A Memory-Based Theory of Beliefs
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Introduction

- Since Lucas (1976), shift towards rational expectations
- Models robust to policy changes but fail to acknowledge cognitive biases
- Tversky and Kahneman (1974) posit that common ‘belief biases’ can be explained by agents following the representativeness heuristic
    - conjunction/disjunction fallacy
    - base rate neglect
    - insensitivity to sample size...
- Recently, those biases have been revisited in micro as well as in macro
- We contribute to this body of research by introducing a model where memory shapes beliefs

- The human brain can store an estimated 2.5 million gigabytes of data, while we can only hold 7±2 items in our working memory (Miller, 1956)
- ⇒ recall process from long-term to short-term memory is the real bottleneck
- Bordalo et al. (2022) build a model where memory shapes beliefs through how similar a hypothesis is known data
- Our approach differs on at least 3 counts:

	Bordalo et al. (2022)	Our Model
Object of focus	Contents of memories	Counts of memories
Sample space	$\Omega = \{0, 1\}^F$	Any finite Ω
Aim	Illustrate specific biases	Fully characterize beliefs consistent with model

Introduction

Related Literature

- Belief biases/Representativeness heuristic:
    - Implications of biases: Tversky and Kahneman (1974), Benjamin (2018), Zhao (2018)
    - Evidence of importance: Bordalo et al. (2018), Bordalo et al. (2020), Bianchi et al. (2021), L'Huillier et al. (2021)
- Memory and beliefs:
    - Related models: Bordalo et al. (2022), Gennaioli and Shleifer (2010)
    - Psychological evidence: Schacter and Scarry (2001), Kahana (2012)

Introduction

Research Question

What beliefs are consistent with the limited recall of past observations?

Our tentative answer:

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	Beliefs	Recall Process	Conclusion
Model 1	Depend on relevant information only	Defined on sample space	Bayesian beliefs

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Our tentative answer:

	Beliefs	Recall Process	Conclusion
Model 1	Depend on relevant information only	Defined on sample space	Bayesian beliefs
Model 2	Depend on both relevant and irrelevant information	Defined on σ-algebra	Non-Bayesian beliefs*

Framework

Notation

- Ω: finite sample space
- $\Sigma = 2^{\Omega}$: power set
- $\Sigma^* = \Sigma \setminus \{\emptyset\}$
- N: number of realizations observed by agent
- $m : \Sigma \to \mathbb{Z}_+$: a memory database, where $m(A)$ is the number of realizations equal to members of A recorded in the DM's long-term memory
- $m(\{\emptyset\}) \equiv 0$ and $m(\Omega) = N$
- For A and B disjoint, we have $m(A \cup B) = m(A) + m(B)$

Framework

Example

- Consider A the event "inflation is high" and B the event "interest rates are low"
- $m(A)$ is the number of times the agent has observed high inflation in the past
- $m(B)$ is the number of times the agent has observed low interest rates in the past
- $m(A \cap B)$ is the count of simultaneous observations of high inflation and low interest rates

Framework

Conditional Beliefs

- We assume the econometrician observes conditional beliefs of an agent over Σ

Definition 1 (Conditional Beliefs)

A function $b : \Sigma \times \Sigma^* \to [0, 1]$ is called a conditional belief function

- In our example: $b(A|B)$ is the agent's reported belief of high inflation given that the interest rates are low
- Note that if $B = \Omega$, then the belief is unconditional

Framework

Recall Process

- We assume the agent has a fixed number $n \in \mathbb{N}$ of 'slots' in their working memory
- Each slot is filled with a past observation of the agent's long-term memory
- This recall process is stochastic: the probability that an observation is successfully sampled is given by some conditional probability $p_{A|B}$
- Our 2 models make different assumptions on $p_{A|B}$

Framework

Working Memory

- Let $W_{A|B}$ be the number of successful recalls of A conditional on B, then
$$W_{A|B} \sim \text{Binomial}(n, p_{A|B})$$
- If A conditional on B fails to be recalled, A^c conditional on B is recalled
- Thus, $W_{A^c|B} = n - W_{A|B}$ where $W_{A^c|B}$ is also Binomial

Framework

Beliefs

- After resolving recalls, the agent forms a conditional belief Π such that
$$\Pi_{A|B} = \frac{W_{A|B}}{W_{A|B} + W_{A^c|B}} = \frac{W_{A|B}}{n}$$
- As $W_{A|B}$ follows a binomial distribution, we have:
$$\mathbb{E}[W_{A|B}] = np_{A|B}$$
- Thus, we can obtain the expected conditional belief function:
$$\mathbb{E}[\Pi_{A|B}] = p_{A|B}$$
- We focus on expected beliefs and leave the randomness introduced by the sampling process to future work

Model I: Subjective Bayesianism

Research Question

What beliefs are consistent with the limited recall of past observations?

