EXERCISE 6.3

BASIC

1 Prove that a particle where $x = a\cos(nt + \alpha) + c$ is in Simple Harmonic Motion

2 A particle moves in SHM about the centre of motion $x = -1$, with amplitude 2 and period $\frac{3\pi}{2}$. Find a possible equation of motion.

3 A particle moves in SHM about the origin, with a period of $\frac{\pi}{3}$ seconds and amplitude 2 metres. Find

i The maximum and minimum displacement

ii The maximum and minimum velocity

iii The maximum and minimum acceleration

4 A particle moves in SHM about the origin, with displacement given by $x = 2\sin\left(t - \frac{\pi}{4}\right)$.

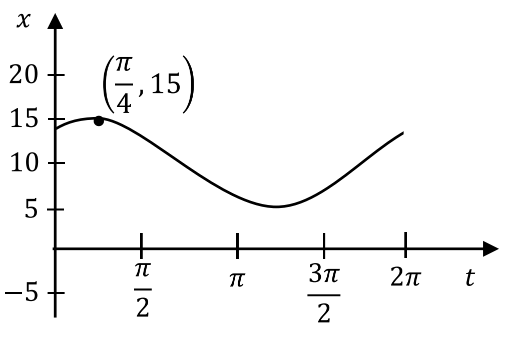
i What is its initial displacement?

ii What is its initial velocity?

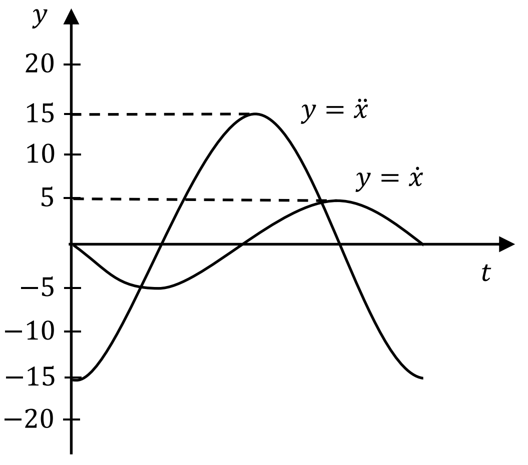
iii Once its velocity is first zero, how long does it take to reach the origin for the second time?

5 A particle moves in SHM centred about the origin. When $x = 2$ the particle is at rest. When $x = 1$ the velocity of the particle is 3. Given the equation of motion is $x = a\sin(nt)$ find the values of a and n.

6 The graph below shows the displacement of a particle in SHM. Its equation of motion is given by $x = a\cos(nt + \alpha) + c$. Find the values of a, n, α and c.



7 The graph below shows part of the graph of the velocity and acceleration of a particle in SHM, with the horizontal scale missing.



i Find the value of n

ii Sketch the displacement of the particle onto the graph.

8 A particle moves in SHM with $x = 2\cos t - 2$. Sketch displacement and acceleration on the same axes.

MEDIUM

9 A particle is undergoing simple harmonic motion on the x-axis about the origin. It is initially at its extreme positive position. The amplitude of motion is 18 and the particle returns to its initial position every 5 seconds.

i Write down an equation for the position of the particle at time t seconds.

ii How long does it take the particle to move from a rest position to the point halfway between the rest position and the equilibrium position?

10 Two particles oscillate horizontally. The displacement of the first is given by $x = 3\sin 4t$ and the displacement of the second is given by $x = a\sin nt$. In one oscillation, the second particle covers twice the distance of the first particle, but in half the time. What are the values of a and n?

11 The displacement, in metres, of a particle from a fixed point in time t, in seconds, $t \geq 0$, is given by $x = 2\cos 3t$. How many oscillations does the particle make per second?

12 The tide can be modelled using simple harmonic motion. At a particular location, the high tide is 9 metres and the low tide is 1 metre. At this location the tide completes 2 full periods every 25 hours. Let t be the time in hours after the first high tide today.

i Explain why the tide can be modelled by the function $x = 5 + 4\cos\left(\frac{4\pi}{25}t\right)$

ii The first high tide tomorrow is at 2 am. What is the earliest time tomorrow at which the tide is increasing at the fastest rate?

13 A particle is oscillating between A and B, 7 m apart, in Simple Harmonic Motion. The time for a particle to travel from B to A and back is 3 seconds. Find the velocity and acceleration at M, the midpoint of OB where O is the centre of AB.

14 A particle moving in simple harmonic motion oscillates about a fixed point O in a straight line with a period of 10 seconds. The maximum displacement of P from O is 5 m. Which of the following statements are true?

