EXERCISE 1.4

BASIC

1 Prove x is even if and only if x^2 is even.

2 Prove the following statement is false: If $a - b > 0$, where a, b are real, then $a^2 - b^2 > 0$

3 Prove the following statement is false: There are no prime numbers divisible by 7

4 Prove the following statement is false: ∃ a real number x, $-x^2 + 2x - 2 \geq 0$

5 Prove the following statement is false: There is a Pythagorean Triad where the two smallest numbers are even and the largest number is odd.

6 Prove or disprove the following statement: The sum of the squares of three consecutive even numbers is divisible by 4

7 Prove or disprove the following statement: ∃ a real number n such that $3^n + 4^n < 5^n$

MEDIUM

8 Prove for integral x, x^2 is divisible by 9 if and only if x is a multiple of 3.

9 Prove that if m, n are integers that $m^2 - n^2$ is even iff at least one of the sum and difference of m and n are even.

10 Prove the following statement is false: $|2x + 5| \leq 9 \Rightarrow |x| \leq 4$

11 Prove or disprove that if x and y are irrational and $x \neq y$, then xy is irrational.

12 Prove that a number is divisible by 6 if and only if it is divisible by 2 and 3.

13 Prove that the sum of two integers is even if and only if they have the same parity (both odd or both even).

CHALLENGING

14 Prove that a number is divisible by 4 if and only if the last two digits are a multiple of 4.

15 Prove that a three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

1 If x is even, let $x = 2k$ for integral k.
$x^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
$= 2p$ for integral p
$\therefore$ if x is even then x^2 is even

Conversely, we will show that if x^2 is even then x is even using proof by contrapositive.
Suppose x is odd
Let $x = 2k + 1$ for integral k
$x^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2p + 1 \text{ for integral } p \text{ since } k \text{ is integral}$
$\therefore$ if x is odd x^2 is odd, hence if x^2 is even then x is even by contrapositive.
$\therefore x$ is even if and only if x^2 is even □

2 Let $a = -2$ $b = -3$
$\therefore a - b = -2 - (-3) = 1 > 0$
$(-2)^2 - (-3)^2 < 0$, so the statement is false.

3 7 is prime and it is divisible by 7, so the statement is false.

4 $\text{a} < 0$
$\Delta = 2^2 - 4(-1)(-2) < 0$
The quadratic is negative definite, so the statement is false.

5 Here we will use a proof by contradiction to prove it is false for all real numbers.

Let the triad of odd numbers be a, b and c, such that $a = 2i$,
$b = 2j$ and $c = 2k + 1$ for i, j, k integral.

$\therefore (2i)^2 + (2j)^2 = (2k+1)^2$
$4i^2 + 4j^2 = 4k^2 + 4k + 1$
$4(i^2 + j^2) = 4(k^2 + k) + 1$
$4m = 4n + 1 \text{ for integral } m, n \text{ since } i, j, k \text{ are integral}$

Now the LHS is a multiple of 4 yet the RHS isn't so we have a contradiction, so there is no Pythagorean Triad where the two smallest numbers are even and the largest number is odd.

6 Let the three consecutive even numbers be $2k, 2k + 2$ and $2k + 4$ for integral k.

$(2k)^2 + (2k+2)^2 + (2k+4)^2$
$= 4k^2 + 4k^2 + 8k + 4 + 4k^2 + 16k + 16$
$= 12k^2 + 24k + 20$
$= 4(3k^2 + 6k + 5)$
$= 4p \text{ for integral } p \text{ since } k \text{ is integral}$
The statement is true.

7 Let $n = 3$ $3^3 + 4^3 = 27 + 64 = 91 < 5^3$
The statement is true.

8 Prove that if x^2 is divisible by 9 then x is a multiple of 3 by contrapositive
Suppose x is not a multiple of 3
Let $x = 3k + j$ for integral k and $j = 1,2$

$$\begin{aligned}\therefore x^2 &= (3k+j)^2 \\ &= 9k^2 + 6jk + j^2 \\ &= 3(3k^2 + 2k) + j^2 \\ &= 3p + 1 \text{ or } 3p + 4 \ \text{ for integral } p \text{ since } k \text{ is integral, which are not multiples of } 9\end{aligned}$$

$\therefore$ if x is not a multiple of 3 then x^2 is not a multiple of 9
$\therefore$ if x^2 is a multiple of 9 then x is a multiple of 3 by contrapositive.

Conversely, if x is a multiple of 3 let $x = 3j$ for integral j

$$\begin{aligned}x^2 &= (3j)^2 \\ &= 9j^2 \\ &= 9p \text{ for integral } p \text{ since } j \text{ is integral}\end{aligned}$$

$\therefore x^2$ is divisible by 9.