Our tentative answer:

	Beliefs	Recall Process	Conclusion
Model 1	**Depend on relevant information only**	**Depends on sample space**	**Bayesian beliefs**
Model 2	Depend on both relevant and irrelevant information	Depends on σ-algebra	Non-Bayesian beliefs*

Model I: Subjective Bayesianism

- Simplified recall model where recall only depends on one elementary recall function

Definition 2 (Elementary Recall Function)

A function $r : \Omega \mapsto \mathbb{R}_+$ is called an elementary recall function

- Implication is subjective Bayesian agent
- We introduce 3 belief axioms:
  1. Unitarity of unconditional beliefs
  2. Finite additivity of unconditional beliefs
  3. Bayesian conditional beliefs

Model I: Subjective Bayesianism

Axioms

Axiom 1 (Unitarity of unconditional beliefs)

We say that b satisfies unitarity of unconditional beliefs if $b(\Omega|\Omega) = 1$.

- Second Kolmogorov axiom (first is implied by definition of b)

Model I: Subjective Bayesianism

Axioms

Axiom 2 (Finite additivity of unconditional beliefs)

Let $A_1, A_2, \ldots, A_N$ be any disjoint sets of Σ, then we say that b satisfies finite additivity of unconditional beliefs if

$$b\left(\bigcup_{i=1}^{N} A_i \middle| \Omega\right) = \sum_{i=1}^{N} b(A_i|\Omega)$$

- Third Kolmogorov axiom, belief of union of disjoint sets is the sum of beliefs of each disjoint set
- Reported belief of observing high inflation and medium inflation is equal to belief of observing high inflation + belief of observing medium inflation

Model I: Subjective Bayesianism

Axioms

Axiom 3 (Bayesian conditional beliefs)

We say that b is Bayesian with respect to unconditional beliefs if for all sets $A \in \Sigma, B \in \Sigma^*$

$$b(A|B) = \frac{b(A \cap B|\Omega)}{b(B|\Omega)}$$

- Beliefs must satisfy Bayes rule with respect to unconditional beliefs
- 3 axioms combined: belief must be a probability distribution and must be Bayesian
- Does not imply that unconditional probabilities are correct with respect to some objective probabilities
- Distortions can be applied to unconditional beliefs and still satisfy those 3 axioms

Model I: Subjective Bayesianism

Recall Process

- The probability of recall takes the form:

$$p_{A|B} = \frac{\sum_{x \in A \cap B} r(x)}{\sum_{y \in B} r(y)}$$

- Recall process only depends on one single univariate function of the sample space
- Each elementary event has its own “strength of recall”

Definition 3 (r generates b)

An elementary recall function $r : \Omega \mapsto \mathbb{R}_+$ generates belief b if for all $A \in \Sigma$, and $B \in \Sigma^*$, we have:

$$b(A|B) = \frac{\displaystyle\sum_{x \in A \cap B} r(x)}{\displaystyle\sum_{y \in B} r(y)}$$

Model I: Subjective Bayesianism

Representation Theorem

Theorem 4

The following are equivalent:

1. *There exists* $r : \Omega \to \mathbb{R}_+$ *that generates* b.
2. b *satisfies unitarity of unconditional beliefs, finite additivity of unconditional beliefs and Bayesian conditional beliefs.*

Proof.

In Appendix A. □

- Beliefs which follow this recall process must be subjective Bayesian
- If conditional beliefs do not reflect true conditional probabilities, then unconditional beliefs must also disagree with unconditional probabilities

Model I: Subjective Bayesianism

Representation Theorem

Corollary 5

r is unique up to a positive linear transformation.

- Beliefs are insensitive to absolute counts of each events, only relative counts matter

Corollary 6

Let b be conditional beliefs generated by some recall function r. $b(x|\Omega) = m(x)/m(\Omega)$ if and only if $r(x) = \alpha m(x)$ for some $\alpha > 0$.

- Unconditional beliefs reflect memory frequencies if and only if recall is proportional to count
- In the long-run, we can assume agents observe enough realization so frequencies in the memory converge to true probabilities; but we can still observe biases

Model II: Interference of Irrelevant Data

Research Question

What beliefs are consistent with the limited recall of past observations?