If initially the particle is at O moving to the right then 27 second later P will be:

(I) moving with a decreasing displacement

(II) moving with a decreasing speed

(III) moving with a decreasing acceleration

CHALLENGING

15 At the start of the observation yesterday, the upper deck of a ship, anchored at Sydney Wharf was 1.2 metres above the wharf at $6{:}13$ am, when the tide was at its lowest level. At $12{:}03$ pm at the following high tide the last observation record shows that the upper deck was 2.6 metres above the wharf. Considering that the tide moves in simple harmonic motion, find:

i At what time during the observation period, was the upper deck exactly 2 metres above the wharf?

ii What was the maximum rate at which the tide increased during this period of observation?

16 A particle moves in SHM with period T about a centre O. Its displacement at any time t is given by $x = A\sin nt$, where A is the amplitude.

i Draw a neat sketch of one period of this displacement-time equation, showing all intercepts.

ii Show that $\dot{x} = \dfrac{2\pi A}{T}\cos\left(\dfrac{2\pi t}{T}\right)$

iii The point P lies D units on the positive side of O. Let V be the velocity of the particle when it first passes through P. Show that the first time the particle is at P after passing through O is $t = \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

iv Show that the time between the first two occasions when the particle passes through P is $\dfrac{T}{\pi}\tan^{-1}\left(\dfrac{VT}{2\pi D}\right)$. You may assume that $\tan^{-1}x + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ for $x > 0$

1

$x = a\cos(nt+\alpha)+c$
$\dot{x} = -an\sin(nt+\alpha)$
$\ddot{x} = -an^2\cos(nt+\alpha)$
$\quad = -n^2(a\cos(nt+\alpha)+c-c)$
$\quad = -n^2(x-c)$
$\therefore$ a particle where $= a\cos(nt+\alpha)+c$ is in SHM

2

$c=-1, a=2$
$T = \frac{2\pi}{n} = \frac{3\pi}{2} \rightarrow n = \frac{4}{3}$
One possible equation of motion is
$x = 2\sin\left(\frac{4t}{3}\right)-1.$
Alternative solutions would swap the sine for cosine, and replace $\frac{4t}{3}$ with $\frac{4t}{3}+\alpha$ for any angle α

3

$\frac{2\pi}{n} = \frac{\pi}{3} \rightarrow n = 6$
i $x = 2\sin(6t)$
$\therefore -2 \le x \le 2$ since $-1 \le \sin\theta \le 1$
ii $\dot{x} = 12\cos(6t)$
$\therefore -12 \le \dot{x} \le 12$ since $-1 \le \cos\theta \le 1$
iii $\ddot{x} = -72\sin(2t)$
$\therefore -72 \le \ddot{x} \le 72$ since $-1 \le \sin\theta \le 1$

4

i Let $t=0$
$x_0 = 2\sin\left(0-\frac{\pi}{4}\right)$
$\quad = 2\times\left(-\frac{1}{\sqrt{2}}\right)$
$\quad = -\sqrt{2}$

ii $\dot{x} = 2\cos\left(t-\frac{\pi}{4}\right)$
Let $t=0$
$\dot{x}_0 = 2\cos\left(0-\frac{\pi}{4}\right)$
$\quad = 2\times\frac{1}{\sqrt{2}}$
$\quad = \sqrt{2}$

iii To get from one end of the motion to the other end of the motion then back to the centre takes three quarters of the period
$t = \frac{3}{4}\times\frac{2\pi}{1}$
$\quad = \frac{3\pi}{2}$ sec

5

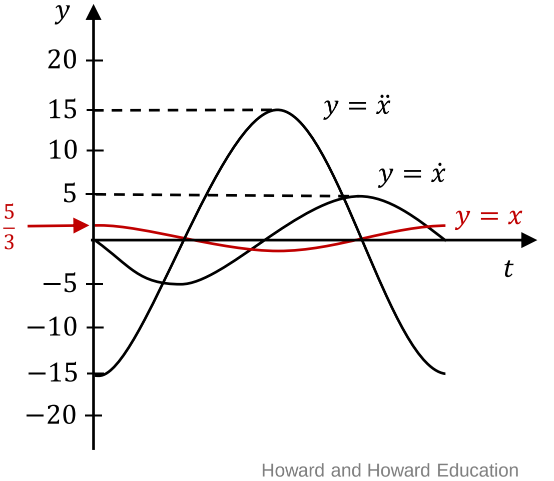
$a=2$
$x = 2\sin(nt)$
$\dot{x} = 2n\cos(nt)$
Let $x=1, \dot{x}=3$
$\therefore 1 = 2\sin(nt) \rightarrow \sin(nt) = \frac{1}{2} \quad (1)$
$3 = 2n\cos(nt) \rightarrow \cos(nt) = \frac{3}{2n} \quad (2)$
$\sin^2(nt)+\cos^2(nt) = 1$
$\therefore \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2n}\right)^2 = 1$
$\frac{1}{4}+\frac{9}{4n^2} = 1$
$n^2+9 = 4n^2$
$3n^2 = 9$
$n^2 = 3$
$n = \sqrt{3}$