$\therefore x^2$ is divisible by 9 if and only if x is a multiple of 3 □

9 If $m^2 - n^2$ is even then $(m+n)(m-n)$ is even, using the difference of two squares.
$\therefore$ At least one of $m + n$ and $m - n$ is even, since two odd numbers have an odd product,
$\therefore$ If $m^2 - n^2$ is even at least one of the sum and difference of m and n are even.

Conversely, if at least one of $m + n$ and $m - n$ are even then $(m+n)(m-n)$ is even, since the product of two even numbers or an even and an odd number is even.
$\therefore m^2 - n^2$ is even, using the difference of two squares.
$\therefore$ If at least one of the sum and difference of m and n are even then $m^2 - n^2$ is even.

$\therefore m^2 - n^2$ is even iff at least one of the sum and difference of m and n are even. □

10 Let $x = -6$
$|2(-6) + 5| = 7 \le 9$ yet $|-6| > 4$, so the statement is false.

11 Let $x = \sqrt{2}, y = 2\sqrt{2} \therefore xy = 2 \times 2 = 4$
$\therefore$ the statement is false.

12 Let x be divisible by 6
$\therefore x = 6m$ for integral m
$\therefore x = 2 \times 3 \times m$
$\therefore$ if a number is divisible by 6 then it is divisible by 2 and 3.

Conversely, we will use contrapositive to show that if a number is not divisible by 6 then it is not divisible by 2 and 3.
Let $x = 6m + k$ where k is not a multiple of 6

$$\begin{aligned}&= 2 \times 3 \times \left(m + \frac{k}{6}\right) \\ &\neq 2 \times 3 \times p \ \text{ for integral } p \text{ since } k \text{ is not a multiple of } 6\end{aligned}$$

$\therefore$ if a number is not divisible by 6 it is not divisible by 2 and 3.
$\therefore$ if a number is divisible by 2 and 3 then it is divisible by 6

$\therefore$ a number is divisible by 6 if and only if it is divisible by 2 and 3

13 Let $a = 2m + j, b = 2n + j$ for integral m, n and $j = 0,1$
$a + b = 2m + 2n + 2j$
$= 2(m + n + j)$
$= 2p$ for integral p
$\therefore a + b$ is even if a, b have the same parity
Conversely, we will show by contradiction that if two numbers have the same parity then their sum must be even.
Suppose a, b have opposite parity and their sum is even (*)
Let $a = 2m + j, b = 2n + k$ for integral m, n and $j, k = 0,1$ and $j \neq k$
$a + b = 2m + 2n + j + k$
$= 2(m + n) + 1$
$= 2p + 1$ for integral p
$\therefore a + b$ is odd #
This contradicts (*) as $a + b$ cannot be odd and even.
$\therefore$ if two numbers have the same parity then their sum must be even.
$\therefore$ the sum of two integers is even if and only if they have the same parity (both odd or both even)

14 Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $b, c \leq 9$
If the last two digits are a multiple of 4 then $10b + c = 4m$ for integral m
$\therefore x = 4(25a) + 4m$
$= 4(25a + m)$
$= 4p$ for integral p since a, m are integral
$\therefore$ if the last two digits are a multiple of 4 then the number is divisible by 4.
Conversely, we will show by contrapositive that if a number is divisible by 4 then the last two digits are a multiple of 4.
If the last two digits are not a multiple of 4 then $10b + c = 4m + k$ for integral m, k with k not a multiple of 4.
$\therefore x = 4(25a) + 4m + k$
$= 4(25a + m) + k$
$\neq 4p$ for integral p since a, m are integral and k is not a multiple of 4
$\therefore$ if the last two digits are not a multiple of 4 then the number is not divisible by 4.
$\therefore$ if the number is divisible by 4 then the last two digits are a multiple of 4
$\therefore$ a number is divisible by 4 if and only if the last two digits are a multiple of 4.

15 Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $\mathrm{a}, b, c \leq 9$
If the sum of the digits is divisible by 9 then $a + b + c = 9m$ for integral m
$\therefore x = 100a + 10b + c$
$= 99a + a + 9b + b + c$
$= 9(11a + b) + a + b + c$
$= 9p + 9m$ for integral p since a, b are integral
$= 9q$ for integral q since p, m are integral
$\therefore$ if the sum of the digits is divisible by 9 then the three digit number is divisible by 9
Conversely, we will show by contrapositive that if the three digit number is divisible by 9 then the sum of the digits is 9.
Suppose the sum of the digits is not divisible by 9 then $a + b + c = 9m + k$ for integral m, and k not a multiple of 9
$\therefore x = 100a + 10b + c$
$= 99a + a + 9b + b + c$
$= 9(11a + b) + a + b + c$
$= 9p + 9m + k$ for integral p since a, b are integral
$\neq 9q$ for integral q since p, m are integral and k is not a multiple of 9
$\therefore$ if the sum of the digits is not divisible by 9 then the three digit number is not divisible by 9
$\therefore$ if the three digit sum is divisible by 9 then the sum of the digits is divisible by 9
$\therefore$ a three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.