Our tentative answer:

	Beliefs	Recall Process	Conclusion
Model 1	Depend on relevant information only	Depends on sample space	Bayesian beliefs
Model 2	**Depend on both relevant and irrelevant information**	**Depends on σ-algebra**	**Non-Bayesian beliefs****

Model II: Interference of Irrelevant Data

- Beliefs biases can come from interference of irrelevant data
- Bordalo et al. (2022) find strong supporting experimental evidence of this
- To allow for this recall process to reflect this, we make 4 natural assumptions on the recall function and look at their implications for beliefs

Model II: Interference of Irrelevant Data

Relevant v. Irrelevant Data

- Agents try to recall past observations and use those to compute frequencies and then report belief $b(A|B)$
- In particular, let $\mathcal{A}_{A,B}$ be the cross partition of collections $\{A, A^c\}$ and $\{B, B^c\}$, then:

$$\mathcal{A}_{A,B} = \{A \cap B, A^c \cap B, A \cap B^c, A^c \cap B^c\}$$

- In this model we assume that the probability that an observation is recalled depends on where it belongs in $\mathcal{A}_{A,B}$

Relevant information	Irrelevant information
$A \cap B$	$A \cap B^c$
$A^c \cap B$	$A^c \cap B^c$

Model II: Interference of Irrelevant Data

Relevant v. Irrelevant Data

- In this model, the recall process works as follows:
  1. The agent tries to recall past observations pertaining to A given B
  2. If a recalled observation is a member of $A \cap B^c$ or $A^c \cap B^c$, it is deemed irrelevant, and therefore discarded
  3. The agent continues to sample until they fill the n-th slot in their working memory
  4. The agent forms and reports belief $b(A|B)$: it is the ratio of the number of observations of type $A \cap B$ recalled over n
- We allow for irrelevant information to have an impact on the recall probabilities of relevant information but we do not allow it to be sampled

Model II: Interference of Irrelevant Data

Relevant v. Irrelevant Data

- We assume that the probability that an element of $A \cup B$ is recalled only depends on the number of observations in each set in the cross-partition $\mathcal{A}_{A,B}$
- Formally, with this recall process, we have

$$p_{A|B} = p^1(\aleph_{A,B})$$

where $\aleph_{A,B} = (m(A \cap B), m(A^c \cap B), m(A \cap B^c), m(A^c \cap B^c))$ and $p^i : \mathbb{Z}_+^4 \mapsto \mathbb{R}_+, \forall i \in \{1, 2, 3, 4\}$

- Here, p^i denotes the probability of successfully recalling a member from i-th set in $\mathcal{A}_{A,B}$
- We can think of this recall process as a contest between 4 types of memories that attempt to be sampled, each $m(\cdot)$ is their exerted effort and p^i is their probability to succeed

Model II: Interference of Irrelevant Data

Recall Process

Assumption 1

$\sum_{i=1}^{4} p^i(\aleph_{A,B}) = 1$ and $p^i(\aleph_{A,B}) \geq 0$ for all $i \in \{1, \ldots, 4\}$ and all $\aleph_{A,B}$. For some event $X_i \in \mathcal{A}$, if $m(X_i) > 0$, then $p^i(\aleph_{A,B}) > 0$.

- The recall process must satisfy properties of a probability distribution
- When an observation has a strictly positive count in the long-term memory, then the probability of being sampled should be strictly positive

Model II: Interference of Irrelevant Data

Recall Process

Assumption 2

For all $i \in \{1, \ldots, 4\}$, $p^i(\aleph_{A,B})$ is nondecreasing in $m(X_i)$ and nonincreasing in $m(X_j)$, for all $j \neq i$.

- An observation's probability of being sampled is nondecreasing in own effort but nonincreasing in every other memory's effort
- Probability of recalling A given B is nondecreasing in $m(A \cap B)$ but nonincreasing in $m(A^c \cap B)$ (among others)

Model II: Interference of Irrelevant Data

Recall Process

Assumption 3

If π is a permutation such that $\pi(x_i) = x_j \Rightarrow \pi(x_j) = x_i$ (involution) and $\pi(x_1) = x_2 \Rightarrow \pi(x_3) = x_4$; we have: $p^{\pi(i)}(\aleph_{\pi(X)}) = p^i(m(X_{\pi(1)}), \ldots, m(X_{\pi(4)})), \forall i \in \{1, 2, 3, 4\}$