6

The difference between the peak and the trough is $15-5=10$, so $a=5$.
The motion has period 2π, so $\frac{2\pi}{n} = 2\pi \rightarrow n = 1$
The curve is the cosine curve shifted vertically by 10 and right by $\frac{\pi}{4}$,
so $\frac{\alpha}{1} = -\frac{\pi}{4} \rightarrow \alpha = -\frac{\pi}{4}$
The centre of motion is the average of the peak and the trough, so $c = \frac{15+5}{2} = 10$.

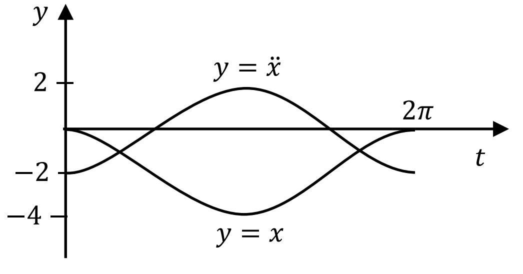
7

i The amplitudes are 5 and 15 respectively, and the amplitude of $\ddot{x}$ is the amplitude of $\dot{x}$ multiplied by n, so $n=3$
ii The graph of displacement is the reflection of the graph of acceleration over the t-axis, but with amplitude $\frac{1}{n}$ that of velocity, so $\frac{5}{3}$.



8 Displacement is a cosine curve with an amplitude of 2, centre of motion -2 and $T = 2\pi$ since $n = 1$. So it is a cosine curve stretched vertically by a factor of 2 then moved down 2 units.

Acceleration is given by $\ddot{x} = -(x+2) = -x-2$ since $n = 1$ and $c = -2$, so it is the displacement curve reflected over the x-axis then moved down 2 units. Notice that it is centred about 0, not 3, as acceleration is proportional to the distance from the centre of motion.



9 **i**

$$n = \frac{2\pi}{T} = \frac{2\pi}{5}$$

$at\ t = 0$ we want $\cos(nt + \alpha) = 1$

$(\text{ or } \sin(nt + \alpha) = 1)$

$$x = 18\cos\frac{2\pi}{5}t$$

$$\left(\text{or } 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)\right)$$

ii

The particle is at rest at the extreme position, so at time $t = 0$.

$$9 = 18\cos\frac{2\pi}{5}t$$

$$\cos\frac{2\pi}{5}t = \frac{1}{2}$$

$$\frac{2\pi}{5}t = \frac{\pi}{3}$$

$$t = \frac{5}{6}\text{ sec}$$

10 Twice the distance means that the amplitude is doubled. Half the time means that the angle velocity (n) is doubled.

$\therefore a_2 = 6, n_2 = 8$

11

$$f = \frac{1}{T} = \frac{n}{2\pi} = \frac{3}{2\pi}$$

12 **i**

$$c = \frac{9+1}{2} = 5$$

$$a = \frac{1}{2}(9-1) = 4$$

The period of motion is

$$\frac{2\pi}{n} = \frac{25}{2} \rightarrow n = \frac{4\pi}{25}$$

$\therefore\ x = 5 + 4\cos\left(\frac{4\pi}{25}t\right)$ if $t = 0$ at high tide

ii

The tide increases fastest as it rises past its centre value, so $\frac{3}{4}$ of a period after 2 am:

$$2\text{ am} + \frac{3}{4} \times 12\frac{1}{2}\text{ hours} = 11{:}22{:}30\text{ am}$$

13 $a = \frac{7}{2}, \frac{2\pi}{n} = 3 \ \rightarrow n = \frac{2\pi}{3}$, at $M\ x = \frac{7}{4}$.

Let $x = \frac{7}{2}\sin\left(\frac{2\pi t}{3}\right)$

$$\dot{x} = \frac{7\pi}{3}\cos\left(\frac{2\pi t}{3}\right)$$

At M:

$$\frac{7}{2}\sin\left(\frac{2\pi t}{3}\right) = \frac{7}{4}$$

$$\sin\left(\frac{2\pi t}{3}\right) = \frac{1}{2}$$

$$\frac{2\pi t}{3} = \frac{\pi}{6}$$

$$t = \frac{1}{4}$$

$$\dot{x}_M = \frac{7\pi}{3}\cos\left(\frac{2\pi\left(\frac{1}{4}\right)}{3}\right) = \frac{7\pi}{3}\cos\frac{\pi}{6} = \frac{7\pi}{3} \times \frac{\sqrt{3}}{2}$$

$= \frac{7\sqrt{3}\pi}{6}$ ms^{-1}

$$\ddot{x}_M = -n^2x = -\left(\frac{2\pi}{3}\right)^2 \times \frac{7}{4} = -\frac{7\pi^2}{9}$$

14 The period is 10 seconds, so after 27 seconds P will have completed to full cycles and be almost halfway through the third. Since it started at O moving right it will be to the right of O moving towards O when $t = 27$.

