EXERCISE 4.9

BASIC

1 If $I_n = \int x^n e^{2x}\,dx$ prove that $I_n = \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2} I_{n-1}$

2 If $I_n = \int \sin^n x\,dx$ prove that $I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

3 If $I_n = \int \cot^n x\,dx$ prove that $I_n = -\dfrac{1}{n-1}\cot^{n-1} x - I_{n-2}$

4 In Question 1 we saw that if $I_n = \int x^n e^{2x}\,dx$ then $I_n = \dfrac{x^n e^{2x}}{2} - \dfrac{n}{2} I_{n-1}$. Find $\int x^2 e^{2x}\,dx$

MEDIUM

5 If $I_n = \int_1^{e^2} (\log_e x)^n\,dx$ prove that $I_n = 2^n e^2 - nI_{n-1}$

6 **i** Let $I_n = \int_0^x \sec^n t\,dt$ where $0 \le x \le \dfrac{\pi}{2}$. Show that $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

ii Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$

7 Prove $\int x \ln^n x\,dx = \dfrac{x^2 \ln^n x}{2} - \dfrac{n}{2}\int x \ln^{n-1} x\,dx$ and hence find $\int x \ln^2 x\,dx$

8 If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ prove that $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

9 If $I_n = \int x^n \sqrt{2x+1}\,dx$ prove that $I_n = \dfrac{x^n\sqrt{(2x+1)^3}}{2n+3} - \dfrac{n}{2n+3} I_{n-1}$

CHALLENGING

10 Find a recurrence relationship for $I_n = \int_0^1 \dfrac{x^n}{(x^2+1)^2}\,dx$

11 Find a recurrence relationship for $I_m = \int_0^1 x^m (x^2-1)^5\,dx$

12 Find a recurrence relationship for $I_n = \int_{-3}^0 x^n \sqrt{x+3}\,dx$

13 Find a recurrence relationship for $I_n = \int \dfrac{dx}{\sin^n x}$

1

$$I_n = \int x^n e^{2x}\,dx$$

$u = x^n$	$\dfrac{dv}{dx} = e^{2x}$
$\dfrac{du}{dx} = nx^{n-1}$	$v = \dfrac{1}{2}e^{2x}$

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2}\int x^{n-1}e^{2x}\,dx$$

$$= \frac{x^n e^{2x}}{2} - \frac{n}{2}I_{n-1}$$

2

$$I_n = \int \sin^n x\,dx$$

$u = \sin^{n-1} x$	$dv = \sin x\ dx$
$\dfrac{du}{dx} = (n-1)\sin^{n-2} x\cos x$	$v = -\cos x$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos x\cos x\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$\therefore I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}I_{n-2}$$

3

$$I_n + I_{n-2}$$

$$= \int (\cot^n x + \cot^{n-2} x)\,dx$$

$$= \int \cot^{n-2} x\,(\cot^2 x + 1)\,dx$$

$$= \int \operatorname{cosec}^2 x\cot^{n-2} x\ dx$$

$$= -\frac{1}{n-1}\cot^{n-1} x$$

$$\therefore I_n = -\frac{1}{n-1}\cot^{n-1} x - I_{n-2}$$

4

$$I_0 = \int x^0 e^{2x}\,dx$$

$$= \int e^{2x}\,dx$$

$$= \frac{1}{2}e^{2x} + c$$

$$\int x^2 e^{2x}\,dx = I_2$$

$$= \frac{x^2e^{2x}}{2} - \frac{2}{2}I_{2-1}$$

$$= \frac{x^2e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{1}{2}I_0\right)$$

$$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x} + c$$

5

$$I_n = \int_1^{e^2} (\log_e x)^n\,dx$$

$$u = (\log_e x)^n \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = n(\log_e x)^{n-1} \times \frac{1}{x} \qquad v = x$$