- Partial anonymity condition implies that the following have to be equal:
    1. $p^2(\aleph_{A,B})$: when reporting belief $b(A|B)$, the probability that an event in $A^c \cap B$ is sampled
    2. $p^1(\aleph_{A^c,B})$: when reporting belief $b(A^c|B)$, the probability that an event in $A^c \cap B$ is sampled

Model II: Interference of Irrelevant Data

Recall Process

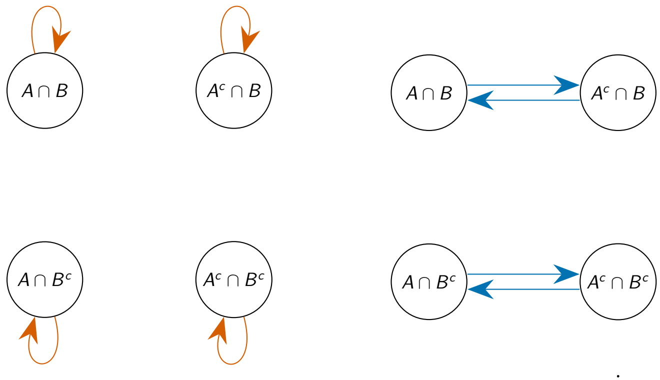

Figure: Graphical representation of Assumption 3

- Arrows indicate anonymous permutations in our model

Model II: Interference of Irrelevant Data

Recall Process

Assumption 4

$p^i_K(\aleph_{A,B}) = \frac{p^i(\aleph_{A,B})}{\sum_{k \in K} p^k(\aleph_{A,B})}, \forall i \in K$ *and* $\forall K \subseteq \{1,2,3,4\}$ *with* $|K| \geq 2$.

- This condition defines our re-sampling: the probability of a member of $A \cap B$ being sampled, after re-sampling, is a normalized version of the general sampling of $A \cap B$, with only relevant information
- Irrelevant information has an impact on each probability of recall (as seen in assumption 2), but cannot be sampled
- In this model, we have $K = \{1, 2\}$

Model II: Interference of Irrelevant Data

Axioms

- We now discuss the empirical content of this model and state 4 axioms on the beliefs of agents

Axiom 4 (Unitarity of Complements)

For all A and $B \subseteq \Omega$:

$$b(A|B) + b(A^c|B) = 1$$

- We do not require unconditional beliefs to sum to one, but complements must sum to one
- Consequence of anonymity condition: $p^1_K(\aleph_{A,B}) = p^2_K(\aleph_{A^c,B})$

Model II: Interference of Irrelevant Data

Axioms

Axiom 5 (Subset Certainty)

$\forall A, B \subseteq \Omega,\ \textit{if}\ B \subseteq A,\ \textit{then}\ b(A|B) = 1$

- If the condition B is a subset of the event A then the conditional belief must be equal to 1

Model II: Interference of Irrelevant Data

Axioms

Definition 7 (Covering Order)

We say A' is greater than A in the B-covering order, and we write $A' \succeq_B A$, if $A \subseteq A'$ and $A' \setminus A \subseteq B$.

- A' is greater than A in the B-covering order if A' contains "more of" B than A and elements not in B are held constant

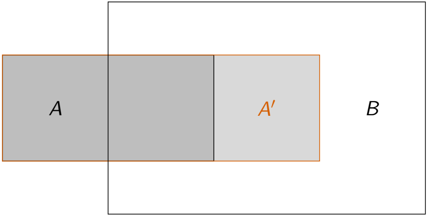

Figure: Graphical representation of $A' \succeq_B A$

Model II: Interference of Irrelevant Data

Axioms

Axiom 6 (Monotonicity in Covering Orders)

Let A, A', B, and B' be events in Σ. If $A' \succeq_B A$, and $B' \succeq_A B$ then:

$$b(A|B) \leq b(A'|B')$$

- It is straightforward to show that the condition that $A' \succeq_B A$, and $B' \succeq_A B$ implies that $A \cap B \subseteq A' \cap B'$, while $A \cup B = A' \cup B'$
- $\Rightarrow$if two events A' and B' have more in common than A and B, while their union is held fixed, then the conditional belief of A' given B' must be greater than the conditional belief of A and B

Model II: Interference of Irrelevant Data

Axioms

Axiom 7 (Union-Preservation of Ordering)

For all events A, A', B and C such that $A, A' \subseteq B$, $C \subseteq B \setminus (A \cup A')$, we have that if $b(A|B) \leq b(A'|B)$, then $b(A \cup C|B) \leq b(A' \cup C|B)$.