P is moving left so displacement is decreasing, ∴ (I) is true

P is moving left at an increasing speed, so velocity is becoming move negative, ∴ (II) is true

P is getting closer to the origin, so the magnitude of the acceleration is decreasing. Since acceleration is negative it is increasing, ∴ (III) is false.

15 **i** $a = \dfrac{2.6 - 1.2}{2} = 0.7$

$T = 2 \times (12{:}03 - 6{:}13) = 700$ min

$n = \dfrac{2\pi}{T} = \dfrac{\pi}{350}$; $c = \dfrac{2.6 + 1.2}{2} = 1.9$

$\therefore x = 1.9 - 0.7\cos\left(\dfrac{\pi}{350}t\right)$

where t is measured in minutes from $6{:}13$am.

$$\therefore 2 = 1.9 - 0.7\cos\left(\frac{\pi}{350}t\right)$$

$$\cos\left(\frac{\pi}{350}t\right) = -\frac{1}{7}$$

$$\frac{\pi}{350}t = \cos^{-1}\left(-\frac{1}{7}\right)$$

$$t = \frac{350}{\pi}\left(\cos^{-1}\left(-\frac{1}{7}\right)\right)$$

$$= 190.97$$

$= 3$ hours 11 minutes

The water is 2.0 m high at $9{:}24$ am

ii Maximum rate of increase when it passes the equilibrium point, which is when $t = 175$

$x = 1.9 - 0.7\cos\left(\dfrac{\pi}{350}t\right)$

$\dfrac{dx}{dt} = \dfrac{0.7\pi}{350}\sin\left(\dfrac{\pi}{350}t\right)$

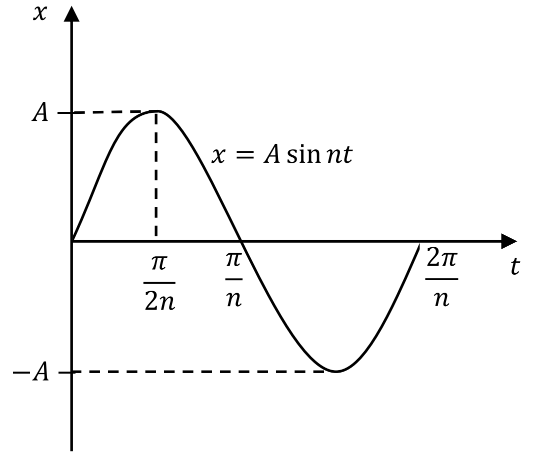
when $t = 175$

$\dfrac{dx}{dt} = \dfrac{0.7\pi}{350}\sin\left(\dfrac{\pi}{350} \times 175\right)$

$= 0.00628$ m/min

$= 37.7$ cm/hour

16 **i**



ii

$T = \dfrac{2\pi}{n} \;\rightarrow\; n = \dfrac{2\pi}{T}$

$x = A\sin nt$

$\quad = A\sin\left(\dfrac{2\pi}{T}t\right)$

$\dot{x} = \dfrac{2\pi A}{T}\cos\left(\dfrac{2\pi t}{T}\right)$

iii

Let $x = D$

$$D = A\sin\left(\frac{2\pi t}{T}\right)$$

$$\frac{D}{A} = \sin\left(\frac{2\pi t}{T}\right) \qquad (1)$$

Let $\dot{x} = V$

$$V = \frac{2\pi A}{T}\cos\left(\frac{2\pi t}{T}\right)$$

$$\frac{VT}{2\pi A} = \cos\left(\frac{2\pi t}{T}\right) \qquad (2)$$

$(1) \div (2)$:

$$\frac{2\pi D}{VT} = \tan\left(\frac{2\pi t}{T}\right)$$

$$\frac{2\pi t}{T} = \tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

$$t = \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

This is the first time that the particle passes through P since all variables are positive.

iv

The particle will pass through P again when t_2 and $\frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$ add to half the period.

$\therefore t_2 = \dfrac{T}{2} - \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

The difference between the times is:

$$\left(\frac{T}{2} - \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)\right) - \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

$$= \frac{T}{2} - \frac{T}{\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

$$= \frac{T}{2} - \frac{T}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{VT}{2\pi D}\right)\right)$$

$$= \frac{T}{2} - \frac{T}{2} + \frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi D}\right)$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi D}\right)$$