$$= \Big[x(\log_e x)^n\Big]_1^{e^2} - n\int_1^{e^2} (\log_e x)^{n-1}\,dx$$

$$= e^2 \times 2^n - 0 - nI_{n-1}$$

$$\therefore I_n = 2^n e^2 - nI_{n-1}$$

6

i $I_n = \int_0^x \sec^n t\,dt$

$$u = \sec^{n-2} t \qquad \frac{dv}{dt} = \sec^2 t$$

$$\frac{du}{dt} = (n-2)\sec^{n-3} t \sec t \tan t \qquad v = \tan t$$

$$= (n-2)\sec^{n-2} t \tan t$$

$$= \Big[\sec^{n-2} t \tan t\Big]_0^x - (n-2)\int \sec^{n-2} t \tan^2 t\,dt$$

$$= \sec^{n-2} x \tan x - 0 - (n-2)\int \sec^{n-2} t\,(\sec^2 t - 1)\,dt$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n t\,dt + (n-2)\int \sec^{n-2} t\,dt$$

$$\therefore I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

ii $\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$

$$= I_4$$

$$= \frac{\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}}{3} + \frac{2}{3}I_2$$

$$= 2^2 \times \frac{\sqrt{3}}{3} + \frac{2}{3}\left(\frac{1 \times \tan\frac{\pi}{3}}{1} + 0\right)$$

$$= \frac{4\sqrt{3}}{3} + \frac{2}{3}(\sqrt{3})$$

$$= 2\sqrt{3}$$

7

$$I_n = \int x \ln^n x \, dx = \frac{x^2 \ln^n x}{2} - \frac{n}{2} \int \frac{\ln x}{x} \times x^2 \, dx$$

$$u = \ln^n x \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{n \ln^{n-1} x}{x} \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2 \ln^n x}{2} - \frac{n}{2} \int x \ln^{n-1} x \, dx$$

$$\therefore I_n = \frac{x^2 \ln^n x}{2} - \frac{n}{2} I_{n-1}$$

$$I_0 = \int x \, dx = \frac{x^2}{2} + c$$

$$I_2 = \frac{x^2 \ln^2 x}{2} - \frac{2}{2} I_1$$

$$= \frac{x^2 \ln^2 x}{2} - \left(\frac{x^2 \ln x}{2} - \frac{1}{2} I_0 \right)$$

$$= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2} \left(\frac{x^2}{2} + c \right)$$

$$= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + c$$

8

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$

$$u = x^n \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = \sin x$$

$$u = x^{n-1} \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = (n-1)x^{n-2} \qquad v = -\cos x$$

$$= \Big[x^n \sin x \Big]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2} \right)^n - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2} \right)^n + n \left\{ \Big[x^{n-1} \cos x \Big]_0^{\frac{\pi}{2}} - (n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx \right\}$$

$$= \left(\frac{\pi}{2} \right)^n + n(0 - 0) - n(n-1) I_{n-2}$$

$$\therefore I_n = \left(\frac{\pi}{2} \right)^n - n(n-1) I_{n-2}$$

9

$$u = x^n \qquad \frac{dv}{dx} = (2x+1)^{\frac{1}{2}}$$
$$\frac{du}{dx} = nx^{n-1} \qquad v = \frac{1}{3}(2x+1)^{\frac{3}{2}}$$

$$I_n = \int x^n\sqrt{2x+1}\,dx$$

$$= \frac{x^n(2x+1)^{\frac{3}{2}}}{3} - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int (2x+1)x^{n-1}(2x+1)^{\frac{1}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}\int 2x \times x^{n-1}(2x+1)^{\frac{1}{2}}\,dx - \frac{n}{3}\int x^{n-1}(2x+1)^{\frac{1}{2}}\,dx$$

$$= \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}\int x^n\sqrt{2x+1}\,dx - \frac{n}{3}\int x^{n-1}\sqrt{2x+1}\,dx$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{2n}{3}I_n - \frac{n}{3}I_{n-1}$$