- If, given the same conditional B, a belief is greater than another, then this order is preserved when adding more events from B into A and A'

Model II: Interference of Irrelevant Data

Representation Theorem

Proposition 1

If p and m generate b, then b satisfies unitarity of complements, subset certainty, monotonicity in covering orders and union preservation of ordering.

Proof.

In Appendix A □

Conjecture 1

If b satisfies unitarity of complements, subset certainty, monotonicity in covering orders and union-preservation of ordering, then there exist p and m that generate b.

- Proposition 1, together with Conjecture 1 constitute our representation theorem

Next Steps

- We developed a model of belief formation through memory where recall plays the central role
- We derive beliefs from this model and offer a representation of subjective Bayesian beliefs through a specific recall process
- We discussed the importance of departures from Bayesianism through interference of irrelevant data and developed assumptions on recall and attempt to characterize the induced beliefs

Next Steps

- Next steps:
  1. Prove the converse of Proposition 1 (in progress)
  2. Verify which common belief biases can be captured by this model
  3. Find functional forms for p that generate some selected belief biases, such as characteristic overreaction
  4. Potentially look at a dynamic version of this model where realizations could be forgotten and replaced and see the impact on belief biases and their evolution

Appendix A: Proofs I

Theorem 1. The following are equivalent:

1. There exists $r : \Omega \to \mathbb{R}_+$ that generates b.
2. b satisfies unitarity of conditional beliefs, finite additivity of unconditional beliefs and Bayesian conditional beliefs.

Proof. Consider $A \in \Sigma$ and $B \in \Sigma^*$, we want to show that r generates b if and only if b satisfies axioms 1-3.

1. We can start by assuming r generates conditional belief b:

$$b(\Omega|\Omega) = \frac{\sum_{x \in \Omega} r(x)}{\sum_{y \in \Omega} r(y)} = 1$$

Appendix A: Proofs II

Thus satisfying axiom 1.

Now consider $A_1, A_2, \ldots A_N$ sequence of N disjoint sets in Σ, then:

$$b\left(\bigcup_{i=1}^{N} A_i \middle| \Omega\right) = \frac{\sum_{i=1}^{N}\sum_{x\in A_i} r(x)}{\sum_{y\in\Omega} r(y)}$$

$$= \sum_{i=1}^{N} \frac{\sum_{x\in A_i} r(x)}{\sum_{y\in\Omega} r(y)}$$

Appendix A: Proofs III

$$= \sum_{i=1}^{N} b(A_i|\Omega)$$

which proves that axiom 2 holds. Finally, using the following:

$$b(A|B) = \frac{\sum\limits_{x \in A \cap B} r(x)}{\sum\limits_{y \in B} r(y)}, \qquad b(B|A) = \frac{\sum\limits_{x \in A \cap B} r(x)}{\sum\limits_{y \in A} r(y)}$$

$$b(A|\Omega) = \frac{\sum\limits_{x \in A} r(x)}{\sum\limits_{y \in \Omega} r(y)}, \qquad b(B|\Omega) = \frac{\sum\limits_{x \in B} r(x)}{\sum\limits_{y \in \Omega} r(y)}$$

Appendix A: Proofs IV

We can compute:

$$\frac{b(B|A)b(A|\Omega)}{b(B|\Omega)} = \frac{\left(\sum_{x\in A\cap B} r(x)\right)\left(\sum_{x\in A} r(x)\right)\left(\sum_{y\in\Omega} r(y)\right)}{\left(\sum_{y\in A} r(y)\right)\left(\sum_{y\in\Omega} r(y)\right)\left(\sum_{x\in B} r(x)\right)}$$

$$= \frac{\sum_{x\in A\cap B} r(x)}{\sum_{y\in B} r(y)} = b(A|B)$$

Appendix A: Proofs V

and thus axiom 3 holds.

2. Now, we want to show that if b satisfies axioms 1-3, then there exists $r : \Omega \to \mathbb{R}_+$ that generates b. Let b be conditional beliefs satisfying axioms 1-3, then let us define r so that $r(x) \equiv b(x|\Omega)$ for all $x \in \Omega$. We will first show that r generates unconditional beliefs $b(A|\Omega)$ by induction on the cardinality of A, and then show that it also generates all the conditional beliefs as well. Let $A \subseteq \Omega$. For $|A| = 1$, letting $x \in A$, r generates b if

$$b(A|\Omega) = \frac{r(x)}{\sum_{y \in \Omega} r(y)}$$

Appendix A: Proofs VI

by definition, we have:

$$b(A|\Omega) = \frac{r(x)}{\sum_{y\in\Omega} b(y|\Omega)}$$

applying finite additivity, we get:

$$b(A|\Omega) = \frac{r(x)}{b\left(\bigcup_{y\in\Omega}\{y\}\middle|\Omega\right)}$$
$$= \frac{r(x)}{b(\Omega|\Omega)}$$

