$$= \frac{5}{40}$$
$$\alpha = 7°8'$$

8 **i**

$$x = 14t\cos\theta \rightarrow t = \frac{x}{14\cos\theta}$$
$$y = 14\left(\frac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\frac{x}{14\cos\theta}\right)^2$$
$$= \tan\theta\, x - \frac{4.9}{14^2}\sec^2\theta\, x^2$$
$$= \tan\theta\, x - \frac{1}{40}(1 + \tan^2\theta)x^2$$
$$= mx - \frac{1 + m^2}{40}x^2$$

ii

Let $x = 10, y = h$:

$$h = 10m - (1 + m^2) \times \frac{5}{2}$$
$$2h = 20m - 5 - 5m^2$$
$$5m^2 - 20m + 5 + 2h = 0$$
$$m = \frac{20 \pm \sqrt{(-20)^2 - 4(5)(5 + 2h)}}{2(5)}$$
$$= \frac{20 \pm \sqrt{300 - 40h}}{10}$$
$$= 2 \pm \sqrt{3 - 0.4h}$$

iii

$$2 - \sqrt{3 - 0.4 \times 3.9} \le m \le 2 - \sqrt{3 - 0.4 \times 5.9}$$
$$0.8 \le m \le 1.2$$

iv)

Let $y = 0$:

$$0 = mx - \frac{1 + m^2}{40}x^2$$
$$x\left(m - \frac{1 + m^2}{40}x\right) = 0$$
$$\therefore x = 0 \text{ or } m - \frac{1 + m^2}{40}x = 0$$
$$\frac{1 + m^2}{40}x = m$$
$$x = \frac{40m}{1 + m^2} \text{ metres}$$

$m = 0.8$ Range $= \dfrac{40(0.8)}{1 + 0.8^2} = 19.51\ m$

$m = 1$ $\text{Range}_{\max} = \dfrac{40(1)}{1 + 1^2} = 20\ m$

$m = 1.2$ Range $= \dfrac{40(1.2)}{1 + 1.2^2} = 19.67\ m$

$m = 2.8$ Range $= \dfrac{40(2.8)}{1 + 2.8^2} = 12.67\ m$

$m = 3.2$ Range $= \dfrac{40(3.2)}{1 + 3.2^2} = 11.39\ m$

The paintball can land for 49 cm from 19.51 to 20 m, or 128 cm from 11.39 to 12.67 m.

9 **i**

$$x = Vt\cos\theta \ \rightarrow t = \frac{x}{V\cos\theta}$$

$$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$= -\frac{gx^2}{2V^2\cos^2\theta} + \frac{\sin\theta\, x}{\cos\theta}$$

$$= x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

ii

Let P be $\left(\frac{D}{\sqrt{2}}, -\frac{D}{\sqrt{2}}\right)$, since it lies on the line $y = -x$, and $OP = D$.

$$\therefore -\frac{D}{\sqrt{2}} = \frac{D}{\sqrt{2}}\tan\theta - \frac{g\left(\frac{D}{\sqrt{2}}\right)^2}{2V^2}\sec^2\theta$$

$$-\frac{D}{\sqrt{2}} = \frac{D}{\sqrt{2}}\tan\theta - \frac{gD^2}{4V^2}\sec^2\theta$$

$$-\frac{D}{\sqrt{2}}\cos^2\theta = \frac{D}{\sqrt{2}}\sin\theta\cos\theta - \frac{gD^2}{4V^2}$$

$$\frac{gD^2}{4V^2} - \frac{D}{\sqrt{2}}(\sin\theta\cos\theta + \cos^2\theta) = 0$$

$$D\left(\frac{gD}{4V^2} - \frac{1}{\sqrt{2}}\cos\theta(\sin\theta + \cos\theta)\right) = 0$$

$$\therefore D = 0 \ \ or \ \ \frac{gD}{4V^2} = \frac{1}{\sqrt{2}}\cos\theta(\sin\theta + \cos\theta)$$

$$D = 2\sqrt{2}\frac{V^2}{g}\cos\theta(\sin\theta + \cos\theta)$$

iii

$$\frac{dD}{d\theta} = 2\sqrt{2}\frac{V^2}{g}\left[(\cos\theta + \sin\theta)(-\sin\theta) + (\cos\theta)(-\sin\theta + \cos\theta)\right]$$

$$= 2\sqrt{2}\frac{V^2}{g}\left[-\cos\theta\sin\theta - \sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right]$$

$$= 2\sqrt{2}\frac{V^2}{g}\left[\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\right]$$

$$= 2\sqrt{2}\frac{V^2}{g}\left[\cos 2\theta - \sin 2\theta\right]$$

iv

$$\frac{dD^2}{d\theta^2} = 2\sqrt{2}\frac{V^2}{g}\left[-2\sin 2\theta - 2\cos 2\theta\right]$$

$$= -4\sqrt{2}\frac{V^2}{g}\left[\sin 2\theta + \cos 2\theta\right] < 0 \quad \text{for } 0 \le \theta < \frac{\pi}{4}$$

$\therefore D$ has a maximum value

Let $\frac{dD}{d\theta} = 0$:

$$2\sqrt{2}\frac{V^2}{g}\left[\cos 2\theta - \sin 2\theta\right] = 0$$

$\cos 2\theta = \sin 2\theta$

$\tan 2\theta = 1$

$2\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{8}$

The maximum value of D occurs when $\theta = \frac{\pi}{8}$