$$\frac{2n+3}{3}I_n = \frac{x^n\sqrt{(2x+1)^3}}{3} - \frac{n}{3}I_{n-1}$$

$$\therefore I_n = \frac{x^n\sqrt{(2x+1)^3}}{2n+3} - \frac{n}{2n+3}I_{n-1}$$

10

$$u = \frac{1}{2}x^{n-1} \qquad \frac{dv}{dx} = 2x(x^2+1)^{-2}$$
$$\frac{du}{dx} = \frac{1}{2}(n-1)x^{n-2} \qquad v = -(x^2+1)^{-1}$$

$$I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx, \qquad \text{for } n = 0, 1, 2\ldots$$

$$= -\frac{1}{2}\left[\frac{x^{n-1}}{x^2+1}\right]_0^1 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= -\frac{1}{4} - 0 + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}(x^2+1)}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}\int_0^1 \frac{x^n}{(x^2+1)^2}\,dx + \frac{n-1}{2}\int_0^1 \frac{x^{n-2}}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{4} + \frac{n-1}{2}I_n + \frac{n-1}{2}I_{n-2}$$

$$\therefore \frac{3-n}{2}I_n = -\frac{1}{4} + \frac{n-1}{2}I_{n-2}$$

$$I_n = \frac{1}{2(n-3)} - \frac{n-1}{n-3}I_{n-2}$$

11

$$I_m = \int_0^1 x^m (x^2-1)^5\,dx$$

$$u = \frac{1}{2}x^{m-1} \qquad \frac{dv}{dx} = 2x(x^2-1)^5$$

$$\frac{du}{dx} = \frac{m-1}{2}x^{m-2} \qquad v = \frac{(x^2-1)^6}{6}$$

$$= \frac{1}{12}\left[x^{m-1}(x^2-1)^6\right]_0^1 - \frac{(m-1)}{12}\int_0^1 x^{m-2}(x^2-1)^6\,dx$$

$$= 0 - \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)(x^2-1)^5\,dx$$

$$= -\frac{m-1}{12}\int_0^1 x^m(x^2-1)^5\,dx + \frac{m-1}{12}\int_0^1 x^{m-2}(x^2-1)^5\,dx$$

$$\therefore \frac{m+11}{12}I_m = \frac{m-1}{12}I_{m-2}$$

$$\therefore I_m = \frac{m-1}{m+11}I_{m-2}$$

12

$$I_n = \int_{-3}^0 x^n\sqrt{x+3}\,dx$$

$$u = x^n \qquad \frac{dv}{dx} = (x+3)^{\frac{1}{2}}$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = \frac{2}{3}(x+3)^{\frac{3}{2}}$$

$$= \frac{2}{3}\left[x^n(x+3)^{\frac{3}{2}}\right]_{-3}^0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)^{\frac{3}{2}}\,dx$$

$$= 0 - \frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)(x+3)^{\frac{1}{2}}\,dx$$

$$= -\frac{2n}{3}\int_{-3}^0 x^n\sqrt{x+3}\,dx - 2n\int_{-3}^0 x^{n-1}\sqrt{x+3}\,dx$$

$$\therefore I_n = -\frac{2n}{3}I_n - 2nI_{n-1}$$

$$\frac{2n+3}{3}I_n = -2nI_{n-1}$$

$$I_n = -\frac{6n}{2n+3}I_{n-1}$$

13

$$u = \operatorname{cosec}^{n-2} x \qquad \frac{dv}{dx} = \operatorname{cosec}^2 x$$

$$\frac{du}{dx} = (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x) \qquad v = -\cot x$$

$$I_n = \int \frac{dx}{\sin^n x}$$

$$= \int \operatorname{cosec}^n x\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$$

$$= -\frac{\cos x}{\sin x} \times \frac{1}{\sin^{n-2} x} - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2)\int \operatorname{cosec}^n x\, dx + (n-2)\int \operatorname{cosec}^{n-2} x\, dx$$

$$\therefore I_n = -\frac{\cos x}{\sin^{n-1} x} - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = -\frac{\cos x}{\sin^{n-1} x} + (n-2)I_{n-2}$$

$$\therefore I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}$$