Appendix A: Proofs VII

Using unitarity of unconditional beliefs, we have:

$$b(A|\Omega) = r(x)$$

as $b(\Omega|\Omega) = 1$

and thus r generates unconditional beliefs $b(A|\Omega)$. Now suppose that for all $A \subseteq \Omega$ with $|A| = k$, r generates $b(A|\Omega)$, that is :

$$b(A|\Omega) = \sum_{x \in A} r(x)$$

Appendix A: Proofs VIII

Now, consider a set $A' \subseteq \Omega$ of cardinality $|A'| = k + 1$, we need to show:

$$b(A'|\Omega) = \sum_{x \in A'} r(x)$$

because A' is of cardinality k, there exists $A \subseteq \Omega$ of cardinality k and $x \in \Omega$ such that $A' = A \cup \{x\}$ and by axiom 2, we have:

$$\begin{aligned} b(A'|\Omega) &= b(x|\Omega) + b(A|\Omega) \\ &= r(x) + \sum_{y \in A} r(y) \quad \text{(by our induction hypothesis)} \\ &= \sum_{y \in A'} r(x) \end{aligned}$$

Appendix A: Proofs IX

to complete the proof, we consider any conditional belief $b(A|B)$ with $|B|$ an arbitrary number. Then, by using our definition for $r(x)$ we can write:

$$\frac{\sum_{x \in A \cap B} r(x)}{\sum_{y \in B} r(y)} = \frac{\sum_{x \in A \cap B} b(x|\Omega)}{\sum_{y \in B} b(y|\Omega)} = \frac{b(A \cap B|\Omega)}{b(B|\Omega)}$$

As b satisfies axiom 3, we have:

$$\frac{\sum_{x \in A \cap B} r(x)}{\sum_{y \in B} r(y)} = b(A|B)$$

Appendix A: Proofs X

which completes the proof.

Corollary 1. r is unique up to a positive linear transformation.

Proof. Let r and $\tilde{r}$ both generate conditional beliefs b. We want to show that there exists $\alpha > 0$ such that $r(x) = \alpha \tilde{r}(x)$ for all $x \in \Omega$. To that effect, let $x \in \Omega$. Because both r and $\tilde{r}$ represent b, we have:

$$\frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{\tilde{r}(x)}{\sum_{y \in \Omega} \tilde{r}(y)} = b(x|\Omega)$$

Rearranging, we get:

$$r(x) = \frac{\sum_{y \in \Omega} r(y)}{\sum_{y \in \Omega} \tilde{r}(y)} \tilde{r}(x)$$

Appendix A: Proofs XI

Letting $\alpha = \frac{\sum_{y \in \Omega} r(y)}{\sum_{y \in \Omega} \tilde{r}(y)}$ proves the claim. **Corollary 2.** Let b be conditional beliefs generated by some recall function r. $b(x|\Omega) = m(x)/m(\Omega)$ if and only if $r(x) = \alpha m(x)$ for some $\alpha > 0$.

1. *Proof.* Assume $b(x|\Omega) = m(x)/m(\Omega)$ for all $x \in \Omega$. Then, because r generates b, we have, for all $x \in \Omega$:

$$\frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{m(x)}{m(\Omega)}$$

Equivalently, we can write:

$$r(x) = \frac{m(x)}{m(\Omega)} \sum_{y \in \Omega} r(y)$$

Appendix A: Proofs XII

which proves the first part of the claim.

2. Let us assume there exists $\alpha > 0$ such that $r(x) = \alpha m(x)$. We have:

$$b(x|\Omega) = \frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{\alpha m(x)}{\sum_{y \in \Omega} \alpha m(y)} = \frac{m(x)}{m(\Omega)}$$

- Let r and $\tilde{r}$ both generate conditional beliefs b

Appendix A: Proofs XIII

- WTS that there exists $\alpha > 0$ such that $r(x) = \alpha \tilde{r}(x)$ for all $x \in \Omega$
- Let $x \in \Omega$, as both r and $\tilde{r}$ represent b, we have:

$$\frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{\tilde{r}(x)}{\sum_{y \in \Omega} \tilde{r}(y)} = b(x|\Omega)$$

Rearranging, we get:

$$r(x) = \frac{\sum_{y \in \Omega} r(y)}{\sum_{y \in \Omega} \tilde{r}(y)} \tilde{r}(x)$$

Appendix A: Proofs XIV

- Letting $\alpha = \frac{\sum_{y \in \Omega} r(y)}{\sum_{y \in \Omega} \tilde{r}(y)}$ proves the claim.

