EXERCISE 3.1

BASIC

1 Prove by induction that $2^n + 1$ is divisible by 3 for all odd integers.

2 Prove by induction that the square of an even number is even.

MEDIUM

3 Prove by induction that the product of n even integers is even for $n \geq 2$.

4 Prove by induction that $\sum_{r=1}^{n} 4r + 4^r = 2n(n+1) + \frac{4^{n+1} - 4}{3}$ for $n \geq 1$.

5 Prove by induction that $3^{2n} - 4^n$ is divisible by 5 if n is a positive odd number.

6 Prove by induction that $4^n + 5^n$ is divisible by 9 if n is a positive odd number.

7 Prove by induction that $x^n - y^n$ is divisible by $x - y$, $(x \neq y)$ for integral x, y with n a positive integer.

8 Prove by induction that $4^{n+1} + 6^n$ is divisible by 10 when n is even

9 Prove by induction that $6n + 6 < 2^n \text{ for } n \geq 6$

CHALLENGING

10 Prove by induction that $n^2 < 4^n$ for n a positive integer.

11 Prove by induction that $12^n > 7^n + 5^n$ *for* $n \geq 2$

12 Prove by induction for positive integers n that $1! \times 3! \times 5! \times \ldots \times (2n-1)! \geq (n!)^n$

13 Prove by induction for $n \geq 2$ that $1^3 + 2^3 + \ldots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3$

1 Let $P(n)$ represent the proposition.

$P(1)$ is true since $2^1 + 1 = 3$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $2^k + 1 = 3m$ for integral m

RTP $P(k+2)$ $\quad 2^{k+2} + 1 = 3p$ for integral p

$$\begin{aligned}
\text{LHS} &= 2^{k+1} + 1 \\
&= 2^2\left(2^k + 1\right) - 3 \\
&= 4(3m) - 3 \qquad \text{from } P(k) \\
&= 3(4m - 1) \\
&= 3p \qquad \text{for integral } p \text{ since } m \text{ is integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction $\square$

2 Let $P(n)$ represent the proposition.

$P(2)$ is true since $2^2 = 4$ which is even

If $P(k)$ is true for some arbitrary even $k \geq 2$ then $k^2 = 2m$ for integral m

RTP $P(k+2)$ $\quad (k+2)^2 = 2p$ for integral p

$$\begin{aligned}
\text{LHS} &= k^2 + 4k + 4 \\
&= 2m + 4k + 4 \qquad \text{from } P(k) \\
&= 2(m + 2k + 2) \\
&= 2p \qquad \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction $\square$

3 Let $P(n)$ represent the proposition, and the even numbers be $2j_1, 2j_2, \ldots, 2j_n$ for integral $j_1, j_2, \ldots j_n$.

$P(2)$ is true since $(2j_1)(2j_2) = 4j_1j_2 = 2(2j_1j_2)$ which is even since j_1, j_2 are integral.

If $P(k)$ is true for some arbitrary $k \geq 2$ then $(2\text{j}_1)(2j_2)\ldots(2j_k) = 2m$ for integral m

RTP $P(k+1)$ $\quad (2\text{j}_1)(2j_2)\ldots(2j_k)(2j_{k+1}) = 2p$ for integral p

$$\begin{aligned}
\text{LHS} &= (2\text{j}_1)(2j_2)\ldots(2j_k)(2j_{k+1}) \\
&= (2m)(2j_{k+1}) \qquad \text{from } P(k) \\
&= 4mj_{k+1} \\
&= 2(2mj_{k+1}) \\
&= 2p \qquad \text{for integral } p \text{ since } m \text{ and } j_{k+1} \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction $\square$

4 Let $P(n)$ represent the proposition.

$P(1)$ is true since $4(1)+4^1=8; 2(1)(1+1)+\frac{4^{1+1}-4}{3}=8$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $\sum_{r=1}^{k} 4r+4^r=2k(k+1)+\frac{4^{k+1}-4}{3}$

RTP $P(k+1)$ $\sum_{r=1}^{k+1} 4r+4^r=2(k+1)(k+2)+\frac{4^{k+2}-4}{3}$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k} 4r+4^r+4(k+1)+4^{k+1} \\
&= 2k(k+1)+\frac{4^{k+1}-4}{3}+4(k+1)+4^{k+1} \qquad \text{from } P(k) \\
&= 2k(k+1)+4(k+1)+\frac{4^{k+1}-4}{3}+4^{k+1} \\
&= 2(k+1)\big((k+1)+1\big)+\frac{4^{k+1}-4+3\times 4^{k+1}}{3} \\
&= 2(k+1)(k+2)+\frac{4\times 4^{k+1}-4}{3} \\
&= 2(k+1)(k+2)+\frac{4^{k+2}-4}{3} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

5 Let $P(n)$ represent the proposition.

$P(1)$ is true since $3^{2(1)}-4^1=5$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $3^{2k}-4^k=5m$ for integral m

RTP $P(k+2)$ $3^{2(k+2)}-4^{k+2}=5p$ for integral p

$$\begin{aligned}
\text{LHS} &= 3^{2k+4}-4^{(k+2)} \\
&= 81\left(3^{2k}\right)-16\left(4^k\right) \\
&= 81\left(3^{2k}-4^k\right)+65\left(4^k\right) \\
&= 9(5m)+65\left(4^k\right) \quad \text{from } P(k) \\
&= 5\left(9m+13\times 4^k\right) \\
&= 5p \qquad \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction □

6 Let $P(n)$ represent the proposition.

$P(1)$ is true since $4^1+5^1=9$

If $P(k)$ is true for some arbitrary odd $k \geq 1$ then $4^k+5^k=9m$ for integral m

RTP $P(k+2)$ $4^{k+2}+5^{k+2}=9p$ for integral p

LHS $= 16 \cdot 4^k + 25 \cdot 5^k$

$= 16(4^k + 5^k) + 9 \cdot 5^k$

$= 16(9m) + 9 \cdot 5^k$ from $P(k)$

$= 9(16m + 5^k)$

$= 9p$ for integral p since m and k are integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for odd $n \geq 1$ by induction □

7 Let $P(n)$ represent the proposition.

$P(1)$ is true since $x^1 - y^1 = x - y$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $x^k - y^k = m(x-y)$ for integral m

RTP $P(k+1)$ $x^{k+1} - y^{k+1} = p(x-y)$ for integral p

LHS $= x^{k+1} - y^{k+1}$

$= x \cdot x^k - y \cdot y^k$

$= x(x^k - y^k) - (y - x)y^k$

$= x(m(x-y)) + (x-y)y^k$ from $P(k)$

$= (x-y)(mx + y^k)$

$= p(x-y)$ for integral p since m, x, y and k are integral

$=$ RHS

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

8 Let $P(n)$ represent the proposition.

$P(2)$ is true since $4^{2+1} + 6^2 = 100 = 10(10)$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $4^{k+1} + 6^k = 10m$ for integral m

RTP $P(k+2)$ $4^{k+3} + 6^{k+2} = 10p$ for integral p

LHS $= 16(4^{k+1}) + 36(6^k)$
$= 16(4^{k+1} + 6^k) + 20(6^k)$
$= 16(10m) + 20(6^k)$ from $P(k)$
$= 10(16m + 2 \times 6^k)$
$= 10p$ for integral p since m and k are integral
$=$ RHS
$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction □

9 Let $P(n)$ represent the proposition.

$P(6)$ is true since $\text{LHS} = 6(6) + 6 = 42; \text{RHS} = 2^6 = 64$

If $P(k)$ is true for some arbitrary $k \geq 6$ then $6k + 6 < 2^k$

RTP $P(k+1)$ $\quad 6k + 12 < 2^{k+1}$

$$\begin{aligned} \text{LHS} &= 6k + 6 + 6 \\ &< 2^k + 6 \quad \text{from } P(k) \\ &< 2^k + 2^k \quad \text{for } k \geq 6 \\ &= 2^{k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 6$ by induction □

10 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1^2 < 4^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $k^2 < 4^k$

RTP $P(k+1)$ $\quad (k+1)^2 < 4^{k+1}$

$$\begin{aligned} \text{LHS} &= (k+1)^2 \\ &= k^2 + 2k + 1 \\ &< 4^k + 2k + 1 \quad \text{from } P(k) \\ &< 4^k + 3 \times 4^k \quad \text{since } 2k + 1 < 3\left(4^k\right) \text{ for } k \geq 1 \\ &= 4\left(4^k\right) \\ &= 4^{k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction □

11 Let $P(n)$ represent the proposition.

$P(2)$ is true since $\text{LHS} = 12^2 = 144; \text{RHS} = 7^2 + 5^2 = 74$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $12^k > 7^k + 5^k$

RTP $P(k+1)$ $\quad 12^{k+1} > 7^{k+1} + 5^{k+1}$

$$\begin{aligned} \text{LHS} &= 12(12^k) \\ &> 12\left(7^k + 5^k\right) \quad \text{from } P(k) \\ &= 12(7^k) + 12(5^k) \\ &> 7\left(7^k\right) + 5\left(5^k\right) \\ &= 7^{k+1} + 5^{k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 2$ by induction □

12 Let $P(n)$ represent the proposition.

$P(1)$ is true since $1! \geq \left((1)!\right)^1$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $1! \times 3! \times 5! \times \ldots \times (2k-1)! \geq (k!)^k$

RTP $P(k+1)$ $\quad 1! \times 3! \times 5! \times \ldots \times (2k-1)! \times (2k+1)! \geq ((k+1)!)^{k+1}$

$$\begin{aligned}
\text{LHS} &= 1! \times 3! \times 5! \times \ldots \times (2k-1)! \times (2k+1)! \\
&\geq (k!)^k \times (2k+1)! \quad \text{from } P(k) \\
&= (k!)^k \cdot \left(\underbrace{(2k+1)\cdot(2k)\cdot(2k-1)\ldots(k+2)}_{k \text{ terms}}(k+1)k!\right) \\
&\geq (k!)^k \cdot \left((k+1)^k (k+1)!\right) \\
&= \left((k+1)k!\right)^k \cdot (k+1)! \\
&= \left((k+1)!\right)^k \cdot (k+1)! \\
&= \left((k+1)!\right)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

13 Let $P(n)$ represent the proposition.

$P(2)$ is true since $1^3 < \frac{2^4}{4} < 1^3 + 2^3 \;\rightarrow\; 1 < 4 < 9$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3$

RTP $P(k+1)$ $\quad 1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$$1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3 \quad \text{from } P(k)$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1^3 + 2^3 + \ldots + k^3 + k^3$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4}$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} < 1^3 + 2^3 + \ldots + k^3 + k^3 + 3k^2 + 3k + 1$$

$$1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$