Back to Corollary 1

1. Assume $b(x|\Omega) = m(x)/m(\Omega)$ for all $x \in \Omega$. Then, because r generates b, we have, for all $x \in \Omega$:

$$\frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{m(x)}{m(\Omega)}$$

Equivalently, we can write:

$$r(x) = \frac{m(x)}{m(\Omega)} \sum_{y \in \Omega} r(y)$$

which proves the first part of the claim.

Appendix A: Proofs XV

2. Let us assume there exists $\alpha > 0$ such that $r(x) = \alpha m(x)$. We have:

$$b(x|\Omega) = \frac{r(x)}{\sum_{y \in \Omega} r(y)} = \frac{\alpha m(x)}{\sum_{y \in \Omega} \alpha m(y)} = \frac{m(x)}{m(\Omega)}$$

Back to Corollary 2 **Proposition 1.** If p and m generate b, then b satisfies unitarity of complements, subset certainty, monotonicity in covering orders and union-preservation of ordering.

Appendix A: Proofs XVI

1. *Proof.* Let b be some conditional beliefs, and p and m be a CSF and a memory database that generate b.
2. Let us show that b satisfies unitarity of complements. Let A and B be two events. We have:

$$\begin{aligned}
b(A|B) + b(A^C|B) &= \frac{p_1(\aleph_{A,B})}{p_1(\aleph_{A,B}) + p_2(\aleph_{A,B})} + \frac{p_1(\aleph_{A^C,B})}{p_1(\aleph_{A^C,B}) + p_2(\aleph_{A^C,B})} \\
&= \frac{p_1(\aleph_{A,B}) + p_2(\aleph_{A^C,B})}{p_1(\aleph_{A,B}) + p_2(\aleph_{A^C,B})} \\
&\qquad \text{(by the properties of } p\text{)} \\
&= 1.
\end{aligned}$$

Appendix A: Proofs XVII

3. Let us show that b satisfies Subset Certainty. Let A and B be two events such that $A \subseteq B$. We have:

$$\begin{aligned} b(A|B) &= 1 - b(A^C|B) \\ &= 1 - \frac{p_1(\aleph_{A^C,B})}{p_1(\aleph_{A^C,B}) + p_2(\aleph_{A^C,B})} \\ &= 1 - 0 = 1. \qquad\qquad (\text{as } A^C \cap B = \emptyset) \end{aligned}$$

4. Let us show that b satisfies monotonicity in covering orders.
   1. Let A, A', B, and B' be events such that $A' \succeq_B A$ and $B' \succeq_A B$.
   2. First, we will show that $A \cap B \subseteq A' \cap B'$. Let $\omega \in A \cap B$. Because $A \subseteq A'$, $\omega \in A'$. Moreover, because $A' \setminus A \subseteq B$, we must have $\omega \in B$ and so, $\omega \in B'$.

Appendix A: Proofs XVIII

3. Second, we want to show that $A'^C \cap B' \subseteq A^C \cap B$. Let $\omega \in A'^C \cap B'$. This implies that $\omega \notin A'$ and thus $\omega \notin A$. On the other hand, $\omega \in B'$, and so, by $B' \succeq_A B$, $\omega \notin B' \setminus B$, otherwise $B' \succeq_A B$ would imply $\omega \in A$, a contradiction. But then we must have $\omega \in B$.
4. Third, we need to show that $A' \cap B'^C \subseteq A \cap B^C$. Let $\omega \in A' \cap B'^C$. By $B \subseteq B'$, $\omega \in B'^C$ implies $\omega \in B^C$. But then, we have that $\omega \in A'$ and $\omega \notin B$, and so $\omega \in A$, otherwise a contradiction with $A' \setminus A \subseteq B$ would arise.
5. Fourth, let us show what $A'^C \cap B'^C \subseteq A^C \cap B^C$. Let $\omega \in A'^C \cap B'^C$. For the sake of contradiction, assume $\omega \notin A^C \cap B^C$. If $\omega \notin A^C$, then $\omega \in A$. But then, $\omega \in A$ and $\omega \notin A'$, a contradiction. Alternatively, if $\omega \notin B^C$, this implies that $\omega \notin B$, a contradiction with $\omega \notin B$. Therefore, $\omega \in A^C \cap B^C$

Appendix A: Proofs XIX

6. From the four previous points, we can conclude:

$$m_{A\cap B} \leq m_{A'\cap B'}$$
$$m_{A^c\cap B} \geq m_{A'^c\cap B'}$$
$$m_{A\cap B^c} \geq m_{A'\cap B'^c}$$
$$m_{A^c\cap B^c} \geq m_{A'^c\cap B'^c}$$

7. Therefore, by the properties of p, it must be that:

$$p_1(\aleph_{A,B}) \leq p_1(\aleph_{A',B'}).$$

Or, in other words, $b(A|B) \leq b(A'|B')$.

5. Let us show that b satisfies union-preservation of ordering.
   1. Let A, A', B and C be events such that $A, A' \subseteq B$, $C \subseteq B \setminus (A \cup A')$.

Appendix A: Proofs XX

2. Because $A, A' \subseteq B$, we have that $A \cap B^C = A' \cap B^C = \emptyset$, and $A^C \cap B^C = A'^C \cap B^C = B^C$.
3. Therefore, $b(A|B) \leq b(A'|B)$ implies:

$$p(m_{A\cap B}, m_{A^C\cap B}, 0, m_{B^C}) \leq p(m_{A'\cap B}, m_{A'^C\cap B}, 0, m_{B^C})$$

4. But then, by the properties of p, this implies:

$$m(A \cap B) \leq m(A' \cap B) \tag{1}$$

$$m(A^C \cap B) \geq m(A'^C \cap B) \tag{2}$$

5. In turn, because $C \subseteq B \setminus (A \cup A')$, this implies:

$$m((A \cup C) \cap B) \leq m((A' \cup C) \cap B) \tag{3}$$

$$m((A \cup C)^C \cap B) \geq m((A' \cup C)^C \cap B) \tag{4}$$

$$m((A \cup C) \cap B^C) = m((A \cup C) \cap B^C) = 0 \tag{5}$$

$$m((A \cup C)^C \cap B^C) = m((A \cup C)^C \cap B^C) = m(B^C) \tag{6}$$

Appendix A: Proofs XXI

6. And so, again by property of p, we have that:

$$b(A \cup C|B) \leq b(A' \cup C|B)$$

as desired.

Appendix B: Characteristic Overreaction I

Following the definition of diagnostic expectations from Bordalo et al. (2018), we write a more general formulation called "characteristic overreaction".

Definition 8 (Characteristic Event)

An event $A \in \Sigma$ is characteristic of $B \in \Sigma^*$ if

$$\frac{m(A \cap B)}{m(B)} > \frac{m(A \cap (B^C))}{m(B^C)}$$

Definition 9 (Overreaction)

A conditional belief $b(A|B)$ is an overreaction with respect to m if

$$b(A|B) > \frac{m(A \cap B)}{m(B)}$$

Appendix B: Characteristic Overreaction II

Definition 10 (Characteristic Overreaction)

A conditional belief function $b : \Sigma \times \Sigma^* \to [0, 1]$ exhibits characteristic overreaction with respect to some memory database m, if, for all $A \in \Sigma$ and $B \in \Sigma^*$, the following is true:

$$\frac{m(A \cap B)}{m(B)} > \frac{m(A \cap (B^C))}{m(B^C)} \Rightarrow b(A|B) > \frac{m(A \cap B)}{m(B)}$$

Proposition 2

If an agent is subjective Bayesian, then they cannot display overreaction to characteristic events.

1. *Proof.* Let A and B be two events such that A is characteristic of B, and $B \neq \emptyset$ and $A \neq B$.

Appendix B: Characteristic Overreaction III

2. Because the DM is Bayesian, we have that: $b(A|B) = b(A \cap B|B)$.
3. For the DM to display characteristic over-reaction, it must be that $b(A|B) > p(A|B)$.
4. Consider event $C = A \cup B^C$. C cannot be characteristic of B, because $p(C|B) = p(A \cup B|B) < 1$ and $p(C|B^C) = p(B^C|B^C) = 1$.
5. But then because the DM is Bayesian, $b(C|B) = b(A \cap B|B) > p(A|B) = p(C|B)$, which implies that the DM is overreacting to C!
6. This implies that the DM is not overreacting to C^C given B, which is a characteristic event.
7. Therefore, the DM cannot display overreaction to characteristic events.

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